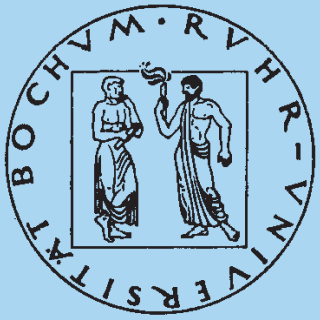

RUHR-UNIVERSITÄT BOCHUM

Wojciech Pietraszkiewicz

Introduction to the Non-Linear Theory of Shells

Heft Nr. 10

Mitteilungen
aus dem
Institut für Mechanik

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Zusammenfassung

Die vorliegende Arbeit wurde vom Autor im Wintersemester 1976/77 an der Ruhr-Universität Bochum als Vorlesung über nichtlineare Schalentheorie vorgetragen.

Kapitel 1 gibt eine Einführung in die Grundlagen der Differentialgeometrie der Flächen und der Kurven auf Flächen. Kapitel 2 beinhaltet eine exakte dreidimensionale Theorie der Flächenverformung. In Kapitel 3 wird die Verformung einer Schale unter Zugrundelegung der Kirchhoff-Love'schen Hypothese behandelt. Es wird die Struktur verschiedener Verformungstensoren analysiert und eine Theorie endlicher Rotationen für Schalenprobleme entwickelt. Die Anwendung dieser Theorie auf Elemente der Randkurve erlaubt im Kapitel 4 die Formulierung dreier verschiedener Typen von geometrischen Randbedingungen. In Kapitel 5 werden auf der Grundlage des im zweidimensionalen angewandten Prinzips der virtuellen Arbeit oder mit Hilfe direkter Integration von Gleichungen der dreidimensionalen Kontinuumsmechanik verschiedene Formen von Gleichgewichtsbedingungen und natürlichen Randbedingungen behandelt. In den beiden letzten Kapiteln werden für homogenes, isotropes, elastisches Material verschiedene vereinfachte Formen der nichtlinearen Schalen- und Plattentheorie diskutiert. Diese Beziehungen werden unter der Voraussetzung kleiner Dehnungen oder mit Hilfe zusätzlicher Vereinfachungen hergeleitet.

Summary

These are lecture notes on the non-linear shell theory delivered by the author during AY 1976/77 at the Ruhr-Universität Bochum.

An elementary introduction to differential geometry of a surface and surface curves is given in chapter 1. An exact theory of deformation of a surface in three-dimensional Euclidean space is presented in chapter 2. In chapter 3 the shell deformation is discussed under the Kirchhoff-Love constraints. The structure of various strain tensors is analyzed and the theory of finite rotations in shells is developed. The total finite rotations of the boundary material elements described in chapter 4 allow to construct three types of geometric boundary conditions. In chapter 5 various forms of equilibrium equations and natural boundary conditions are discussed either on the basis of a two-dimen-

sional virtual work principle or using direct integration of three-dimensional equations of continuum mechanics.

Various reduced forms of the non-linear relations of shells and plates are discussed in the last two chapters for homogeneous isotropic elastic material. These relations are derived either within the small strain theory or with additionally restricted rotations.

Streszczenie

Praca zawiera notatki wykładów z nieliniowej teorii powłok prowadzonych przez autora w RA 1976/77 w Uniwersytecie w Bochum(RFN).

Rozdział 1 zawiera elementarny wstęp do geometrii rożniczkowej powierzchni oraz geometrii krzywych na powierzchni. W rozdziale 2 podano ścisłą teorię deformacji powierzchni w trójwymiarowej przestrzeni Euklidesowej. W rozdziale 3 rozważono deformację powłoki z więzami typu Kirchhofa - Love'a , przedyskutowano strukturę różnych tensorów odkształcenia oraz podano teorię obrotów skończonych w powłoce. Całkowity obrót skończony elementu materialnego brzegu powłoki zbudowany został w rozdziale 4. Umożliwił on skonstruowanie trzech typów geometrycznych warunków brzegowych. Różne postacie równań równowagi i naturalnych warunków brzegowych zostały wyprowadzone w rozdziale 5, zarówno przy pomocy dwuwymiarowej zasady prac wirtualnych jak i poprzez bezpośrednie całkowanie równań trójwymiarowych mechaniki ośrodka ciągłego.

Różne uproszczone postacie zależności nieliniowych dla powłok i płyt przedyskutowano w dwóch ostatnich rozdziałach dla jednorodnego izotropowego materiału sprężystego. Zależności uproszczone wyprowadzono zarówno przy założeniu małych odkształceń jak i przy dodatkowym ograniczeniu obrotów elementów materialnych powłoki.

PREFACE

The rapid development of computer-oriented solution techniques makes it possible to calculate complex thin-walled structures with an appropriate accuracy. It is currently recongnized that the non-linear effects play an important role in the rational and economic design of modern thin shell structures and have to be taken into account. These effects may be associated with large displacements and rotations, non-linear material behaviour, loss of stability , dependence of external forces on deformation, non-classical boundary conditions etc.

The literature of the subject is growing rapidly, particularly in so far as numerical solutions of various non-linear shell problems are concerned. Development of various aspects of the non-linear theory of shells has been presented, for example, in the works of MUSHTARI and GALIMOV [34], KOITER [23,6], WOŹNIAK [37], NAGHDI [4,5], KOITER and SIMMONDS [22], GALIMOV [7], WEMPNER [38] and the author [9,12] where additional references may be found.

Here we shall discuss the non-linear relations which govern static problems of thin shells. It is assumed that the behaviour of a thin shell can be described with a sufficient accuracy by the behaviour of the shell middle surface. This is accomplished by imposing the Kirchhoff-Love constraints on a deformation of the shell.

An exact theory of deformation of a surface in three-dimensional Euclidean space is presented in chapter 2. In chapter 3 the shell deformation is decomposed into a rigid-body translation, a pure stretch along principal directions of strain followed by a rigid-body rotation of the principal directions. The structure of various strain tensors and of the finite rotation tensor is discussed. An equivallent description of the rotations by means of a finite rotation vector is developed. The total finite rotations of the boundary material elements are analysed in chapter 4. This allows us to construct three types of geometric boundary conditions: in terms of displacements, in terms of the total finite rotation vector and the elongation of the boundary contour as well as in terms of four combinations of the strain measures at the boundary. In chapter 5 various forms of equilibrium equations and natural boundary conditions are discussed. In their derivation either the two-dimensional virtual work principle or direct integration of three-dimensional equations of continuum mechanics over

the shell thickness are used. Kinematics of deformation and various shell relations presented in chapters 3 to 5 are exact under K - L constraints.

Various reduced forms of the non-linear equations are discussed in the last two chapters for homogeneous isotropic elastic shells. These relations are derived either within the small strain theory or with additionally restricted rotations. Under small elastic strains the change of the shell thickness during deformation has also been taken into consideration by the proper formulation of the constitutive equations.

A clear distinction shall be made between Lagrangean and Eulerian descriptions. Roughly speaking, in the Lagrangean description all quantities and equations are related to the known reference (or undeformed) shell configuration, while in the Eulerian description they are related to the unknown deformed state. In contrast to most of general works on the non-linear shell theory, attention is focused here primarily on the Lagrangean shell equations developed in the author's papers [10,11,12]. In order to stress the important differences between these two descriptions we use absolute tensor notation in definitions of strain and stress measures of the shell and in presentation of the various shell equations.

The importance of the finite rotations in any theory of thin bodies has long been recognized. Unfortunately, the literature is not free from confusion about the analytical representation of the rotations. There are many works in which the linearized rotations or displacement gradients are used, apparently on the intuitive grounds, to describe finite rotations of the shell material elements also within the non-linear range of deformation. A general theory of finite rotations in shells has been developed recently in the author's thesis [12]. This theory is exact at the shell middle surface. Here we develope independently the theory of finite rotations in shells subject to K - L constraints, which is a particular case of the theory given in [12]. The notion of a finite rotation happens to be extremely helpful in deriving the new variants of geometrical boundary conditions, in obtaining various modified shell equations as well as in providing a clear and consistent classification of various approximate variants of geometrically non-linear theory of shells and plates.

Most of the non-linear shell relations are derived here in detail in order to make the subject self-contained and easy to follow for a reader with some mathematical training. For this reason we found it necessary

to provide the reader with an elementary introduction to differential geometry of a surface and surface curves. References in the text are primarily to those original papers which have been used in preparation of this work or have influenced the author's research, the results of which are presented here.

This work has been prepared as lecture notes during the Academic Year 1976/77, when the author had been Visiting Professor at the Ruhr-Universität Bochum. The lectures were delivered to a group of assistents and research staff of the Institut für Mechanik and the Institut für Konstruktiven Ingenieurbau RUB. The author would like to take this opportunity to thank Professor H. Stumpf, Professor W. B. Krätzig and the other participants in the seminars for stimulating discussions, Professor Wempner for his comments on chapter 1 to 5 of the manuscript, Dipl.-Ing. R. Schmidt for checking most of the formulae and Frau Mönikes for an outstanding job in typewriting.

The author would welcome any critical remarks on the subject presented here, which may be sent to the following address: Instytut Maszyn Przepływowych PAN, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland.

Bochum, April 1977 W. P.

CONTENTS

Chapter 1

ELEMENTS OF DIFFERENTIAL GEOMETRY OF A SURFACE

Differential geometry of a surface has been developed in detail in many mathematical monographs. The most extensive treatise on the subject was published by KAGAN [1]. An excellent introduction to differential geometry of a surface, within the scope necessary in shell theory, may be found in the books of GREEN and ZERNA [2] and CHERNYKH [3]. Some useful relations may also be found in works of NAGHDI [4,5], KOITER [6], GALIMOV [7] and the author [8].

Here we present an elementary introduction to differential geometry of a surface and surface curves. We begin by introducing the surface base vectors and components of the metric, alternation and curvature tensors. Surface vectors and tensors are defined in absolute notation as elements of some linear space. Then the rules of covariant differentiation at the surface, the role of the Riemann-Christoffel tensor and the Codazzi-Gauss equations are discussed. The notion of physical components allow us to derive some useful geometrical relations for a surface curve, differentiation rules along the curve and in the direction normal to the curve, as well as to relate the results to the geometry of a curve in space. Finally, the lines of principal curvatures at the surface and physical components of various coordinate lines are discussed.

All geometrical relations collected here will be used in the remaining parts of these notes. The reader already familiar with differential geometry of a surface may omit this chapter and use the formulae for reference purposes only.

1.1. Geometry of a surface

A *surface* M in three-dimensional Euclidean point space E can be defined by three scalar functions of two parameters

$$x^1 = f^1(\theta^1,\theta^2) \quad , \quad x^2 = f^2(\theta^1,\theta^2) \quad , \quad x^3 = f^3(\theta^1,\theta^2) \qquad (1.1.1)$$

or one vector function, Fig. 1,

$$\underset{\sim}{r} = \overrightarrow{OM} = \underset{\sim}{f}(\theta^1,\theta^2) = \sum_{k=1}^{3} f^k(\theta^1,\theta^2)\, i_k \equiv x^k \underset{\sim}{i}_k \qquad (1.1.2)$$

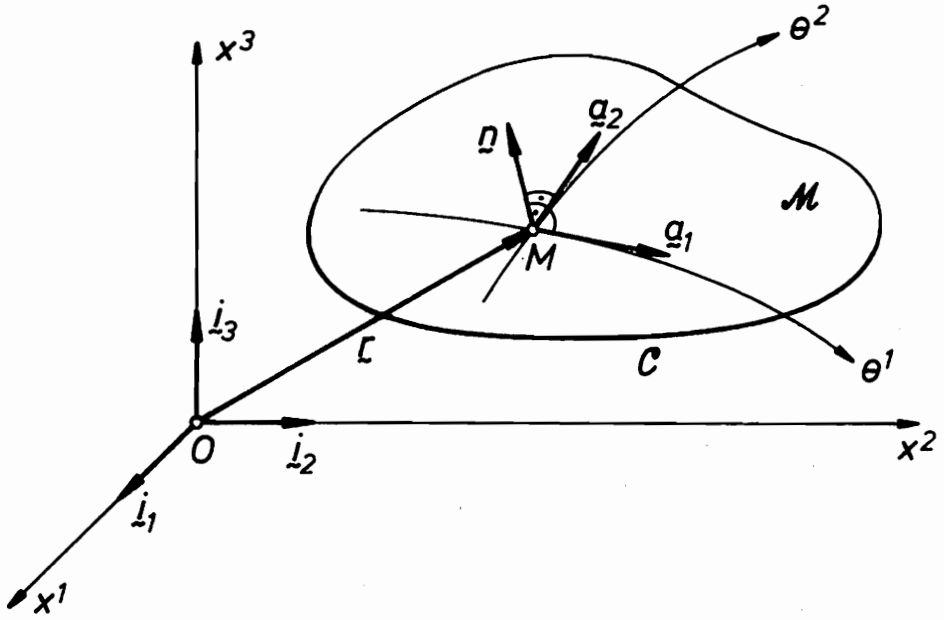

Fig. 1

where the *summation convention* over the repeated index k has been used. The vector $\underset{\sim}{r}$ is known as the *position vector* of the surface points $M \in \mathcal{M}$, while x^k (k = 1,2,3) and θ^α (α = 1,2) are called *cartesian* and *curvilinear* Gaussian coordinates of the surface, respectively.

Two vectors, tangent to the surface at M, and defined by

$$\underset{\sim}{a}_\alpha = \frac{\partial \underset{\sim}{f}(\theta^\beta)}{\partial \theta^\alpha} \equiv \underset{\sim}{r}_{,\alpha} = x^k{}_{,\alpha} \underset{\sim}{i}_k \qquad (1.1.3)$$

are called the *covariant base vectors* of the surface, or shortly $\underset{\sim}{a}_\alpha$ - basis.

The vector $\underset{\sim}{n}$ defined by

$$\underset{\sim}{n} = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} = n^k \underset{\sim}{i}_k \qquad (1.1.4)$$

is called the *unit normal* to the surface, since

$$\underset{\sim}{n} \cdot \underset{\sim}{n} = 1 \quad , \quad \underset{\sim}{n} \cdot \underset{\sim}{a}_\alpha = 0 \qquad (1.1.5)$$

The vector product × in (1.1.4) is taken according to the right-hand rule.

The base vectors $\underset{\sim}{a}_\alpha$ define at $M \in \mathcal{M}$ a two-dimensional linear vector space V in which we can introduce the *contravariant base vectors* $\underset{\sim}{a}^\alpha$, dual to those of $\underset{\sim}{a}_\alpha$, by the relations:

$$\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = \delta^\alpha_\beta \tag{1.1.6}$$

where

$$\delta^\alpha_\beta = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases} . \tag{1.1.7}$$

It is easy to see that the vectors $\underset{\sim}{a}^\alpha$ satisfying the equations (1.1.6) have the form, Fig. 2,

$$\underset{\sim}{a}^1 = \frac{\underset{\sim}{a}_2 \times \underset{\sim}{n}}{(\underset{\sim}{a}_1 \times \underset{\sim}{a}_2) \cdot \underset{\sim}{n}} \quad , \quad \underset{\sim}{a}^2 = \frac{\underset{\sim}{n} \times \underset{\sim}{a}_1}{(\underset{\sim}{a}_1 \times \underset{\sim}{a}_2) \cdot \underset{\sim}{n}} \tag{1.1.8}$$

The coefficients defined by

$$a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta = x^k{}_{,\alpha} x^l{}_{,\beta} \delta_{kl} \tag{1.1.9}$$

are known as the *covariant* components of the surface *metric tensor*. The *contravariant* components $a^{\alpha\beta}$ of the surface *metric tensor* may be obtained from the following set of linear algebraic equations:

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\beta \tag{1.1.10}$$

According to (1.1.10),

$$a^{11} = \frac{a_{22}}{a} \quad , \quad a^{12} = a^{21} = -\frac{a_{12}}{a} \quad , \quad a^{22} = \frac{a_{11}}{a}$$
$$a = |a_{\alpha\beta}| = a_{11} a_{22} - (a_{12})^2 \quad . \tag{1.1.11}$$

It follows now from (1.1.6), (1.1.8) and (1.1.10) that

$$\underset{\sim}{a}^\alpha = a^{\alpha\beta} \underset{\sim}{a}_\beta = \frac{\partial \theta^\alpha}{\partial x^k} \underset{\sim}{i}^k$$
$$a^{\alpha\beta} = \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta \quad , \quad \underset{\sim}{a}_\alpha = a_{\alpha\beta} \underset{\sim}{a}^\beta \quad . \tag{1.1.12}$$

These equations relate the covariant and contravariant basic vectors.

In various geometrical formulae it is convenient to make use of surface quantities

$$\epsilon_{\alpha\beta} = (\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta)\cdot\underset{\sim}{n} \quad , \quad \epsilon^{\alpha\beta} = (\underset{\sim}{a}^\alpha \times \underset{\sim}{a}^\beta)\cdot\underset{\sim}{n} \tag{1.1.13}$$

which are known as *covariant* and *contravariant* components of a surface *alternation tensor*, respectively.

Using some algebraic transformations it is shown that

$$[(\underset{\sim}{a}_1 \times \underset{\sim}{a}_2)\cdot\underset{\sim}{n}]^2 = \begin{vmatrix} \underset{\sim}{a}_1\cdot\underset{\sim}{a}_1 & \underset{\sim}{a}_1\cdot\underset{\sim}{a}_2 & \underset{\sim}{a}_1\cdot\underset{\sim}{n} \\ \underset{\sim}{a}_2\cdot\underset{\sim}{a}_1 & \underset{\sim}{a}_2\cdot\underset{\sim}{a}_2 & \underset{\sim}{a}_2\cdot\underset{\sim}{n} \\ \underset{\sim}{n}\cdot\underset{\sim}{a}_1 & \underset{\sim}{n}\cdot\underset{\sim}{a}_2 & \underset{\sim}{n}\cdot\underset{\sim}{n} \end{vmatrix} = a > 0 \tag{1.1.14}$$

and for the alternation tensor components we obtain

$$\begin{aligned} &\epsilon_{12} = -\epsilon_{21} = +\sqrt{a} \quad , \quad \epsilon_{11} = \epsilon_{22} = 0 \\ &\epsilon^{12} = -\epsilon^{21} = +\frac{1}{\sqrt{a}} \quad , \quad \epsilon^{11} = \epsilon^{22} = 0 \end{aligned} \tag{1.1.15}$$

or

$$\epsilon_{\alpha\beta} = \sqrt{a}\, e_{\alpha\beta} \quad , \quad \epsilon^{\alpha\beta} = \frac{1}{\sqrt{a}}\, e^{\alpha\beta} \tag{1.1.16}$$

where

$$e_{\alpha\beta} = e^{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = 1, \beta = 2 \\ -1 \text{ if } \alpha = 2, \beta = 1 \\ 0 \text{ if } \alpha = \beta \end{cases} . \tag{1.1.17}$$

With the help of $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ it is possible to derive many identities and formulae, for example,

$$\begin{cases} \epsilon_{\alpha\beta}\epsilon^{\lambda\mu} = \delta^\lambda_\alpha \delta^\mu_\beta - \delta^\lambda_\beta \delta^\mu_\alpha \\ \epsilon^{\alpha\beta}\epsilon_{\alpha\lambda} = \delta^\beta_\lambda \quad , \quad \epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2 \end{cases} \tag{1.1.18}$$

$$\begin{cases} \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}a_{\alpha\beta} = a^{\lambda\mu} \quad , \quad \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}a^{\alpha\beta} = a_{\lambda\mu} \\ a^{\alpha\lambda}a^{\beta\mu}\epsilon_{\alpha\beta} = \epsilon^{\lambda\mu} \quad , \quad a_{\alpha\lambda}a_{\beta\mu}\epsilon^{\alpha\beta} = \epsilon_{\lambda\mu} \end{cases} \tag{1.1.19}$$

$$\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta = \epsilon_{\alpha\beta}\underset{\sim}{n} \quad , \quad \underset{\sim}{a}^\alpha \times \underset{\sim}{a}^\beta = \epsilon^{\alpha\beta}\underset{\sim}{n}$$

$$\underset{\sim}{n} \times \underset{\sim}{a}_\alpha = \epsilon_{\alpha\beta}\underset{\sim}{a}^\beta \quad , \quad \underset{\sim}{n} \times \underset{\sim}{a}^\alpha = \epsilon^{\alpha\beta}\underset{\sim}{a}_\beta \quad . \tag{1.1.20}$$

$$\underset{\sim}{n} = \frac{1}{2}\epsilon^{\alpha\beta}\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\underset{\sim}{a}^\alpha \times \underset{\sim}{a}^\beta$$

These formulae are very useful in general discussion of various geometrical relations at a surface.

Differentiating the unit normal $\underset{\sim}{n}$ with respect to surface coordinates, we obtain two vectors tangent to the surface at M, $\underset{\sim}{n}_{,\alpha} \in V$. The coefficients defined by

$$b_{\alpha\beta} = -\underset{\sim}{n}_{,\alpha} \cdot \underset{\sim}{a}_\beta = -\underset{\sim}{n}_{,\beta} \cdot \underset{\sim}{a}_\alpha = \underset{\sim}{n} \cdot \underset{\sim}{a}_{\alpha,\beta} \tag{1.1.21}$$

are known as the *covariant* components of a surface *curvature tensor*. Associated with them are two invariants:

$$K = \frac{1}{2}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}b_{\alpha\beta}b_{\lambda\mu} = \frac{b}{a} \quad , \quad b = |b_{\alpha\beta}| \tag{1.1.22}$$

$$H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} \quad . \tag{1.1.23}$$

These are called the *Gaussian* and *mean curvatures*, respectively.

The metric tensor components allow us to calculate lengths, angles and areas at the surface. The curvature tensor components allow us to calculate curvatures and torsions of curves at the surface. The geometrical meaning of $a_{\alpha\beta}$ and $b_{\alpha\beta}$ will be discussed in more detail in § 1.4. and § 1.6.

1.2. Surface vectors and tensors

Any surface vector $\underset{\sim}{v} \in V$ may be represented by its components in $\underset{\sim}{a}_\alpha$ or $\underset{\sim}{a}^\alpha$ -bases, respectively, according to linear relations

$$\underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha = v_\alpha \underset{\sim}{a}^\alpha$$
$$v^\alpha = \underset{\sim}{v} \cdot \underset{\sim}{a}^\alpha, \; v_\alpha = \underset{\sim}{v} \cdot \underset{\sim}{a}_\alpha \tag{1.2.1}$$

where v^α and v_α are called *contravariant* and *covariant* components of the surface vector $\underset{\sim}{v}$, respectively. From (1.1.12) we obtain

$$v^\alpha = a^{\alpha\beta}v_\beta \; , \quad v_\alpha = a_{\alpha\beta}v^\beta \tag{1.2.2}$$

what establishes a law for raising and lowering of indices of the vector components. The physical meaning of these components is obvious from Fig. 2.

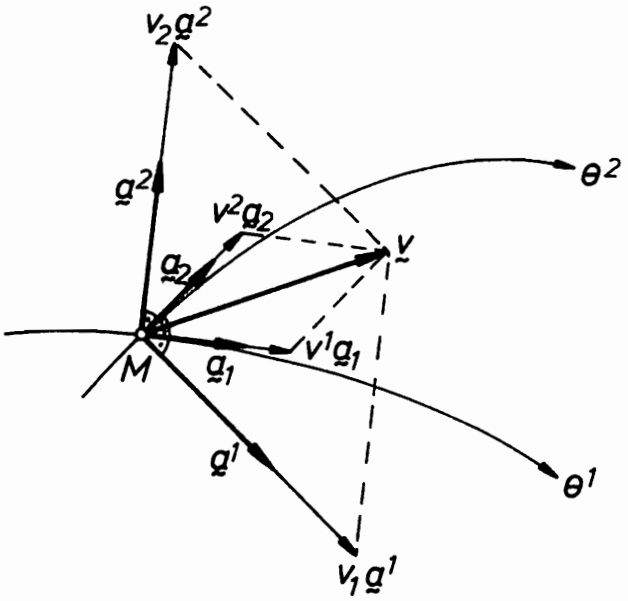

Fig. 2

We shall often have to deal with physical or geometrical quantities which at each surface point M may be represented by more than two of their scalar components. To discuss them let us consider the *tensor product* $\otimes$ of two vectors $\underset{\sim}{u},\underset{\sim}{v} \in V$ to be a linear transformation $\underset{\sim}{L} \equiv \underset{\sim}{u} \otimes \underset{\sim}{v}$, $\underset{\sim}{L} : V \to V$ such that

$$\underset{\sim}{L}\underset{\sim}{w} = (\underset{\sim}{u} \otimes \underset{\sim}{v})\underset{\sim}{w} = \underset{\sim}{u}(\underset{\sim}{v} \cdot \underset{\sim}{w}) \tag{1.2.3}$$

for any $w \in V$.

All such linear transformations and their linear combinations form a linear space $T_2 \equiv V \otimes V$ called the surface *tensor space* of the second order. Elements of the space $\underset{\sim}{T} \in T_2$ are called *surface tensors of the second order*.

It follows from (1.2.1) and (1.2.3) that

$$\begin{aligned} \underset{\sim}{L}\underset{\sim}{w} &= (u^\alpha \underset{\sim}{a}_\alpha \otimes v^\beta \underset{\sim}{a}_\beta) w^\lambda \underset{\sim}{a}_\lambda = u^\alpha \underset{\sim}{a}_\alpha (v^\beta w^\lambda a_{\beta\lambda}) \\ &= u^\alpha v^\beta w_\beta \underset{\sim}{a}_\alpha \end{aligned} \tag{1.2.4}$$

and we see that $L^{\alpha\beta} = u^\alpha v^\beta$ play the role of *components* of $\underset{\sim}{L}$ with respect to tensor basis $\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$.

Expressing vectors in (1.2.3) in terms of other components and basic vectors, different equivallent representations of the linear transformation $\underset{\sim}{L}$ may be found. In general, an arbitrary tensor $\underset{\sim}{T} \in \mathcal{T}_2$ can be represented by various linear combinations

$$\begin{aligned} \underset{\sim}{T} &= T^{\alpha\beta}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta = T_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = \\ &= T^{\alpha}_{\cdot\beta}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\beta = T^{\cdot\beta}_{\alpha}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\beta \end{aligned} \tag{1.2.5}$$

where $a_\alpha \otimes a_\beta$, $a^\alpha \otimes a^\beta$ and $a_\alpha \otimes a^\beta$, $a_\alpha \otimes a^\beta$ are the *covariant, contravariant and mixed base tensors*, respectively, and $T^{\alpha\beta}$, $T_{\alpha\beta}$, $T^{\alpha}_{\cdot\beta}$, $T^{\cdot\beta}_{\alpha}$ are contravariant, covariant and mixed *components* of $\underset{\sim}{T}$ with respect to the appropriate base tensors, respectively.

Let $\underset{\sim}{T}, \underset{\sim}{S} \in \mathcal{T}_2$. In what follows we shall use the following tensor operations:

transposition $\quad \underset{\sim}{T}^T = T^{\alpha\beta}\underset{\sim}{a}_\beta \otimes \underset{\sim}{a}_\alpha = T^{\beta\alpha}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$

contraction $\quad \mathrm{tr}\underset{\sim}{T} = T^{\alpha}_{\cdot\alpha}$

addition $\quad \underset{\sim}{T} \pm \underset{\sim}{S} = (T^{\alpha\beta} \pm S^{\alpha\beta})\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$ (1.2.6)

multiplication $\quad \underset{\sim}{T}\underset{\sim}{S} = T^{\alpha\lambda}S^{\cdot\beta}_{\lambda}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$

scalar product $\quad \underset{\sim}{T} \cdot \underset{\sim}{S} = T^{\alpha\beta}S_{\alpha\beta}$

A tensor $\underset{\sim}{T} \in \mathcal{T}_2$ for which $\underset{\sim}{T} = \underset{\sim}{T}^T$ or $\underset{\sim}{T} = -\underset{\sim}{T}^T$ is called *symmetric* or *skew-symmetric*, respectively.

Any tensor $\underset{\sim}{T} \in \mathcal{T}_2$ can be decomposed uniquely into its symmetric and skew-symmetric part

$$\begin{aligned} \underset{\sim}{T} &= (T^{(\alpha\beta)} + T^{[\alpha\beta]})\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta \\ T^{(\alpha\beta)} &= \frac{1}{2}(T^{\alpha\beta} + T^{\beta\alpha}) \;, \quad T^{[\alpha\beta]} = \frac{1}{2}(T^{\alpha\beta} - T^{\beta\alpha}) \;. \end{aligned} \tag{1.2.7}$$

In exactly the same way tensors of higher-order may be discussed. For example, linear transformation $\underset{\sim}{T} \otimes \underset{\sim}{v} : \mathcal{V} \to \mathcal{T}_2$ such that for any $\underset{\sim}{w} \in \mathcal{V}$

$$(\underset{\sim}{T} \otimes \underset{\sim}{v})\,\underset{\sim}{w} = \underset{\sim}{T}(\underset{\sim}{v} \cdot \underset{\sim}{w}) \tag{1.2.8}$$

is an element of a surface tensor space of the third order, etc.

Within this absolute notation the *metric tensor* and the *curvature tensor* may be defined by

$$\underset{\sim}{a} = a_{\alpha\beta}\underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} = a^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta}$$
$$\underset{\sim}{b} = b_{\alpha\beta}\underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} \qquad (1.2.9)$$

and it is evident from (1.1.9), (1.1.12) and (1.1.21) that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are *symmetric*. Similarily, the *alternation tensor* have the form

$$\underset{\sim}{\epsilon} = \epsilon_{\alpha\beta}\underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} = \epsilon^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta} \qquad (1.2.10)$$

and is *skew-symmetric*, which follows from (1.1.13).

The geometry of a surface may be developed entirely in this absolute tensor notation, although in modern works on shell theory the component notation is used almost exclusively. Still, all formulae are independent of surface coordinate system θ^{α} specified in (1.1.1). Thus if $\theta'^{\alpha} \equiv (\theta'^{1}, \theta'^{2})$ is another system of curvilinear surface coordinates, such that transformations

$$\theta'^{\alpha} = \theta'^{\alpha}(\theta^{1}, \theta^{2}) \quad , \quad \theta^{\beta} = \theta^{\beta}(\theta'^{1}, \theta'^{2}) \qquad (1.2.11)$$

are single-valued and reversible, then at the same point $M \in \mathcal{M}$ we obtain, Fig. 3,

$$\underset{\sim}{a}'_{\alpha} = \frac{\partial}{\partial\theta'^{\alpha}}\,\underset{\sim}{r}[\theta^{\beta}(\theta'^{\alpha})] = \frac{\partial\theta^{\beta}}{\partial\theta'^{\alpha}}\,\underset{\sim}{a}_{\beta} \qquad (1.2.12)$$

$$d\theta'^{\alpha} = \frac{\partial\theta'^{\alpha}}{\partial\theta^{\beta}}\,d\theta^{\beta} \qquad (1.2.13)$$

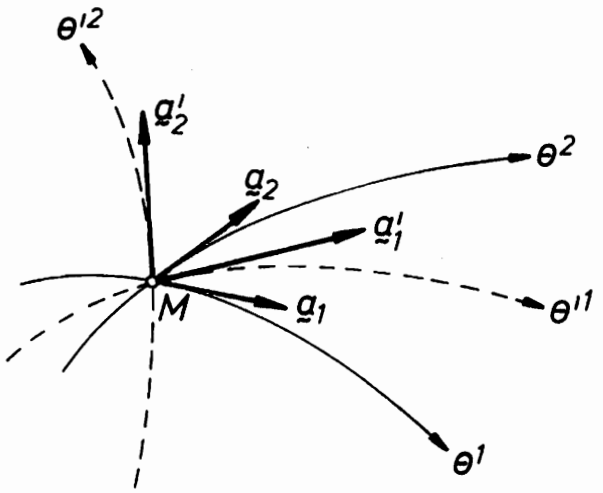

Fig. 3

The transformation of the type of (1.2.12) is called *covariant*, while of the type of (1.2.13) - *contravariant*. Transformations (1.2.12) and (1.2.13) are linear ones, while that of (1.2.11) may not be linear, in general.

In what follows, position of index (subscript or superscript) shows us the transformation rule for the quantity under a change of the surface coordinate system (1.2.11). For example, the transformation rule of basic vectors $\underset{\sim}{a}^{\alpha}$ is of contravariant type

$$\underset{\sim}{a}'^{\alpha} = \frac{\partial \theta'^{\alpha}}{\partial \theta^{\beta}} \underset{\sim}{a}^{\beta} \tag{1.2.14}$$

which is verified directly from (1.1.6) and (1.2.12)

$$\underset{\sim}{a}'^{\alpha} \cdot \underset{\sim}{a}'_{\lambda} = \frac{\partial \theta'^{\alpha}}{\partial \theta^{\beta}} \frac{\partial \theta^{\mu}}{\partial \theta'^{\lambda}} \underset{\sim}{a}^{\beta} \cdot \underset{\sim}{a}_{\mu} =$$

$$= \frac{\partial \theta'^{\alpha}}{\partial \theta'^{\lambda}} = \delta^{\alpha}_{\lambda} \tag{1.2.15}$$

The transformation rules for various components of metric and alternation tensors $\underset{\sim}{a}$ and $\underset{\sim}{\epsilon}$ follow from the position of their indices as well, which may be verified by similar direct transformations. For any scalar field, $\alpha : M \to R$, we obtain

$$\alpha = \alpha(M) = \alpha(\theta^{\alpha}) = \alpha'(\theta'^{\alpha}) \tag{1.2.16}$$

For any vector field, $\underset{\sim}{v} : M \to V$, from (1.2.1) we obtain

$$\begin{aligned} \underset{\sim}{v} = \underset{\sim}{v}(M) &= v^{\beta} \underset{\sim}{a}_{\beta} = v'^{\alpha} \underset{\sim}{a}'_{\alpha} \\ &= v_{\beta} \underset{\sim}{a}^{\beta} = v'_{\alpha} \underset{\sim}{a}'^{\alpha} \end{aligned} \tag{1.2.17}$$

and it follows from (1.2.12) and (1.2.14) that

$$v'^{\alpha} = \frac{\partial \theta'^{\alpha}}{\partial \theta^{\beta}} v^{\beta} \quad , \quad v'_{\alpha} = \frac{\partial \theta^{\beta}}{\partial \theta'^{\alpha}} v_{\beta} \tag{1.2.18}$$

For any tensor field of the second order, $\underset{\sim}{T} : M \to T_2$, we obtain

$$\underset{\sim}{T} = \underset{\sim}{T}(M) = T^{\lambda\mu} \underset{\sim}{a}_{\lambda} \otimes a_{\mu} = T'^{\alpha\beta} \underset{\sim}{a}'_{\alpha} \otimes \underset{\sim}{a}'_{\beta} \tag{1.2.19}$$

from which it follows that

$$T'^{\alpha\beta} = \frac{\partial\theta'^{\alpha}}{\partial\theta^{\lambda}} \frac{\partial\theta'^{\beta}}{\partial\theta^{\mu}} T^{\lambda\mu} \tag{1.2.20}$$

and in a similar way

$$T'_{\alpha\beta} = \frac{\partial\theta^{\lambda}}{\partial\theta'^{\alpha}} \frac{\partial\theta^{\mu}}{\partial\theta'^{\beta}} T_{\lambda\mu} \quad , \quad T'^{\cdot\beta}_{\alpha} = \frac{\partial\theta^{\lambda}}{\partial\theta'^{\alpha}} \frac{\partial\theta'^{\beta}}{\partial\theta^{\mu}} T^{\cdot\mu}_{\lambda} \text{ etc.} \tag{1.2.21}$$

If a quantity appears in some relations only through its components in a specified coordinate system, the transformation rules (1.2.16), (1.2.18), (1.2.20) and (1.2.21) may be used to verify whether these components form a tensor of the zeroth order (scalar), of the first order (vector) or of the second order. The same rule may be applied to higher-order tensors as well. For example, mixed components $A^{\alpha}_{\cdot\beta\gamma}$ of a surface tensor $\underset{\sim}{A}$ of the third order should transform according to the following rule

$$A'^{\alpha}_{\cdot\beta\gamma} = \frac{\partial\theta'^{\alpha}}{\partial\theta^{\lambda}} \frac{\partial\theta^{\mu}}{\partial\theta'^{\beta}} \frac{\partial\theta^{\rho}}{\partial\theta'^{\gamma}} A^{\lambda}_{\cdot\mu\rho} \tag{1.2.22}$$

1.3. Covariant differentiation

Differentiating the basic vectors $\underset{\sim}{a}_{\alpha}$, with the help of (1.1.3), (1.1.9) and (1.1.20) we obtain

$$\begin{aligned} \underset{\sim}{a}_{\alpha,\beta} = \underset{\sim}{r}_{,\alpha\beta} &= \Gamma_{\lambda.\alpha\beta}\underset{\sim}{a}^{\lambda} + b_{\alpha\beta}\underset{\sim}{n} \\ &= \Gamma^{\mu}_{\alpha\beta}\underset{\sim}{a}_{\mu} + b_{\alpha\beta}\underset{\sim}{n} \end{aligned} \tag{1.3.1}$$

where

$$\begin{aligned} \Gamma_{\lambda.\alpha\beta} = \underset{\sim}{a}_{\lambda} \cdot \underset{\sim}{a}_{\alpha,\beta} &= x^{l}_{,\lambda} x^{k}_{,\alpha\beta} \delta_{lk} \\ &= \frac{1}{2}\left(a_{\lambda\alpha,\beta} + a_{\lambda\beta,\alpha} - a_{\alpha\beta,\lambda}\right) \end{aligned} \tag{1.3.2}$$

$$\Gamma^{\mu}_{\alpha\beta} = a^{\lambda\mu}\Gamma_{\lambda.\alpha\beta} = \underset{\sim}{a}^{\mu} \cdot \underset{\sim}{a}_{\alpha,\beta} = -\underset{\sim}{a}^{\mu}_{,\alpha} \cdot \underset{\sim}{a}_{\beta} \tag{1.3.3}$$

are called the *surface Christoffel symbols* of the first and second kind, respectively. In general, these symbols are not components of a surface tensor. It follows, for example, from (1.3.3), (1.3.2), (1.1.9) and (1.1.12) that transformation rule for $\Gamma^{\mu}_{\alpha\beta}$ is

$$\Gamma'^{\kappa}_{\lambda\rho} = \underset{\sim}{a}^{\prime\kappa} \cdot \underset{\sim}{a}'_{\lambda,\rho} = \frac{\partial\theta'^{\kappa}}{\partial\theta^{\mu}} \underset{\sim}{a}^{\mu} \cdot \left[\frac{\partial^2\theta^{\alpha}}{\partial\theta'^{\lambda}\partial\theta'^{\rho}} \underset{\sim}{a}_{\alpha} + \frac{\partial\theta^{\alpha}}{\partial\theta'^{\lambda}} \frac{\partial\theta^{\beta}}{\partial\theta'^{\rho}} \underset{\sim}{a}_{\alpha,\beta}\right] =$$

$$= \frac{\partial\theta'^{\kappa}}{\partial\theta^{\mu}} \left[\frac{\partial\theta^{\alpha}}{\partial\theta'^{\lambda}} \frac{\partial\theta^{\beta}}{\partial\theta'^{\rho}} \Gamma^{\mu}_{\alpha\beta} + \frac{\partial^2\theta^{\mu}}{\partial\theta'^{\lambda}\partial\theta'^{\rho}}\right] \tag{1.3.4}$$

what differs from transformation rule (1.2.16) of a third-order tensor.

Let us differentiate a surface vector field $\underset{\sim}{v} = \underset{\sim}{v}(M)$, $\underset{\sim}{v} \in V$, along coordinate lines

$$\begin{aligned} \underset{\sim}{v}_{,\beta} &= \frac{\partial}{\partial\theta^{\beta}} (v^{\alpha}\underset{\sim}{a}_{\alpha}) = (v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\lambda\beta} v^{\lambda})\underset{\sim}{a}_{\alpha} + b_{\alpha\beta}v^{\alpha}\underset{\sim}{n} \\ &= v^{\alpha}|_{\beta}\underset{\sim}{a}_{\alpha} + b_{\alpha\beta}v^{\alpha}\underset{\sim}{n} = v_{\alpha|\beta}\,\underset{\sim}{a}^{\beta} + b^{\alpha}_{\beta}v_{\alpha}\underset{\sim}{n} \end{aligned} \tag{1.3.5}$$

where the operation defined by

$$v^{\alpha}|_{\beta} = v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\lambda\beta}v^{\lambda}, \quad v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}v_{\lambda} \tag{1.3.6}$$

is called the *covariant derivative* of the surface vector components.

Similarly, differentiating a surface tensor field $\underset{\sim}{T} = \underset{\sim}{T}(M)$, $\underset{\sim}{T} \in T_2$, along coordinate lines we obtain

$$\begin{aligned} \underset{\sim}{T}_{,\lambda} &= \frac{\partial}{\partial\theta^{\lambda}} (T^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta}) = \\ &= T^{\alpha\beta}|_{\lambda}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta} + b_{\alpha\lambda}T^{\alpha\beta}\underset{\sim}{n} \otimes \underset{\sim}{a}_{\beta} + b_{\beta\lambda}T^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{n} \\ &= T_{\alpha\beta|\lambda}\underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta} + b^{\alpha}_{\lambda}T_{\alpha\beta}\underset{\sim}{n} \otimes \underset{\sim}{a}^{\beta} + b^{\beta}_{\lambda}T_{\alpha\beta}\underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{n} \end{aligned} \tag{1.3.7}$$

where *covariant derivatives* of tensor components are defined by

$$\begin{aligned} T^{\alpha\beta}|_{\lambda} &= T^{\alpha\beta}{}_{,\lambda} + \Gamma^{\alpha}_{\kappa\lambda}T^{\kappa\beta} + \Gamma^{\beta}_{\kappa\lambda}T^{\alpha\kappa} \\ T_{\alpha\beta}|_{\lambda} &= T_{\alpha\beta,\lambda} - \Gamma^{\kappa}_{\alpha\lambda}T_{\kappa\beta} - \Gamma^{\kappa}_{\beta\lambda}T_{\alpha\kappa} \end{aligned} \tag{1.3.8}$$

In exactly the same way we can derive covariant derivatives of various components of higher-order tensor fields, for example

$$A^{\alpha}_{\cdot\beta\mu|\lambda} = A^{\alpha}_{\cdot\beta\mu,\lambda} + \Gamma^{\alpha}_{\kappa\lambda} A^{\kappa}_{\cdot\beta\mu} - \Gamma^{\kappa}_{\beta\lambda} A^{\alpha}_{\cdot\kappa\mu} - \Gamma^{\kappa}_{\mu\lambda} A^{\alpha}_{\cdot\beta\kappa} \tag{1.3.9}$$

While partial derivatives $v_{\alpha,\beta}$, $T^{\alpha\beta}{}_{,\lambda}$ etc. are not, in general, the components of a surface tensor, the covariant derivatives are. For example, using (1.3.6), (1.3.4) and (1.2.18) we obtain

$$\begin{aligned} v'_{\mu|\rho} &= \frac{\partial}{\partial\theta'^{,\rho}} \left(\frac{\partial\theta^{\alpha}}{\partial\theta'^{,\mu}} v_{\alpha} \right) - \Gamma'^{,\kappa}_{\mu\rho} \frac{\partial\theta^{\lambda}}{\partial\theta'^{,\kappa}} v_{\lambda} = \frac{\partial^2\theta^{\alpha}}{\partial\theta'^{,\mu}\partial\theta'^{,\rho}} v_{\alpha} + \\ &+ \frac{\partial\theta^{\alpha}}{\partial\theta'^{,\mu}} \frac{\partial\theta^{\beta}}{\partial\theta'^{,\rho}} v_{\alpha,\beta} - \frac{\partial\theta^{\alpha}}{\partial\theta'^{,\mu}} \frac{\partial\theta^{\beta}}{\partial\theta'^{,\rho}} \Gamma^{\lambda}_{\alpha\beta} v_{\lambda} - \frac{\partial^2\theta^{\lambda}}{\partial\theta'^{,\mu}\partial\theta'^{,\rho}} v_{\lambda} = \\ &= \frac{\partial\theta^{\alpha}}{\partial\theta'^{,\mu}} \frac{\partial\theta^{\beta}}{\partial\theta'^{,\rho}} v_{\alpha|\beta} \end{aligned} \tag{1.3.10}$$

The covariant derivative of a sum or product of tensor components follow the *usual rules of ordinary differentiation*, for example

$$\begin{aligned} &(T_{\alpha\beta} \pm S_{\alpha\beta})_{|\lambda} = T_{\alpha\beta|\lambda} \pm S_{\alpha\beta|\lambda} \\ &(T^{\alpha}_{\cdot\kappa} A^{\kappa}_{\cdot\mu\nu})_{|\lambda} = T^{\alpha}_{\cdot\kappa|\lambda} A^{\kappa}_{\cdot\mu\nu} + T^{\alpha}_{\cdot\kappa} A^{\kappa}_{\cdot\mu\nu|\lambda} \text{ , etc.} \end{aligned} \tag{1.3.11}$$

In particular, it is possible to prove that

$$a_{\alpha\beta|\lambda} = a^{\alpha\beta}{}_{|\lambda} = \epsilon_{\alpha\beta|\lambda} = \epsilon^{\alpha\beta}{}_{|\lambda} = 0 \tag{1.3.12}$$

what means that the metric and alternation tensor components may be treated as *constants* under covariant differentiation.

In future applications we shall have to deal with *spatial vectors* $\underset{\sim}{w} \in W \equiv V \times N$, where N is a one-dimensional vector space, orthogonal to V, having the unit normal $\underset{\sim}{n}$ as its basis, and $\times$ is the Cartesian product operation.

The spatial vector $\underset{\sim}{w}$ is expressible in terms of its components in the spatial basis $\underset{\sim}{a}_{\alpha}, \underset{\sim}{n}$ or $\underset{\sim}{a}^{\alpha}, \underset{\sim}{n}$ according to

$$\underset{\sim}{w} = w^{\alpha}\underset{\sim}{a}_{\alpha} + \underset{\sim}{w}\underset{\sim}{n} = w_{\alpha}\underset{\sim}{a}^{\alpha} + w\underset{\sim}{n} \tag{1.3.13}$$

and using (1.3.6) we obtain

$$\begin{aligned}\underset{\sim}{w}_{,\beta} &= (w^{\alpha}|_{\beta} - b^{\alpha}_{\beta}w)\underset{\sim}{a}_{\alpha} + (w_{,\beta} + b_{\alpha\beta}w^{\alpha})\underset{\sim}{n} = \\ &= (w_{\alpha|\beta} - b_{\alpha\beta}w)\underset{\sim}{a}^{\alpha} + (w_{,\beta} + b^{\alpha}_{\beta}w_{\alpha})\underset{\sim}{n}\end{aligned} \tag{1.3.14}$$

The subsequent partial differentiation of a surface tensor field is always interchangable. The subsequent covariant differentiation of the tensor field components may not be interchangable, in general.

Let us apply twice the covariant differentiation to covariant components of a surface vector field. Using (1.3.6) and (1.3.8) we obtain

$$\begin{aligned}v_{\beta|\lambda\mu} &= v_{\beta,\lambda\mu} - \Gamma^{\kappa}_{\beta\lambda,\mu}v_{\kappa} - \Gamma^{\kappa}_{\beta\lambda}v_{\kappa,\mu} - \Gamma^{\rho}_{\beta\mu}(v_{\rho,\lambda} - \Gamma^{\kappa}_{\rho\lambda}v_{\kappa}) - \\ &\quad - \Gamma^{\rho}_{\lambda\mu}(v_{\beta,\rho} - \Gamma^{\kappa}_{\beta\rho}v_{\kappa}) \\ v_{\beta|\mu\lambda} &= v_{\beta,\mu\lambda} - \Gamma^{\kappa}_{\beta\lambda,\mu} - \Gamma^{\kappa}_{\beta\mu}v_{\kappa,\lambda} - \Gamma^{\rho}_{\beta\lambda}(v_{\rho,\mu} - \Gamma^{\kappa}_{\rho\mu}v_{\kappa}) - \\ &\quad - \Gamma^{\rho}_{\mu\lambda}(v_{\beta,\rho} - \Gamma^{\kappa}_{\beta\rho}v_{\kappa})\end{aligned} \tag{1.3.15}$$

Hence

$$v_{\beta|\lambda\mu} - v_{\beta|\mu\lambda} = R^{\kappa}_{\cdot\beta\lambda\mu}v_{\kappa} \tag{1.3.16}$$

where mixed components of the *surface Riemann-Christoffel tensor* are defined by

$$R^{\kappa}_{\cdot\beta\lambda\mu} = \Gamma^{\kappa}_{\beta\mu,\lambda} - \Gamma^{\kappa}_{\beta\lambda,\mu} + \Gamma^{\rho}_{\beta\mu}\Gamma^{\kappa}_{\rho\lambda} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\kappa}_{\rho\mu} \tag{1.3.17}$$

According to (1.3.2)

$$\Gamma_{\alpha.\beta\lambda} = a_{\alpha\beta,\lambda} - \Gamma_{\beta.\alpha\lambda} \tag{1.3.18}$$

and using this relation we obtain *covariant components* of the Riemann-Christoffel tensor after following transformations:

$$R_{\alpha\beta\lambda\mu} = a_{\alpha\kappa}\Gamma^{\kappa}_{\beta\mu,\lambda} - a_{\alpha\kappa}\Gamma^{\kappa}_{\beta\lambda,\mu} + \Gamma^{\rho}_{\beta\mu}\Gamma_{\alpha.\rho\lambda} - \Gamma^{\rho}_{\beta\lambda}\Gamma_{\alpha.\rho\mu} =$$

$$= \Gamma_{\alpha.\beta\mu,\lambda} - \Gamma_{\alpha.\beta\lambda,\mu} - a_{\alpha\kappa,\lambda}\Gamma^{\kappa}_{\beta\mu} + a_{\alpha\kappa,\mu}\Gamma^{\kappa}_{\beta\lambda} +$$

$$+ \Gamma^{\rho}_{\beta\mu}(a_{\alpha\rho,\lambda} - \Gamma_{\rho.\alpha\lambda}) - \Gamma^{\rho}_{\beta\lambda}(a_{\alpha\rho,\mu} - \Gamma_{\rho.\alpha\mu}) =$$

$$= \Gamma_{\alpha.\beta\mu,\lambda} - \Gamma_{\alpha.\beta\lambda,\mu} + \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa.\beta\lambda} - \Gamma^{\kappa}_{\alpha\lambda}\Gamma_{\kappa.\beta\mu} =$$

$$= \frac{1}{2}(a_{\alpha\mu,\beta\lambda} + a_{\beta\lambda,\alpha\mu} - a_{\alpha\lambda,\beta\mu} - a_{\beta\mu,\alpha\lambda}) +$$

$$+ \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa.\beta\lambda} - \Gamma^{\kappa}_{\alpha\lambda}\Gamma_{\kappa.\beta\mu} \quad . \tag{1.3.19}$$

It follows from (1.3.19) that the tensor has the following *symmetry conditions*

$$R_{\alpha\beta\lambda\mu} = -R_{\beta\alpha\lambda\mu} = -R_{\alpha\beta\mu\lambda} = R_{\lambda\mu\alpha\beta} \tag{1.3.20}$$

and hence, has in fact only one independent component R_{1212}.

Applying twice the covariant differentiation to components of a surface tensor field we obtain similar relations, for example

$$T_{\alpha\beta|\lambda\mu} - T_{\alpha\beta|\mu\lambda} = R^{\kappa}_{.\alpha\lambda\mu}T_{\kappa\beta} + R^{\kappa}_{.\beta\lambda\mu}T_{\alpha\kappa} \quad . \tag{1.3.21}$$

From (1.3.1), (1.1.20) and (1.3.6) we find that

$$\underset{\sim}{a}_{\beta|\lambda} = b_{\beta\lambda}\underset{\sim}{n} \quad , \quad \underset{\sim}{n}_{,\mu} = \underset{\sim}{n}_{|\mu} = -b^{\kappa}_{\mu}\underset{\sim}{a}_{\kappa} \tag{1.3.22}$$

Applying (1.3.16) to $\underset{\sim}{a}_{\beta}$ and using (1.3.22) we obtain one vector relation

$$\underset{\sim}{a}_{\beta|\lambda\mu} - \underset{\sim}{a}_{\beta|\mu\lambda} = R^{\kappa}_{.\beta\lambda\mu}\underset{\sim}{a}_{\kappa} =$$

$$= (b_{\beta\lambda|\mu} - b_{\beta\mu|\lambda})\underset{\sim}{n} + (b_{\beta\mu}b^{\kappa}_{\lambda} - b_{\beta\lambda}b^{\kappa}_{\mu})\underset{\sim}{a}_{\kappa} \tag{1.3.23}$$

or three scalar relations

$$b_{\beta\lambda|\mu} = b_{\beta\mu|\lambda} \tag{1.3.24}$$

$$b_{\alpha\lambda}b_{\beta\mu} - b_{\alpha\mu}b_{\beta\lambda} = R_{\alpha\beta\lambda\mu} \tag{1.3.25}$$

The relations (1.3.24) and (1.3.25) are known as *Codazzi equations* and *Gauss equation* of the surface, respectively. As the surface has been defined in (1.1.1) by three scalar components of the position vector $\underset{\sim}{r}$, the equations (1.3.24) and (1.3.25) express just three conditions, which six components of $a_{\alpha\beta}$ and $b_{\alpha\beta}$ must satisfy. And conversely, an arbitrarily chosen set of six functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of surface coordinates θ^{α} describes a surface in the three-dimensional Euclidean space if and only if the functions satisfy the Codazzi-Gauss equations.

Using (1.1.21) and (1.3.25) we find as well

$$\begin{aligned} &\frac{1}{4}\,\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}R_{\alpha\beta\lambda\mu} = K \\ &R_{\alpha\beta\lambda\mu} = \epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\,K \end{aligned} \qquad (1.3.26)$$

These relations together with (1.3.16) and (1.3.21) show us that the subsequent covariant differentiations on a surface are *interchangable* if and only if the Gaussian curvature K of the surface becomes *equal to zero*. If K = 0 the surface becomes a *two-dimensional Euclidean space*. With $K \neq 0$ the surface constitutes a *two-dimensional Riemanian space*.

In (1.1.22) we have defined the Gaussian curvature K in terms of components of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The relation (1.3.26), together with (1.3.19), shows that the Gaussian curvature may be calculated entirely from the metric tensor components.

Various alternative forms of the Codazzi-Gauss equations may also be derived. For example, if we contract skew-symmetric indices in (1.3.24), (1.3.25) and (1.3.19) by using the alternation tensor, then

$$\begin{aligned} &\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}b_{\beta\lambda|\mu} = 0 \\ &\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\,(\Gamma_{\alpha,\beta\mu,\lambda} + \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa,\beta\lambda} + b_{\alpha\mu}b_{\beta\lambda}) = 0 \end{aligned} \qquad (1.3.27)$$

This form of the *Codazzi-Gauss equations* will be particularily suitable in deriving compatibility conditions in shell theory.

1.4. Physical components

When using tensor notation it is necessary at the end of transformations to introduce such components of vectors and tensors which have definite physical meaning. For this reason let us introduce the *unit base vectors* $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, tangent to the surface coordinate lines θ^1 and θ^2, Fig. 4, defined by

$$\underset{\sim}{e}_1 = \frac{\underset{\sim}{a}_1}{A_1} \quad , \quad \underset{\sim}{e}_2 = \frac{\underset{\sim}{a}_2}{A_2} \tag{1.4.1}$$

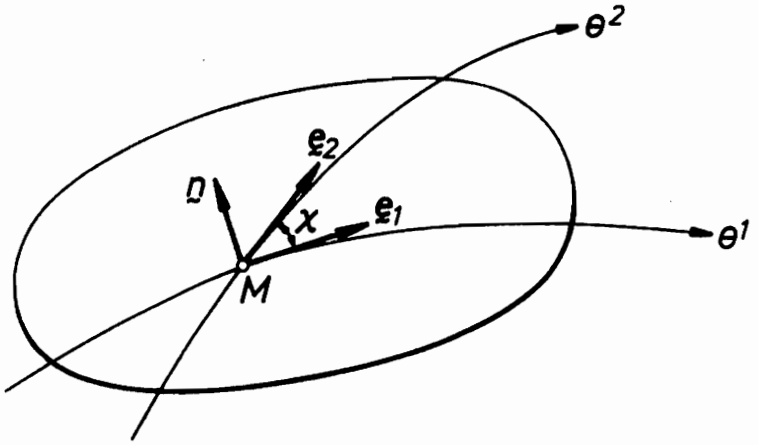

Fig. 4

and the angle between them let us denote by $\chi(\theta^1,\theta^2)$. The length scale parameters

$$A_1 = |\underset{\sim}{a}_1| \quad , \quad A_2 = |\underset{\sim}{a}_2| \tag{1.4.2}$$

are known as the *Lame parameters* of the coordinate system θ^1,θ^2. In terms of these parameters the metric tensor components have the form

$$\begin{gathered} a_{11} = (A_1)^2 \ , \ a_{12} = A_1A_2 \cos\chi \ , \ a_{22} = (A_2)^2 \\ a = (A_1)^2(A_2)^2 \sin^2\chi \\ a^{11} = \frac{1}{(A_1)^2\sin^2\chi} \quad , \quad a^{12} = -\frac{\cos\chi}{A_1A_2\sin^2\chi} \ , \\ a^{22} = \frac{1}{(A_2)^2\sin^2\chi} \end{gathered} \tag{1.4.3}$$

Using (1.4.1) in (1.3.2) and (1.3.3) it is possible to calculate all Christoffel symbols entirely in terms of A_1, A_2 and χ, for example

$$\Gamma^1_{11} = \frac{A_2 A_{1,1} + A_1 A_{1,2} \cos\chi - \cos\chi\,(A_1 A_2 \cos\chi)_{,1}}{A_1 A_2 \sin^2\chi}$$

$$\Gamma^1_{12} = \frac{A_{1,2} - \cos\chi\, A_{2,1}}{A_1 \sin^2\chi} \quad , \quad \Gamma^2_{12} = \frac{A_{2,1} - \cos\chi\, A_{1,2}}{A_2 \sin^2\chi} \text{ , etc.} \tag{1.4.4}$$

By the *physical components* of a vector or a tensor we mean its components with respect to the unit base vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ (not projections upon them!)

For any vector $\underset{\sim}{v} \in V$ we obtain

$$\underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha = v_\alpha \underset{\sim}{a}^\alpha = v_{<\alpha>} \underset{\sim}{e}_\alpha \tag{1.4.5}$$

from which

$$v_{<\alpha>} = A_\alpha v^\alpha = A_\alpha a^{\alpha\beta} v_\beta \quad , \quad \not\Sigma_\alpha \tag{1.4.6}$$

Similarly, for any tensor $\underset{\sim}{T} \in T_2$

$$\underset{\sim}{T} = T^{\alpha\beta} \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta = T_{\alpha\beta} \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = T_{<\alpha\beta>} \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta \tag{1.4.7}$$

from which

$$T_{<\alpha\beta>} = A_\alpha A_\beta T^{\alpha\beta} = A_\alpha A_\beta a^{\alpha\lambda} a^{\beta\mu} T_{\lambda\mu} \quad , \quad \not\Sigma_{\alpha,\beta} \tag{1.4.8}$$

In particular, it follows from (1.4.3) and (1.1.16) that

$$\begin{aligned} &a_{<11>} = a_{<22>} = \frac{1}{\sin^2\chi} \quad , \; a_{<12>} = -\frac{\cos\chi}{\sin^2\chi}, \\ &\epsilon_{<12>} = -\epsilon_{<21>} = \frac{1}{\sin\chi} \quad , \; \epsilon_{<11>} = \epsilon_{<22>} = 0 \end{aligned} \tag{1.4.9}$$

and using these relations together with (1.4.4) we obtain the expressions for physical components of covariant derivatives of vector and tensor components. For example

$$v_{<i|2>} = A_1 A_2 a^{2\lambda}(v^1{}_{,\lambda} + \Gamma^1_{\mu\lambda} v^\mu) =$$

$$= A_1 A_2 a^{2\lambda}\left[\left(\frac{v_{<1>}}{A_1}\right)_{,\lambda} + \Gamma^1_{1\lambda}\left(\frac{v_{<1>}}{A_1}\right) + \Gamma^1_{2\lambda}\left(\frac{v_{<2>}}{A_2}\right)\right] \qquad (1.4.10)$$

$$T_{<11|2>} = (A_1)^2 A_2 a^{2\lambda}\left[\left\{\frac{T_{<11>}}{(A_1)^2}\right\}_{,\lambda} + 2\Gamma^1_{1\lambda}\frac{T_{<11>}}{(A_1)^2} + \Gamma^1_{2\lambda}\frac{1}{A_1 A_2}(T_{<12>} + T_{<21>})\right].$$

It is important to note that, in general non-orthogonal coordinate system, it is also possible to use dual unit vectors $\underset{\sim}{e}^\alpha = \frac{\underset{\sim}{a}^\alpha}{|\underset{\sim}{a}^\alpha|}$ as bases for definition of physical components of vectors and tensors. However, $\underset{\sim}{e}^\alpha \neq \underset{\sim}{e}_\alpha$ in general, and such physical components defined with respect to $\underset{\sim}{e}^\alpha$ would differ from those defined with respect to $\underset{\sim}{e}_\alpha$. In these lectures we will use the definition of physical components with respect to $\underset{\sim}{e}_\alpha$ basis.

In *orthogonal coordinate system* $\underset{\sim}{e}_\alpha \equiv \underset{\sim}{e}^\alpha$, $\cos\chi = 0$, $\sin\chi = 1$. In this case all fomulae become much simpler:

$$a_{11} = (A_1)^2 \quad , \quad a_{22} = (A_2)^2 \quad , \quad a_{12} = a^{12} = 0$$

$$a^{11} = \frac{1}{(A_1)^2} \quad , \quad a^{22} = \frac{1}{(A_2)^2} \quad , \quad a = (A_1)^2 (A_2)^2 \qquad (1.4.11)$$

$$\Gamma^1_{11} = \frac{1}{A_1} A_{1,1} \quad , \quad \Gamma^1_{12} = \frac{1}{A_1} A_{1,2} \quad , \quad \Gamma^2_{12} = \frac{1}{A_2} A_{2,1} \quad , \text{ etc.} \qquad (1.4.12)$$

$$v_{<\alpha>} = A_\alpha v^\alpha = \frac{1}{A_\alpha} v_\alpha$$

$$T_{<\alpha\beta>} = A_\alpha A_\alpha T^{\alpha\beta} = \frac{1}{A_\alpha A_\beta} T_{\alpha\beta} = \frac{A_\alpha}{A_\beta} T^\alpha{}_{\cdot\beta} \quad , \quad \not\Sigma_{\alpha,\beta} \qquad (1.4.13)$$

$$a_{<11>} = a_{<22>} = 1 \quad , \quad a_{<12>} = 0$$

$$\epsilon_{<12>} = -\epsilon_{<21>} = 1 \quad , \quad \epsilon_{<11>} = \epsilon_{<22>} = 0 \qquad (1.4.14)$$

$$v_{<1|2>} = \frac{1}{A_2} v_{<1>,2} - \frac{1}{A_1 A_2} A_{2,1} v_{<2>} \quad ,$$

$$T_{<11|2>} = \frac{1}{A_2} T_{<11>,2} - \frac{1}{A_1 A_2} A_{2,1}(T_{<12>} + T_{<21>}) \quad , \text{ etc.} \qquad (1.4.15)$$

Because of this relative simplicity of all geometrical relations, orthogonal coordinates are frequently used in the literature.

1.5. Geometry of a surface curve

Let $\theta^\alpha = \theta^\alpha(s)$ be the equations of an arbitrary curve C on the surface M, Fig. 5, where s is length along the curve.

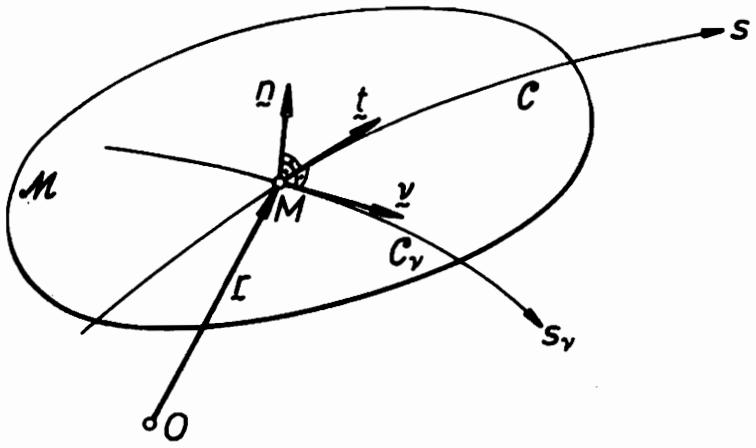

Fig. 5

At each point on the curve let us define the triad of orthonormal vectors $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$, Fig. 5: $\underset{\sim}{t}$ is the *unit tangent* to C and $\underset{\sim}{\nu}$ is the *outward unit normal* lying in a plane tangent to M at M. Then the following relations hold:

$$\begin{aligned} \underset{\sim}{\nu} &= \nu_\alpha \underset{\sim}{a}^\alpha = \nu^\beta \underset{\sim}{a}_\beta = \underset{\sim}{t} \times \underset{\sim}{n} = \epsilon^{\beta\alpha} t_\alpha \underset{\sim}{a}_\beta \\ \underset{\sim}{t} &= t_\alpha \underset{\sim}{a}^\alpha = t^\beta \underset{\sim}{a}_\beta = \underset{\sim}{n} \times \underset{\sim}{\nu} = \epsilon^{\alpha\beta} \nu_\alpha \underset{\sim}{a}_\beta \end{aligned} \tag{1.5.1}$$

If by s_ν we denote the length along a curve C_ν orthogonal to C at M, defined by the equations $\theta^\beta = \theta^\beta(s_\nu)$, then

$$\begin{aligned} \underset{\sim}{t} &= \frac{d\underset{\sim}{r}}{ds} = \frac{\partial\theta^\alpha}{\partial s} \underset{\sim}{a}_\alpha = \frac{\partial s}{\partial\theta^\beta} \underset{\sim}{a}^\beta \\ \underset{\sim}{\nu} &= \frac{d\underset{\sim}{r}}{ds_\nu} = \frac{\partial\theta^\alpha}{\partial s_\nu} \underset{\sim}{a}_\alpha = \frac{\partial s_\nu}{\partial\theta^\beta} \underset{\sim}{a}^\beta \end{aligned} \tag{1.5.2}$$

$$\underset{\sim}{a}_\alpha = \nu_\alpha \underset{\sim}{\nu} + t_\alpha \underset{\sim}{t}$$

Using these formulae it is easy to find the relations between the vector and tensor components with respect to base vectors $\underset{\sim}{a}_\alpha$ or $\underset{\sim}{a}^\alpha$ and those with respect to $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$.

Let physical components of $\underset{\sim}{v} \in V$ and $\underset{\sim}{T} \in T_2$ with respect to orthonormal basis $\underset{\sim}{\nu}$, $\underset{\sim}{t}$ be denoted by

$$v_\nu = \underset{\sim}{v} \cdot \underset{\sim}{\nu} = v^\alpha \nu_\alpha = v_\beta \nu^\beta$$
$$v_t = \underset{\sim}{v} \cdot \underset{\sim}{t} = v^\alpha t_\alpha = v_\beta t^\beta \qquad (1.5.3)$$

$$T_{\nu\nu} = \underset{\sim}{\nu} \cdot \underset{\sim}{T\nu} = \nu_\alpha T^{\alpha\beta} \nu_\beta = \nu^\alpha T_{\alpha\beta} \nu^\beta = \ldots$$
$$T_{\nu t} = \underset{\sim}{\nu} \cdot \underset{\sim}{Tt} = \nu_\alpha T^{\alpha\beta} t_\beta = \nu^\alpha T_\alpha{}^\beta t_\beta = \ldots \qquad (1.5.4)$$
$$T_{tt} = \underset{\sim}{t} \cdot \underset{\sim}{Tt} = t_\alpha T^{\alpha\beta} t_\beta = t_\alpha T^\alpha{}_\beta t^\beta = \ldots$$

Using (1.5.3), (1.5.4) and (1.5.1) for vector and tensor components with respect to base vectors $\underset{\sim}{a}_\alpha$ or $\underset{\sim}{a}^\alpha$ we obtain

$$v_\alpha = \nu_\alpha v_\nu + t_\alpha v_t \,, \quad v^\alpha = \nu^\alpha v_\nu + t^\alpha v_t \,,$$
$$T_{\alpha\beta} = \nu_\alpha \nu_\beta T_{\nu\nu} + \nu_\alpha t_\beta T_{\nu t} + t_\alpha \nu_\beta T_{t\nu} + t_\alpha t_\beta T_{tt}, \ \ldots \,, \qquad (1.5.5)$$

Let us derive expressions for the derivatives of the triad $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ along the curve C. Differentiating with respect to s the identities

$$\underset{\sim}{\nu} \cdot \underset{\sim}{\nu} = \underset{\sim}{t} \cdot \underset{\sim}{t} = \underset{\sim}{n} \cdot \underset{\sim}{n} = 1$$
$$\underset{\sim}{\nu} \cdot \underset{\sim}{t} = \underset{\sim}{t} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \underset{\sim}{\nu} = 0 \qquad (1.5.6)$$

we obtain

$$\frac{d\underset{\sim}{\nu}}{ds} = \kappa_t \underset{\sim}{t} - \tau_t \underset{\sim}{n}$$
$$\frac{d\underset{\sim}{t}}{ds} = \sigma_t \underset{\sim}{n} - \kappa_t \underset{\sim}{\nu} \qquad (1.5.7)$$
$$\frac{d\underset{\sim}{n}}{ds} = \tau_t \underset{\sim}{\nu} - \sigma_t \underset{\sim}{t}$$

where it follows from (1.5.1) and (1.1.21) that

$$\sigma_t = \underset{\sim}{n} \cdot \frac{d\underset{\sim}{t}}{ds} = -\underset{\sim}{t} \cdot \frac{d\underset{\sim}{n}}{ds} = -t^\alpha \underset{\sim}{a}_\alpha \cdot \underset{\sim}{n}_{,\beta} t^\beta =$$
$$= t^\alpha t^\beta b_{\alpha\beta} = b_{tt} \qquad (1.5.8)$$

$$\tau_t = \underset{\sim}{\nu} \cdot \frac{d\underset{\sim}{n}}{ds} = -\underset{\sim}{n} \cdot \frac{d\underset{\sim}{\nu}}{ds} = \nu^\alpha \underset{\sim}{a}_\alpha \cdot \underset{\sim}{n}_{,\beta} t^\beta =$$
$$= -\nu^\alpha t^\beta b_{\alpha\beta} = -b_{\nu t} \qquad (1.5.9)$$

$$\kappa_t = \underset{\sim}{t}\cdot\frac{d\underset{\sim}{\nu}}{ds} = -\underset{\sim}{\nu}\cdot\frac{d\underset{\sim}{t}}{ds} = t_\alpha \underset{\sim}{a}^\alpha \cdot (\nu^\lambda|_\beta \underset{\sim}{a}_\lambda + b_{\lambda\beta}\nu^\lambda \underset{\sim}{n})\, t^\beta =$$
$$= t_\alpha \nu^\alpha|_\beta t^\beta = -\nu_\alpha t^\alpha|_\beta t^\beta \quad . \tag{1.5.10}$$

The quantity σ_t is known as the *normal curvature*, τ_t as the *geodesic torsion* and κ_t as the *geodesic curvature* of the surface curve C, respectively.

Defining the vector

$$\underset{\sim}{\omega}_t = \sigma_t \underset{\sim}{\nu} + \tau_t \underset{\sim}{t} + \kappa_t \underset{\sim}{n} \tag{1.5.11}$$

differential relations (1.5.7) can be written in more symmetric form

$$\frac{d\underset{\sim}{\nu}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{\nu} \quad , \quad \frac{d\underset{\sim}{t}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{t} \quad , \quad \frac{d\underset{\sim}{n}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{n} \tag{1.5.12}$$

The vector $\underset{\sim}{\omega}_t$ is the rate of rotation of the triad $(\underset{\sim}{\nu}, \underset{\sim}{t}, \underset{\sim}{n})$ as it advances along the curve C.

In exactly the same way we may obtain differentiation rules of $\underset{\sim}{\nu}, \underset{\sim}{t}, \underset{\sim}{n}$ in direction of the outward unit normal $\underset{\sim}{\nu}$ to be

$$\frac{d\underset{\sim}{\nu}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{\nu} \quad , \quad \frac{d\underset{\sim}{t}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{t} \quad , \quad \frac{d\underset{\sim}{n}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{n} \tag{1.5.13}$$

where

$$\underset{\sim}{\omega}_\nu = \tau_\nu \underset{\sim}{\nu} + \sigma_\nu \underset{\sim}{t} + \kappa_\nu \underset{\sim}{n} \tag{1.5.14}$$

$$\begin{aligned} \sigma_\nu &= \nu^\alpha \nu^\beta b_{\alpha\beta} = b_{\nu\nu} \\ \tau_\nu &= -t^\alpha \nu^\beta b_{\alpha\beta} = -b_{t\nu} = \tau_t \\ \kappa_\nu &= \nu_\alpha t^\alpha|_\beta \nu^\beta = -t_\alpha \nu^\alpha|_\beta \nu^\beta \end{aligned} \tag{1.5.15}$$

The minus sign in (1.5.13) results because the orthonormal triad $\underset{\sim}{t}, \underset{\sim}{\nu}, \underset{\sim}{n}$ and triad $\underset{\sim}{\nu}, \underset{\sim}{t}, \underset{\sim}{n}$ have opposite orientations with respect to C_ν and with respect to C, respectively.

In order to explain the geometrical meaning of the quantities just introduced for the surface curve C, let us remaind here some relations valid for the same curve C in three-dimensional Euclidean space.

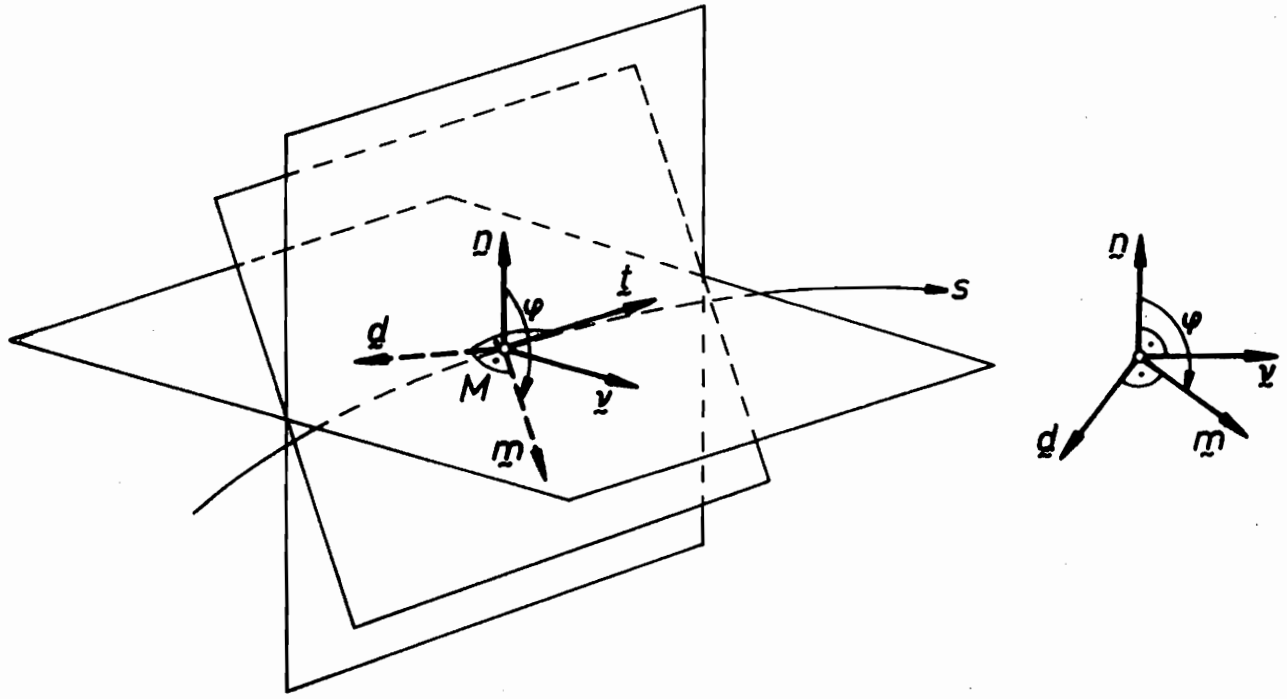

Fig. 6

At any point M of the curve C, described by a vector function $\underset{\sim}{r} = \underset{\sim}{r}(s)$, Fig. 6, we can associate the *unit tangent* vector $\underset{\sim}{t} = \underset{\sim}{r}_{,s}$, the *principal unit normal* $\underset{\sim}{m} \equiv \frac{\underset{\sim}{t}_{,s}}{|\underset{\sim}{t}_{,s}|}$ and the *unit binormal* vector $\underset{\sim}{d} \equiv \underset{\sim}{t} \times \underset{\sim}{m}$. Differentiating these unit vectors we obtain the *Frenet equations* of the curve

$$\frac{d\underset{\sim}{t}}{ds} = \sigma \underset{\sim}{m} \quad , \quad \frac{d\underset{\sim}{m}}{ds} = \tau \underset{\sim}{d} - \sigma \underset{\sim}{t} \quad ; \quad \frac{d\underset{\sim}{d}}{ds} = - \tau \underset{\sim}{m} \tag{1.5.16}$$

where

$$\begin{aligned} \sigma &= \underset{\sim}{m} \cdot \frac{d\underset{\sim}{t}}{ds} = - \underset{\sim}{t} \cdot \frac{d\underset{\sim}{m}}{ds} \\ \tau &= \underset{\sim}{d} \cdot \frac{d\underset{\sim}{m}}{ds} = - \underset{\sim}{m} \cdot \frac{d\underset{\sim}{d}}{ds} \end{aligned} \tag{1.5.17}$$

The scalars σ and τ are known as the *principal curvature* and *principal torsion* of the curve C, respectively. Note that $\sigma \geq 0$, as the principal unit normal $\underset{\sim}{m}$ always is directed inward, toward the concave side of the curve C. The points of C at which $\sigma = 0$ are called *straightening points* and at them direction of $\underset{\sim}{m}$ is not unique.

It follows from (1.5.7) and (1.5.16) that

$$\sigma \underset{\sim}{m} = \sigma_b \underset{\sim}{n} - \kappa_t \underset{\sim}{\nu} \tag{1.5.18}$$

This gives a geometrical meaning to σ_t and $-\kappa_t$; they are components of the principal curvature vector $\sigma\underset{\sim}{m}$ of the curve C with respect to $\underset{\sim}{n}$ and $\underset{\sim}{\nu}$, respectively. If φ is the angle between $\underset{\sim}{n}$ and $\underset{\sim}{m}$, Fig. 6, then from (1.5.18) we obtain

$$\sigma_t = \sigma \cos\varphi \quad , \quad \kappa_t = -\sigma \sin\varphi$$
$$\sigma = \sqrt{\sigma_t^2 + \kappa_t^2} \tag{1.5.19}$$

Keeping in mind that

$$\underset{\sim}{m} = \underset{\sim}{n} \cos\varphi + \underset{\sim}{\nu} \sin\varphi$$
$$\underset{\sim}{d} = -\underset{\sim}{n} \sin\varphi + \underset{\sim}{\nu} \cos\varphi \tag{1.5.20}$$

let us differentiate $\underset{\sim}{d}$ and make use of (1.5.7) and (1.5.19) to obtain

$$\frac{d\underset{\sim}{d}}{ds} = -(\tau_t \underset{\sim}{\nu} - \sigma_t \underset{\sim}{t})\sin\varphi + (\kappa_t \underset{\sim}{t} - \tau_t \underset{\sim}{n})\cos\varphi -$$
$$- (\underset{\sim}{n}\cos\varphi + \underset{\sim}{\nu}\sin\varphi)\frac{d\varphi}{dt} =$$
$$= -(\tau_t + \frac{d\varphi}{dt})\underset{\sim}{m} \tag{1.5.21}$$

When this relation is compared with (1.5.16) we obtain

$$\tau = \tau_t + \frac{d\varphi}{dt} \tag{1.5.22}$$

This gives further geometrical meaning to τ_t as well.

1.6. Curvatures of coordinate lines

The values of σ_t and κ_t at each point $M \in M$ depend, according to (1.5.8) and (1.5.9), on the direction $\underset{\sim}{t}$ of the surface curve C. Let us find the *principal directions* for which the normal curvature σ_t assumes an extremal value. Using the method of Langrangean multipliers, the problem of finding the extremal values of $b_{\alpha\beta}t^\alpha t^\beta$, under the condition $a_{\alpha\beta}t^\alpha t^\beta = 1$, leads to the problem of finding the extremal values of the function of two parameters

$$F(t^\alpha) = b_{\alpha\beta}t^\alpha t^\beta - \sigma_t(a_{\alpha\beta}t^\alpha t^\beta - 1) \tag{1.6.1}$$

from which we obtain the following set of homogeneous equations

$$\frac{1}{2}\frac{\partial F}{\partial t^{\alpha}} = (b_{\alpha\beta} - \sigma_t a_{\alpha\beta})t^{\beta} = 0 \tag{1.6.2}$$

which have a nontrivial solution if

$$\det(b_{\alpha\beta} - \sigma_t a_{\alpha\beta}) = 0 \tag{1.6.3}$$

or

$$\sigma_t^2 - 2\sigma_t H + K = 0 \quad . \tag{1.6.4}$$

The roots of this equation

$$\begin{aligned} \sigma_I &= (\sigma_t)_I = H + \sqrt{H^2 - K} \\ \sigma_{II} &= (\sigma_t)_{II} = H - \sqrt{H^2 - K} \end{aligned} \tag{1.6.5}$$

are called the *principal curvatures* of the surface at a point $M \in M$. The principal curvatures are always *real*; using (1.1.22), (1.1.23) and (1.1.18) we obtain

$$2H = b_1^1 + b_2^2 \quad , \quad K = b_1^1 b_2^2 - b_2^1 b_1^2 \tag{1.6.6}$$

and then

$$4(H^2 - K) = (b_1^1 - b_2^2)^2 + 4b_2^1 b_1^2 \geq 0 \quad . \tag{1.6.7}$$

Note that $H^2 - K$ may be equal to zero if and only if

$$b_1^1 = b_2^2 \quad \text{and} \quad b_2^1 = b_1^2 = 0 \tag{1.6.8}$$

and only then $\sigma_I = \sigma_{II} = H$. Such surface points are called *spherical* (or umbilics), and any direction at the point is principal.

At any surface point, except spherical, the two principal curvatures (1.6.5) assume *different values* such that

$$\sigma_I \sigma_{II} = K \quad , \quad \frac{1}{2}(\sigma_I + \sigma_{II}) = H \quad . \tag{1.6.9}$$

It is possible now to give geometrical meaning to K and H depicted in Fig. 7.

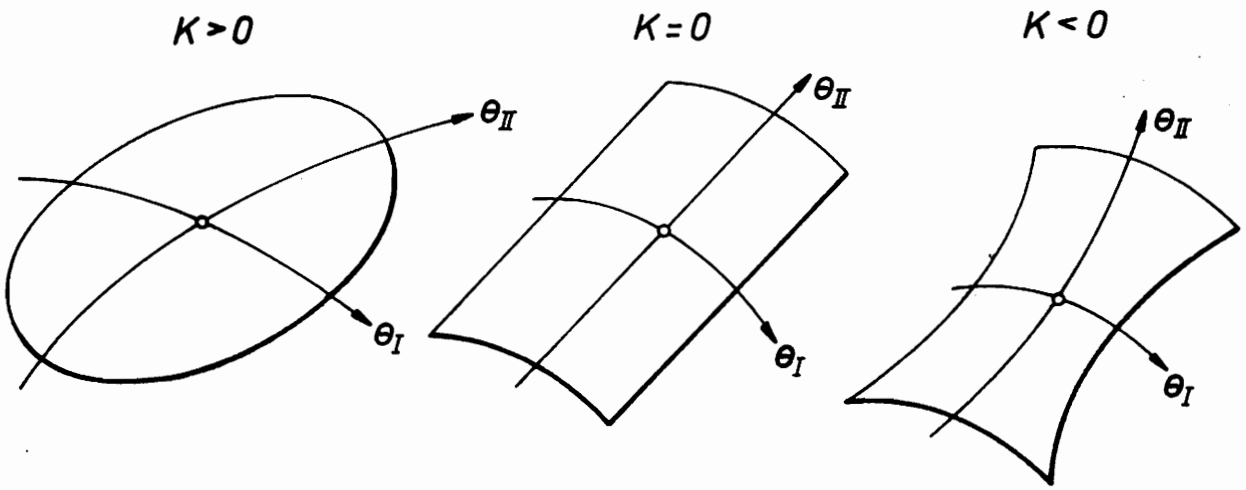

Fig. 7

The point $M \in \mathbb{M}$ is called *eliptic* when $K > 0$, *parabolic* when $K = 0$ and *hyperbolic* when $K < 0$. A surface consisting entirely of one type of points mentioned above is called the surface of *positive, zeroth* or *negative Gaussian curvature*, respectively.

If $t^{\alpha}_{(I)}$ and $t^{\alpha}_{(II)}$ are two principal directions corresponding to σ_I and σ_{II}, respectively, then using (1.6.2) we find

$$(b_{\alpha\beta} - \sigma_I a_{\alpha\beta}) t^{\beta}_{(I)} t^{\alpha}_{(II)} = (b_{\alpha\beta} - \sigma_{II} a_{\alpha\beta}) t^{\beta}_{(II)} t^{\alpha}_{(I)} = 0 \qquad (1.6.10)$$

or

$$(\sigma_{II} - \sigma_I) a_{\alpha\beta} t^{\alpha}_{(I)} t^{\beta}_{(II)} = (\sigma_{II} - \sigma_I) \underset{\sim}{t}_{(II)} \cdot \underset{\sim}{t}_{(I)} = 0 \qquad (1.6.11)$$

and this means that, unless the point is spherical, the two *principal directions* are *orthogonal*.

If we multiply (1.6.3) by ν^{α} then

$$b_{\alpha\beta} \nu^{\alpha} t^{\beta} = \sigma_t a_{\alpha\beta} \nu^{\alpha} t^{\beta} = \sigma_t \underset{\sim}{\nu} \cdot \underset{\sim}{t} = 0 \qquad (1.6.12)$$

and this means that in principal directions the *geodesic torsions* of the surface **curves** *vanish*,

$$\tau_t = \tau_\nu = 0 \quad . \qquad (1.6.13)$$

A curve on a surface, whose tangent at each point is along the principal direction, is called a *line of principal curvature*.

It is easily seen now from (1.6.11) and (1.6.12), that the necessary and sufficient conditions for the coordinate curves on M to be lines of principal curvatures are

$$a_{12} = b_{12} = 0 \quad . \tag{1.6.14}$$

Such coordinate systems are used frequently in shell literature.

Let us now discuss in more detail the physical components of curvatures of the coordinate lines on a surface M.

Consider an *orthogonal* coordinate system θ^1, θ^2 on the surface, and let C be an arbitrary curve, Fig. 8.

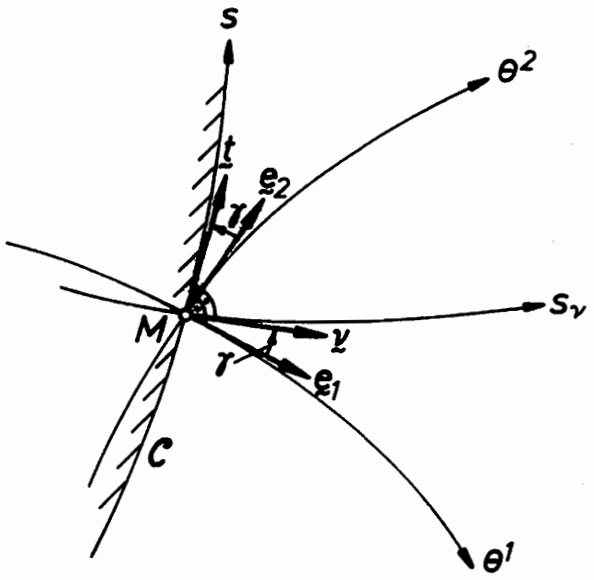

Fig. 8

Then using (1.4.6) we obtain

$$\begin{aligned} t_{<\alpha>} &= A_\alpha t^\alpha = \frac{A_\alpha \partial\theta^\alpha}{\partial s} = \frac{ds_\alpha}{ds} \\ \nu_{<\alpha>} &= A_\alpha \nu^\alpha = \frac{A_\alpha \partial\theta^\alpha}{\partial s_\nu} = \frac{ds_\alpha}{ds_\nu} \end{aligned} \quad . \tag{1.6.15}$$

From Fig. 8 we obtain another representation for these components

$$\begin{aligned} t_{<1>} &= -\sin\gamma \ , \quad t_{<2>} = \cos\gamma \\ \nu_{<1>} &= \cos\gamma \ , \quad \nu_{<2>} = \sin\gamma \end{aligned} \tag{1.6.16}$$

Using definitions (1.4.8) of the physical components, we obtain from (1.5.8), (1.5.9) and (1.5.15) the following relations for normal curvatures and geodesic torsions

$$\sigma_t = b_{<\alpha\beta>} t_{<\alpha>} t_{<\beta>} \quad , \quad \sigma_\nu = b_{<\alpha\beta>} \nu_{<\alpha>} \nu_{<\beta>} \quad ,$$
$$\tau_t = \tau_\nu = -b_{<\alpha\beta>} \nu_{<\alpha>} t_{<\beta>} \quad . \tag{1.6.17}$$

With the help of (1.6.16), these become

$$\begin{aligned} \sigma_t &= b_{<11>} \sin^2\gamma - 2b_{<12>} \sin\gamma \cos\gamma + b_{<22>} \cos^2\gamma \\ \sigma_\nu &= b_{<11>} \cos^2\gamma + 2b_{<12>} \sin\gamma \cos\gamma + b_{<22>} \sin^2\gamma \\ \tau_t &= \tau_\nu = (b_{<11>} - b_{<22>}) \sin\gamma \cos\gamma + b_{<12>} (\sin^2\gamma - \cos^2\gamma) \end{aligned} \tag{1.6.18}$$

In particular, for $\gamma = 0$ we obtain from (1.6.18) the formulae for orthogonal coordinate lines:

$$\begin{aligned} \sigma_t = \sigma_2 &= b_{<22>} = -\frac{1}{R_2} \\ \sigma_\nu = \sigma_1 &= b_{<11>} = -\frac{1}{R_1} \\ \tau_t = \tau_\nu &= b_{<12>} = -\frac{1}{R_{12}} \end{aligned} \tag{1.6.19}$$

where R_1 and R_2 are the *radii of normal curvatures* and R_{12} the *radius of geodesic torsion of the orthogonal coordinate lines*, respectively. The sign convention adopted here is compatible with our definition of the surface unit normal $\underset{\sim}{n}$ with respect to the curve principal unit normal $\underset{\sim}{m}$, Fig. 6.

In analogy to (1.6.19) we can define *radii of curvatures* of an arbitrary surface curve C to be

$$\sigma_\nu = -\frac{1}{R_\nu} \quad , \quad \sigma_t = -\frac{1}{R_t} \quad , \quad \tau_t = \tau_\nu = -\frac{1}{R_{\nu t}} \tag{1.6.20}$$

and using (1.6.19) the relations (1.6.18) become

$$\frac{1}{R_\nu} = \frac{\cos^2\gamma}{R_1} - \frac{\sin 2\gamma}{R_{12}} + \frac{\sin^2\gamma}{R_2}$$

$$\frac{1}{R_{\nu t}} = \frac{\cos 2\gamma}{R_{12}} + \frac{1}{2}\sin 2\gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (1.6.21)$$

$$\frac{1}{R_t} = \frac{\sin^2\gamma}{R_1} + \frac{\sin 2\gamma}{R_{12}} + \frac{\cos^2\gamma}{R_2}$$

If θ^I, θ^{II} are *lines of principal curvatures* then from (1.6.20) and (1.6.14) we obtain

$$\sigma_I = -\frac{1}{R_I} \quad , \quad \sigma_{II} = -\frac{1}{R_{II}} \quad , \quad \tau_I = \tau_{II} = \tau = 0 \qquad (1.6.22)$$

where R_I and R_{II} are the *principal radii of curvatures*.

Let γ be an angle between the θ^1 coordinate line and θ^I principal line. Using (1.6.21) for radii of the *orthogonal coordinate lines* θ^1, θ^2 we obtain the relations

$$\frac{1}{R_1} = \frac{\cos^2\gamma}{R_I} + \frac{\sin^2\gamma}{R_{II}}$$

$$\frac{1}{R_{12}} = \frac{1}{2}\sin 2\gamma\left(\frac{1}{R_I} - \frac{1}{R_{II}}\right) \qquad (1.6.23)$$

$$\frac{1}{R_2} = \frac{\sin^2\gamma}{R_I} + \frac{\cos^2\gamma}{R_{II}}$$

For general *non-orthogonal* system of coordinates θ^1, θ^2 it is also possible to use the notion of radii of curvatures. However, the resulting relations in terms of principal radii of curvatures become more complex, therefore we do not discuss them here.

Chapter 2

DEFORMATION OF A SURFACE

By a surface deformation we understand a one-to-one mapping between two surfaces in three-dimensional Euclidean space. One surface is called the *reference* (or undeformed) surface while the second is called the *deformed* one.

Two different descriptions of deformation may be used: Lagrangean or Eulerian. In the Lagrangean description geometry of the reference surface is supposed to be known and all geometric quantities of deformed surface are expressed in terms of geometric quantities of the reference surface and displacement vector between these two surfaces. In the Eulerian description geometry of the reference surface is expressed in terms of deformed surface geometry and displacements.

In this chapter exact relations valid for arbitrary smooth deformation of a surface are discussed in both Lagrangean and Eulerian descriptions. The presentation of Lagrangean description is that suggested by KOITER [6] and by our previous papers [8,10,11,12]. The presentation of Eulerian description follow the authors papers [9,12]. Various exact formulae for linearized strain and rotation quantities as well as for the non-linear surface strain measures are obtained in terms of displacements. Compatibility conditions for the strain measures as well as transformation rules for covariant surface differentiation are derived.

2.1. Deformed surface geometry

Consider two surfaces M and $\bar{M}$ in three dimensional Euclidean space, Fig. 9. A one-to-one mapping

$$\bar{M} = \chi(M) \quad \text{or} \quad \underset{\sim}{\bar{r}} = \chi(\underset{\sim}{r}) \tag{2.1.1}$$

which is non-singular and has a unique inverse, is called a *deformation* of the surface M. We call M to be the *reference* (or undeformed) surface while $\bar{M}$ to be *deformed* one.

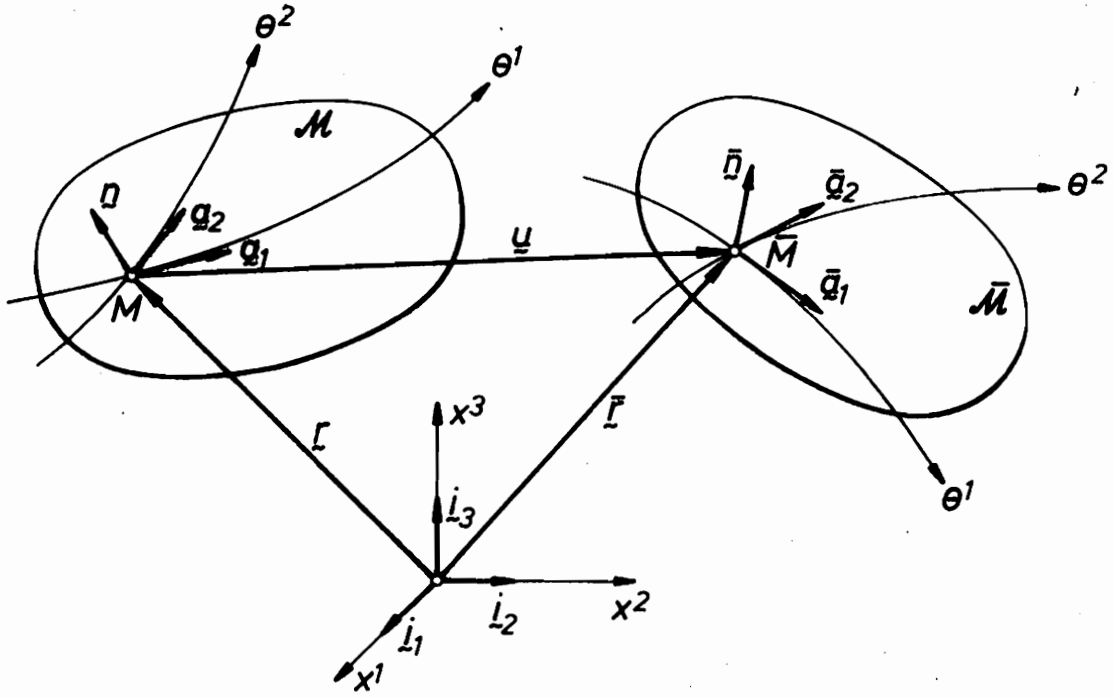

Fig. 9

Using (1.1.2) and (2.1.1) we obtain

$$\bar{\underset{\sim}{r}} = \bar{f}(\theta^1,\theta^2) = \bar{x}^k \underset{\sim}{i}_k$$
$$= \underset{\sim}{r}(\theta^\alpha) + \underset{\sim}{u}(\theta^\alpha) \qquad (2.1.2)$$

where θ^α are the surface *convected coordinates* and the vector $\underset{\sim}{u} = \bar{\underset{\sim}{r}} - \underset{\sim}{r}$ is known as *displacement vector* of the surface points under deformation (2.1.1).

Let us identify all geometric quantities of the reference surface M with those already defined in chapter 1. For the deformed surface similar geometric quantities at $\bar{M} \in \bar{M}$, related to $M \in M$ by (2.1.1), will be distinquished by a dash:
$\bar{\underset{\sim}{a}}_\alpha$, $\bar{\underset{\sim}{n}}$, $\bar{\underset{\sim}{a}}^\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{b}^{\alpha\beta}$, $\bar{H}$, $\bar{K}$, $\bar{\epsilon}_{\alpha\beta}$, $\bar{\epsilon}^{\alpha\beta}$, $\bar{\Gamma}_{\lambda,\alpha\beta}$, $\bar{\Gamma}^\kappa_{\alpha\beta}$, $\bar{R}_{\alpha\beta\lambda\mu}$ etc.
Covariant differentiation with respect to deformed surface metric will be denoted by a double stroke $(\ )_{||\alpha}$, for example $\bar{u}_{\alpha||\beta}$, $\bar{b}^\alpha_{\beta||\lambda}$ etc.

Let us express all geometric quantities of the *deformed* surface $\bar{M}$ in terms of geometric quantities of the *reference* surface M and displacement components with respect to the reference basis $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$. Such an approach is called *Lagrangean*.

Thus if

$$\underset{\sim}{u} = \underset{\sim}{u}(M) = \bar{\underset{\sim}{r}}[\chi(M)] - \underset{\sim}{r}(M) = \\ = u^{\alpha}\underset{\sim}{a}_{\alpha} + w\underset{\sim}{n} = u_{\alpha}\underset{\sim}{a}^{\alpha} + w\underset{\sim}{n} \tag{2.1.3}$$

then from (2.1.2), (1.1.3) and (1.1.21) we obtain

$$\bar{\underset{\sim}{a}}_{\alpha} = \bar{x}^{k}{}_{,\alpha}\underset{\sim}{i}_{k} = (x^{k}{}_{,\alpha} + u^{k}{}_{,\alpha})\underset{\sim}{i}_{k} = \\ = \underset{\sim}{a}_{\alpha} + \underset{\sim}{u}_{,\alpha} = l^{\lambda}{}_{\alpha}\underset{\sim}{a}_{\lambda} + \varphi_{\alpha}\underset{\sim}{n} = l_{\lambda\alpha}\underset{\sim}{a}^{\lambda} + \varphi_{\alpha}\underset{\sim}{n} \tag{2.1.4}$$

where

$$l_{\alpha\beta} = a_{\alpha\lambda}l^{\lambda}{}_{\cdot\beta} = a_{\alpha\beta} + \varphi_{\alpha\beta} \\ \varphi_{\alpha\beta} = u_{\alpha|\beta} - b_{\alpha\beta}w = \theta_{\alpha\beta} - \omega_{\alpha\beta} \\ \varphi_{\alpha} = w_{,\alpha} + b^{\lambda}_{\alpha}u_{\lambda} \tag{2.1.5}$$

The components φ_{α} are known as the *linearized rotations of the normal* to the surface. The quantities $\varphi_{\alpha\beta}$ are the components of a surface tensor. The components of its symmetric part denoted by

$$\theta_{\alpha\beta} = \frac{1}{2}(\varphi_{\alpha\beta} + \varphi_{\beta\alpha}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w \tag{2.1.6}$$

are called the *linearized surface strain tensor* components, the components of its skew-symmetric part with an opposite sign defined by

$$\omega_{\alpha\beta} = \frac{1}{2}(\varphi_{\beta\alpha} - \varphi_{\alpha\beta}) = \frac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) = \epsilon_{\alpha\beta}\varphi \tag{2.1.7}$$

are called the *linearized in-surface rotations*, and the surface invariant φ defined by

$$\varphi = \frac{1}{2}\epsilon^{\alpha\beta}\omega_{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta}u_{\beta|\alpha} \tag{2.1.8}$$

is called the *linearized rotation around the normal.*

A vector $\underset{\sim}{\phi}$ defined by

$$\underset{\sim}{\phi} = \epsilon^{\beta\alpha}\varphi_{\alpha}\underset{\sim}{a}_{\beta} + \varphi\underset{\sim}{n} \tag{2.1.9}$$

is known as the *linearized rotation vector.*

In terms of $\underset{\sim}{\phi}$ the formula (2.1.4) may be written as

$$\bar{\underset{\sim}{a}}_\alpha - \underset{\sim}{a}_\alpha = \underset{\sim}{u}_{,\alpha} = \theta_{\lambda\alpha}\underset{\sim}{a}^\lambda + \underset{\sim}{\phi} \times \underset{\sim}{a}_\alpha \qquad (2.1.10)$$

In terms of these linearized quantities we can define also the componenets $\mu_{\alpha\beta}$ of the *linearized tensor of change of curvatures* of the surface by the relation

$$\mu_{\alpha\beta} = -\frac{1}{2}\,(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha \omega_{\beta\lambda} + b^\lambda_\beta \omega_{\alpha\lambda}) \qquad (2.1.11)$$

The name "linearized" associated with all these Lagrangean quantities comes from the fact that, for very small strains, displacements and rotations the vector $\underset{\sim}{\phi}$ describes the small rotations of the material elements, while $\theta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are small surface strain measures; these are supposed to be the "best" strain measures in the linear theory of shells [13]. All linearized quantities are linear functions of displacements u_α,w. However, for arbitrarily large deformations, different quantities will be needed to describe finite stretches and finite rotations of the surface.

2.2. Lagrangean surface strain measures

The geometry of deformed surface $\bar{M}$ is described by components of the metric tensor $\bar{a}_{\alpha\beta}$ and the curvature tensor $\bar{b}_{\alpha\beta}$, which at $\bar{M}$ should satisfy the Codazzi-Gauss equations. To describe the surface deformation let us define the following *surface strain measures*

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \quad , \quad \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) \qquad (2.2.1)$$

The quantities $\gamma_{\alpha\beta}$ are known as components of the *surface Lagrangean strain tensor* while $\kappa_{\alpha\beta}$ are called the components of the *Lagrangean tensor of change of curvature* of the surface.

It should be noted that the definition of the strain tensor components is generally accepted in shell literature. However,the definition of the changes of surface curvature varies depending on the author. One important difference come from a sign convention adopted. Our sign convention used in series of papers [8-12] agrees with that of GALIMOV [7] and the one used in linear shell theory by GREEN and ZERNA [2], CHERNYKH [3] and NAGHDI [4]. It is opposite to that used by KOITER [6] and KOITER and SIMMONDS [16]. SANDERS [14] and BUDIANSKY [15] overcame the sign convention difficulties by defining with opposite sign their curvature tensor $b_{\alpha\beta}$.

Besides, any functionally independent combination of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ may be chosen as a measure for the surface curvature changes. In what follows we shall use occasionally the Lagrangean tensor $\underset{\sim}{\rho}$ defined by

$$\rho_{\alpha\beta} = \kappa_{\alpha\beta} + \frac{1}{2}\,(b^{\lambda}_{\alpha}\gamma_{\lambda\beta} + b^{\lambda}_{\beta}\gamma_{\lambda\alpha}) \tag{2.2.2}$$

as the *tensor of change of curvature* of the surface. This tensor has the feature, that in the limit for small strains, displacements and rotations it gives us the measure for the linear shell theory (2.1.11) which is supposed to be the "best" choice according to [13]. It should be noted, however, that KRÄTZIG [40] has recently discussed the structure of the linear shell theory with the help of diagrams by TONTI [41,42]. Such diagrams have been constructed in [40] when the linear parts of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ have been chosen as the basic strain measures of the linear shell theory. In what follows we shall use $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ as the preferred strain measures for the non-linear shell theory, although the TONTI diagrams for the non-linear shell theory have not been constructed yet.
Using (2.1.4) and (2.1.5) we obtain

$$\begin{aligned}\bar{a}_{\alpha\beta} &= \bar{\underset{\sim}{a}}_{\alpha}\cdot\bar{\underset{\sim}{a}}_{\beta} = 1^{\lambda}_{\cdot\alpha}1_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta}\\ &= a_{\alpha\beta} + \varphi_{\alpha\beta} + \varphi_{\beta\alpha} + \varphi^{\lambda}_{\cdot\alpha}\varphi_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta}\end{aligned} \tag{2.2.3}$$

and from (2.2.1) we find *general* and *exact formulae* for $\gamma_{\alpha\beta}$ in terms of linearized quantities

$$\begin{aligned}\gamma_{\alpha\beta} &= \frac{1}{2}\,(1^{\lambda}_{\cdot\alpha}1_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta} - a_{\alpha\beta}) =\\ &= \frac{1}{2}\,(\varphi_{\alpha\beta} + \varphi_{\beta\alpha} + \varphi^{\lambda}_{\cdot\alpha}\varphi_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta}) =\\ &= \theta_{\alpha\beta} + \frac{1}{2}\,(\theta^{\lambda}_{\alpha} - \omega^{\lambda}_{\cdot\alpha})\,(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + \frac{1}{2}\,\varphi_{\alpha}\varphi_{\beta}\end{aligned} \tag{2.2.4}$$

Similar relations for $\kappa_{\alpha\beta}$ happen to be more complicated. First let us note that using (1.1.13) and (1.1.15) we obtain

$$\bar{\epsilon}_{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\,\epsilon_{\alpha\beta} \quad , \qquad \bar{\epsilon}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta} \tag{2.2.5}$$

For any surface tensor $\bar{\underset{\sim}{T}} \in \bar{T}_2$ on deformed surface $\bar{M}$ we have the following relation

$$\bar{T} \equiv |\bar{T}_{\alpha\beta}| = |\bar{a}_{\alpha\lambda}\bar{T}^{\lambda}_{\cdot\beta}| = |\bar{a}_{\alpha\lambda}||\bar{T}^{\lambda}_{\cdot\beta}| =$$

$$= \bar{a}\,\frac{1}{2}\,\bar{\epsilon}^{\alpha\lambda}\bar{\epsilon}^{\beta\mu}\bar{T}_{\alpha\beta}\bar{T}_{\lambda\mu} \tag{2.2.6}$$

where (1.1.18) has been used. Applying (2.2.6) to the components $\bar{a}_{\alpha\beta}$ and using (2.2.5) and (2.2.1) we obtain

$$\begin{aligned}\frac{\bar{a}}{a} &= \frac{1}{2}\,\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\bar{a}_{\alpha\beta}\bar{a}_{\lambda\mu} = \\ &= \frac{1}{2}\,\epsilon_{\alpha\lambda}\epsilon^{\beta\mu}(\delta^{\alpha}_{\beta} + 2\gamma^{\alpha}_{\beta})(\delta^{\lambda}_{\mu} + 2\gamma^{\lambda}_{\mu}) = \\ &= 1 + 2\gamma^{\alpha}_{\alpha} + 2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha})\end{aligned} \tag{2.2.7}$$

Thus the surface invariant $\bar{a}/a$ is expressed entirely in terms of the surface strain tensor invariants.

The *unit normal* $\underset{\sim}{\bar{n}}$ of deformed surface $\bar{M}$ follows from (1.1.20):

$$\begin{aligned}\underset{\sim}{\bar{n}} &= \frac{1}{2}\,\bar{\epsilon}^{\alpha\beta}\underset{\sim}{\bar{a}}_{\alpha} \times \underset{\sim}{\bar{a}}_{\beta} = \\ &= n^{\alpha}\underset{\sim}{a}_{\alpha} + n\,\underset{\sim}{n} = n_{\alpha}\underset{\sim}{a}^{\alpha} + n\,\underset{\sim}{n}\end{aligned} \tag{2.2.8}$$

We can obtain the components from (2.1.4), (2.2.5) and (1.1.20):

$$\begin{aligned}n_{\mu} &= \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}\varphi_{\alpha}1^{\lambda}_{\cdot\beta} = \sqrt{\frac{a}{\bar{a}}}\,(\varphi_{\lambda}1^{\lambda}_{\cdot\mu} - \varphi_{\mu}1^{\lambda}_{\cdot\lambda}) \\ n &= \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}1^{\lambda}_{\cdot\alpha}1^{\mu}_{\cdot\beta} = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,(1^{\lambda}_{\cdot\lambda}1^{\mu}_{\cdot\mu} - 1^{\lambda}_{\cdot\mu}1^{\mu}_{\cdot\lambda})\end{aligned} \tag{2.2.9}$$

The *curvature tensor* components $\bar{b}_{\alpha\beta}$ of the deformed surface may be found from

$$\begin{aligned}\bar{b}_{\alpha\beta} &= \underset{\sim}{\bar{n}} \cdot \underset{\sim}{\bar{a}}_{\alpha,\beta} = \underset{\sim}{\bar{n}} \cdot (\underset{\sim}{\bar{a}}_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\underset{\sim}{\bar{a}}_{\lambda}) = \\ &= \underset{\sim}{\bar{n}} \cdot \underset{\sim}{\bar{a}}_{\alpha|\beta} = \underset{\sim}{\bar{n}} \cdot \underset{\sim}{\bar{a}}_{\beta|\alpha}\end{aligned} \tag{2.2.10}$$

Differentiating (2.1.4) with the help of (1.3.16) we obtain

$$\underset{\sim}{\bar{a}}_{\alpha|\beta} = d^{\lambda}_{\cdot\alpha\beta}\underset{\sim}{a}_{\lambda} + d_{\alpha\beta}\underset{\sim}{n} = \underset{\sim}{\bar{a}}_{\beta|\alpha} \tag{2.2.11}$$

where

$$d^{\lambda}_{\cdot\alpha\beta} = 1^{\lambda}_{\cdot\alpha|\beta} - b^{\lambda}_{\beta}\varphi_{\alpha} = d^{\lambda}_{\cdot\beta\alpha}$$
$$= u^{\lambda}|_{\alpha\beta} - b^{\lambda}_{\beta}b^{\mu}_{\alpha}u_{\mu} - b^{\lambda}_{\alpha}w_{,\beta} - b^{\lambda}_{\beta}w_{,\alpha} - b^{\lambda}_{\alpha|\beta}w \qquad (2.2.1)$$

$$d_{\alpha\beta} = \varphi_{\alpha|\beta} + b_{\lambda\beta}1^{\lambda}_{\cdot\alpha} = d_{\beta\alpha}$$
$$= b_{\alpha\beta} + w|_{\alpha\beta} - b^{\lambda}_{\alpha}b_{\lambda\beta}w + b^{\lambda}_{\alpha}u_{\lambda|\beta} + b^{\lambda}_{\beta}u_{\lambda|\alpha} + b^{\lambda}_{\alpha|\beta}u_{\lambda} \qquad (2.2.13)$$

Substituting now (2.2.11) and (2.2.8) into (2.2.10) we obtain from (2.2.1) the following *general* and *exact formulae* for $\kappa_{\alpha\beta}$

$$\kappa_{\alpha\beta} = -(nd_{\alpha\beta} + n_{\lambda}d^{\lambda}_{\cdot\alpha\beta} - b_{\alpha\beta}) \qquad (2.2.14)$$

where in terms of linearized quantities various terms have the following expressions [6] (see also (7.3.8))

$$\sqrt{\frac{\bar{a}}{a}}\,n = 1 + \theta^{\kappa}_{\kappa} + \frac{1}{2}(\theta^{\kappa}_{\kappa})^2 - \frac{1}{2}\theta^{\lambda}_{\mu}\theta^{\mu}_{\lambda} + \varphi^2 \qquad (2.2.15)$$

$$\sqrt{\frac{\bar{a}}{a}}\,n_{\mu} = -(1 + \theta^{\kappa}_{\kappa})\varphi_{\mu} + \varphi^{\lambda}(\theta_{\lambda\mu} - \omega_{\lambda\mu}) \qquad (2.2.16)$$

$$d_{\alpha\beta} = b_{\alpha\beta} + \frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + \frac{1}{2}b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + \frac{1}{2}b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha}) \qquad (2.2.17)$$

$$d^{\lambda}_{\cdot\alpha|\beta} = \theta^{\lambda}_{\alpha|\beta} + \theta^{\lambda}_{\beta|\alpha} - \theta_{\alpha\beta}|^{\lambda} - b_{\alpha\beta}\varphi^{\lambda} \qquad (2.2.18)$$

An expression for $\sqrt{\frac{\bar{a}}{a}}$ follows from (2.2.7) and (2.2.4).

It is easily seen from (2.2.15) to (2.2.18) that, when expressed in terms of linearized quantities, the components $\kappa_{\alpha\beta}$ become extremely complicated *non-rational* (square root!) functions. They become even more complex if expressed in terms of displacements u_{α}, w and nobody as yet has tried to use them without approximation. Nevertheless, the formulae (2.2.14) and (2.2.4) are *exact* and form the general basis for various approximate variants of the non-linear Kirchhoff-Love type shell theory.

2.3. Lagrangean compatibility conditions

Six components $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ of the metric and curvature tensors are not independent from each other, but have to satisfy the Codazzi-Gauss equations (1.3.27), which for deformed surface will take the form:

$$\begin{aligned} &\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}\bar{b}_{\beta\lambda}\|_{\mu} = 0 \\ &\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}(\bar{\Gamma}_{\alpha.\beta\mu,\lambda} + \bar{\Gamma}^{\kappa}_{\alpha\mu}\bar{\Gamma}_{\kappa.\beta\lambda} + \bar{b}_{\alpha\mu}\bar{b}_{\beta\lambda}) = 0 \end{aligned} \tag{2.3.1}$$

From (1.3.2), (1.3.3) and (2.2.1) we obtain

$$\begin{aligned} \bar{\Gamma}_{\lambda.\alpha\beta} &= \frac{1}{2}(\bar{a}_{\lambda\alpha,\beta} + \bar{a}_{\lambda\beta,\alpha} - \bar{a}_{\alpha\beta,\lambda}) = \\ &= \frac{1}{2}(a_{\lambda\alpha,\beta} + a_{\lambda\beta,\alpha} - a_{\alpha\beta,\lambda}) + (\gamma_{\lambda\alpha,\beta} + \gamma_{\lambda\beta,\alpha} - \gamma_{\alpha\beta,\lambda}) \pm \\ &\pm \Gamma^{\kappa}_{\lambda\beta}\gamma_{\kappa\alpha} \pm \Gamma^{\kappa}_{\alpha\beta}\gamma_{\lambda\kappa} \pm \Gamma^{\kappa}_{\lambda\alpha}\gamma_{\kappa\beta} \pm \Gamma^{\kappa}_{\beta\alpha}\gamma_{\lambda\kappa} \pm \Gamma^{\kappa}_{\alpha\lambda}\gamma_{\kappa\beta} \pm \Gamma^{\kappa}_{\beta\lambda}\gamma_{\alpha\kappa} = \\ &= \Gamma_{\lambda.\alpha\beta} + (\gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda}) + 2\Gamma^{\kappa}_{\alpha\beta}\gamma_{\lambda\kappa} = \\ &= (a_{\lambda\kappa} + 2\gamma_{\lambda\kappa})\Gamma^{\kappa}_{\alpha\beta} + (\gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda}) \end{aligned} \tag{2.3.2}$$

or

$$\begin{aligned} \bar{\Gamma}_{\lambda.\alpha\beta} &= \bar{a}_{\lambda\kappa}\Gamma^{\kappa}_{\alpha\beta} + \gamma_{\lambda\alpha\beta} \\ \bar{\Gamma}^{\kappa}_{\alpha\beta} &= \Gamma^{\kappa}_{\alpha\beta} + \bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta} \end{aligned} \tag{2.3.3}$$

where

$$\begin{aligned} \gamma_{\lambda\alpha\beta} &= \gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda} = \\ &= \gamma_{\lambda\alpha,\beta} + \gamma_{\lambda\beta,\alpha} - \gamma_{\alpha\beta,\lambda} - 2\Gamma^{\kappa}_{\alpha\beta}\gamma_{\lambda\kappa} \end{aligned} \tag{2.3.4}$$

are the components of a surface tensor, and the contravariant components $\bar{a}^{\kappa\lambda}$ of the deformed surface metric tensor can be found from (1.1.19) to be

$$\bar{a}^{\kappa\lambda} = \bar{\epsilon}^{\kappa\alpha}\bar{\epsilon}^{\lambda\beta}\bar{a}_{\alpha\beta} = \frac{a}{\bar{a}}(a^{\kappa\lambda} + 2\epsilon^{\kappa\alpha}\epsilon^{\lambda\beta}\gamma_{\alpha\beta}) \tag{2.3.5}$$

Taking partial derivatives of (2.3.3) and writing them, as far as possible, in terms of covariant derivatives we obtain

$$\bar{\Gamma}_{\alpha.\beta\mu,\lambda} = (\bar{a}_{\alpha\kappa}\Gamma^{\kappa}_{\beta\mu} + \gamma_{\alpha\beta\mu})_{,\lambda} =$$

$$= (\bar{a}_{\alpha\kappa,\lambda} \pm \Gamma^{\nu}_{\alpha\lambda}\bar{a}_{\nu\kappa} \pm \Gamma^{\nu}_{\beta\mu}\bar{a}_{\alpha\nu})\Gamma^{\kappa}_{\beta\mu} + \bar{a}_{\alpha\kappa}\Gamma^{\kappa}_{\beta\mu,\lambda} +$$

$$+ \gamma_{\alpha\beta\mu,\lambda} \pm \Gamma^{\kappa}_{\alpha\lambda}\gamma_{\kappa\beta\mu} \pm \Gamma^{\kappa}_{\beta\lambda}\gamma_{\alpha\kappa\mu} \pm \Gamma^{\kappa}_{\mu\lambda}\gamma_{\alpha\beta\kappa} = \qquad (2.3.6)$$

$$= (2\gamma_{\alpha\kappa|\lambda} + \Gamma^{\nu}_{\alpha\lambda}\bar{a}_{\nu\kappa} + \Gamma^{\nu}_{\kappa\lambda}\bar{a}_{\alpha\nu})\Gamma^{\kappa}_{\beta\mu} + \bar{a}_{\alpha\kappa}\Gamma^{\kappa}_{\beta\mu,\lambda} +$$

$$+ \gamma_{\alpha\beta\mu|\lambda} + \Gamma^{\kappa}_{\alpha\lambda}\gamma_{\kappa\beta\mu} + \Gamma^{\kappa}_{\beta\lambda}\gamma_{\alpha\kappa\mu} + \Gamma^{\kappa}_{\mu\lambda}\gamma_{\alpha\beta\kappa}$$

and from (2.3.3) it follows that

$$\bar{\Gamma}^{\kappa}_{\alpha\mu}\bar{\Gamma}_{\kappa.\beta\lambda} = (\Gamma^{\kappa}_{\alpha\mu} + \bar{a}^{\kappa\rho}\gamma_{\rho\alpha\mu})(\bar{a}_{\kappa\nu}\Gamma^{\nu}_{\beta\lambda} + \gamma_{\kappa\beta\lambda}) = \qquad (2.3.7)$$

$$= \bar{a}_{\kappa\nu}\Gamma^{\kappa}_{\alpha\mu}\Gamma^{\nu}_{\beta\lambda} + \Gamma^{\kappa}_{\alpha\mu}\gamma_{\kappa\beta\lambda} + \Gamma^{\kappa}_{\beta\lambda}\gamma_{\kappa\alpha\mu} + \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\kappa\beta\lambda}$$

Contracting some terms in (2.3.6) and (2.3.7) , which are symmetric in α,β or λ,μ indices inside the brackets of (2.3.1), and using (1.3.17) we obtain

$$\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}(\bar{\Gamma}_{\alpha.\beta\mu,\lambda} + \bar{\Gamma}^{\kappa}_{\alpha\mu}\bar{\Gamma}_{\kappa.\beta\lambda}) =$$

$$= \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[\bar{a}_{\alpha\kappa}(\Gamma^{\kappa}_{\beta\mu,\lambda} + \Gamma^{\nu}_{\beta\mu}\Gamma^{\kappa}_{\nu\lambda}) + \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\kappa\beta\lambda} + 2\Gamma^{\kappa}_{\beta\mu}\gamma_{\alpha\kappa|\lambda} +$$

$$+ \gamma_{\alpha\beta|\mu\lambda} + \gamma_{\alpha\mu|\beta\lambda} - \gamma_{\beta\mu|\alpha\lambda} + \Gamma^{\kappa}_{\beta\lambda}(\gamma_{\alpha\kappa|\mu} + \gamma_{\alpha\mu|\kappa} - \gamma_{\kappa\mu|\alpha} + \gamma_{\kappa\alpha|\mu} + \gamma_{\kappa\mu|\alpha} - \gamma_{\alpha\mu|\kappa})] =$$

$$= \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[2\gamma_{\alpha\mu|\beta\lambda} + \frac{1}{2}\bar{a}_{\alpha\kappa}R^{\kappa}_{.\beta\lambda\mu} + \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\kappa\beta\lambda}] \qquad (2.3.8)$$

But from (2.2.5), (1.3.24) and (1.3.25) it follows that

$$\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}\cdot\frac{1}{2}(a_{\alpha\kappa} + 2\gamma_{\alpha\kappa})R^{\kappa}_{.\beta\lambda\mu} = \qquad (2.3.9)$$

$$= \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}(-b_{\alpha\mu}b_{\beta\lambda}) + 2\frac{a}{\bar{a}}K\gamma^{\kappa}_{\kappa}$$

and the first term on the right of (2.3.9) is similar to the final term in the Gauss equation (2.3.1)

With the help of (2.3.3) the Codazzi equations (2.3.1) may be transformed as follows

$$\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}\bar{b}_{\beta\lambda||\mu} = \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[\bar{b}_{\beta\lambda,\mu} - \bar{\Gamma}^{\kappa}_{\beta\mu}\bar{b}_{\kappa\lambda} - \bar{\Gamma}^{\kappa}_{\lambda\mu}\bar{b}_{\beta\kappa}] =$$

$$= \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[(\kappa_{\beta\lambda} - b_{\beta\lambda})_{,\mu} - (\Gamma^{\kappa}_{\beta\mu} + \bar{a}^{\kappa\nu}\gamma_{\nu\beta\mu})(\kappa_{\kappa\lambda} - b_{\kappa\lambda})] = \qquad (2.3.10)$$

$$= \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[\kappa_{\beta\lambda|\mu} - b_{\beta\lambda|\mu} + \bar{a}^{\kappa\nu}(b_{\kappa\lambda} - \kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}]$$

Finally, with the help of (2.3.8), (2.3.9) and (2.3.10), the Codazzi-Gauss equations (2.3.1) lead to the following exact *compatibility conditions* upon $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu} + \bar{a}^{\kappa\nu}(b_{\kappa\lambda} - \kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}] = 0$$

$$K\gamma^{\kappa}_{\kappa} + \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\kappa_{\beta\lambda} + \frac{1}{2}(\kappa_{\alpha\mu}\kappa_{\beta\lambda} - \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda})] = 0 \qquad (2.3.11)$$

As the surface deformation has been defined (2.1.2) entirely by three components of displacement vector $\underset{\sim}{u}$, the compatibility conditions (2.3.11) assert that the six components $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are not independent. An arbitrarily chosen set of six functions $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ of the surface coordinates θ^{α} may describe a deformation of a surface in three-dimensional Euclidean space if and only if the functions satisfy the compatibility conditions.

It is important to note here, that while the conditions (2.3.11) are *quadratic* with respect to $\kappa_{\alpha\beta}$, they are quite *complex* with respect to $\gamma_{\alpha\beta}$. It follows from (2.3.5) and (2.2.7) that we have polynomials of the second degree with respect to $\gamma_{\alpha\beta}$ standing in the denominator of last expressions of (2.3.11). That makes the conditions (2.3.11) very difficult to handle and nobody has tried, as yet, to use them without approximation. Nevertheless, the conditions (2.3.11) are *exact* and may serve as a basis for various approximate variants of the non-linear Kirchhoff-Love type shell theory.

When we express all quantities in (2.3.11) in terms of *linearized* quantities (2.1.5) to (2.1.11), three exact relations between the linearized quantities may be obtained. One would expect them to be extremely complicated. However, it was shown recently by the author [17,18], that in terms of linearized quantities the compatibility conditions (2.3.11) can be reduced *exactly* to the following simple form

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[(\mu_{\beta\lambda} - \frac{1}{2} b^{\kappa}_{\beta}\theta_{\kappa\lambda} - \frac{1}{2} b^{\kappa}_{\lambda}\theta_{\kappa\beta})_{|\mu} - b^{\kappa}_{\lambda}(\theta_{\kappa\beta|\mu} + \theta_{\kappa\mu|\beta} - \theta_{\beta\mu|\kappa})] = 0$$

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\theta_{\alpha\mu|\beta\lambda} + b_{\alpha\mu}\mu_{\beta\lambda}] = 0 \qquad (2.3.12)$$

It is interesting to note here, that these condition are *linear* both in $\theta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ and have exactly the same form as in the linear theory of shells. Within the linear theory [3,4,6], many restrictions have been used to show the conditions are satisfied (small strains, small displacements and rotations). In [17,18] we have shown that the conditions (2.3.9) are satisfied exactly for an arbitrary deformation of the surface in space, without any limitations of strains, displacements or rotations. This may be verified by direct substitution (2.1.6) and (2.1.11) into (2.3.12).

2.4. Eulerian description of deformation

Deformation of a surface has been described as yet entirely in terms of Lagrangean quantities defined with respect to the *reference surface* geometry.

It is possible, however, to describe deformation of a surface in a dual way, in terms of quantities defined with respect to *deformed surface* $\bar{M}$. Such a description of surface deformation is called *Eulerian*.

Let us express all geometrical quantities of the reference surface M in terms of geometrical quantities of deformed surface $\bar{M}$ and displacement vector components with respect to deformed basis

$$\begin{aligned} \underset{\sim}{\bar{u}} &= \underset{\sim}{\bar{u}}(\bar{M}) = \underset{\sim}{\bar{r}}(\bar{M}) - \underset{\sim}{r}[M(\bar{M})] = \\ &= \bar{u}^{\alpha}\underset{\sim}{\bar{a}}_{\alpha} + \bar{w}\underset{\sim}{\bar{n}} = \bar{u}_{\alpha}\underset{\sim}{\bar{a}}^{\alpha} + \bar{w}\underset{\sim}{\bar{n}} \end{aligned} \qquad (2.4.1)$$

Then in a similar way as for Lagrangean quantities, we obtain the following *relations for various Eulerian quantities*

$$\begin{aligned} \underset{\sim}{a}_{\alpha} &= \bar{l}^{\lambda}_{.\alpha}\underset{\sim}{\bar{a}}_{\lambda} + \bar{\varphi}_{\alpha}\underset{\sim}{\bar{n}} = \bar{l}_{\lambda\alpha}\underset{\sim}{\bar{a}}^{\lambda} + \bar{\varphi}_{\alpha}\underset{\sim}{\bar{n}} \\ \underset{\sim}{n} &= \bar{n}_{\alpha}\underset{\sim}{\bar{a}}^{\alpha} + \bar{n}\underset{\sim}{\bar{n}} = \bar{n}^{\alpha}\underset{\sim}{\bar{a}}_{\alpha} + \bar{n}\underset{\sim}{\bar{n}} \end{aligned} \qquad (2.4.2)$$

where

$$\bar{I}_{\alpha\beta} = \bar{a}_{\alpha\lambda}\bar{I}^{\lambda}_{\cdot\beta} = \bar{a}_{\alpha\beta} + \bar{\Phi}_{\alpha\beta}$$
$$\bar{\Phi}_{\alpha\beta} = -\bar{u}_{\alpha||\beta} + \bar{b}_{\alpha\beta}\bar{w} = \bar{\theta}_{\alpha\beta} - \bar{\omega}_{\alpha\beta} \qquad (2.4.3)$$
$$\bar{\varphi}_{\alpha} = -\bar{w}_{,\alpha} - \bar{b}^{\lambda}_{\alpha}\bar{u}_{\lambda}$$

$$\bar{\theta}_{\alpha\beta} = -\frac{1}{2}(\bar{u}_{\alpha||\beta} + \bar{u}_{\beta||\alpha}) + \bar{b}_{\alpha\beta}\bar{w}$$
$$\bar{\omega}_{\alpha\beta} = +\frac{1}{2}(\bar{u}_{\alpha||\beta} - \bar{u}_{\beta||\alpha}) = \bar{\epsilon}_{\alpha\beta}\bar{\varphi} \qquad (2.4.4)$$
$$\bar{\varphi} = \frac{1}{2}\bar{\epsilon}^{\alpha\beta}\bar{\omega}_{\alpha\beta} = \frac{1}{2}\bar{\epsilon}^{\alpha\beta}\bar{u}_{\alpha||\beta}$$

$$\bar{n}_{\mu} = \sqrt{\frac{\bar{a}}{a}}\,\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}_{\lambda\mu}\bar{\varphi}_{\alpha}\bar{I}^{\lambda}_{\cdot\beta} = \sqrt{\frac{\bar{a}}{a}}\,(\bar{\varphi}_{\lambda}\bar{I}^{\lambda}_{\cdot\mu} - \bar{\varphi}_{\mu}\bar{I}^{\lambda}_{\cdot\lambda})$$
$$\bar{n} = \frac{1}{2}\sqrt{\frac{\bar{a}}{a}}\,\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}_{\lambda\mu}\bar{I}^{\lambda}_{\cdot\alpha}\bar{I}^{\mu}_{\cdot\beta} = \frac{1}{2}\sqrt{\frac{\bar{a}}{a}}\,(\bar{I}^{\lambda}_{\cdot\lambda}\bar{I}^{\mu}_{\cdot\mu} - \bar{I}^{\lambda}_{\cdot\mu}\bar{I}^{\mu}_{\cdot\lambda}) \qquad (2.4.5)$$

For an invariant $a/\bar{a}$, from a formula dual do that used in (2.2.7), we obtain

$$\frac{a}{\bar{a}} = \frac{1}{2}\bar{\epsilon}^{\alpha\lambda}\bar{\epsilon}^{\beta\mu}a_{\alpha\beta}a_{\lambda\mu}$$
$$= 1 - 2\bar{\gamma}^{\alpha}_{\alpha} + 2(\bar{\gamma}^{\alpha}_{\alpha}\bar{\gamma}^{\beta}_{\beta} - \bar{\gamma}^{\alpha}_{\beta}\bar{\gamma}^{\beta}_{\alpha}) \qquad (2.4.6)$$

where for the *Eulerian strain measures* we have the formulae

$$\bar{\gamma}_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) =$$
$$= \frac{1}{2}(\bar{a}_{\alpha\beta} - \bar{I}^{\lambda}_{\cdot\alpha}\bar{I}_{\lambda\beta} - \bar{\varphi}_{\alpha}\bar{\varphi}_{\beta}) \qquad (2.4.7)$$

$$\bar{\kappa}_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) =$$
$$= -[\bar{b}_{\alpha\beta} - \bar{n}(\bar{\varphi}_{\alpha||\beta} + \bar{b}^{\lambda}_{\beta}\bar{I}_{\lambda\alpha}) - \bar{n}_{\lambda}(\bar{I}^{\lambda}_{\cdot\alpha||\beta} - \bar{b}^{\lambda}_{\beta}\bar{\varphi}_{\alpha})] \qquad (2.4.8)$$

These strain measures have to satisfy some Eulerian compatibility conditions. These conditions may be obtained along similar lines as the Lagrangean ones (2.3.11). In Codazzi-Gauss equations (2.3.1) we need to express all geometrical quantities of the reference surface M in terms of geometrical quantities of deformed surface $\bar{M}$ and Eulerian strain measures $\bar{\gamma}_{\alpha\beta}$ and $\bar{\kappa}_{\alpha\beta}$. The transformations are quite similar to those presented before in (2.3.2) to (2.3.11). In particular, for Christoffel symbols we

obtain

$$\bar{\Gamma}_{\lambda.\alpha\beta} = a_{\lambda\kappa}\bar{\Gamma}^{\kappa}_{\alpha\beta} - \bar{\gamma}_{\lambda\alpha\beta}$$
$$\bar{\Gamma}^{\kappa}_{\alpha\beta} = \bar{\Gamma}^{\kappa}_{\alpha\beta} - a^{\kappa\lambda}\bar{\gamma}_{\lambda\alpha\beta} \tag{2.4.9}$$

where

$$\bar{\gamma}_{\lambda\alpha\beta} = \bar{\gamma}_{\lambda\alpha}||_{\beta} + \bar{\gamma}_{\lambda\beta}||_{\alpha} - \bar{\gamma}_{\alpha\beta}||_{\lambda}$$
$$a^{\kappa\lambda} = \frac{\bar{a}}{a}(\bar{a}^{\kappa\lambda} - 2\bar{\epsilon}^{\kappa\alpha}\bar{\epsilon}^{\lambda\beta}\bar{\gamma}_{\alpha\beta}) \tag{2.4.10}$$

With the help of (2.4.7) to (2.4.9) some terms in the Codazzi-Gauss equations (1.3.27) may be transformed to the form

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}b_{\beta\lambda}|_{\mu} = \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\{\bar{\kappa}_{\beta\lambda}||_{\mu} + \bar{b}_{\beta\lambda}||_{\mu} + a^{\kappa\nu}(\bar{b}_{\kappa\lambda} + \bar{\kappa}_{\kappa\lambda})\bar{\gamma}_{\nu\beta\mu}\}$$
$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}(\Gamma_{\alpha.\beta\mu,\lambda} + \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa.\beta\lambda}) = -2\frac{\bar{a}}{a}\bar{K}\bar{\gamma}^{\kappa}_{\kappa} + \tag{2.4.11}$$
$$+ \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}(-2\bar{\gamma}_{\alpha\mu}||_{\beta\lambda} - \bar{b}_{\alpha\mu}\bar{b}_{\beta\lambda} + a^{\kappa\nu}\bar{\gamma}_{\kappa\alpha\mu}\bar{\gamma}_{\kappa\beta\lambda})$$

After some additional transformations we obtain the following exact *Eulerian compatibility conditions* for $\bar{\gamma}_{\alpha\beta}$ and $\bar{\kappa}_{\alpha\beta}$ to be satisfied

$$\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[\bar{\kappa}_{\beta\lambda}||_{\mu} + a^{\kappa\nu}(\bar{b}_{\kappa\lambda} + \bar{\kappa}_{\kappa\lambda})\bar{\gamma}_{\nu\beta\mu}] = 0$$
$$\bar{K}\bar{\gamma}^{\kappa}_{\kappa} + \bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}[\bar{\gamma}_{\alpha\mu}||_{\beta\lambda} + \bar{b}_{\alpha\mu}\bar{\kappa}_{\beta\lambda} - \frac{1}{2}(\bar{\kappa}_{\alpha\mu}\bar{\kappa}_{\beta\lambda} + a^{\kappa\nu}\bar{\gamma}_{\kappa\alpha\mu}\bar{\gamma}_{\nu\beta\lambda})] = 0 \tag{2.4.12}$$

It is easy to see that these conditions are really dual to the Lagrangean ones, and could have been written at once, keeping in mind the opposite sign in defintions of Lagrangean and Eulerian strain measures with respect to the metric and curvature tensor components, respectively.

Both descriptions of the surface deformation - Lagrangean and Eulerian - are equivallent from mathematical point of view. However, from physical point of view they are not equivallent, as in most engineering problems it is only the *reference* surface configuration which is *known in advance*. The deformed surface configuration is the one we try to find by solving the problem and, in general, its geometry is not known before the problem is actually solved. Thus from engineering

point of view the *Lagrangean description is preferable* and we will use here, as far as possible, the Lagrangean description.

2.5. Transformations of covariant derivatives

According to (1.3.12) at the deformed surface $\bar{M}$ we have as well

$$\bar{\epsilon}^{\alpha\beta}||_{\gamma} = \bar{\epsilon}_{\alpha\beta}||_{\gamma} = 0 \tag{2.5.1}$$

Using (2.2.5), (2.3.3) and (1.3.12) we obtain

$$\begin{aligned}\bar{\epsilon}^{\alpha\beta}||_{\gamma} &= \bar{\epsilon}^{\alpha\beta}{}_{,\gamma} + \bar{\Gamma}^{\alpha}_{\kappa\gamma}\bar{\epsilon}^{\kappa\beta} + \bar{\Gamma}^{\beta}_{\kappa\gamma}\bar{\epsilon}^{\alpha\kappa} = \\ &= \left(\sqrt{\frac{a}{\bar{a}}}\right)_{,\gamma}\epsilon^{\alpha\beta} + \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}{}_{,\gamma} + \sqrt{\frac{a}{\bar{a}}}\,[(\Gamma^{\alpha}_{\kappa\gamma} + \bar{a}^{\alpha\lambda}\gamma_{\lambda\kappa\gamma})\epsilon^{\kappa\beta} + \\ &+ (\Gamma^{\beta}_{\kappa\gamma} + \bar{a}^{\beta\lambda}\gamma_{\lambda\kappa\gamma})\epsilon^{\alpha\kappa}] = 0\end{aligned} \tag{2.5.2}$$

Contracting (2.5.2) by $\epsilon_{\alpha\beta}$ we obtain

$$2\left(\sqrt{\frac{a}{\bar{a}}}\right)_{,\gamma} + \sqrt{\frac{a}{\bar{a}}}\,(\bar{a}^{\alpha\lambda}\delta^{\kappa}_{\alpha} + \bar{a}^{\beta\lambda}\delta^{\kappa}_{\beta})\gamma_{\lambda\kappa\gamma} = 0 \tag{2.5.3}$$

and transforming in a similar way the second of (2.5.1) we finally find

$$\begin{aligned}\left(\sqrt{\frac{a}{\bar{a}}}\right)_{,\gamma} &= -\sqrt{\frac{a}{\bar{a}}}\,\bar{a}^{\kappa\lambda}\gamma_{\lambda\kappa\gamma} \\ \left(\sqrt{\frac{\bar{a}}{a}}\right)_{,\gamma} &= +\sqrt{\frac{\bar{a}}{a}}\,\bar{a}^{\kappa\lambda}\gamma_{\lambda\kappa\gamma}\end{aligned} \tag{2.5.4}$$

In these lecture notes we shall use frequently some vector and tensor fields defined at points of the reference surface, $\underset{\sim}{v} \in V$, $\underset{\sim}{T} \in T_2$, and at points of the deformed surface $\bar{\underset{\sim}{v}} \in \bar{V}$, $\bar{\underset{\sim}{T}} \in \bar{T}_2$, which are connected by deformation (2.1.1). The components of these fields with respect to appropriate bases will obey the following rules of transformation:

$$\begin{aligned}\bar{v}^{\alpha} &= \sqrt{\frac{a}{\bar{a}}}\,v^{\alpha} \quad , \quad \bar{T}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,T^{\alpha\beta} \\ \bar{v}_{\alpha} &= \sqrt{\frac{a}{\bar{a}}}\,v_{\alpha} \quad , \quad \bar{T}_{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,T_{\alpha\beta}\end{aligned} \tag{2.5.5}$$

Then with the help of (2.5.4) and (2.3.3) we obtain the following *transformation rules of covariant differentiation*

$$\bar{v}^{\alpha}\|_{\gamma} = (\sqrt{\frac{\bar{a}}{a}}\, v^{\alpha})\|_{\gamma} =$$

$$= (\sqrt{\frac{\bar{a}}{a}})_{,\gamma} v^{\alpha} + \sqrt{\frac{\bar{a}}{a}}\, v^{\alpha}{}_{,\gamma} + \sqrt{\frac{\bar{a}}{a}}\,(\Gamma^{\alpha}_{\kappa\gamma} + \bar{a}^{\alpha\lambda}\gamma_{\lambda\kappa\gamma})v^{\kappa} =$$

$$= \sqrt{\frac{\bar{a}}{a}}\,[v^{\alpha}|_{\gamma} + (\bar{a}^{\alpha\lambda}v^{\kappa} - \bar{a}^{\kappa\lambda}v^{\alpha})\gamma_{\lambda\kappa\gamma}] \qquad (2.5.6)$$

$$\bar{T}^{\alpha\beta}\|_{\gamma} = (\sqrt{\frac{\bar{a}}{a}}\, T^{\alpha\beta})\|_{\gamma} = (\sqrt{\frac{\bar{a}}{a}})_{,\gamma} T^{\alpha\beta} + \sqrt{\frac{\bar{a}}{a}}\, T^{\alpha\beta}{}_{,\gamma} +$$

$$+ \sqrt{\frac{\bar{a}}{a}}\,[(\Gamma^{\alpha}_{\kappa\gamma} + \bar{a}^{\alpha\lambda}\gamma_{\lambda\kappa\gamma})T^{\kappa\beta} + (\Gamma^{\beta}_{\kappa\gamma} + \bar{a}^{\beta\lambda}\gamma_{\lambda\kappa\gamma})T^{\alpha\kappa}] = \qquad (2.5.7)$$

$$= \sqrt{\frac{\bar{a}}{a}}[T^{\alpha\beta}|_{\gamma} + (\bar{a}^{\alpha\lambda}T^{\kappa\beta} + \bar{a}^{\beta\lambda}T^{\alpha\kappa} - \bar{a}^{\kappa\lambda}T^{\alpha\beta})\gamma_{\lambda\kappa\gamma}]$$

and similarly

$$(\sqrt{\frac{\bar{a}}{a}}\, v_{\alpha})\|_{\gamma} = \sqrt{\frac{\bar{a}}{a}}\,[v_{\alpha|\gamma} - \bar{a}^{\kappa\lambda}(\gamma_{\lambda\alpha\gamma}v_{\kappa} + \gamma_{\lambda\kappa\gamma}v_{\alpha})] \qquad (2.5.8)$$

$$(\sqrt{\frac{\bar{a}}{a}}\, T_{\alpha\beta})\|_{\gamma} = \sqrt{\frac{\bar{a}}{a}}\,[T_{\alpha\beta|\gamma} - \bar{a}^{\kappa\lambda}(\gamma_{\lambda\alpha\gamma}T_{\kappa\beta} + \gamma_{\lambda\beta\gamma}T_{\alpha\kappa} + \gamma_{\lambda\kappa\gamma}T_{\alpha\beta})] \qquad (2.5.9)$$

In particular case when $\gamma = \alpha$ and $T^{\alpha\beta} = T^{\beta\alpha}$ the *transformation rules* become

$$(\sqrt{\frac{\bar{a}}{a}}\, v^{\alpha})\|_{\alpha} = \sqrt{\frac{\bar{a}}{a}}\, v^{\alpha}|_{\alpha} \qquad (2.5.10)$$

$$(\sqrt{\frac{\bar{a}}{a}}\, T^{\alpha\beta})\|_{\beta} = \sqrt{\frac{\bar{a}}{a}}\,[T^{\alpha\beta}|_{\beta} + \bar{a}^{\alpha\lambda}(2\gamma_{\lambda\kappa|\beta} - \gamma_{\kappa\beta|\lambda})T^{\kappa\beta}] \qquad (2.5.11)$$

It is now quite obvious how to obtain similar relations with $\sqrt{\frac{\bar{a}}{a}}$ as a multiplier in (2.5.5). In the same way one may obtain appropriate dual formulae as well, but in our mainly Lagrangean approach we shall not need them.

Chapter 3

DEFORMATION OF A THIN SHELL

In the previous chapter an exact theory of deformation of a surface has been developed without any reference to shell theory. Any shell is a three-dimensional body in which one dimension is much smaller than two other dimensions. Our intuition suggests, that the thinner the shell the better its three-dimensional behaviour may be approximated by two-dimensional quantities describing deformation of its middle surface. It will be shown in chapter 6, that for small elastic strains and under some assumptions about external surface forces, the strain energy function of the "first approximation" shell theory is indeed expressible in terms of strain measures of the shell middle surface only. Unfortunately, under unrestricted (large) strains such a proof is not yet available and perhaps impossible. Thus, in order to express the shell deformation in terms of quantities defined at its middle surface, some reasonable constraints upon the shell deformation should be introduced. Then methods of constrained continua by WOŹNIAK [43] may give us some estimates of reaction forces due to introduced constraints. We expect these reaction forces to be small in interior domains of the shell.

In this chapter a non-linear deformation theory of a thin shell is developed with the help of Kirchhoff-Love constraints. Under these constraints deformation of a shell is completely described by deformation of the shell middle surface. Geometrical properties of a normal coordinate system in the shell are discussed and Lagrangean and Eulerian descriptions of shell deformation are developed. Then the polar decomposition theorem is used to decompose deformation of a neighbourhood about a point of the middle surface into rigid-body translation, pure stretch along principal directions of strain and rigid-body rotation of principal directions of strain. Lagrangean and Eulerian descriptions differ only by different order of these elementary deformations. The structure of various Lagrangean shell strain tensors is discussed and exact formulae for the finite rotation tensor as well as for the finite rotation vector are derived. By using explicitly the notion of finite rotations we are able to derive here many interesting formulae most of which have not appeared

in the shell literature before.

In the authors thesis [12] general theory of finite rotations in shells has been developed under more relaxed constraints "material elements, normal to the reference shell middle surface, remain straight and tangent at deformed middle surface points, to the real three-dimensional deformed material fibres" . Most of results of this chapter have already been obtained in [12] as a particular case of more general results. Nevertheless, here we derive the relations valid unter K - L constraints independently from the very beginning.

3.1. Normal coordinates

Let us consider a body S consisting of particles $X,Y,\ldots \in S$, [19]. A onte-to-one correspondence between the particles $X \in S$ and points $P \in P \subset E$, in three-dimensional Euclidean space is called the *configuration* of the body S, $P = \kappa(X)$, $\kappa : S \to P \subset E$. If one dimension of the region $P \subset E$, occupied by the body in a configuration κ, is much smaller than its other dimensions, then such a body is called *a shell*.

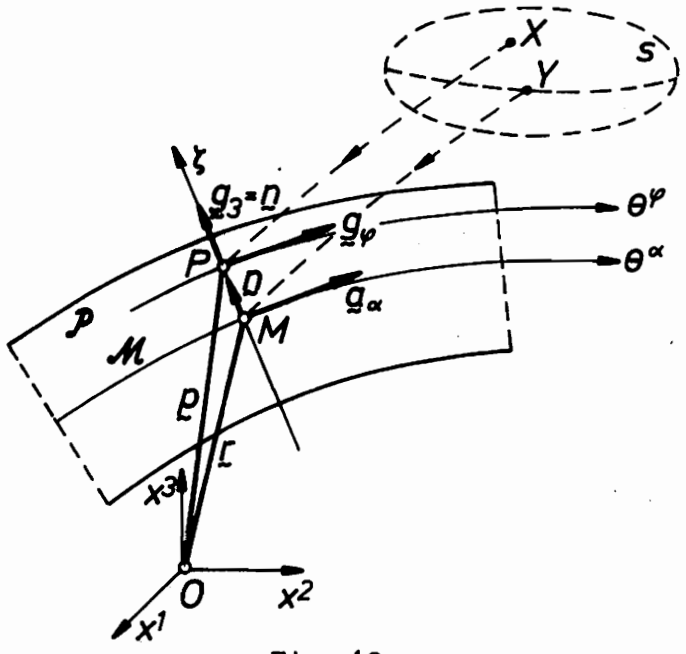

Fig. 10

Let us introduce in the region P the *normal curvilinear coordinate system* $\{\theta^i\}$, $i = 1,2,3$, such that a surface defined by $\theta^3 = 0$ is the *middle surface* M of the region, and $\theta^3 \equiv \zeta$ is the *perpendicular distance* from the surface M, $-\frac{h}{2} \le \zeta \le \frac{h}{2}$, where h is the *thickness* of the shell, Fig. 10. The set of particles $Y = \kappa^{-1}(M)$, $M \in M$, form a material surface called the *middle surface* of a shell S. In normal coordinates we can

easily obtain the following relations [12]

$$\begin{cases} \underset{\sim}{p} = \underset{\sim}{r} + \zeta \underset{\sim}{n} \\ \underset{\sim}{g}_{\varphi} = \underset{\sim}{p}_{,\varphi} = \mu^{\alpha}_{\varphi} \underset{\sim}{a}_{\alpha} \quad , \quad \underset{\sim}{g}_{3} = \underset{\sim}{p}_{,3} = \underset{\sim}{n} \\ \underset{\sim}{g}^{\psi} \cdot \underset{\sim}{g}_{\varphi} = \delta^{\psi}_{\varphi} \quad , \quad \underset{\sim}{g}^{\psi} = \mu^{\psi}_{\beta} \underset{\sim}{a}^{\beta} = g^{\psi\varphi} \underset{\sim}{g}_{\varphi} \quad , \quad \underset{\sim}{g}^{3} = \underset{\sim}{n} \end{cases} \tag{3.1.1}$$

$$\begin{cases} \mu^{\alpha}_{\varphi} = \underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{g}_{\varphi} = \delta^{\alpha}_{\varphi} - \zeta \delta^{\beta}_{\varphi} b^{\alpha}_{\beta} \\ \mu^{\psi}_{\beta} = \underset{\sim}{g}^{\psi} \cdot \underset{\sim}{a}_{\beta} = a_{\alpha\beta} \mu^{\alpha}_{\varphi} g^{\varphi\psi} = \\ \quad = \delta^{\psi}_{\beta} + \zeta \delta^{\psi}_{\alpha} b^{\alpha}_{\beta} + \zeta^{2} \delta^{\psi}_{\alpha} b^{\alpha}_{\lambda} b^{\lambda}_{\beta} + \ldots \\ \mu = |\mu^{\alpha}_{\varphi}| = \sqrt{\frac{g}{a}} = 1 - 2\zeta H + \zeta^{2} K \end{cases} \tag{3.1.2}$$

$$\begin{cases} q_{\varphi\psi} = \underset{\sim}{g}_{\varphi} \cdot \underset{\sim}{g}_{\psi} = \mu^{\alpha}_{\varphi} \mu^{\beta}_{\psi} a_{\alpha\beta} = \\ \quad = \delta^{\alpha}_{\varphi} \delta^{\beta}_{\psi} (a_{\alpha\beta} - 2\zeta b_{\alpha\beta} + \zeta^{2} b^{\lambda}_{\alpha} b_{\lambda\beta}) \\ g^{\varphi\psi} = \underset{\sim}{g}^{\varphi} \cdot \underset{\sim}{g}^{\psi} = \mu^{\varphi}_{\alpha} \mu^{\psi}_{\beta} a^{\alpha\beta} = g^{\varphi\theta} g^{\psi\sigma} g_{\theta\sigma} = \\ \quad = \delta^{\varphi}_{\alpha} \delta^{\psi}_{\beta} (a^{\alpha\beta} + 2\zeta b^{\alpha\beta} + 3\zeta^{2} b^{\alpha}_{\lambda} b^{\lambda\beta} + \ldots) \\ g_{33} = g^{33} = 1 \quad , \quad g_{\varphi 3} = g^{\varphi 3} = 0 \\ g = |g_{ij}| = |g_{\varphi\psi}| \end{cases} \tag{3.1.3}$$

Here $\underset{\sim}{p}$ is the *position vector* of a point $P \in P$, $\underset{\sim}{g}_{\varphi}$ or $\underset{\sim}{g}^{\psi}$, $\varphi,\psi = 1,2$, are the *covariant* or *contravariant base vectors* of a surface M_{ζ} at distance ζ from M, and $g_{\varphi\psi}$ or $g^{\varphi\psi}$ are the *covariant* or *contravariant* components of the surface M_{ζ} *metric tensor*. Besides, $\underset{\sim}{g}_{i}$ or $\underset{\sim}{g}^{j}$ as well as g_{ij} or g^{ij} describe metric properties of three-dimensional Euclidean space E in the normal coordinates θ^{i}.

In a way similar to that used for the surface M, we may introduce in E spatial alternation tensor components ϵ_{ijk}, ϵ^{ijk}, spatial Christoffel symbols $G_{l.ij}$, G^{k}_{ij}, spatial covariant differentiation $(\ )_{;k}$ of spatial tensor components and other geometrical quantities, and obtain the following relations [2,3]:

$$
\begin{aligned}
&\epsilon_{ijk} = (\underset{\sim}{g}_i \times \underset{\sim}{g}_j)\cdot \underset{\sim}{g}_k \quad , \quad \epsilon^{ijk} = (\underset{\sim}{g}^i \times \underset{\sim}{g}^j)\cdot \underset{\sim}{g}^k \\
&G_{l.ij} = \underset{\sim}{g}_l \cdot \underset{\sim}{g}_{i,j} = \frac{1}{2}\,(g_{li,j} + g_{lj,i} - g_{ij,l}) \\
&G^k_{ij} = g^{kl} G_{l.ij} = \underset{\sim}{g}^k \cdot \underset{\sim}{g}_{i,j} = - \underset{\sim}{g}_i \cdot \underset{\sim}{g}^k{}_{,j} \\
&\underset{\sim}{v}_{,k} = v^i{}_{;k}\underset{\sim}{g}_i = v_{i;k}\underset{\sim}{g}^i \\
&v_{i;k} = v_{i,k} - G^m_{ik} v_m \,, \\
&T^i_{\cdot j;k} = T^i_{\cdot j,k} + G^i_{mk} T^m_{\cdot j} - G^m_{jk} T^i_{\cdot m}
\end{aligned}
\tag{3.1.4}
$$

It is important to note here, that for components of various quantities defined at point $P \in \mathcal{P}$ out of the middle surface M we have used different Greek indices $\varphi, \psi, \theta, \sigma = 1,2$, in order to distinguish them from components of similar quantities defined at the middle surface points $M \in M$, for which indices $\alpha, \beta, \lambda, \mu, \kappa = 1,2$ have been used. This convention has been introduced in the author's papers [12,25,35] to be compatible with the notion of absolute tensor analysis.

This convention eliminates possible confusion and allows in the normal and skew coordinate system a simpler description of tensor fields in shells [12,25,35]. In the convention adopted by GREEN and ZERNA [2] or NAGHDI [4] spatial tensor components with respect to $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$ are distinguished by a bar from those with respect to $\underset{\sim}{g}_i$, leaving the index unaltered. Then this bar is omitted in all surface relations, which occasionally may cause some confusion.

The *spatial metric tensor* $\underset{\sim}{1} \in L_2 = \mathcal{W} \otimes \mathcal{W}$ in the normal coordinate system may be represented by components with respect to various basic vectors

$$
\begin{aligned}
\underset{\sim}{1} &= \underset{\sim}{1}^T = g_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^i = \underset{\sim}{g}_i \otimes \underset{\sim}{g}^i = \\
&= a_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta + \underset{\sim}{n} \otimes \underset{\sim}{n} = \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\alpha + \underset{\sim}{n} \otimes \underset{\sim}{n} = \\
&= \mu^\alpha_\varphi \underset{\sim}{a}_\alpha \otimes \underset{\sim}{g}^\varphi + \underset{\sim}{n} \otimes \underset{\sim}{n} = \mu^\psi_\beta \underset{\sim}{g}_\psi \otimes \underset{\sim}{a}^\beta + \underset{\sim}{n} \otimes \underset{\sim}{n}
\end{aligned}
\tag{3.1.5}
$$

from which we see that μ^α_φ and μ^ψ_β are just components of the spatial metric tensor with respect to mixed basis. It is convenient to introduce a *translation tensor* $\underset{\sim}{g} \in L_2$

$$\underset{\sim}{g} \equiv \delta^{\varphi}_{\alpha}\underset{\sim}{g}_{\varphi} \otimes \underset{\sim}{a}^{\alpha} + \underset{\sim}{n} \otimes \underset{\sim}{n} \quad , \quad \underset{\sim}{g}^{-1} = \delta^{\beta}_{\psi}\underset{\sim}{a}_{\beta} \otimes \underset{\sim}{g}^{\psi} + \underset{\sim}{n} \otimes \underset{\sim}{n} \tag{3.1.6}$$

with the help of which we obtain the following formulae:

$$\begin{cases} \underset{\sim}{g} = \underset{\sim}{1} - \zeta\underset{\sim}{b} \\ \underset{\sim}{g}^2 = \underset{\sim}{g}\underset{\sim}{g} = \underset{\sim}{1} - 2\zeta\underset{\sim}{b} + \zeta^2\underset{\sim}{b}^2 \end{cases} \tag{3.1.7}$$

$$\begin{cases} \underset{\sim}{g}_{\varphi} = \delta^{\alpha}_{\varphi}\underset{\sim}{g}\underset{\sim}{a}_{\alpha} \ , \ \underset{\sim}{g}^{\psi} = \mu^{\psi}_{\beta}\underset{\sim}{g}^{-1}\underset{\sim}{a}^{\beta} \\ \underset{\sim}{g}_3 = \underset{\sim}{g}^3 = \underset{\sim}{g}\underset{\sim}{n} \end{cases} \tag{3.1.8}$$

These relations allow us to express in coordinate-free notation quantities at $P \in \mathcal{P}$ in terms of those defined at $M \in \mathcal{M}$ and the distance ζ.

3.2. Lagrangean shell strain measures

In the *Lagrangean* description, the displacement of a particle $X \in \mathcal{S}$ from its position $P = \kappa(X)$ in the reference shell configuration to its position $\bar{P} = \bar{\kappa}(X)$ in the deformed one, Fig. 11, is described by a *spatial displacement vector* referred to the reference basis

$$\underset{\sim}{v} = \underset{\sim}{v}(P) = \bar{\underset{\sim}{p}}[\bar{P}(P)] - \underset{\sim}{p}(P) = v^i\underset{\sim}{g}_i = v_j\underset{\sim}{g}^j \tag{3.2.1}$$

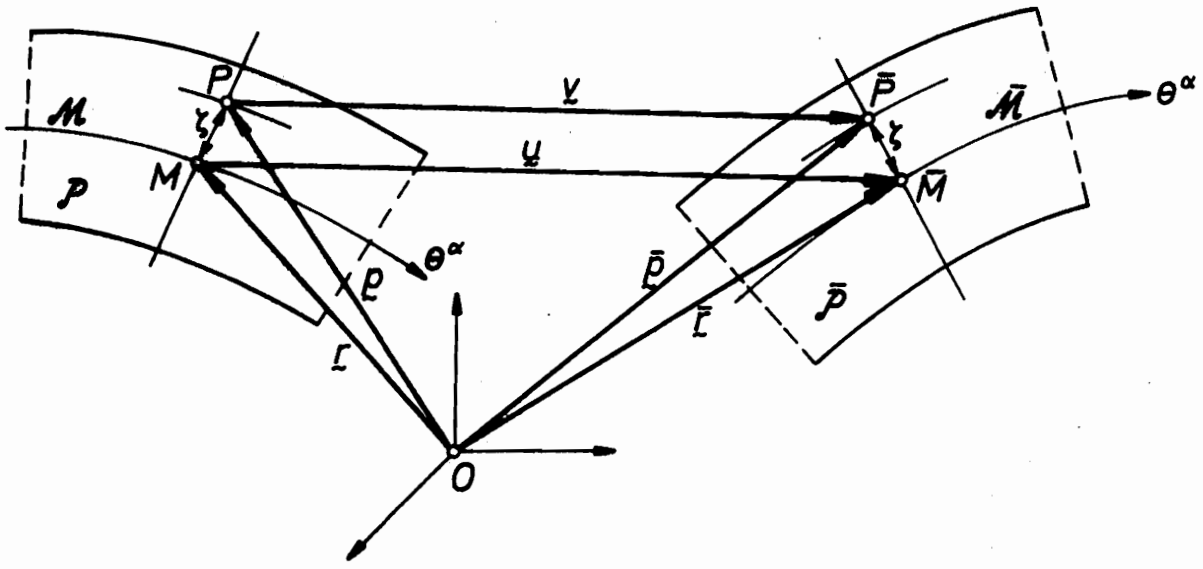

Fig. 11

Thus for differentials

$$d\bar{\underset{\sim}{p}} = \bar{\underset{\sim}{p}}_{,i} d\theta^i = (\underset{\sim}{g}_i + \underset{\sim}{v}_{,i}) d\theta^i = \underset{\sim}{F}\, d\underset{\sim}{p} \tag{3.2.2}$$

where a tensor $\underset{\sim}{F} \in L_2$ has the form

$$\underset{\sim}{F} = \underset{\sim}{F}(P) = \underset{\sim}{1} + \operatorname{grad} \underset{\sim}{v} = \underset{\sim}{g}_i \otimes \underset{\sim}{g}^i + \underset{\sim}{v}_{,i} \otimes \underset{\sim}{g}^i = \bar{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}^i \tag{3.2.3}$$

This tensor is called the *spatial deformation gradient tensor*. It fully describes deformation of a particle neighbourhood during shell deformation from the reference to deformed configuration, respectively.

According to *Kirchhoff-Love constraints*, material elements of the shell, which are normal to the reference shell middle surface M, remain normal to the deformed shell middle surface $\bar{M}$ and do not change their length. Thus at the shell middle surface, under Kirchhoff-Love constraints, the *shell deformation gradient tensor* $\underset{\sim}{G} \in L_2$ has the following form

$$\underset{\sim}{G} = \underset{\sim}{G}(M) \equiv \underset{\sim}{F}(P)\Big|_{P = M} = \bar{\underset{\sim}{a}}_\alpha \otimes \underset{\sim}{a}^\alpha + \bar{\underset{\sim}{n}} \otimes \underset{\sim}{n} \tag{3.2.4}$$

Using (3.2.4), (3.1.6) and (3.1.7) we obtain the following expressions for spatial quantities in terms of quantities defined at the shell middle surface:

$$\underset{\sim}{F} = \bar{\underset{\sim}{g}}\,\underset{\sim}{G}\,\underset{\sim}{g}^{-1} = (\underset{\sim}{G} - \zeta \bar{\underset{\sim}{b}}\,\underset{\sim}{G})\,\underset{\sim}{g}^{-1} \tag{3.2.5}$$

$$\underset{\sim}{v} = \underset{\sim}{u} + \zeta \underset{\sim}{\beta} \tag{3.2.6}$$

where under K - L constraints

$$\underset{\sim}{\beta} = (\underset{\sim}{G} - \underset{\sim}{1})\underset{\sim}{n} = \bar{\underset{\sim}{n}} - \underset{\sim}{n} = n^\alpha \underset{\sim}{a}_\alpha + (n - 1)\underset{\sim}{n} = n_\alpha \underset{\sim}{a}^\alpha + (n - 1)\underset{\sim}{n} \tag{3.2.7}$$

It is shown in continuum mechanics [19], that in the *Lagrangean description* of strain we use mainly *Green strain tensor* $\underset{\sim}{E} \in L_2$ defined by

$$\underset{\sim}{E} = \underset{\sim}{E}(P) = \frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{1}) = E_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \tag{3.2.8}$$

$$E_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i} + g^{kl}v_{k;i}v_{l;j}) = \frac{1}{2}(\bar{g}_{ij} - g_{ij}) \tag{3.2.9}$$

It follows form (3.2.5) that under K - L constraints the tensor $\underset{\sim}{E}$ may be expressed in terms of quantities defined at the shell middle surface according to the relation:

$$\underset{\sim}{E} = \underset{\sim}{g}^{-1}(\underset{\sim}{\gamma} + \zeta\underset{\sim}{\kappa} + \zeta^2\underset{\sim}{\nu})\underset{\sim}{g}^{-1} \tag{3.2.10}$$

where

$$\begin{aligned}
\underset{\sim}{\gamma} &= \frac{1}{2}(\underset{\sim}{G}^T\underset{\sim}{G} - \underset{\sim}{1}) = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta})\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \\
\underset{\sim}{\kappa} &= -(\underset{\sim}{G}^T\underset{\sim}{\bar{b}}\underset{\sim}{G} - \underset{\sim}{b}) = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \\
\underset{\sim}{\nu} &= \frac{1}{2}(\underset{\sim}{G}^T\underset{\sim}{b}^2\underset{\sim}{G} - \underset{\sim}{b}^2) = \frac{1}{2}(\bar{b}^\lambda_\alpha\bar{b}_{\lambda\beta} - b^\lambda_\alpha b_{\lambda\beta})\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta
\end{aligned} \tag{3.2.11}$$

It is easy to recognize that $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$ are the *Lagrangean strain measures* of the middle surface (2.2.1). The third strain measure $\underset{\sim}{\nu}$ is *not independent*, and may be expressed entirely in terms of $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$ by the relation

$$\underset{\sim}{\nu} = \frac{1}{2}[(\underset{\sim}{b} - \underset{\sim}{\kappa})(\underset{\sim}{1} + 2\underset{\sim}{\gamma})^{-1}(\underset{\sim}{b} - \underset{\sim}{\kappa}) - \underset{\sim}{b}^2] \tag{3.2.12}$$

where the first two of (3.2.11) have been used.

Another representation for $\underset{\sim}{\nu}$, which does not contain an inverse operation of tensors, has been obtained by the author [12] with the help of *Caley-Hamilton theorem* applied to the curvature tensor $\underset{\sim}{b}$,

$$\begin{aligned}
&\underset{\sim}{b}^2 - (\mathrm{tr}\,\underset{\sim}{b})\underset{\sim}{b} + (\det\underset{\sim}{b})\underset{\sim}{a} = \underset{\sim}{0} \\
&\mathrm{tr}\,\underset{\sim}{b} = b^\alpha_\alpha = 2H \quad , \quad \det\underset{\sim}{b} = |b^\alpha_\beta| = K
\end{aligned} \tag{3.2.13}$$

Then for the components $\nu_{\alpha\beta}$ from (3.2.11) we obtain

$$\begin{aligned}
\nu_{\alpha\beta} = &-K\gamma_{\alpha\beta} - H\kappa_{\alpha\beta} - \frac{1}{2}(\bar{K} - K)a_{\alpha\beta} + (\bar{H} - H)b_{\alpha\beta} - \\
&-(\bar{K} - K)\gamma_{\alpha\beta} - (\bar{H} - H)\kappa_{\alpha\beta}
\end{aligned} \tag{3.2.14}$$

where

$$\bar{H} - H = \frac{1}{2}\frac{a}{\bar{a}}\left[-\kappa_\alpha^\alpha - 4H(\gamma_\alpha^\alpha + \hat{\gamma}^{\alpha\beta}\gamma_{\alpha\beta}) + 2\hat{\gamma}^{\alpha\beta}(b_{\alpha\beta} - \kappa_{\alpha\beta})\right]$$
$$\bar{K} - K = \frac{1}{2}\frac{a}{\bar{a}}\left[\hat{\kappa}^{\alpha\beta}b_{\alpha\beta} - 4K(\gamma_\alpha^\alpha + \hat{\gamma}^{\alpha\beta}\gamma_{\alpha\beta}) - \hat{\kappa}^{\alpha\beta}(b_{\alpha\beta} - \kappa_{\alpha\beta})\right] \tag{3.2.15}$$

in which

$$\hat{\gamma}^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\gamma_{\lambda\mu} = a^{\alpha\beta}\gamma_\lambda^\lambda - \gamma^{\alpha\beta}$$
$$\hat{\kappa}^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\kappa_{\lambda\mu} = a^{\alpha\beta}\kappa_\lambda^\lambda - \kappa^{\alpha\beta} \tag{3.2.16}$$

Thus, under the K - L constraints, strains in a shell are described entirely by two strain measures $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$ of its middle surface.

3.3. Eulerian shell strain measures

In the *Eulerian description* we use the spatial displacement vector referred to the deformed basis

$$\bar{\underset{\sim}{v}} = \bar{\underset{\sim}{v}}(\bar{P}) = \bar{\underset{\sim}{p}}(\bar{P}) - \underset{\sim}{p}[P(\bar{P})] = \bar{v}^i\bar{\underset{\sim}{g}}_i = \bar{v}_j\bar{\underset{\sim}{g}}^j \tag{3.3.1}$$

Thus for differentials

$$d\underset{\sim}{p} = d\bar{\underset{\sim}{p}} - (\overline{\text{grad}}\ \bar{\underset{\sim}{v}})\ d\bar{\underset{\sim}{p}} = \underset{\sim}{F}^{-1}d\bar{\underset{\sim}{p}} \tag{3.3.2}$$

where

$$\underset{\sim}{F}^{-1} = \underset{\sim}{F}^{-1}(\bar{P}) = \bar{\underset{\sim}{1}} - \overline{\text{grad}}\ \bar{\underset{\sim}{v}} = \bar{\underset{\sim}{g}}_i \otimes \bar{\underset{\sim}{g}}^i - \bar{\underset{\sim}{v}}_{,i} \otimes \bar{\underset{\sim}{g}}^i = \underset{\sim}{g}_i \otimes \bar{\underset{\sim}{g}}^i \tag{3.3.3}$$

$$\underset{\sim}{F}^{-1}\underset{\sim}{F} = \underset{\sim}{1}\ , \quad \underset{\sim}{F}\underset{\sim}{F}^{-1} = \bar{\underset{\sim}{1}} \tag{3.3.4}$$

Thus, according to (3.2.4),

$$\underset{\sim}{G}^{-1} = \underset{\sim}{G}^{-1}(\bar{M}) \equiv \underset{\sim}{F}^{-1}(\bar{P})\Big|_{\bar{P} = \bar{M}} = \underset{\sim}{a}_\alpha \otimes \bar{\underset{\sim}{a}}^\alpha + \underset{\sim}{n} \otimes \bar{\underset{\sim}{n}} \tag{3.3.5}$$

In the Eulerian approach strains are described [19] in terms of the *Almansi strain tensor* $\bar{\underset{\sim}{E}}$ defined by

$$\bar{\underset{\sim}{E}} = \bar{\underset{\sim}{E}}(\bar{P}) = \frac{1}{2}\,[\bar{\underset{\sim}{I}} - (\underset{\sim}{F}^{-1})^{T}\underset{\sim}{F}^{-1}] = \bar{E}_{ij}\bar{\underset{\sim}{g}}^{i} \otimes \bar{\underset{\sim}{g}}^{j} \tag{3.3.6}$$

$$\bar{E}_{ij} = \frac{1}{2}\,(\bar{g}_{ij} - g_{ij}) = \frac{1}{2}\,(\bar{v}_{i;|j} + \bar{v}_{j;|i} - \bar{g}^{kl}\bar{v}_{k;|i}\bar{v}_{l;|j}) \tag{3.3.7}$$

In terms of surface quantities

$$\underset{\sim}{F}^{-1} = (\bar{\underset{\sim}{g}}\,\underset{\sim}{G}\,\underset{\sim}{g}^{-1})^{-1} = (\underset{\sim}{G}^{-1} - \zeta\,\underset{\sim}{b}\,\underset{\sim}{G}^{-1})\bar{\underset{\sim}{g}}^{-1} \tag{3.3.8}$$

and (3.3.6) takes now the form

$$\bar{\underset{\sim}{E}} = (\bar{\underset{\sim}{g}}^{-1})^{T}(\bar{\underset{\sim}{\gamma}} + \zeta\bar{\underset{\sim}{\kappa}} + \zeta^{2}\bar{\underset{\sim}{\nu}})\bar{\underset{\sim}{g}}^{-1} \tag{3.3.9}$$

where

$$\begin{aligned}
\bar{\underset{\sim}{\gamma}} &= \frac{1}{2}\,[\bar{\underset{\sim}{I}} - (\underset{\sim}{G}^{-1})^{T}\underset{\sim}{G}^{-1}] = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta})\bar{\underset{\sim}{a}}^{\alpha} \otimes \bar{\underset{\sim}{a}}^{\beta} \\
\bar{\underset{\sim}{\kappa}} &= -[\bar{\underset{\sim}{b}} - (\underset{\sim}{G}^{-1})\underset{\sim}{b}\,\underset{\sim}{G}^{-1}] = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})\bar{\underset{\sim}{a}}^{\alpha} \otimes \bar{\underset{\sim}{a}}^{\beta} \\
\bar{\underset{\sim}{\nu}} &= \frac{1}{2}\,[\bar{\underset{\sim}{b}}^{2} - (\underset{\sim}{G}^{-1})^{T}\underset{\sim}{b}^{2}\underset{\sim}{G}^{-1}] = \frac{1}{2}\,(\bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta} - b^{\lambda}_{\alpha}b_{\lambda\beta})\bar{\underset{\sim}{a}}^{\alpha} \otimes \bar{\underset{\sim}{a}}^{\beta}
\end{aligned} \tag{3.3.10}$$

and $\bar{\underset{\sim}{\nu}}$ is expressible in terms of $\bar{\underset{\sim}{\gamma}}$ and $\bar{\underset{\sim}{\kappa}}$ by relations dual to (3.2.12) or (3.2.14).

By comparison of (3.2.11) and (3.3.10) the following *transformation rules* between Eulerian and Lagrangean strain measures are established

$$\begin{aligned}
\underset{\sim}{\gamma} &= \underset{\sim}{G}^{T}\bar{\underset{\sim}{\gamma}}\,\underset{\sim}{G} \quad , \quad \bar{\underset{\sim}{\gamma}} = (\underset{\sim}{G}^{-1})^{T}\underset{\sim}{\gamma}\,\underset{\sim}{G}^{-1} \\
\underset{\sim}{\kappa} &= \underset{\sim}{G}^{T}\bar{\underset{\sim}{\kappa}}\,\underset{\sim}{G} \quad , \quad \bar{\underset{\sim}{\kappa}} = (\underset{\sim}{G}^{-1})^{T}\underset{\sim}{\kappa}\,\underset{\sim}{G}^{-1} \\
\underset{\sim}{\nu} &= \underset{\sim}{G}^{T}\bar{\underset{\sim}{\nu}}\,\underset{\sim}{G} \quad , \quad \bar{\underset{\sim}{\nu}} = (\underset{\sim}{G}^{-1})^{T}\underset{\sim}{\nu}\,\underset{\sim}{G}^{-1}
\end{aligned} \tag{3.3.11}$$

It is important to note here, that in convected coordinates the Lagrangean and Eulerian strain measures, when given by their components, have exactly *the same formal definitions*, (2.2.1) and (2.4.7), (2.4.8). However, these components are related to *different tensor basis*, which is obvious from tensor definitions (3.2.11) and (3.2.10). Unfortunately, according to convention usually applied together with convected coordinates,

the same Greek indices are used to describe vector and tensor components associated with *the same material particle* with respect both to reference and deformed bases. Most papers on the non-linear shell theory are written entirely in terms of component notation, and the important geometrical difference between Lagrangean and Eulerian description of strain is thus hidden from view. Here we shall use occasionally this absolute tensor notation to show this difference explicitly.

3.4. Polar decomposition of shell deformation

The fundamental feature of any non-linear shell deformation is the occurence of finite rotations of the material elements. Finite rotations may appear even within small strains, in the case of thin bodies: bars, thin-walled bars, plates and shells.

In the non-linear continuum mechanics the rotational part of deformation, described by a finite rotation tensor $\underset{\sim}{R}$, has been discussed by TRUESDELL and NOLL [19]. An equivallent description of rotational part of continuum deformation, by means of finite rotation vector $\underset{\sim}{\Omega}$, has been used recently by SHAMINA [20] in her discussion of various compatibility conditions in continuum mechanics. An exact theory of finite rotations in shells, in terms of the finite rotation tensor $\underset{\sim}{R}$ and the finite rotation vector $\underset{\sim}{\Omega}$, has been developed in the author's thesis [12]. Here we present the special case of [12] compatible with Kirchhoff-Love constraints.

According to (3.2.5) and (3.2.7), the deformation gradient tensor $\underset{\sim}{G}$ provides complete information about the shell non-linear deformation compatible with K - L constraints. As $\underset{\sim}{G}$ is non-singular ($\in L_2^N$), then according to polar decomposition theorem [19] it can be uniquely represented by the following two formulas

$$\underset{\sim}{G} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R} \quad , \quad \underset{\sim}{G}^{-1} = \underset{\sim}{U}^{-1}\underset{\sim}{R}^{T} = \underset{\sim}{R}^{T}\underset{\sim}{V}^{-1} \tag{3.4.1}$$

where $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are right and left *stretch tensors*, respectively, and $\underset{\sim}{R}$ is the *finite rotation tensor*. The tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are positive definite ($\in L_2^+$) and symmetric ($\in L_2^S$) while $\underset{\sim}{R}$ is the proper orthogonal tensor ($\in L_2^{O+}$)

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{G}^{T}\underset{\sim}{G}} \quad , \quad \underset{\sim}{V} = \sqrt{\underset{\sim}{G}\underset{\sim}{G}^{T}} \quad , \quad \underset{\sim}{U} = \underset{\sim}{R}^{T}\underset{\sim}{V}\underset{\sim}{R} \tag{3.4.2}$$

$$\underset{\sim}{R}^T\underset{\sim}{R} = \underset{\sim}{R}\underset{\sim}{R}^T = \underset{\sim}{1} \quad , \quad \underset{\sim}{R}^T = \underset{\sim}{R}^{-1} \quad , \quad \det \underset{\sim}{R} = +1 \tag{3.4.3}$$

Decomposition of $\underset{\sim}{G}$ in terms of $\underset{\sim}{U}$ in (3.4.1) is compatible with the Lagrangean description of deformation; in terms of $\underset{\sim}{V}$ it is compatible with the Eulerian one.

From (3.2.4) and (3.4.1) we obtain

$$\underset{\sim}{\bar{a}}_\alpha = \underset{\sim}{G}\underset{\sim}{a}_\alpha = \underset{\sim}{R}\overset{V}{\underset{\sim}{a}}_\alpha = \underset{\sim}{V}\overset{*}{\underset{\sim}{a}}_\alpha \quad , \quad \underset{\sim}{\bar{n}} = \underset{\sim}{G}\underset{\sim}{n} = \underset{\sim}{R}\underset{\sim}{n} \tag{3.4.4}$$

$$\underset{\sim}{\bar{a}}^\alpha = (\underset{\sim}{G}^{-1})^T\underset{\sim}{a}^\alpha = \underset{\sim}{R}\overset{V\alpha}{\underset{\sim}{a}} = \underset{\sim}{V}^{-1}\overset{*\alpha}{\underset{\sim}{a}} \tag{3.4.5}$$

where

$$\overset{V}{\underset{\sim}{a}}_\alpha = \underset{\sim}{U}\underset{\sim}{a}_\alpha \quad , \quad \overset{V\alpha}{\underset{\sim}{a}} = \underset{\sim}{U}^{-1}\underset{\sim}{a}^\alpha \tag{3.4.6}$$

$$\overset{*}{\underset{\sim}{a}}_\alpha = \underset{\sim}{R}\underset{\sim}{a}_\alpha \quad , \quad \overset{*\alpha}{\underset{\sim}{a}} = \underset{\sim}{R}\underset{\sim}{a}^\alpha \tag{3.4.7}$$

The *intermediate Lagrangean basis* $\overset{V}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ is obtained by stretching the reference basis $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$ with the help of stretch tensor $\underset{\sim}{U}$. The *intermediate Eulerian basis* $\overset{*}{\underset{\sim}{a}}$, $\underset{\sim}{\bar{n}}$ is obtained by rigid-body rotation of the reference basis $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$ by means of the finite rotation tensor $\underset{\sim}{R}$. Thus for these tensors we obtain the following exact formulae:

$$\begin{aligned}
&\underset{\sim}{U} = \overset{V}{\underset{\sim}{a}}_\alpha \otimes \underset{\sim}{a}^\alpha + \underset{\sim}{n} \otimes \underset{\sim}{n} \quad , \quad \underset{\sim}{U}^{-1} = \underset{\sim}{a}_\alpha \otimes \overset{V\alpha}{\underset{\sim}{a}} + \underset{\sim}{n} \otimes \underset{\sim}{n} \\
&\underset{\sim}{V} = \underset{\sim}{\bar{a}}_\alpha \otimes \overset{*\alpha}{\underset{\sim}{a}} + \underset{\sim}{\bar{n}} \otimes \underset{\sim}{\bar{n}} \quad , \quad \underset{\sim}{V}^{-1} = \overset{*}{\underset{\sim}{a}}_\alpha \otimes \underset{\sim}{\bar{a}}^\alpha + \underset{\sim}{\bar{n}} \otimes \underset{\sim}{\bar{n}} \\
&\underset{\sim}{R} = \underset{\sim}{\bar{a}}_\alpha \otimes \overset{V\alpha}{\underset{\sim}{a}} + \underset{\sim}{\bar{n}} \otimes \underset{\sim}{n} = \overset{*}{\underset{\sim}{a}}_\alpha \otimes \underset{\sim}{a}^\alpha + \underset{\sim}{\bar{n}} \otimes \underset{\sim}{n}
\end{aligned} \tag{3.4.8}$$

In what follows we shall use the intermediate basis $\overset{V}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ compatible with Lagrangean description of deformation. The basis $\overset{*}{\underset{\sim}{a}}$, $\underset{\sim}{n}$ was used by SIMMONDS and DANIELSON [22,23], who took $\underset{\sim}{\Omega}$ as an independent parameter of the shell deformation. The relations between $\underset{\sim}{\Omega}$ and displacements have not been discussed in [22, 23].

3.5. Lagrangean surface strain tensors

It follows from (3.2.11) and (3.4.2) that

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{1} - 2\underset{\sim}{\gamma}} \quad , \quad \underset{\sim}{\gamma} = \frac{1}{2}\,(\underset{\sim}{U}^2 - \underset{\sim}{1}) \tag{3.5.1}$$

and from (2.2.1) it follows that $\underset{\sim}{U}$ becomes the non-rational square-root function when expressed in terms of displacements.

Let us discuss now the structure of the Lagrangean surface strain measures. As the tensor $\underset{\sim}{U} = \underset{\sim}{U}(M)$ is *positive definite and symmetric* then, according to a spectral decomposition theorem [36] it has *three real and positive eigenvalues* U_r, $r = 1,2,3$ in three orthogonal principal directions defined by triad of unit vectors $\underset{\sim}{k}_r$, which satisfy the set of equations

$$\underset{\sim}{U}\,\underset{\sim}{k}_r - U_r\underset{\sim}{k}_r = \underset{\sim}{0} \quad , \quad \underset{\sim}{k}_r \cdot \underset{\sim}{k}_s = \delta_{rs} \tag{3.5.2}$$

It is seen from (3.4.8) that $\underset{\sim}{n}$ is one of the principal directions, say $\underset{\sim}{k}_3 \equiv \underset{\sim}{n}$, and the eigenvalue in this direction is equal to 1. Thus in this spatial orthonormal basis tensor $\underset{\sim}{U}$ has the following diagonal form

$$\underset{\sim}{U} = U_1\underset{\sim}{k}_1 \otimes \underset{\sim}{k}_1 + U_2\underset{\sim}{k}_2 \otimes \underset{\sim}{k}_2 + \underset{\sim}{n} \otimes \underset{\sim}{n} \tag{3.5.3}$$

The tensor $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}(M)$ is *symmetric* as well, although in general it is neither non-singular nor positive definite. Thus [36] it has *three real eigenvalues* γ_r in three principal direction which, according to (3.5.1), are the same as for $\underset{\sim}{U}$. Thus $\underset{\sim}{\gamma}$ may be put in the following diagonal form

$$\underset{\sim}{\gamma} = \gamma_1\underset{\sim}{k}_1 \otimes \underset{\sim}{k}_1 + \gamma_2\underset{\sim}{k}_2 \otimes \underset{\sim}{k}_2 \tag{3.5.4}$$

where

$$\gamma_\rho = \frac{1}{2}\,(U_\rho^2 - 1) \quad , \quad U_\rho = \sqrt{1 + 2\gamma_\rho} \quad , \quad \rho = 1,2 \tag{3.5.5}$$

and the third eigenvalue of $\underset{\sim}{\gamma}$ is equal to 0, according to K - L constraints.

In what follows we shall use a *modified Lagrangean surface strain tensor* $\overset{v}{\underset{\sim}{\gamma}}$ defined in terms of $\underset{\sim}{U}$ or $\underset{\sim}{\gamma}$ by

$$\overset{v}{\underset{\sim}{\gamma}} = \underset{\sim}{U} - \underset{\sim}{1} = \sqrt{\underset{\sim}{1} + 2\underset{\sim}{\gamma}} - \underset{\sim}{1} = \overset{v}{\gamma}_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \tag{3.5.6}$$

It is important to note here, that a numerous relations which are *non-rational* in terms of $\underset{\sim}{\gamma}$ become *polynomials* when expressed in terms of $\overset{v}{\underset{\sim}{\gamma}}$. For example, using (3.4.6), (3.5.6) and (3.2.11) we obtain in terms of $\overset{v}{\underset{\sim}{\gamma}}$

$$\begin{aligned}
\overset{v}{\underset{\sim}{a}}_\alpha &= \underset{\sim}{a}_\alpha + \overset{v}{\gamma}_{\alpha\beta}\underset{\sim}{a}^\beta = (\delta_\alpha^\beta + \overset{v}{\gamma}{}_\alpha^\beta)\underset{\sim}{a}_\beta \\
\bar{a}_{\alpha\beta} &= (\delta_\alpha^\lambda + \overset{v}{\gamma}{}_\alpha^\lambda)(\delta_\beta^\mu + \overset{v}{\gamma}{}_\beta^\mu)a_{\lambda\mu} \\
2\gamma_{\alpha\beta} &= 2\overset{v}{\gamma}_{\alpha\beta} + \overset{v}{\gamma}{}_\alpha^\lambda\overset{v}{\gamma}_{\lambda\beta}
\end{aligned} \tag{3.5.7}$$

and from (1.1.13) and (2.2.5)

$$\begin{aligned}
\bar{\epsilon}_{\alpha\beta} &= \sqrt{\frac{\bar{a}}{a}}\epsilon_{\alpha\beta} = (\delta_\alpha^\lambda + \overset{v}{\gamma}{}_\alpha^\lambda)(\delta_\beta^\mu + \overset{v}{\gamma}{}_\beta^\mu)\epsilon_{\lambda\mu} \\
\sqrt{\frac{\bar{a}}{a}} &= \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(\delta_\alpha^\lambda + \overset{v}{\gamma}{}_\alpha^\lambda)(\delta_\beta^\mu + \overset{v}{\gamma}{}_\beta^\mu) \\
\bar{\epsilon}^{\lambda\mu} &= \sqrt{\frac{a}{\bar{a}}}\epsilon^{\lambda\mu} = \frac{a}{\bar{a}}(\delta_\alpha^\lambda + \overset{v}{\gamma}{}_\alpha^\lambda)(\delta_\beta^\mu + \overset{v}{\gamma}{}_\beta^\mu)\epsilon^{\alpha\beta}
\end{aligned} \tag{3.5.8}$$

which enable us to obtain as well

$$\begin{aligned}
\bar{a}^{\alpha\beta} &= \frac{a}{\bar{a}}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}(\delta_\lambda^\gamma + \overset{v}{\gamma}{}_\lambda^\gamma)(\delta_\mu^\rho + \overset{v}{\gamma}{}_\mu^\rho)a_{\gamma\rho} \\
\overset{v}{\underset{\sim}{a}}{}^\alpha &= \sqrt{\frac{a}{\bar{a}}}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(\delta_\beta^\mu + \overset{v}{\gamma}{}_\beta^\mu)\underset{\sim}{a}^\lambda
\end{aligned} \tag{3.5.9}$$

Additional results are given in [12]. Note that all of these formulae are polynomials in terms of $\overset{v}{\underset{\sim}{\gamma}}$.

3.6. Finite rotation tensor

From (3.4.8), using (2.1.4), (2.2.8) and (3.5.7) we obtain the following general expression for $\underset{\sim}{R}$ in terms of displacements

$$\underset{\sim}{R} = \bar{a}^{\alpha\beta}(\underset{\sim}{a}_\alpha + \underset{\sim}{u}_{,\alpha}) \otimes (\underset{\sim}{a}_\beta + \overset{v}{\gamma}_{\beta\lambda}\underset{\sim}{a}^\lambda) + (n^\alpha\underset{\sim}{a}_\alpha + n_3\underset{\sim}{n}) \otimes \underset{\sim}{n} \tag{3.6.1}$$

According to the spectral decomposition theorem [35] the proper orthogonal tensor $\underset{\sim}{R}$ has only *one real eigenvalue* equal to +1 and two complex conjugate eigenvalues. Let $\underset{\sim}{e}_1$ be unit vector of the principal direction corresponding to real eigenvalue of $\underset{\sim}{R}$. Choosing arbitrarily a unit vector $\underset{\sim}{e}_2 \perp \underset{\sim}{e}_1$ we may define uniquely a third unit vector $\underset{\sim}{e}_3 = \underset{\sim}{e}_1 \times \underset{\sim}{e}_2$ as well, and in this orthonormal right-handed basis $\underset{\sim}{e}_r$

the finite rotation tensor takes the form [12]

$$\underset{\sim}{R} = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \cos\omega\,(\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3) - \sin\omega\,(\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3 - \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_2) \qquad (3.6.2)$$

Direction defined by $\underset{\sim}{e}_1$ is called the *axis of rotation* of $\underset{\sim}{R}$ and the angle ω, $|\omega| < \pi$, is called the *angle of finite rotation* around the axis of rotation.

It is easy to show geometrically how tensor $\underset{\sim}{R}$ acts on a vector $\underset{\sim}{v}$, Fig. 12.

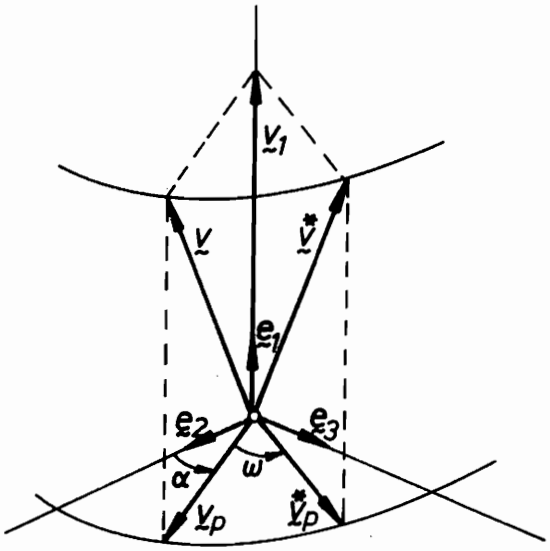

Fig. 12

A vector $\underset{\sim}{v}$ may be decomposed into $\underset{\sim}{v}_1$ along e_1 and $\underset{\sim}{v}_p \perp \underset{\sim}{e}_1$. Then if $\alpha \equiv \sphericalangle(\underset{\sim}{e}_2, \underset{\sim}{v}_p)$ the vector $\underset{\sim}{v}$ has the form

$$\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_p = v_1\underset{\sim}{e}_1 + v_p(\cos\alpha\,\underset{\sim}{e}_2 + \sin\alpha\,\underset{\sim}{e}_3) \qquad (3.6.3)$$

If the vector $\underset{\sim}{v}$ is acted on by the finite rotation tensor $\underset{\sim}{R}$, then we obtain a new vector $\overset{*}{\underset{\sim}{v}}$ which in the frame defined by $\underset{\sim}{e}_r$ takes the form

$$\begin{aligned}\overset{*}{\underset{\sim}{v}} = \underset{\sim}{R}\,\underset{\sim}{v} &= v_1\underset{\sim}{e}_1 + v_p\cos\alpha\,(\cos\omega\,\underset{\sim}{e}_2 + \sin\omega\,\underset{\sim}{e}_3) + \\ &+ v_p\sin\alpha\,(\cos\omega\,\underset{\sim}{e}_3 - \sin\omega\,\underset{\sim}{e}_3) \\ &= v_1\underset{\sim}{e}_1 + v_p[\cos(\alpha+\omega)\underset{\sim}{e}_2 + \sin(\alpha+\omega)\underset{\sim}{e}_3]\end{aligned} \qquad (3.6.4)$$

It is evident now, that the tensor $\underset{\sim}{R}$ rotates the vector $\underset{\sim}{v}$ through the angle ω around the axes of rotation defined by $\underset{\sim}{e}_1$.

3.7. Finite rotation vector

It is possible to describe finite rotations by means of an equivallent *finite rotation vector* $\underset{\sim}{\Omega}$, which has direction defined by $\underset{\sim}{e}_1 \equiv \underset{\sim}{e}$ and magnitude equal to $|\sin\omega|$. Thus for $|\omega| < \pi$

$$\underset{\sim}{\Omega} \equiv \sin\omega\, \underset{\sim}{e} \tag{3.7.1}$$

In terms of $\underset{\sim}{\Omega}$ for the components of $\underset{\sim}{v}$ we obtain, Fig. 12,

$$\begin{aligned} \underset{\sim}{v}_1 &= \frac{1}{\sin^2\omega}\,(\underset{\sim}{\Omega}\cdot\underset{\sim}{v})\underset{\sim}{\Omega} \\ \underset{\sim}{v}_p &= \underset{\sim}{v} - \frac{1}{\sin^2\omega}\,(\underset{\sim}{\Omega}\cdot\underset{\sim}{v})\underset{\sim}{\Omega} \end{aligned} \tag{3.7.2}$$

Under rotation through the angle ω the vector $\underset{\sim}{v}_1$ remains unchanged, while $\underset{\sim}{v}_p$ swings into the new vector, Fig. 13,

$$\begin{aligned} \overset{*}{\underset{\sim}{v}}_p &= \cos\omega\,\underset{\sim}{v}_p + \underset{\sim}{\Omega}\times\underset{\sim}{v}_p = \\ &= \cos\omega\,\underset{\sim}{v} + \underset{\sim}{\Omega}\times\underset{\sim}{v} - \frac{\cos\omega}{\sin^2\omega}\,(\underset{\sim}{\Omega}\cdot\underset{\sim}{v})\underset{\sim}{\Omega} \end{aligned} \tag{3.7.3}$$

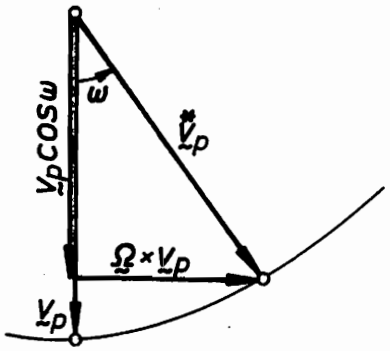

Fig. 13

Using the indentities

$$\begin{aligned} \cos\omega &= 1-2\sin^2\omega/2 \quad , \quad \sin\omega = 2\sin\omega/2\,\cos\omega/2 \\ \underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{v}) &= (\underset{\sim}{\Omega}\cdot\underset{\sim}{v})\underset{\sim}{\Omega} - \sin^2\omega\,\underset{\sim}{v} \end{aligned} \tag{3.7.4}$$

we finally obtain

$$\begin{aligned} \overset{*}{\underset{\sim}{v}} &= \underset{\sim}{v} + \underset{\sim}{\Omega}\times\underset{\sim}{v} + \frac{1}{2\cos^2\omega/2}\,\underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{v}) = \\ &= \cos\omega\,\underset{\sim}{v} + \underset{\sim}{\Omega}\times\underset{\sim}{v} + \frac{1}{2\cos^2\omega/2}\,(\underset{\sim}{\Omega}\cdot\underset{\sim}{v})\underset{\sim}{\Omega} \end{aligned} \tag{3.7.5}$$

In particular, using (3.4.4) we obtain the relations for deformed shell middle surface basic vectors

$$\begin{aligned} \bar{\underset{\sim}{a}}_\alpha &= \overset{v}{\underset{\sim}{a}}_\alpha + \underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_\alpha + \frac{1}{2\cos^2\omega/2}\, \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_\alpha) \\ \bar{\underset{\sim}{n}} &= \underset{\sim}{n} + \underset{\sim}{\Omega} \times \underset{\sim}{n} + \frac{1}{2\cos^2\omega/2}\, \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{n}) \end{aligned} \tag{3.7.6}$$

The finite rotation vector $\underset{\sim}{\Omega}$ is defined uniquely by the tensor $\underset{\sim}{R}$. Recalling that

$$\begin{aligned} \underset{\sim}{r} &= x^k \underset{\sim}{i}_k \quad , \qquad \underset{\sim}{a}_\alpha = x^k{}_{,\alpha} \underset{\sim}{i}_k \\ \underset{\sim}{n} &= \frac{1}{2} \epsilon^{\alpha\beta} x^k{}_{,\alpha} x^l{}_{,\beta} e_{klm} \underset{\sim}{i}^m \end{aligned} \tag{3.7.7}$$

it follows from (3.6.1) that $\underset{\sim}{R}$ may easily be expressed in an orthonormal tensor basis by

$$\underset{\sim}{R} = R_{kl} \underset{\sim}{i}^k \otimes \underset{\sim}{i}^l \tag{3.7.8}$$

where R_{kl} depend only on geometrical parameters of the reference surface M and displacement vector $\underset{\sim}{u}$. Then the orthogonal matrix components R_{kl} allow us to calculate the axis of rotation and the angle of rotation as follows:

$$\begin{aligned} \underset{\sim}{\Omega} &= \sin\omega\, \underset{\sim}{e} = -\frac{1}{2} e^{klm} R_{kl} \underset{\sim}{i}_m \\ \cos\omega &= \frac{1}{2}(\mathrm{tr}\, \underset{\sim}{R} - 1) = \frac{1}{2}(R_{11} + R_{22} + R_{33} - 1) \\ 2\cos^2\omega/2 &= 1 + \cos\omega \quad , \quad \sin\omega = \sqrt{1 - \cos^2\omega} \end{aligned} \tag{3.7.9}$$

These formulae express the components of $\underset{\sim}{\Omega} = \underset{\sim}{\Omega}(\underset{\sim}{u})$ with respect to the orthonormal basis $\underset{\sim}{i}_k$. Its components with respect to the reference surface basis may be found using (1.1.3), (1.1.4) and (1.1.12).

It is possible to express $\underset{\sim}{\Omega}$ in terms of $\underset{\sim}{u}$ and with respect to the surface basis *directly*. The expressions are more convenient for our future purposes.

Let us find first components of $\underset{\sim}{\Omega}$ in the *intermediate* basis $\overset{v}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$. Multiplying (3.7.6) by $\overset{v}{\underset{\sim}{a}}_\alpha$ or $\underset{\sim}{n}$ and using (3.7.5) we obtain

$$\bar{\underset{\sim}{a}}_\alpha \cdot \overset{v}{\underset{\sim}{a}}_\beta = \cos\omega \bar{a}_{\alpha\beta} + \underset{\sim}{\Omega} \cdot (\overset{v}{\underset{\sim}{a}}_\alpha \times \overset{v}{\underset{\sim}{a}}_\beta) + \\ + \frac{1}{2\cos^2\omega/2} (\underset{\sim}{\Omega} \cdot \overset{v}{\underset{\sim}{a}}_\alpha)(\underset{\sim}{\Omega} \cdot \overset{v}{\underset{\sim}{a}}_\beta) \tag{3.7.10}$$

$$\bar{\underset{\sim}{n}} \cdot \overset{v}{\underset{\sim}{a}}_\alpha = \underset{\sim}{\Omega} \cdot (\underset{\sim}{n} \times \overset{v}{\underset{\sim}{a}}_\alpha) + \frac{1}{2\cos^2\omega/2} (\underset{\sim}{\Omega} \cdot \underset{\sim}{n})(\underset{\sim}{\Omega} \cdot \overset{v}{\underset{\sim}{a}}_\alpha) \tag{3.7.11}$$

$$\bar{\underset{\sim}{a}}_\alpha \cdot \underset{\sim}{n} = \underset{\sim}{\Omega} \cdot (\overset{v}{\underset{\sim}{a}}_\alpha \times \underset{\sim}{n}) + \frac{1}{2\cos^2\omega/2} (\underset{\sim}{\Omega} \cdot \overset{v}{\underset{\sim}{a}}_\alpha)(\underset{\sim}{\Omega} \cdot \underset{\sim}{n}) \tag{3.7.12}$$

Contraction of (3.7.10) by $\bar{\epsilon}^{\alpha\beta}$ gives

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{n} = \frac{1}{2} \bar{\epsilon}^{\alpha\beta} \bar{\underset{\sim}{a}}_\alpha \cdot \overset{v}{\underset{\sim}{a}}_\beta \equiv \Omega \tag{3.7.13}$$

and substracting (3.7.12) from (3.7.11) we obtain as well

$$\underset{\sim}{\Omega} \cdot \overset{v\lambda}{\underset{\sim}{a}} = \frac{1}{2} \bar{\epsilon}^{\alpha\beta} (\bar{\underset{\sim}{n}} \cdot \overset{v}{\underset{\sim}{a}}_\alpha - \bar{\underset{\sim}{a}}_\alpha \cdot \underset{\sim}{n}) = \overset{v\lambda}{\Omega} \tag{3.7.14}$$

Then the general expression for $\underset{\sim}{\Omega}$ in terms of displacements follows:

$$\underset{\sim}{\Omega} = \frac{1}{2} \bar{\epsilon}^{\alpha\beta} [(\bar{\underset{\sim}{n}} \cdot \overset{v}{\underset{\sim}{a}}_\alpha - \bar{\underset{\sim}{a}}_\alpha \cdot \underset{\sim}{n}) \overset{v}{\underset{\sim}{a}}_\beta + (\bar{\underset{\sim}{a}}_\alpha \cdot \overset{v}{\underset{\sim}{a}}_\beta) \underset{\sim}{n}] \tag{3.7.15}$$

where various quantities still have to be expressed in terms of $\underset{\sim}{u}$ by (2.2.5), (2.2.7), (2.2.4), (2.2.8), (3.5.7) and (2.1.4).

An equivallent simple formula for $\underset{\sim}{\Omega}$ has also been found in [12] to be

$$\underset{\sim}{\Omega} = \frac{1}{2} (\overset{v}{\underset{\sim}{a}}_\alpha \times \bar{\underset{\sim}{a}}^\alpha + \underset{\sim}{n} \times \bar{\underset{\sim}{n}}) \tag{3.7.16}$$

Both representations (3.7.15) and (3.7.16), after some transformations, give us the following formula for $\underset{\sim}{\Omega}$ with respect to the reference basis $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$, [12],

$$2\underset{\sim}{\Omega} = \epsilon_{\lambda\mu} [n^\lambda - \bar{a}^{\alpha\beta} (\delta^\lambda_\alpha + \overset{v}{\gamma}{}^\lambda_\alpha) \varphi_\beta] \underset{\sim}{a}^\mu + \\ + \epsilon_{\lambda\mu} \bar{a}^{\alpha\beta} (\delta^\lambda_\alpha + \overset{v}{\gamma}{}^\lambda_\alpha) l^\mu_{\cdot\beta} \underset{\sim}{n} \tag{3.7.17}$$

The formula is *exact* under K - L constraints.

Let us show here in more detail that both relations (3.7.16) and (3.7.15) give really the same formula (3.7.17) for $\underset{\sim}{\Omega}$.

With the help of (3.5.7), (2.1.4), (2.2.8) and (1.1.20) for both terms of (3.7.16) we obtain

$$\overset{\vee}{\underset{\sim}{a}}_\alpha \times \bar{\underset{\sim}{a}}^\alpha = (\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)\underset{\sim}{a}_\lambda \times \bar{a}^{\alpha\beta}(1^\mu_{\cdot\beta}\underset{\sim}{a}_\mu + \varphi_\beta \underset{\sim}{n}) =$$

$$= \epsilon_{\lambda\mu}\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)1^\mu_{\cdot\beta}\underset{\sim}{n} - \tag{3.7.18}$$

$$- \epsilon_{\lambda\mu}\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)\varphi_\beta \underset{\sim}{a}^\mu$$

$$\underset{\sim}{n} \times \bar{\underset{\sim}{n}} = \underset{\sim}{n} \times (n^\lambda \underset{\sim}{a}_\lambda + n\,\underset{\sim}{n}) = \epsilon_{\lambda\mu} n^\lambda \underset{\sim}{a}^\mu \tag{3.7.19}$$

Together (3.7.16), (3.7.18) and (3.7.19) give exactly the formula (3.7.17).

Reduction of (3.7.15) to (3.7.17) requires some additional effort. The first term in (3.7.15) is transformed with the help of (2.2.5), (2.2.8), (3.5.7) and third of (3.5.8) as follows:

$$\bar{\epsilon}^{\alpha\beta}(\bar{\underset{\sim}{n}} \cdot \overset{\vee}{\underset{\sim}{a}}_\alpha)\overset{\vee}{\underset{\sim}{a}}_\beta = \sqrt{\frac{a}{\bar{a}}}\, n_\lambda \epsilon^{\alpha\beta}(\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)(\delta^\mu_\beta + \overset{\vee}{\gamma}{}^\mu_\beta)\underset{\sim}{a}_\mu =$$

$$= \sqrt{\frac{a}{\bar{a}}}\, n_\lambda \cdot \sqrt{\frac{\bar{a}}{a}}\,\epsilon^{\lambda\mu}\underset{\sim}{a}_\mu = \epsilon_{\lambda\mu} n^\lambda \underset{\sim}{a}^\mu \tag{3.7.20}$$

Using (2.1.4), (1.1.19), the first of (3.5.8) and the second of (3.5.9) for the second term in (3.7.15) we obtain

$$- \bar{\epsilon}^{\alpha\beta}(\bar{\underset{\sim}{a}}_\alpha \cdot \underset{\sim}{n})\overset{\vee}{\underset{\sim}{a}}_\beta = - \bar{a}^{\alpha\kappa}\bar{a}^{\beta\nu}\bar{\epsilon}_{\kappa\nu}\varphi_\alpha \overset{\vee}{\underset{\sim}{a}}_\beta = - \bar{a}^{\alpha\beta}\varphi_\beta \bar{\epsilon}_{\alpha\nu}\,\bar{\underset{\sim}{a}}^\nu =$$

$$= - \bar{a}^{\alpha\beta}\varphi_\beta \cdot (\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)(\delta^\rho_\nu + \overset{\vee}{\gamma}{}^\rho_\nu)\epsilon_{\lambda\rho} \cdot \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\nu\sigma}\epsilon_{\mu\tau}(\delta^\tau_\sigma + \overset{\vee}{\gamma}{}^\tau_\sigma)\underset{\sim}{a}^\mu =$$

$$= - \bar{a}^{\alpha\beta}\varphi_\beta(\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)\epsilon_{\lambda\rho}\epsilon^{\rho\tau}\epsilon_{\mu\tau}\underset{\sim}{a}^\mu =$$

$$= - \epsilon_{\lambda\mu}\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)\varphi_\beta \underset{\sim}{a}^\mu \tag{3.7.21}$$

Finally, with the help of (1.1.19), the second of (3.5.9), (2.2.5) and (2.1.4) we may transform the third term of (3.7.15) as follows

$$\bar{\epsilon}^{\alpha\beta}(\bar{\underset{\sim}{a}}_\alpha \cdot \overset{\nu}{\underset{\sim}{a}}_\beta)\underset{\sim}{n} = \bar{a}^{\alpha\kappa}\bar{a}^{\beta\nu}\bar{\epsilon}_{\kappa\nu}(\bar{\underset{\sim}{a}}_\alpha \cdot \overset{\nu}{\underset{\sim}{a}}_\beta)\underset{\sim}{n} = \bar{a}^{\kappa\beta}\bar{\epsilon}_{\kappa\nu}(\bar{\underset{\sim}{a}}_\beta \cdot \overset{\nu}{\underset{\sim}{a}}^\nu)\underset{\sim}{n} =$$

$$= \bar{a}^{\kappa\beta}\bar{\epsilon}_{\kappa\nu}\left[\bar{\underset{\sim}{a}}_\beta \cdot \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\nu\alpha}\epsilon_{\mu\lambda}(\delta^\lambda_\alpha + \overset{\nu}{\gamma}{}^\lambda_\alpha)\underset{\sim}{a}{}^\mu\right]\underset{\sim}{n} =$$

$$= -\,\bar{a}^{\alpha\beta}\left[\bar{\underset{\sim}{a}}_\beta \cdot \epsilon_{\mu\lambda}(\delta^\lambda_\alpha + \overset{\nu}{\gamma}{}^\lambda_\alpha)\underset{\sim}{a}{}^\mu\right]\underset{\sim}{n} =$$

$$= \epsilon_{\lambda\mu}\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha + \overset{\nu}{\gamma}{}^\lambda_\alpha)1^\mu_{\cdot\beta}\,\underset{\sim}{n} \qquad (3.7.22)$$

The latter together with (3.7.21) and (3.7.20) gives us exactly the formula (3.7.17).

Thus we have prooved that both vector relations (3.7.15) and (3.7.16), expressing finite rotation vector $\underset{\sim}{\Omega}$ in terms of displacements, are in fact *equivallent* and in the common reference basis $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$ reduce to the same formula (3.7.17).

3.8. Relations in terms of finite rotations

In the authors thesis [12] many useful exact relations for the middle surface have been derived in terms of finite rotation vector $\underset{\sim}{\Omega}$. Here we recall without detail derivation some of these relations valid under K - L constraints. Some of these relations will be useful for our future purposes.

From (3.7.6) and (2.1.4) we obtain

$$\underset{\sim}{u}_{,\beta} = \overset{\nu}{\gamma}_{\alpha\beta}\underset{\sim}{a}{}^\alpha + (\delta^\lambda_\beta + \overset{\nu}{\gamma}{}^\lambda_\beta)\left[\underset{\sim}{\Omega} \times \underset{\sim}{a}_\lambda + \frac{1}{2\cos^2\omega/2}\,\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{a}_\lambda)\right]$$

$$\overset{\nu}{\gamma}_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{u}_{,\beta} + (\delta^\lambda_\beta + \overset{\nu}{\gamma}{}^\lambda_\beta)\left[\epsilon_{\alpha\lambda}(\underset{\sim}{\Omega} \cdot \underset{\sim}{n}) + \frac{1}{2\cos^2\omega/2}(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\alpha)(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\lambda)\right] \qquad (3.8.1)$$

From definitions (2.1.4) and (2.2.8), together with (3.7.6), we obtain the components of deformation gradient tensor

$$1_{\alpha\beta} = (\delta^\lambda_\beta + \overset{\nu}{\gamma}{}^\lambda_\beta)\left[a_{\lambda\alpha} + \epsilon_{\lambda\alpha}(\underset{\sim}{\Omega} \cdot \underset{\sim}{n}) - \frac{1}{2\cos^2\omega/2}\,(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\lambda)(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\alpha)\right]$$

$$\varphi_\beta = (\delta^\lambda_\beta + \overset{\nu}{\gamma}{}^\lambda_\beta)\left[\epsilon_{\alpha\lambda}(\underset{\sim}{\Omega} \cdot \underset{\sim}{a}{}^\alpha) - \frac{1}{2\cos^2\omega/2}\,(\underset{\sim}{\Omega} \times \underset{\sim}{n})(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\lambda)\right] \qquad (3.8.2)$$

$$n_\alpha = \epsilon_{\alpha\lambda}(\underset{\sim}{\Omega} \cdot \underset{\sim}{a}{}^\lambda) - \frac{1}{2\cos^2\omega/2}\,(\underset{\sim}{\Omega} \times \underset{\sim}{a}_\alpha)(\underset{\sim}{\Omega} \times \underset{\sim}{n})$$

$$n = 1 - \frac{1}{2\cos^2\omega/2}\,(\underset{\sim}{\Omega} \times \underset{\sim}{n})(\underset{\sim}{\Omega} \times \underset{\sim}{n}) \,.$$

Differentiation of $\underset{\sim}{\Omega}$ along convected coordinate lines in three-dimensional continua has been discussed by SHAMINA [20]. In our case of two-dimensional shell middle surface we obtain from [20] the following relation

$$\frac{d\underset{\sim}{\Omega}}{d\theta^\beta} = \cos\omega\,\underset{\sim}{\kappa}_\beta + \frac{1}{2}\,\underset{\sim}{\Omega}\times\underset{\sim}{\kappa}_\beta - \frac{1}{4\cos^2\omega/2}\,\underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{\kappa}_\beta) \tag{3.8.3}$$

where $\underset{\sim}{\kappa}_\beta$ is called the *vector of change of curvature* of the coordinate line θ_β, since

$$\frac{d\omega}{d\theta^\beta} = \underset{\sim}{\kappa}_\beta\cdot\underset{\sim}{e} \quad , \quad \underset{\sim}{\kappa}_\beta \equiv \frac{d\omega}{d\theta^\beta}\,\underset{\sim}{e} \tag{3.8.4}$$

For $\underset{\sim}{\kappa}_\beta$ the following relation in terms of the surface strain measures has been obtained in [12]:

$$\underset{\sim}{\kappa}_\beta = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\lambda\mu}\left[(\kappa_{\beta\lambda} + b_\beta^{\alpha\nu}\gamma_{\alpha\lambda})\overset{\nu}{\underset{\sim}{a}}_\mu + (\gamma_{\beta\mu|\lambda} - \frac{1}{2}\,\overset{\nu}{\gamma}_\mu^{\kappa\nu}\gamma_{\kappa\lambda|\beta})\underset{\sim}{n}\right] \tag{3.8.5}$$

or in terms of the finite rotation vector

$$\underset{\sim}{\kappa}_\beta = \frac{d\underset{\sim}{\Omega}}{d\theta^\beta} + \frac{1}{2\cos^2\omega/2}\,\frac{d\underset{\sim}{\Omega}}{d\theta^\beta}\times\underset{\sim}{\Omega} + \frac{d\omega}{d\theta^\beta}\,\mathrm{tg}\omega/2\,\underset{\sim}{\Omega} \tag{3.8.6}$$

Covariant differentiation of the intermediate basis $\overset{\nu}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ may be expressed in terms of $\underset{\sim}{\kappa}_\beta$ by

$$\begin{aligned}
\overset{\nu}{\underset{\sim}{a}}_{\alpha|\beta} &= (b_{\alpha\beta} - \kappa_{\alpha\beta})\underset{\sim}{n} + \bar{a}^{\lambda\mu}\gamma_{\mu\alpha\beta}\overset{\nu}{\underset{\sim}{a}}_\lambda - \underset{\sim}{\kappa}_\beta\times\overset{\nu}{\underset{\sim}{a}}_\alpha \\
\underset{\sim}{n}_{|\beta} &= [-b_\beta^\lambda + \bar{a}^{\lambda\mu}(\kappa_{\beta\mu} + 2b_\beta^\alpha\gamma_{\alpha\mu})]\overset{\nu}{\underset{\sim}{a}}_\lambda - \underset{\sim}{\kappa}_\beta\times\underset{\sim}{n}
\end{aligned} \tag{3.8.7}$$

From (3.8.5) it is easy to find $\kappa_{\alpha\beta}$ in the form

$$\kappa_{\alpha\beta} = \frac{1}{2}\,(\bar{\epsilon}_{\alpha\lambda}\underset{\sim}{\kappa}_\beta + \bar{\epsilon}_{\beta\lambda}\underset{\sim}{\kappa}_\alpha)\cdot\overset{\nu\lambda}{\underset{\sim}{a}} - \frac{1}{2}\,(b_\alpha^{\lambda\nu}\gamma_{\lambda\beta} + b_\beta^{\lambda\nu}\gamma_{\lambda\alpha}) \tag{3.8.8}$$

This relation, together with (3.8.6), gives us the expression for $\kappa_{\alpha\beta}$ in terms of $\underset{\sim}{\Omega}$.

The relation (3.8.8) suggests the possibility of defining a new *modified tensor of change of curvature* of the shell middle surface $\overset{v}{\underset{\sim}{\rho}}$ with components

$$\overset{v}{\rho}_{\alpha\beta} = \kappa_{\alpha\beta} + \frac{1}{2}\left(b_{\alpha}^{\lambda v}\gamma_{\lambda\beta} + b_{\beta}^{\lambda v}\gamma_{\lambda\alpha}\right) \tag{3.8.9}$$

linear part of which coincides with $\mu_{\alpha\beta}$ defined in (2.1.11). From (3.8.8) we obtain the simple formula:

$$\overset{v}{\rho}_{\alpha\beta} = \frac{1}{2}\left(\bar{\epsilon}_{\alpha\lambda}\underset{\sim}{\kappa}_{\beta} + \bar{\epsilon}_{\beta\lambda}\underset{\sim}{\kappa}_{\alpha}\right)\cdot\overset{v\lambda}{\underset{\sim}{a}} \tag{3.8.10}$$

By means of (3.8.6) the tensor $\overset{v}{\underset{\sim}{\rho}}$ may be expressed in terms of $\underset{\sim}{\Omega}$ as well. The tensor may be of particular interest when some approximate variants of shell deformation under restricted rotations are discussed.

Chapter 4

DEFORMATION OF A SHELL BOUNDARY

Deformation of a shell near the boundary is essentially three-dimensional. When the methods of constrained continua are used [37] the reactive forces due to assumed constraints may be significant in a boundary zone of depth comparable with the shell thickness. Here we are not going to discuss the full deformation of the shell in this boundary zone. This problem for a non-linear shell behaviour has not been solved yet. Within the linear theory of shells, GOLDENWEIZER in several papers (see [44] for example) discussed the problems of the shell boundary zone and KOITER and SIMMONDS [6] have proposed some modifications of the boundary conditions at a free edge.

The goal of this chapter is to examine the implications of K - L constraints on deformation of the shell boundary. This limited goal has been solved only recently by NOVOZHILOV and SHAMINA [21] and more general results under relaxed constraints mentioned in the introduction to chapter 3 has been obtained in the author's thesis [12]. Proper description of deformation of the shell boundary admits the formulation of new variants of the geometrical boundary conditions for interior shell equations.

We begin with describing the geometry of the deformed shell boundary subject to K - L constraints. The total finite rotation of the boundary material element is found as a result of superposition of finite rotation of principal directions of strain (discussed in § 3.7.) on a finite rotation due to pure stretch along principle directions of strains. Geometrical boundary conditions in terms of displacements as well as in terms of the total rotation vector and the elongation of the boundary contour are derived. We discuss here various differential relations at deformed boundary which allow us to describe the deformation of the shell boundary entirely in terms of combinations of strain measures as well.

4.1. Geometry of deformed boundary

Let C be the *boundary curve* at the reference middle surface M, given by

$$\theta^1 = \theta^2(s) \quad , \quad \theta^2 = \theta^2(s) \tag{4.1.1}$$

where s is the *length parameter* of C. We assume here that the reference shell boundary surface ∂P is *orthogonal* to M along C. Hence, position of any $P \in \partial P$ is defined by

$$\underset{\sim}{p} = \underset{\sim}{p}(s,\zeta) = \underset{\sim}{r}(s) + \zeta \underset{\sim}{n}(s) \tag{4.1.2}$$

With C let us associate an *orthonormal* triad $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$, Fig. 14.

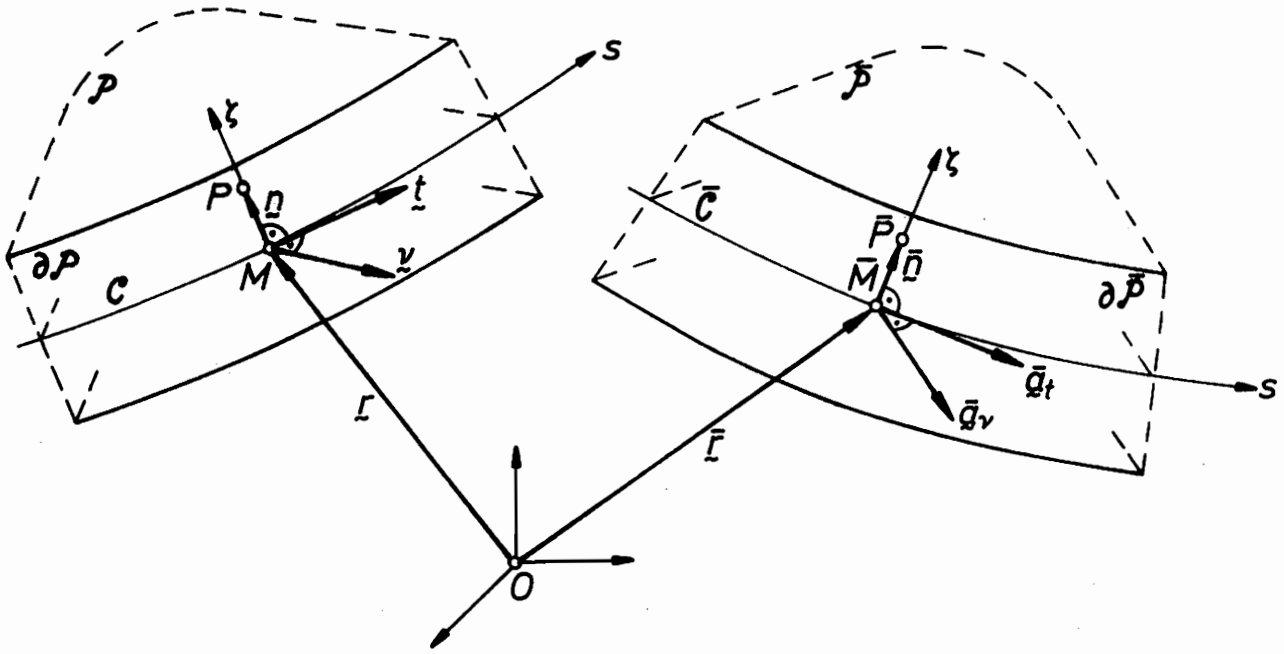

Fig. 14

After a deformation compatible with K - L constraints C transforms into $\bar{C}$ and ∂P into $\partial \bar{P}$, which is still *orthoganal* to $\bar{M}$, and for any $\bar{P} \in \partial \bar{P}$

$$\begin{aligned} \underset{\sim}{\bar{p}} &= \underset{\sim}{\bar{p}}(\bar{P}) = \underset{\sim}{\bar{p}}(s,\zeta) \\ &= \underset{\sim}{\bar{r}}(s) + \zeta \underset{\sim}{\bar{n}}(s) \end{aligned} \tag{4.1.3}$$

In the deformed configuration the orthonormal triad $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ is transformed into an *orthogonal* triad $\underset{\sim}{\bar{a}}_\nu$, $\underset{\sim}{\bar{a}}_t$, $\underset{\sim}{\bar{n}}$ where

$$\bar{\underset{\sim}{a}}_t = \frac{d\bar{\underset{\sim}{r}}}{ds} = t^{\alpha}\bar{\underset{\sim}{a}}_{\alpha} = \underset{\sim}{t} + \frac{d\underset{\sim}{\mu}}{ds} \tag{4.1.4}$$

$$\bar{\underset{\sim}{a}}_{\nu} = \bar{\underset{\sim}{a}}_t \times \bar{\underset{\sim}{n}} = \bar{\epsilon}_{\lambda\alpha} t^{\alpha} \bar{\underset{\sim}{a}}^{\lambda} = \sqrt{\frac{\bar{a}}{a}}\, \nu_{\alpha} \bar{\underset{\sim}{a}}^{\alpha} = \tag{4.1.5}$$

$$= (\underset{\sim}{t} + \frac{d\underset{\sim}{\mu}}{ds}) \times (\underset{\sim}{n} + \underset{\sim}{\beta})$$

These are *not unit* vectors; their lengths are

$$|\bar{\underset{\sim}{a}}_t| = |\bar{\underset{\sim}{a}}_{\nu}| \equiv \bar{a}_t = \tag{4.1.6}$$

$$= \sqrt{(a_{\alpha\beta} + 2\gamma_{\alpha\beta})\, t^{\alpha} t^{\beta}} = \sqrt{1 + 2\gamma_{tt}}$$

According to the polar decomposition theorem (3.4.1) the deformation of the boundary may also be decomposed into a rigid-body translation, the pure strain realized by means of $\underset{\sim}{U}$, $\underset{\sim}{\chi}$ or $\overset{v}{\underset{\sim}{\chi}}$ and the rigid-body rotation realized by means of $\underset{\sim}{R}$ or $\underset{\sim}{\Omega}$. Thus there are such *intermediate vectors* $\overset{v}{\underset{\sim}{a}}_t$, $\overset{v}{\underset{\sim}{a}}_{\nu}$ for which

$$\bar{\underset{\sim}{a}}_t = \overset{v}{\underset{\sim}{a}}_t + \underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_t + \frac{1}{2\cos^2 \omega/2}\, \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_t)$$

$$\bar{\underset{\sim}{a}}_{\nu} = \overset{v}{\underset{\sim}{a}}_{\nu} + \underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_{\nu} + \frac{1}{2\cos^2 \omega/2}\, \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \overset{v}{\underset{\sim}{a}}_{\nu})$$

These vectors follow from (3.5.7) and (1.5.3):

$$\overset{v}{\underset{\sim}{a}}_t = \overset{v}{\underset{\sim}{a}}_{\alpha} t^{\alpha} = (a_{\alpha\beta} + \overset{v}{\gamma}_{\alpha\beta}) t^{\alpha} (\nu^{\beta} \underset{\sim}{\nu} + t^{\beta} \underset{\sim}{t}) =$$

$$= \overset{v}{\gamma}_{\nu t} \underset{\sim}{\nu} + (1 + \overset{v}{\gamma}_{tt}) \underset{\sim}{t} \tag{4.1.8}$$

$$\overset{v}{\underset{\sim}{a}}_{\nu} = \overset{v}{\underset{\sim}{a}}_t \times \underset{\sim}{n} = (1 + \overset{v}{\gamma}_{tt}) \underset{\sim}{\nu} - \overset{v}{\gamma}_{\nu t} \underset{\sim}{t}$$

4.2. Total rotation of a boundary

It follows from (4.1.8) that under the pure stretch of the principal directions of strain, the vectors $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ not only change their lengths, but *suffer rotations* as well. This comes from the fact that directions defined by $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ *do not coincide* in general with principal directions of the surface strain tensor.

Let us denote the *finite rotation vector caused by the pure strain* by $\overset{v}{\underset{\sim}{\Omega}}_t$. Then applying (3.7.16) we obtain

$$\overset{\vee}{\underset{\sim}{\Omega}}_t = \frac{1}{2\sqrt{1+2\gamma_{tt}}}\left(\underset{\sim}{\nu}\times\overset{\vee}{\underset{\sim}{a}}_\nu + \underset{\sim}{t}\times\overset{\vee}{\underset{\sim}{a}}_t\right) =$$

$$= -\frac{\overset{\vee}{\gamma}_{\nu t}}{\sqrt{1+2\gamma_{tt}}}\underset{\sim}{n} \equiv \sin\overset{\vee}{\omega}_t\underset{\sim}{n} \qquad (4.2.1)$$

The axis of the rotation is seen to be along the normal to the surface M and

$$\sin\overset{\vee}{\omega}_t = -\frac{\overset{\vee}{\gamma}_{\nu t}}{\sqrt{1+2\gamma_{tt}}}\quad,\qquad \cos\overset{\vee}{\omega}_t = \frac{1+\overset{\vee}{\gamma}_{tt}}{\sqrt{1+2\gamma_{tt}}}$$

$$2\cos^2\overset{\vee}{\omega}_{t/2} = \frac{\sqrt{1+2\gamma_{tt}}+1+\overset{\vee}{\gamma}_{tt}}{\sqrt{1+2\gamma_{tt}}} \qquad (4.2.2)$$

Transformation of $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ into $\overset{\vee}{\underset{\sim}{a}}_\nu$ and $\overset{\vee}{\underset{\sim}{a}}_t$ may thus now be expressed in the form

$$\overset{\vee}{\underset{\sim}{a}}_\nu = \sqrt{1+2\gamma_{tt}}\left[\underset{\sim}{\nu} + \overset{\vee}{\underset{\sim}{\Omega}}_t\times\underset{\sim}{\nu} + \frac{1}{2\cos^2\overset{\vee}{\omega}_{t/2}}\overset{\vee}{\underset{\sim}{\Omega}}_t\times\left(\overset{\vee}{\underset{\sim}{\Omega}}\times\underset{\sim}{\nu}\right)\right]$$

$$\overset{\vee}{\underset{\sim}{a}}_t = \sqrt{1+2\gamma_{tt}}\left[\underset{\sim}{t} + \overset{\vee}{\underset{\sim}{\Omega}}_t\times\underset{\sim}{t} + \frac{1}{2\cos^2\overset{\vee}{\omega}_{t/2}}\overset{\vee}{\underset{\sim}{\Omega}}_t\times\left(\overset{\vee}{\underset{\sim}{\Omega}}\times\underset{\sim}{t}\right)\right] \qquad (4.2.3)$$

It is seen from (4.2.3) and (4.1.7) that the transformation of $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ into $\bar{\underset{\sim}{a}}_\nu$ and $\bar{\underset{\sim}{a}}_t$ consists of *extension* of these vectors by the factor $\sqrt{1+2\gamma_{tt}}$ and *successive superposition* of two rotations realized by the finite rotation vectors $\overset{\vee}{\underset{\sim}{\Omega}}_t$ and $\underset{\sim}{\Omega}$, respectively. It is convenient to replace these two subsequent rotations by a *single equivallent* rotation realized by an equivallent ***total*** *finite rotation vector* $\underset{\sim}{\Omega}_t$.

Note that the finite rotation vectors are not vectors in the usual sence. The superposition rules for these vectors are different from the usual addition rules of a linear vector space. Rules of superposition of the finite rotation vectors are discussed in detail in some courses on analytical mechanics of rigid-body motion (see LURIE [24] for example). In our notation the *superposition formula* for $\underset{\sim}{\Omega}_t$ takes the form

$$\underset{\sim}{\Omega}_t = \left(1 - \frac{\overset{\vee}{\underset{\sim}{\Omega}}_t\cdot\underset{\sim}{\Omega}}{4\cos^2\overset{\vee}{\omega}_{t/2}\cos^2\omega/2}\right)\left[\cos^2\omega/2\,\overset{\vee}{\underset{\sim}{\Omega}}_t + \cos^2\overset{\vee}{\omega}_{t/2}\underset{\sim}{\Omega} + \frac{1}{2}\underset{\sim}{\Omega}\times\overset{\vee}{\underset{\sim}{\Omega}}_t\right] \qquad (4.2.4)$$

where (4.2.1) and (4.2.2) together with (3.7.17) have to be used.

By means of this total finite rotation vector the transformation formulae for $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ become

$$\begin{aligned}
\bar{\underset{\sim}{a}}_{\nu} &= \sqrt{1+2\gamma_{tt}}\left[\underset{\sim}{\nu} + \underset{\sim}{\Omega}_t \times \underset{\sim}{\nu} + \frac{1}{2\cos^2\omega_{t/2}}\,\underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \underset{\sim}{\nu})\right] \\
\bar{\underset{\sim}{a}}_{t} &= \sqrt{1+2\gamma_{tt}}\left[\underset{\sim}{t} + \underset{\sim}{\Omega}_t \times \underset{\sim}{t} + \frac{1}{2\cos^2\omega_{t/2}}\,\underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \underset{\sim}{t})\right] \qquad (4.2.5) \\
\bar{\underset{\sim}{n}} &= \underset{\sim}{n} + \underset{\sim}{\Omega}_t \times \underset{\sim}{n} + \frac{1}{2\cos^2\omega_{t/2}}\,\underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \underset{\sim}{n})
\end{aligned}$$

4.3. Displacement boundary conditions

According to (4.1.3), (3.2.1) and (3.2.6) the boundary surface $\partial\bar{P}$ is uniquely defined by assuming two vector functions

$$\underset{\sim}{u}(s) = \underset{\sim}{A}(s) \quad , \quad \underset{\sim}{\beta}(s) = \underset{\sim}{B}(s) \quad \text{at } C \qquad (4.3.1)$$

Let

$$\underset{\sim}{\beta} = \beta_{\nu}\bar{\underset{\sim}{a}}_{\nu} + \beta_{t}\bar{\underset{\sim}{a}}_{t} + \beta\bar{\underset{\sim}{n}} \qquad (4.3.2)$$

Then using (3.2.7), (4.1.4) and (4.1.5) we obtain

$$\begin{aligned}
\beta_{\nu} &= \frac{\underset{\sim}{\beta}\cdot\bar{\underset{\sim}{a}}_{\nu}}{1+2\gamma_{tt}} = \frac{(\bar{\underset{\sim}{n}}-\underset{\sim}{n})\cdot\sqrt{\frac{\bar{a}}{a}}\,\nu_{\alpha}\bar{a}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\beta}}{1+2\gamma_{tt}} = \\
&= -\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,\nu_{\alpha}\bar{a}^{\alpha\beta}(\underset{\sim}{u}_{,\beta}\cdot\underset{\sim}{n}) \qquad (4.3.3) \\
\beta_{t} &= \frac{\underset{\sim}{\beta}\cdot\bar{\underset{\sim}{a}}_{t}}{1+2\gamma_{tt}} = \frac{(\bar{\underset{\sim}{n}}-\underset{\sim}{n})\cdot\bar{\underset{\sim}{a}}_{\alpha}t^{\alpha}}{1+2\gamma_{tt}} = -\frac{1}{1+2\gamma_{tt}}\,\frac{d\underset{\sim}{u}}{ds}\cdot\underset{\sim}{n} \qquad (4.3.4)
\end{aligned}$$

From an identity

$$\underset{\sim}{\beta}\cdot\underset{\sim}{\beta} = (\bar{\underset{\sim}{n}}-\underset{\sim}{n})(\bar{\underset{\sim}{n}}-\underset{\sim}{n}) = 2\underset{\sim}{\beta}\cdot\bar{\underset{\sim}{n}} \qquad (4.3.5)$$

with the help of (4.3.2) we obtain as well

$$\beta = 1-\sqrt{1-(1+2\gamma_{tt})(\beta_{\nu}^2+\beta_{t}^2)} \qquad (4.3.6)$$

It follows now from (4.3.4), that $\beta_t(s)$ may be calculated entirely in terms of $\underset{\sim}{u}(s)$, and then using $\beta_\nu(s)$ we may calculate $\beta(s)$ according to (4.3.6). Thus only $\underset{\sim}{u}(s)$ and $\beta_\nu(s)$ are *independent* displacemental variables.

The geometrical boundary conditions of the type

$$\underset{\sim}{u}(s) = \underset{\sim}{A}(s) \quad , \quad \beta_\nu(s) = b(s) \qquad \text{at} \quad C \tag{4.3.7}$$

will be called the *displacement boundary conditions* of the K - L nonlinear theory of shells.

4.4. Kinematical boundary conditions

The deformed boundary surface $\partial\bar{P}$ has been described by the equations (4.1.3). Let us differentiate (4.1.4) with respect to s and ζ to obtain

$$\frac{\partial \underset{\sim}{\bar{p}}}{\partial s} = \underset{\sim}{\bar{a}}_t + \zeta \frac{d}{ds}\underset{\sim}{\bar{n}} \quad , \quad \frac{\partial \underset{\sim}{\bar{p}}}{\partial \zeta} = \underset{\sim}{\bar{n}} \quad , \quad \frac{d\underset{\sim}{\bar{r}}}{ds} = \underset{\sim}{\bar{a}}_t \tag{4.4.1}$$

These differential equations define the same boundary surface $\partial\bar{P}$ *implicitly* with the accuracy up to constant translation. In order to obtain explicitly the boundary surface we need only solve these differential equations. The equations (4.4.1) are defined uniquely if at C the following quantities are specified:

$$\underset{\sim}{\bar{a}}_t(s) = \underset{\sim}{M}(s) \quad , \quad \underset{\sim}{\bar{n}}(s) = \underset{\sim}{L}(s) \tag{4.4.2}$$

where

$$\underset{\sim}{M}(s) = \underset{\sim}{t}(s) + \frac{d\underset{\sim}{A}}{ds} \quad , \quad \underset{\sim}{L}(s) = \underset{\sim}{n}(s) + \underset{\sim}{B}(s) \tag{4.4.3}$$

It follows from (4.2.5) that $\underset{\sim}{\bar{a}}_t$ and $\underset{\sim}{\bar{n}}$ at C may actually be expressed in terms of only four independent quantities: three components of the total finite rotation vector $\underset{\sim}{\Omega}_t$ and the relative elongation γ_{tt} of the boundary contour C. Vector $\underset{\sim}{\Omega}_t$ has been expressed in terms of displacements by (4.2.4) and the expression for γ_{tt} may be found from the relation

$$\gamma_{tt}(s) = \underset{\sim}{t} \cdot \frac{d\underset{\sim}{u}}{ds} + \frac{1}{2}\frac{d\underset{\sim}{u}}{ds} \cdot \frac{d\underset{\sim}{u}}{ds} \tag{4.4.4}$$

Thus, if $\underset{\sim}{\Omega}_t$ and γ_{tt} are specified at the shell boundary C, then $\underset{\sim}{\bar{a}}_t$ and $\underset{\sim}{\bar{n}}$ follow from (4.2.5). That defines completely the right-hand sides of differential equations (4.4.1) of deformed boundary surface $\partial\bar{P}$.

Geometrical boundary conditions of the type

$$\underset{\sim}{\Omega}_t(s) = \underset{\sim}{m}(s) \quad , \quad \gamma_{tt}(s) = 1(s) \text{ at } C. \tag{4.4.5}$$

will be called the *kinematical boundary conditions* of the K - L non-linear theory of shells.

Using (3.7.16), for $\underset{\sim}{\Omega}_t$ we obtain the following formula

$$2\underset{\sim}{\Omega}_t = \frac{1}{\sqrt{1+2\gamma_{tt}}} \{\underset{\sim}{\nu} \times [(\underset{\sim}{t} + \frac{d\underset{\sim}{u}}{ds}) \times (\underset{\sim}{n} + \underset{\sim}{\beta})] + \underset{\sim}{t} \times \frac{d\underset{\sim}{u}}{ds} \} + \underset{\sim}{n} \times \underset{\sim}{\beta} \tag{4.4.6}$$

From (4.4.6) and (4.4.4) it es easy to note the following relations between quantities assumed at the shell boundary contour:

$$2\underset{\sim}{m}(s) = \frac{1}{\sqrt{1+2\,1}} \{\underset{\sim}{\nu} \times [\underset{\sim}{t} \times \underset{\sim}{B} + \frac{d\underset{\sim}{A}}{ds} \times (\underset{\sim}{n} + \underset{\sim}{B})] + \underset{\sim}{t} \times \frac{d\underset{\sim}{A}}{ds} \} + \tag{4.4.7}$$

$$+ \underset{\sim}{n} \times \underset{\sim}{B}$$

$$1(s) = \underset{\sim}{t} \cdot \frac{d\underset{\sim}{A}}{ds} + \frac{1}{2} \frac{d\underset{\sim}{u}}{ds} \cdot \frac{d\underset{\sim}{u}}{ds} \tag{4.4.8}$$

4.5. Differentiation along deformed boundary

Differentiation of the reference triad of vectors $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ along the *reference* boundary contour C follow the relations (1.5.7) or (1.5.11) and (1.5.12).

Let us define the *unit vectors* at the deformed boundary

$$\bar{\underset{\sim}{\nu}} = \frac{\bar{\underset{\sim}{a}}_\nu}{\sqrt{1+2\gamma_{tt}}} \quad , \quad \bar{\underset{\sim}{t}} = \frac{\bar{\underset{\sim}{a}}_{\nu t}}{\sqrt{1+2\gamma_{tt}}} \tag{4.5.1}$$

For the orthonormal triad $\bar{\underset{\sim}{\nu}}$, $\bar{\underset{\sim}{t}}$, $\bar{\underset{\sim}{n}}$ at the *deformed* boundary we have differentiation rules similar to those for $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ to be

$$\frac{d\bar{\underset{\sim}{\nu}}}{ds} = \bar{\underset{\sim}{\omega}}_t \times \bar{\underset{\sim}{\nu}} \; , \quad \frac{d\bar{\underset{\sim}{t}}}{ds} = \bar{\underset{\sim}{\omega}}_t \times \bar{\underset{\sim}{t}} \; , \quad \frac{d\bar{\underset{\sim}{n}}}{ds} = \bar{\underset{\sim}{\omega}}_t \times \bar{\underset{\sim}{n}} \tag{4.5.2}$$

$$\bar{\underset{\sim}{\omega}}_t = \sqrt{1+2\gamma_{tt}}\,(\bar{\sigma}_t \bar{\underset{\sim}{\nu}} + \bar{\tau}_t \bar{\underset{\sim}{t}} + \bar{\kappa}_t \bar{\underset{\sim}{n}}) \tag{4.5.3}$$

where $\bar{\sigma}_t$, $\bar{\tau}_t$ and $\bar{\kappa}_t$ are the *normal curvature*, the *geodesic torsion* and the *geodesic curvature* of *deformed* boundary coutour $\bar{C}$. The additional

factor $\sqrt{1+2\gamma_{tt}}$ in (4.5.3) comes from the fact, that in convected coordinates

$$\frac{d\bar{r}}{ds} = \bar{a}_t = \frac{d\bar{r}}{d\bar{s}}\frac{d\bar{s}}{ds} = \bar{t}\frac{d\bar{s}}{ds} \tag{4.5.4}$$

$$\frac{d\bar{s}}{ds} = |\bar{a}_t| = \sqrt{1+2\gamma_{tt}} \tag{4.5.5}$$

where $\bar{s}$ is the *lenth paramenter* of *deformed* boundary contour $\bar{C}$.

The vectors $\bar{\nu}$, $\bar{t}$, $\bar{n}$ are expressible in terms of ν, t, n by (4.2.5), from which the following identity may be derived

$$\bar{n}\cdot\nu - \bar{\nu}\cdot n = 2\Omega_t \cdot t \tag{4.5.6}$$

Let us differentiate (4.5.6) with respect to the length s along the reference boundary, and use (4.5.2) and (1.5.11) to obtain

$$\begin{aligned} 2\frac{d\Omega_t}{ds}\cdot t = {} & \bar{\omega}_t\cdot(\bar{n}\times\nu - \bar{\nu}\times n) - \\ & - \omega_t\cdot(n\times\bar{\nu} - \nu\times\bar{n}) - 2(\Omega_t\times\omega_t)\cdot t \end{aligned} \tag{4.5.7}$$

Using (4.2.5) and trigonometric identities (3.7.4) we can derive also the following formula:

$$\begin{aligned} \bar{n}\times\nu - \bar{\nu}\times n = {} & 2\cos\omega_t t + \Omega_t\times t - \\ & - \frac{1}{2\cos^2\omega_t/2}\,\Omega_t\times(\Omega_t\times t) \end{aligned} \tag{4.5.8}$$

As the vector $\bar{\omega}_t$ is defined in the *deformed* shell configuration, then by the polar decomposition theorem it has to be expressible in terms of some vector $\overset{v}{\omega}_t$ defined with respect to the reference triad of unit vectors, by the following formula

$$\bar{\omega}_t = \overset{v}{\omega}_t + \Omega_t\times\overset{v}{\omega}_t + \frac{1}{\cos^2\omega_t/2}\,\Omega_t\times(\Omega_t\times\overset{v}{\omega}_t) \tag{4.5.9}$$

where

$$\overset{v}{\omega}_t = \sqrt{1+2\gamma_{tt}}\,(\bar{\sigma}_t\nu + \bar{\tau}_t t + \bar{\kappa}_t n) \tag{4.5.10}$$

After quite long transformations, which are not presented here, carried out with help of trigonometric identities, for both parts of (4.5.7) we obtain

$$\bar{\underset{\sim}{\omega}}_t \cdot (\bar{\underset{\sim}{n}} \times \underset{\sim}{\nu} - \bar{\underset{\sim}{\nu}} \times \underset{\sim}{n}) = [2\cos\omega_t \overset{v}{\underset{\sim}{\omega}}_t + \underset{\sim}{\Omega}_t \times \overset{v}{\underset{\sim}{\omega}}_t - \frac{1}{2\cos^2\omega_{t/2}} \underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \overset{v}{\underset{\sim}{\omega}}_t)] \cdot \underset{\sim}{t} \tag{4.5.11}$$

$$-\underset{\sim}{\omega}_t \cdot (\underset{\sim}{n} \times \bar{\underset{\sim}{\nu}} - \underset{\sim}{\nu} \times \bar{\underset{\sim}{n}}) - 2(\underset{\sim}{\Omega}_t \times \underset{\sim}{\omega}_t) \cdot \underset{\sim}{t} = -[2\cos\omega_t \underset{\sim}{\omega}_t + \underset{\sim}{\Omega}_t \times \underset{\sim}{\omega}_t - \frac{1}{2\cos^2\omega_{t/2}} \underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \underset{\sim}{\omega}_t)] \cdot t \tag{4.5.12}$$

Now, using (4.5.7), (4.5.9), (4.5.11) and (4.5.12), we obtain the formula for differentiation of the total finite rotation vector of the boundary

$$\frac{d\underset{\sim}{\Omega}_t}{ds} = \cos\omega_t \underset{\sim}{k}_t + \frac{1}{2}\underset{\sim}{\Omega}_t \times \underset{\sim}{k}_t - \frac{1}{4\cos^2\omega_{t/2}} \underset{\sim}{\Omega}_t \times (\underset{\sim}{\Omega}_t \times \underset{\sim}{k}_t) \tag{4.5.13}$$

where

$$\underset{\sim}{k}_t = \overset{v}{\underset{\sim}{\omega}}_t - \underset{\sim}{\omega}_t \tag{4.5.14}$$

$\underset{\sim}{k}_t$ is called the *vector of change of boundary curvature* during shell deformation subject to K - L constraints.

Let us note here, that all differentiation formulae depend only on $\underset{\sim}{k}_t$. Really, if components of $\underset{\sim}{k}_t$ are supposed to be known, then $\overset{v}{\underset{\sim}{\omega}}_t$ follows from (4.5.14), $\bar{\underset{\sim}{\omega}}_t$ follows from (4.5.9) and from (4.5.2) derivatives of orthonormal triad $\bar{\underset{\sim}{\nu}}$, $\bar{\underset{\sim}{t}}$, $\bar{\underset{\sim}{n}}$ along deformed boundary contour $\bar{C}$ become known as well.

4.6. Vector of change of the boundary curvature

Let us express components of $\underset{\sim}{\kappa}_t$ in terms of the reference middle surface geometry and surface strain measures. It follows from (1.5.11) and (4.5.10) that

$$\underset{\sim}{k}_t = -k_{tt}\underset{\sim}{\nu} + k_{\nu t}\underset{\sim}{t} - k_{nt}\underset{\sim}{n} \tag{4.6.1}$$

where

$$
\begin{aligned}
-k_{tt} &= \sqrt{1+2\gamma_{tt}}\,\bar{\sigma}_t - \sigma_t \\
k_{vt} &= \sqrt{1+2\gamma_{tt}}\,\bar{\tau}_t - \tau_t \\
-k_{nt} &= \sqrt{1+2\gamma_{tt}}\,\bar{\kappa}_t - \kappa_t
\end{aligned}
\tag{4.6.2}
$$

Let us differentiate with respect to s the deformed triad of vectors at $\bar{C}$. Then

$$
\begin{aligned}
\frac{d}{ds}\,\underset{\sim}{\bar{a}}_t &= \frac{d}{d\theta^\beta}\,(\underset{\sim}{\bar{a}}_\alpha t^\alpha)\,t^\beta = (\underset{\sim}{\bar{a}}_{\alpha,\beta}\,t^\alpha + \underset{\sim}{a}_\alpha t^\alpha{}_{,\beta})\,t^\beta = \\
&= [(\Gamma^\alpha_{\lambda\beta} + \bar{a}^{\alpha\nu}\gamma_{\nu\lambda\beta})\,\underset{\sim}{\bar{a}}_\alpha t^\lambda + \underset{\sim}{\bar{a}}_\alpha t^\alpha{}_{,\beta}]\,t^\beta + (b_{\alpha\beta} - \kappa_{\alpha\beta})\,t^\alpha t^\beta\,\underset{\sim}{n} = \\
&= (t^\alpha|_\beta + \bar{a}^{\alpha\nu}\gamma_{\nu\lambda\beta}t^\lambda)\,t^\beta\,\underset{\sim}{\bar{a}}_\alpha + (\sigma_t - \kappa_{tt})\,\underset{\sim}{\bar{n}}
\end{aligned}
\tag{4.6.3}
$$

$$
\frac{d}{ds}\,\underset{\sim}{\bar{n}} = \underset{\sim}{\bar{n}}_{,\beta}\,t^\beta = -(b_{\alpha\beta} - \kappa_{\alpha\beta})\,\underset{\sim}{\bar{a}}^\alpha t^\beta \tag{4.6.4}
$$

Recalling that

$$
\underset{\sim}{\bar{a}}_t = \sqrt{1+2\gamma_{tt}}\,\underset{\sim}{\bar{t}} \tag{4.6.5}
$$

we have the relations

$$
\frac{d\underset{\sim}{\bar{a}}_t}{ds} = \frac{1}{\sqrt{1+2\gamma_{tt}}}\,\frac{d\gamma_{tt}}{ds}\,\underset{\sim}{\bar{t}} + \sqrt{1+2\gamma_{tt}}\,\frac{d\underset{\sim}{\bar{t}}}{ds} \tag{4.6.6}
$$

$$
\frac{d\underset{\sim}{\bar{t}}}{ds} = \frac{1}{\sqrt{1+2\gamma_{tt}}}\,\frac{d\underset{\sim}{\bar{a}}_t}{ds} - \frac{1}{1+2\gamma_{tt}}\,\frac{d\gamma_{tt}}{ds}\,\underset{\sim}{\bar{t}} \tag{4.6.7}
$$

From (1.5.8), (4.6.7) and (4.6.3) we obtain

$$
\sqrt{1+2\gamma_{tt}}\,\bar{\sigma}_t = \underset{\sim}{\bar{n}} \cdot \frac{d\underset{\sim}{\bar{t}}}{ds} = \frac{1}{\sqrt{1+2\gamma_{tt}}}\,(\sigma_t - \kappa_{tt}) \tag{4.6.8}
$$

and using (1.5.10), (4.1.5) and (4.6.3) we obtain

$$
\begin{aligned}
\sqrt{1+2\gamma_{tt}}\,\bar{\kappa}_t &= -\,\underset{\sim}{\bar{\nu}} \cdot \frac{d\underset{\sim}{\bar{t}}}{ds} = -\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\kappa \underset{\sim}{\bar{a}}^\kappa \cdot \frac{1}{\sqrt{1+2\gamma_{tt}}}\,\frac{d\underset{\sim}{\bar{a}}_t}{ds} = \\
&= +\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,(\kappa_t - \nu_\kappa \bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}t^\alpha t^\beta)
\end{aligned}
\tag{4.6.9}
$$

Finally, using (1.5.9), (4.1.5) and (4.6.4) we obtain

$$\sqrt{1+2\gamma_{tt}}\,\bar{\tau}_t = \bar{\underset{\sim}{\nu}} \cdot \frac{d\bar{\underset{\sim}{n}}}{ds} = -\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\kappa \bar{\underset{\sim}{a}}^{\kappa} \cdot \bar{b}_{\alpha\beta}\bar{\underset{\sim}{a}}^{\alpha} t^\beta =$$

$$= -\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\kappa \bar{a}^{\kappa\alpha}\,(b_{\alpha\beta} - \kappa_{\alpha\beta})\,t^\beta \qquad (4.6.10)$$

Using [12] an equivallent expression for $\bar{\tau}_t$ may be derived in the form

$$\sqrt{1+2\gamma_{tt}}\,\bar{\tau}_t = \frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,[\tau_t + \nu_\lambda \bar{a}^{\lambda\mu}(\kappa_{\mu\beta} + 2b^\kappa_\beta \gamma_{\mu\kappa})\,t^\beta] \qquad (4.6.11)$$

In (4.6.9) and (4.6.10) the invariant $\sqrt{\frac{\bar{a}}{a}}$ and the components $\bar{a}^{\kappa\alpha}$ have to be expressed in terms of strains by (2.2.7) and (2.3.5), respectively.

Thus $\bar{\sigma}_t$, $\bar{\tau}_t$, and $\bar{\kappa}_t$ have been expressed in terms of the reference boundary geometry and the surface strain measures at the boundary. This representation is *exact* under K - L constraints.

4.7. Relations in terms of physical components

The formulae for $\bar{\tau}_t$ and $\bar{\kappa}_t$ are written in terms of tensor components at C. For better understanding of approximations in shell theory, it is convenient to have these formulae written entirely in terms of physical components at C. Such formulae have been obtained in [12] by expressing various tensors in (4.6.9) and (4.6.10) directly in terms of physical components.

We note first, that physical components of the tensor $\gamma_{\lambda\alpha\beta}$ follow from (2.3.4), (1.5.4) and (1.5.5); with the help of (1.5.10), (1.5.15) these components assume the form:

$$\gamma_{\nu tt} = (\gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda})\nu^{\lambda}t^{\alpha}t^{\beta} =$$

$$= 2(\gamma_{\lambda\alpha}\nu^{\lambda}t^{\alpha})_{|\beta}t^{\beta} - (\gamma_{\alpha\beta}t^{\alpha}t^{\beta})_{|\lambda}\nu^{\lambda} -$$

$$- 2(\gamma_{\lambda\alpha}\nu^{\lambda}{}_{|\beta}t^{\alpha} + \gamma_{\lambda\alpha}\nu^{\lambda}t^{\alpha}{}_{|\beta})t^{\beta} + 2\gamma_{\alpha\beta}t^{\alpha}{}_{|\lambda}t^{\beta}\nu^{\lambda} =$$

$$= 2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_{\nu}} - 2(\gamma_{\nu t}\nu_{\lambda}\nu^{\lambda}{}_{|\beta} + \gamma_{tt}t_{\lambda}\nu^{\lambda}{}_{|\beta} +$$

$$+ \gamma_{\nu\nu}\nu_{\alpha}t^{\alpha}{}_{|\beta} + \gamma_{\nu t}t_{\alpha}t^{\alpha}{}_{|\beta})t^{\beta} + 2(\gamma_{\nu t}\nu_{\alpha}t^{\alpha}{}_{|\lambda} + \gamma_{tt}t_{\alpha}t^{\alpha}{}_{|\lambda})\nu^{\lambda} =$$

$$= 2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_{\nu}} + 2\kappa_{t}(\gamma_{\nu\nu} - \gamma_{tt}) + 2\kappa_{\nu}\gamma_{\nu t} \qquad (4.7.1)$$

In a similar way we obtain as well

$$\gamma_{ttt} = (\gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda})t^{\alpha}t^{\beta}t^{\lambda} =$$

$$= \frac{d\gamma_{tt}}{ds} + 2\kappa_{t}\gamma_{\nu t} \qquad (4.7.2)$$

For an invariant $\bar{a}/a$ we obtain

$$\frac{\bar{a}}{a} = 1 + 2(\gamma_{\nu\nu} + \gamma_{tt}) + 4(\gamma_{\nu\nu}\gamma_{tt} - \gamma_{\nu t}^{2}) \qquad (4.7.3)$$

To express $\bar{a}^{\kappa\alpha}$ in terms of physical quantities, we first use (2.3.5) to express it in terms of $\bar{a}_{\alpha\beta}$, for which we can easily obtain

$$\bar{a}_{\alpha\beta} = (1 + 2\gamma_{\nu\nu})\nu_{\alpha}\nu_{\beta} + 2\gamma_{\nu t}(\nu_{\alpha}t_{\beta} + t_{\alpha}\nu_{\beta}) + (1 + 2\gamma_{tt})t_{\alpha}t_{\beta} \qquad (4.7.4)$$

If we use the identities

$$\epsilon^{\alpha\beta} = \nu^{\alpha}t^{\beta} - t^{\alpha}\nu^{\beta}$$

$$t^{\alpha}t_{\alpha} = \nu^{\alpha}\nu_{\alpha} = 1 \quad , \quad t^{\alpha}\nu_{\alpha} = \nu^{\alpha}t_{\alpha} = 0 \quad , \qquad (4.7.5)$$

then from (2.3.5) we obtain

$$\bar{a}^{\kappa\lambda} = \frac{a}{\bar{a}}[\nu^{\kappa}\nu^{\lambda}(1 + 2\gamma_{tt}) - (t^{\kappa}\nu^{\lambda} + \nu^{\kappa}t^{\lambda})2\gamma_{\nu t} + t^{\kappa}t^{\lambda}(1 + 2\gamma_{\nu\nu})] \qquad (4.7.6)$$

If we use these formulae then, according to (4.6.2), we obtain the following relations for the components of the vector of change of boundary curvature [12,21]

$$k_{tt} = \sigma_t\left(1 - \frac{1}{\sqrt{1 + 2\gamma_{tt}}}\right) + \frac{\kappa_{tt}}{\sqrt{1 + 2\gamma_{tt}}} \tag{4.7.7}$$

$$k_{\nu t} = \sqrt{\frac{a}{\bar{a}}}\left[\sqrt{1 + 2\gamma_{tt}}(\tau_t + \kappa_{\nu t}) + 2\gamma_{\nu t}(\sigma_t - \kappa_{tt})\right] - \tau_t \tag{4.7.8}$$

$$k_{nt} = \kappa_t\left(1 - \frac{1}{1 + 2\gamma_{tt}}\sqrt{\frac{a}{\bar{a}}}\right) - \frac{2\gamma_{\nu t}}{1 + 2\gamma_{tt}}\sqrt{\frac{a}{\bar{a}}}\left(\frac{d\gamma_{tt}}{ds} + 2\kappa_t\gamma_{\nu t}\right) + \tag{4.7.9}$$

$$+\sqrt{\frac{a}{\bar{a}}}\left[2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_\nu} + 2\kappa_\nu\gamma_{\nu t} + 2\kappa_t(\gamma_{\nu\nu} - \gamma_{tt})\right]$$

Using [12] an equivallent expression for $k_{\nu t}$ may be derived in the form

$$k_{\nu t} = \tau_t\left(\frac{1}{\sqrt{1 + 2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}} - 1\right) + \frac{1}{\sqrt{1 + 2\gamma_{tt}}}\sqrt{\frac{a}{\bar{a}}}\left\{(1 + 2\gamma_{tt})(\kappa_{\nu t} + \right. \tag{4.7.10}$$

$$\left. + 2\sigma_t\gamma_{\nu t} - 2\tau_t\gamma_{\nu\nu}) - 2\gamma_{\nu t}(\kappa_{tt} + 2\sigma_t\gamma_{tt} - 2\tau_t\gamma_{\nu t})\right\}$$

These expressions are *exact* under K - L constraints.

4.8. Deformational boundary conditions

In (4.4.1) we have presented differential equations which define implicitly the deformed boundary surface $\partial\bar{P}$. Differentiating those equations again with respect to s we obtain

$$\frac{\partial^2\underset{\sim}{\bar{p}}}{\partial s^2} = \frac{d}{ds}\underset{\sim}{\bar{a}}_t + \zeta\frac{d^2}{ds^2}\underset{\sim}{\bar{n}} \quad , \quad \frac{\partial^2\underset{\sim}{\bar{p}}}{\partial s\partial\zeta} = \frac{d}{ds}\underset{\sim}{\bar{n}}_t \quad , \quad \frac{d^2\underset{\sim}{\bar{r}}}{ds^2} = \frac{d}{ds}\underset{\sim}{\bar{a}}_t \tag{4.8.1}$$

These differential equations define implicitly the same boundary surface $\partial\bar{P}$, with accuracy up to translation linearly varying with s. The equations are defined uniquely if at C the following quantities are specified

$$\frac{d}{ds}\underset{\sim}{\bar{a}}_t(s) = \underset{\sim}{P}(s) \quad , \quad \frac{d}{ds}\underset{\sim}{\bar{n}}(s) = \underset{\sim}{Q}(s) \tag{4.8.2}$$

where in terms of $\underset{\sim}{A}(s)$ and $\underset{\sim}{B}(s)$

$$\underset{\sim}{P} = \frac{d\underset{\sim}{t}}{ds} + \frac{d^2\underset{\sim}{A}}{ds^2} \quad , \quad \underset{\sim}{Q} = \frac{d\underset{\sim}{n}}{ds} + \frac{d\underset{\sim}{B}}{ds} \tag{4.8.3}$$

It follows from (4.5.1), (4.5.2) and (4.5.3) that $\frac{d}{ds}\bar{\underset{\sim}{a}}_t$ and $\frac{d}{ds}\bar{\underset{\sim}{n}}$ at $\bar{C}$ are completely defined by the vector $\bar{\underset{\sim}{\omega}}_t$ and the scalar γ_{tt}. Since $\bar{\underset{\sim}{\omega}}_t$ has been expressed in terms of $\underset{\sim}{k}_t$ by (4.5.14) and (4.5.9), in place of $\bar{\underset{\sim}{\omega}}_t$ we may use $\underset{\sim}{k}_t$. In fact, $\underset{\sim}{k}_t$ is more convenient, as it is defined explicitly only in terms of the components of surface strain tensor $\gamma_{\alpha\beta}$ and the components of the tensor of change of surface curvature $\kappa_{\alpha\beta}$, calculated along the reference boundary contour C. Thus the right-hand sides of (4.8.1) are completely defined if the following conditions are specified

$$\underset{\sim}{k}_t(s) = \underset{\sim}{q}(s) \quad , \quad \gamma_{tt}(s) = l(s) \qquad \text{at } C \tag{4.8.4}$$

The conditions are called the *deformational boundary conditions* of the K - L non-linear theory of shells [12,21].

For $\underset{\sim}{p}(s)$ it is possible to obtain the expression in terms of $\underset{\sim}{m}(s)$, which has been used for kinematical boundary conditions (4.3.7). By an analogy to analytical mechanics of rigid-body motion, where the instantaneous angular velocity is expressed by the same relation in terms of the finite rotation vector [24], we obtain for $\underset{\sim}{q}(s)$

$$\underset{\sim}{q}(s) = \frac{d\underset{\sim}{m}}{ds} - \frac{1}{1+\sqrt{1-\underset{\sim}{m}\cdot\underset{\sim}{m}}}\left[\underset{\sim}{m}\cdot\frac{d}{ds}\sqrt{1-\underset{\sim}{m}\cdot\underset{\sim}{m}} + \underset{\sim}{m}\times\frac{d\underset{\sim}{m}}{ds}\right] \tag{4.8.5}$$

It has been noted in [21], that there are no other variants of geometrical boundary conditions beyond those presented in (4.3.7), (4.4.5) and (4.8.4). Indeed, to obtain those boundary conditions we need to differentiate again (4.8.1) with respect to s and ζ. As a result we obtain linear combinations (with coefficients of the type $a+\zeta b$, a,b = const) of the derivatives of $\bar{\underset{\sim}{a}}_t$ and $\bar{\underset{\sim}{n}}$. Therefore, the higher-order derivatives of $\bar{\underset{\sim}{a}}_t$ and $\bar{\underset{\sim}{n}}$ are expressible in terms of the first derivatives, and derivatives of $\bar{\underset{\sim}{p}}(s,\zeta)$ of higher than the second order are expressible in terms of (4.4.1) and (4.8.1).

Chapter 5

BASIC SHELL EQUATIONS

The two-dimensional *equilibrium equations* and the appropriate *natural boundary conditions* for the non-linear theory of shells under K - L constraints may be derived in many ways.

The *Eulerian* equilibrium equations for shells, obtained by *direct integration* over the shell thickness of Eulerian three-demensional equilibrium equations of a continua, have been discussed, for example, in the works of NAGHDI [4,5]. Natural boundary conditions are then constructed by appealing to variational principles. The stress resultant and stress couple tensors defined in this way happen to be *non-symmetric* in general. It is possible, however, to introduce some *symmetric* combinations of these unsymmetric surface tensors and after additional transformations (see [3,7,26] for example) express all shell relations in terms of these symmetric stress and couple resultants. KOITER [6] postulated a two-dimensional *Eulerian virtual work principle*, from which both Eulerian equilibrium equations and natural boundary conditions may be obtained in terms of *symmetric* stress and couple resultants. They are symmetric by definition, being coefficients of symmetric surface strain measures in an invariant virtual work expression. This direct approach is compatible with K - L constraints.

Various forms of the *Lagrangean* equilibrium equations and appropriate natural boundary conditions, in terms of *symmetric* stress and strain measures, have been discussed by the author either on the basis of a Lagragean form of two-dimensional *virtual work principle* [8 - 11] or by *direct integration* of the Lagrangean three-dimensional equations of the continua over the shell thickness in the reference configuration [2,25]. Both ways have been discussed in the authors thesis [12] within relaxed constraints, and results under K - L constraints have been obtained in [12] as a particular case.

Here we begin by postulating an Eulerian two-dimensional virtual work principle, from which Eulerian equilibrium equations and natural boundary conditions in terms of symmetric quantities are derived. Then

we relate the results to those which would follow from direct integration of a three-dimensional stress state in a shell. Transformation rules derived here allow us to introduce various Lagrangean quantities defined with respect to the reference surface geometry. A Lagrangean form of the shell equations is obtained and some possible modified forms are derived. Finally we discuss briefly a form of the two-dimensional constitutive equations for an elastic shell.

5.1. Eulerian virtual work principle

It has been shown in chapter 3 that, under K - L constraints, deformation of a shell is described entirely by deformation of its middle surface. Thus under K - L constraints all external or internal forces and couples should be referred to the shell middle surface.

In the *Eulerian* description all forces and couples are referred to the *deformed* surface geometry.

Let the shell with simply connected middle surface be in *equilibirum* under the surface load $\bar{\underset{\sim}{p}}$, per unit area of *deformed* surface $\bar{M}$, and boundary force $\bar{\underset{\sim}{F}}$ and couple $\bar{\underset{\sim}{K}}$, per unit length of *deformed* boundary $\bar{C}$, Fig. 15.

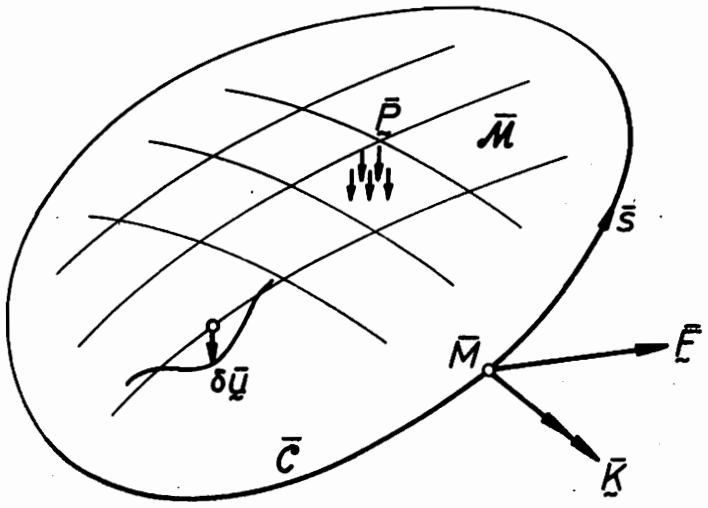

Fig. 15

Then for any *additional* virtual displacement field $\delta\bar{\underset{\sim}{u}}$, subject only to geometrical constraints

$$\delta\bar{\underset{\sim}{u}} = \delta\bar{u}_{\alpha}\,\bar{\underset{\sim}{a}}^{\alpha} + \delta\bar{w}\,\bar{\underset{\sim}{n}} \qquad (5.1.1)$$

there should be two-dimensional *Eulerian* symmetric stress and couple resultant tensors

$$\bar{\underset{\sim}{N}} = \bar{N}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} \otimes \bar{\underset{\sim}{a}}_{\beta} \ , \quad \bar{\underset{\sim}{M}} = \bar{M}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} \otimes \bar{\underset{\sim}{a}}_{\beta} \tag{5.1.2}$$

such that the *Eulerian virtual work principle* takes the form

$$\begin{aligned} &\iint_{\bar{M}} (\bar{N}^{\alpha\beta}\delta\bar{\gamma}_{\alpha\beta} + \bar{M}^{\alpha\beta}\delta\bar{\kappa}_{\alpha\beta})\, d\bar{A} = \\ &= \iint_{\bar{M}} \bar{\underset{\sim}{p}} \cdot \delta\bar{\underset{\sim}{u}}\, d\bar{A} + \int_{\bar{C}} (\bar{\underset{\sim}{F}} \cdot \delta\bar{\underset{\sim}{u}} + \bar{\underset{\sim}{K}} \cdot \delta\bar{\underset{\sim}{\Omega}}_t)\, d\bar{s} \end{aligned} \tag{5.1.3}$$

The left-hand side of (5.1.3) expresses the *internal virtual work* (IVW) done by internal stress and couple resultants on strains due to the virtual deformation of the surface. The right-hand side of (5.1.3) expresses the *external virtual work* (EVW) done by external surface forces $\bar{\underset{\sim}{p}}$ and boundary forces $\bar{\underset{\sim}{F}}$ and couples $\bar{\underset{\sim}{K}}$ on virtual displacements and rotations of the surface.

Under virtual displacements (5.1.1) the shell middle surface moves from deformed $\bar{M}$ to an adjacent M' configuration, deformation between $\bar{M}$ and M' being infinitesimal. Thus the virtual strain measures $\delta\bar{\underset{\sim}{\gamma}}$, $\delta\bar{\underset{\sim}{\kappa}}$ and the total virtual rotation vector $\delta\bar{\underset{\sim}{\Omega}}_t$ are *linear* in $\delta\underset{\sim}{u}$. If geometric quantities of M' are denoted by $\underset{\sim}{r}'$, $\underset{\sim}{a}'_{\alpha}$, $\underset{\sim}{a}'^{\alpha}$, $\underset{\sim}{a}'_{\alpha\beta}$, $\underset{\sim}{b}'_{\alpha\beta}$ etc. then with the help of (2.2.4) and (2.2.14) we obtain

$$\delta\bar{\gamma}_{\alpha\beta} = \frac{1}{2}\,(a'_{\alpha\beta} - \bar{a}_{\alpha\beta}) = \delta\bar{\theta}_{\alpha\beta} = \frac{1}{2}\,(\delta\bar{u}_{\alpha||\beta} + \delta\bar{u}_{\beta||\alpha}) - \bar{b}_{\alpha\beta}\delta\bar{w} \tag{5.1.4}$$

$$\begin{aligned} \delta\bar{\kappa}_{\alpha\beta} &= -\,(b'_{\alpha\beta} - \bar{b}_{\alpha\beta}) = \\ &= -\,\delta\bar{w}_{||\alpha\beta} - \bar{b}^{\lambda}_{\alpha}\delta\bar{u}_{\lambda||\beta} - \bar{b}^{\lambda}_{\beta}\delta\bar{u}_{\lambda||\alpha} - \bar{b}^{\lambda}_{\alpha||\beta}\,\delta\bar{u}_{\lambda} + \bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta}\delta\bar{w} \end{aligned} \tag{5.1.5}$$

It follows from (2.1.10) together with (2.2.8), (2.2.9) and (2.1.9) that

$$\delta\bar{\underset{\sim}{a}}_{\alpha} = \delta\bar{\gamma}_{\lambda\alpha}\bar{\underset{\sim}{a}}^{\lambda} + \delta\bar{\underset{\sim}{\Omega}} \times \bar{\underset{\sim}{a}}_{\alpha} = \delta\bar{\theta}_{\lambda\alpha}\bar{\underset{\sim}{a}}^{\lambda} + \delta\bar{\underset{\sim}{\phi}} \times \bar{\underset{\sim}{a}}_{\alpha} \tag{5.1.6}$$

$$\delta\bar{\underset{\sim}{n}} = \delta\bar{\underset{\sim}{\Omega}} \times \bar{\underset{\sim}{n}} = \delta\bar{\underset{\sim}{\phi}} \times \bar{\underset{\sim}{n}} \tag{5.1.7}$$

and using (4.2.4) and (4.2.1) we obtain

$$\delta\bar{\underset{\sim}{\Omega}}_t = \delta\overset{v}{\bar{\underset{\sim}{\Omega}}}_t + \delta\bar{\underset{\sim}{\Omega}} = \delta\bar{\underset{\sim}{\Omega}} - \delta\overset{v}{\bar{\gamma}}_{\nu t}\bar{\underset{\sim}{n}}$$
$$= \delta\bar{\underset{\sim}{\phi}} - \delta\bar{\theta}_{\nu t}\bar{\underset{\sim}{n}} \qquad (5.1.8)$$

Here $\delta\overset{v}{\bar{\underset{\sim}{\Omega}}}_t$ is the virtual rotation vector of the surface boundary due to the pure virtual stretch along principal directions of virtual strains, and

$$\delta\bar{\underset{\sim}{\Omega}} = \delta\bar{\underset{\sim}{\phi}} = \bar{\epsilon}^{\beta\alpha}(\delta\bar{\varphi}_\alpha\bar{\underset{\sim}{a}}_\beta + \frac{1}{2}\,\delta\bar{\omega}_{\beta\alpha}\bar{\underset{\sim}{n}}) \qquad (5.1.9)$$

is the virtual rotation vector due to the rotational part of the virtual deformation.

When $\underset{\sim}{\rho}$ defined in (2.2.2) is used as the tensor of change of the surface curvature, the left-hand side of (5.1.3) takes the form

$$IVW = \iint_{\bar{M}} (\bar{n}^{\alpha\beta}\delta\bar{\gamma}_{\alpha\beta} + \bar{m}^{\alpha\beta}\delta\bar{\rho}_{\alpha\beta})\, d\bar{A} \qquad (5.1.10)$$

where

$$\delta\bar{\rho}_{\alpha\beta} = \delta\bar{\kappa}_{\alpha\beta} + \frac{1}{2}\,(\bar{b}^\lambda_\alpha\delta\bar{\gamma}_{\lambda\beta} + \bar{b}^\lambda_\beta\delta\bar{\gamma}_{\lambda\alpha}) = \delta\bar{\mu}_{\alpha\beta}$$
$$= -\frac{1}{2}\,(\delta\bar{\varphi}_{\alpha||\beta} + \delta\bar{\varphi}_{\beta||\alpha} + \bar{b}^\lambda_\alpha\delta\bar{\omega}_{\beta\lambda} + \bar{b}^\lambda_\beta\delta\bar{\omega}_{\alpha\lambda}) \qquad (5.1.11)$$

The new Eulerian symmetric stress and couple resultant tensors defined in (5.1.10) as coefficients of the virtual strain measures in this invariant virtual work expression, are related to those defined by (5.1.3) according to

$$\bar{N}^{\alpha\beta} = \bar{n}^{\alpha\beta} + \frac{1}{2}(\bar{b}^\alpha_\lambda\bar{m}^{\lambda\beta} + \bar{b}^\beta_\lambda\bar{m}^{\lambda\alpha}) \quad , \quad \bar{M}^{\alpha\beta} = \bar{m}^{\alpha\beta} \qquad (5.1.12)$$

5.2. Eulerian shell equations

Let us introduce (5.1.4) and (5.1.5) into (5.1.3). Keeping in mind the symmetry of $N^{\alpha\beta}$ and $M^{\alpha\beta}$ and using *Stockes theorem* IVW can be transformed as follows:

$$
\begin{aligned}
IVW = & \iint_{\bar{M}} [\bar{N}^{\alpha\beta}(\delta\bar{u}_{\alpha}||_{\beta} - \bar{b}_{\alpha\beta}\delta\bar{w}) + \\
& + \bar{M}_{\alpha\beta}(-\delta\bar{w}||_{\alpha\beta} - 2\bar{b}^{\lambda}_{\alpha}\delta\bar{u}_{\lambda}||_{\beta} - \bar{b}^{\lambda}_{\alpha}||_{\beta}\delta\bar{u}_{\lambda} + \bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta}\delta\bar{w})]\, d\bar{A} = \\
= & \iint_{\bar{M}} [(\bar{N}^{\alpha\beta}\delta\bar{u}_{\alpha})||_{\beta} - \bar{N}^{\alpha\beta}||_{\beta}\delta\bar{u}_{\alpha} - \bar{b}_{\alpha\beta}\bar{N}^{\alpha\beta}\delta\bar{w} - \\
& - (\bar{M}^{\alpha\beta}\delta\bar{w}_{,\alpha})||_{\beta} + \bar{M}^{\alpha\beta}||_{\beta}\delta\bar{w}_{,\alpha} - 2(\bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta}\delta\bar{u}_{\alpha})||_{\beta} + 2(\bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})||_{\beta}\delta\bar{u}_{\alpha} - \\
& - (\bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})||_{\beta}\delta\bar{u}_{\alpha} + \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta}||_{\beta}\delta\bar{u}_{\alpha} + \bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta}\bar{M}^{\alpha\beta}\delta\bar{w}]\, d\bar{A} = \\
= & - \iint_{\bar{M}} \{[(\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})||_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta}||_{\beta}]\delta\bar{u}_{\alpha} + \\
& + [\bar{M}^{\alpha\beta}||_{\alpha\beta} + \bar{b}_{\alpha\beta}(\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})]\delta\bar{w}\}\, d\bar{A} + \\
& + \int_{\bar{C}} [(\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})\delta\bar{u}_{\alpha} + \bar{M}^{\alpha\beta}||_{\alpha}\delta\bar{w} - \bar{M}^{\alpha\beta}\delta\bar{\varphi}_{\alpha}]\bar{\nu}_{\beta}\, d\bar{s}
\end{aligned}
\tag{5.2.1}
$$

As the virtual displacement field $\delta\underset{\sim}{\bar{u}}$ is *arbitrary* within $\bar{M}$, for the surface forces $\underset{\sim}{\bar{p}}$ such that

$$
\underset{\sim}{\bar{p}} = \bar{p}^{\alpha}\underset{\sim}{\bar{a}}_{\alpha} + \bar{p}\,\underset{\sim}{\bar{n}} \tag{5.2.2}
$$

the surface integrals in (5.2.1) and (5.1.3) give us the following *Eulerian equations of equilibrium*

$$
\begin{aligned}
& (\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})||_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta}||_{\beta} + \bar{p}^{\alpha} = 0 \\
& \bar{M}^{\alpha\beta}||_{\alpha\beta} + \bar{b}_{\alpha\beta}(\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta}) + \bar{p} = 0
\end{aligned}
\tag{5.2.3}
$$

to be satisfied within an arbitrary internal region of the surface $\bar{M}$.

An *equivallent* form of the *Eulerian equiblibrium equations*, written in terms of $\bar{n}^{\alpha\beta}$ and $\bar{m}^{\alpha\beta}$, [6], follows from (5.1.12) and (5.2.3):

$$
\begin{aligned}
& [\bar{n}^{\alpha\beta} + \tfrac{1}{2}(\bar{b}^{\beta}_{\lambda}\bar{m}^{\alpha\lambda} - \bar{b}^{\alpha}_{\lambda}\bar{m}^{\lambda\beta})]||_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{m}^{\lambda\beta}||_{\beta} + \bar{p}^{\alpha} = 0 \\
& \bar{m}^{\alpha\beta}||_{\alpha\beta} + \bar{b}_{\alpha\beta}\bar{n}^{\alpha\beta} + \bar{p} = 0
\end{aligned}
\tag{5.2.4}
$$

Let us define the vectors

$$\bar{\underset{\sim}{N}}^{\beta} = \bar{Q}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} + \bar{Q}^{\beta}\bar{\underset{\sim}{n}} \quad , \quad \bar{\underset{\sim}{M}}^{\beta} = \bar{\epsilon}_{\alpha\lambda}\bar{M}^{\alpha\beta}\bar{\underset{\sim}{a}}^{\lambda} \tag{5.2.5}$$

where

$$\bar{Q}^{\alpha\beta} = \bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta} = \bar{n}^{\alpha\beta} + \frac{1}{2}\left(\bar{b}^{\beta}_{\lambda}\bar{m}^{\alpha\lambda} - \bar{b}^{\alpha}_{\lambda}\bar{m}^{\lambda\beta}\right) \tag{5.2.6}$$

$$\bar{Q}^{\beta} = \bar{M}^{\alpha\beta}\|_{\alpha} = \bar{m}^{\alpha\beta}\|_{\alpha} \tag{5.2.7}$$

It is easy to show that (5.2.3) and (5.2.4) are components, referred to the deformed surface basis $\bar{\underset{\sim}{a}}_{\alpha}$, $\bar{\underset{\sim}{n}}$, of the following *vector Eulerian equilibrium equations*

$$\bar{\underset{\sim}{N}}^{\beta}\|_{\beta} + \bar{\underset{\sim}{p}} = \underset{\sim}{0} \tag{5.2.8}$$

Besides, we have the relation

$$\bar{\underset{\sim}{M}}^{\beta}\|_{\beta} + \bar{\underset{\sim}{a}}_{\beta} \times \bar{\underset{\sim}{N}}^{\beta} = \underset{\sim}{0} \tag{5.2.9}$$

which is an *identity* in the present theory. In the case of dynamics, when rotary inertial forces are taken into account, or even in static case, when we want to take into account some external surface moments, the equation equivallent to (5.2.9) appears independently of (5.2.8) as a local form of moment of momentum or, in statics, the independent moment equilibrium condition about an origin O in space, [11,23].

With the help of (5.2.5) and (5.2.1) the Eulerian virtual work principle can be written in vector form

$$\begin{aligned} -\iint_{\bar{M}} \bar{\underset{\sim}{N}}^{\beta}\|_{\beta} \cdot \delta\bar{\underset{\sim}{u}}\, d\bar{A} + \int_{\bar{C}} \left(\bar{\underset{\sim}{N}}^{\beta} \cdot \delta\bar{\underset{\sim}{u}} - \bar{M}^{\alpha\beta}\bar{\underset{\sim}{n}} \cdot \delta\bar{\underset{\sim}{u}}_{,\alpha}\right)\bar{\nu}_{\beta}\, d\bar{s} = \\ = \iint_{\bar{M}} \bar{\underset{\sim}{p}} \cdot \delta\bar{\underset{\sim}{u}}\, d\bar{A} + \int_{\bar{C}} \left(\bar{\underset{\sim}{F}} \cdot \delta\bar{\underset{\sim}{u}} + \bar{\underset{\sim}{K}} \cdot \delta\bar{\underset{\sim}{\Omega}}_{t}\right) d\bar{s} \end{aligned} \tag{5.2.10}$$

where we have used the following transformation

$$\begin{aligned} -\bar{M}^{\alpha\beta}\bar{\underset{\sim}{n}} \cdot \delta\bar{\underset{\sim}{u}}_{,\alpha} &= -\bar{M}^{\alpha\beta}\bar{\underset{\sim}{n}} \cdot \left(\delta\bar{u}_{\lambda}\bar{\underset{\sim}{a}}^{\lambda}|_{\alpha} + \delta\bar{w}|_{\alpha}\bar{\underset{\sim}{n}}\right) \\ &= -\bar{M}^{\alpha\beta}\left(\delta\bar{w}_{,\alpha} + \bar{b}^{\lambda}_{\alpha}\delta\bar{u}_{\lambda}\right) = -\bar{M}^{\alpha\beta}\delta\bar{\varphi}_{\alpha} \end{aligned} \tag{5.2.11}$$

But in terms of physical quantities at $\bar{C}$

$$\delta\bar{\varphi}_\alpha = \bar{\nu}_\alpha \delta\bar{\varphi}_\nu + \bar{t}_\alpha \delta\bar{\varphi}_t \tag{5.2.12}$$

$$\delta\underset{\sim}{\bar{u}} = \delta\bar{u}_\nu \underset{\sim}{\bar{\nu}} + \delta\bar{u}_t \underset{\sim}{\bar{t}} + \delta\bar{w}\,\underset{\sim}{\bar{n}} \tag{5.2.13}$$

where from (2.1.5), (1.5.8) and (1.5.9) we obtain

$$\begin{aligned}\delta\bar{\varphi}_t &= \bar{t}^\alpha(\delta\bar{w}_{,\alpha} + \bar{b}_\alpha^\lambda \delta\bar{u}_\lambda) \\ &= \frac{d}{d\bar{s}}(\delta\bar{w}) + \bar{\sigma}_t \delta\bar{u}_t - \bar{\tau}_t \delta\bar{u}_\nu\end{aligned} \tag{5.2.14}$$

Keeping only linear terms in (4.3.3) we have as well

$$\delta\bar{\varphi}_\nu = -\delta\bar{\beta}_\nu \tag{5.2.15}$$

With the help of (5.2.14) and (5.2.15) we make the following transformation of the last term in (5.2.1)

$$\begin{aligned}-\int_{\bar{C}} \bar{M}^{\alpha\beta}\delta\bar{\varphi}_\alpha \bar{\nu}_\beta \, d\bar{s} &= \int_{\bar{C}} [\bar{\tau}_t \bar{M}_{t\nu}\delta\bar{u}_\nu - \bar{\sigma}_t \bar{M}_{t\nu}\delta\bar{u}_t - \\ &- \frac{d}{d\bar{s}}(\bar{M}_{t\nu}\delta\bar{w}) + \frac{d}{d\bar{s}}(\bar{M}_{t\nu})\delta\bar{w} + \bar{M}_{\nu\nu}\delta\bar{\beta}_\nu]\, d\bar{s} = \\ &= \int_{\bar{C}} [\frac{d}{d\bar{s}}(\bar{M}_{t\nu}\underset{\sim}{\bar{n}}) \cdot \delta\underset{\sim}{\bar{u}} + \bar{M}_{\nu\nu}\delta\bar{\beta}_\nu]\, d\bar{s} + \\ &+ \sum_{i=1}^{N} [\bar{M}_{t\nu}(\bar{s}_i + 0) - \bar{M}_{t\nu}(\bar{s}_i - 0)]\delta\bar{w}(\bar{s}_i)\end{aligned} \tag{5.2.16}$$

where the term outside the line integral describes *jump discontinuities* of $\bar{M}_{t\nu}$ at corner points of $\bar{C}$ labelled by $\bar{s} = \bar{s}_i$, $i = 1,2...N$, Fig. 16.

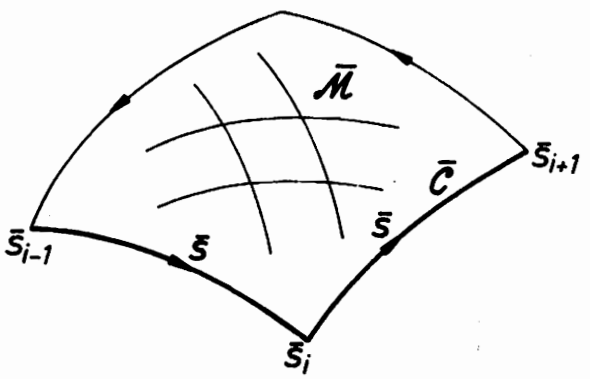

Fig. 16

If we put (5.2.16) into (5.2.1) or the left side of (5.2.10), the line integral becomes

$$\int_{\bar{C}} (\underset{\sim}{\bar{P}}_\nu \cdot \delta\underset{\sim}{\bar{u}} + \bar{M}_{\nu\nu}\delta\bar{\beta}_\nu)\, d\bar{s} + \sum_{i=1}^{N} [\bar{M}_{t\nu}(\bar{s}_i + 0) - \bar{M}_{t\nu}(\bar{s}_i - 0)]\delta\bar{w}(\bar{s}_i) \quad (5.2.17)$$

where

$$\begin{aligned} \underset{\sim}{\bar{P}}_\nu &= \bar{N}^{\alpha\beta}\bar{\nu}_\beta + \frac{d}{d\bar{s}}(\bar{M}_{t\nu}\underset{\sim}{\bar{n}}) \\ &= \bar{P}_{\nu\nu}\underset{\sim}{\bar{\nu}} + \bar{P}_{t\nu}\underset{\sim}{\bar{t}} + \bar{P}_{n\nu}\underset{\sim}{\bar{n}} \end{aligned} \quad (5.2.18)$$

$$\begin{aligned} \bar{P}_{\nu\nu} &= (\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})\bar{\nu}_\alpha\bar{\nu}_\beta + \bar{\tau}_t\bar{M}_{t\nu} \\ \bar{P}_{t\nu} &= (\bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})\bar{t}_\alpha\bar{\nu}_\beta - \bar{\sigma}_t\bar{M}_{t\nu} \\ \bar{P}_{n\nu} &= \bar{M}^{\alpha\beta}\|_\alpha\bar{\nu}_\beta - \frac{d}{d\bar{s}}\bar{M}_{t\nu} \end{aligned} \quad (5.2.19)$$

The vector $\underset{\sim}{\bar{P}}_\nu$ is called the *effective internal force* at the boundary $\bar{C}$.

Using (1.5.4), (1.5.8) to (1.5.10) and (1.5.15) we can express (5.2.19) entirely in terms of physical quantities.

$$\begin{aligned} \bar{P}_{\nu\nu} &= \bar{N}_{\nu\nu} - \bar{\sigma}_\nu\bar{M}_{\nu\nu} + 2\bar{\tau}_t\bar{M}_{t\nu} \\ \bar{P}_{t\nu} &= \bar{N}_{t\nu} + \bar{\tau}_t\bar{M}_{\nu\nu} - 2\bar{\sigma}_t\bar{M}_{t\nu} \\ \bar{P}_{n\nu} &= \frac{d\bar{M}_{\nu\nu}}{d\bar{s}_\nu} + \frac{d\bar{M}_{t\nu}}{d\bar{s}} + \bar{\kappa}_t(\bar{M}_{\nu\nu} - \bar{M}_{tt}) \end{aligned} \quad (5.2.20)$$

In terms of physical components of $\bar{n}^{\alpha\beta}$ and $\bar{m}^{\alpha\beta}$ according to (5.1.12) we would obtain

$$\begin{aligned} \bar{P}_{\nu\nu} &= \bar{n}_{\nu\nu} + \bar{\tau}_t\bar{m}_{t\nu} \\ \bar{P}_{t\nu} &= \bar{n}_{t\nu} + \frac{1}{2}(\bar{\sigma}_\nu - 3\bar{\sigma}_t)\bar{m}_{t\nu} + \frac{1}{2}\bar{\tau}_t(\bar{m}_{\nu\nu} - \bar{m}_{tt}) \\ \bar{P}_{n\nu} &= \frac{d\bar{m}_{\nu\nu}}{d\bar{s}_\nu} + \frac{d\bar{m}_{t\nu}}{d\bar{s}} + \bar{\kappa}_t(\bar{m}_{\nu\nu} - \bar{m}_{tt}) \end{aligned} \quad (5.2.21)$$

In order to transform the line integral of EVW in (5.1.3) consider the boundary forces and couples of the form

$$\bar{\underset{\sim}{F}} = \bar{F}^{\alpha}\bar{\underset{\sim}{a}}_{\alpha} + \bar{F}\bar{\underset{\sim}{n}} = \bar{F}_{\nu}\bar{\underset{\sim}{\nu}} + \bar{F}_{t}\bar{\underset{\sim}{t}} + \bar{F}\underset{\sim}{n} \tag{5.2.22}$$

$$\bar{\underset{\sim}{K}} = \bar{\epsilon}_{\alpha\lambda}\bar{K}^{\alpha}\bar{\underset{\sim}{a}}^{\lambda} = -\bar{K}_{t}\bar{\underset{\sim}{\nu}} + \bar{K}_{\nu}\bar{\underset{\sim}{t}}. \tag{5.2.23}$$

We assume here $\bar{\underset{\sim}{K}}$ to have no component along $\bar{\underset{\sim}{n}}$, but take into account its component along $\bar{\underset{\sim}{\nu}}$. In most papers [4,7] the last one is assumed to be zero as well.

With (5.2.23), (5.1.8) and (5.1.9) we obtain

$$\int_{\bar{C}} \bar{\underset{\sim}{K}} \cdot \delta\bar{\underset{\sim}{\Omega}}_{t}\, d\bar{s} = -\int_{\bar{C}} \bar{K}^{\alpha}\delta\bar{\varphi}_{\alpha}\, d\bar{s} \tag{5.2.24}$$

This relation may be transformed in exactly the same way as in (5.2.16). That for the line integral of EVW in (5.1.3) gives us

$$\int_{\bar{C}} (\bar{\underset{\sim}{R}} \cdot \delta\bar{\underset{\sim}{u}} + \bar{K}_{\nu}\delta\bar{\beta}_{\nu})\, d\bar{s} + \sum_{i=1}^{N} [\bar{K}_{t}(\bar{s}_{i}+0) - \bar{K}_{t}(\bar{s}_{i}-0)]\,\delta\bar{w}(\bar{s}_{i}) \tag{5.2.25}$$

where the *effective external force* at $\bar{C}$ is defined by

$$\begin{aligned} \bar{\underset{\sim}{R}} &= \bar{\underset{\sim}{F}} + \frac{d}{d\bar{s}}(\bar{K}_{t}\bar{\underset{\sim}{n}}) \\ &= \bar{R}_{\nu}\bar{\underset{\sim}{\nu}} + \bar{R}_{t}\bar{\underset{\sim}{t}} + \bar{R}_{n}\bar{\underset{\sim}{n}} \end{aligned} \tag{5.2.26}$$

$$\begin{aligned} \bar{R}_{\nu} &= \bar{F}_{\nu} + \bar{\tau}_{t}\bar{K}_{t} \\ \bar{R}_{t} &= \bar{F}_{t} - \bar{\sigma}_{t}\bar{K}_{t} \\ \bar{R}_{n} &= \bar{F} + \frac{d}{d\bar{s}}\bar{K}_{t} \end{aligned} \tag{5.2.27}$$

The term outside the line integral describes again the *jump discontinuities* of $\bar{K}_{\nu}$ at the corner points of $\bar{C}$, Fig. 16.

It has been shown in § 4.3., that four scalar quantities $\underset{\sim}{u}$ and β_{ν} define uniquely the deformed shell boundary surface $\partial\bar{P}$. All line integrals have been expressed here in terms of variations of these quantities. Thus

for unspecified geometrical constraints of the shell boundary from (5.2.17) and (5.2.25) we obtain the following *Eulerian natural boundary conditions*

$$\underset{\sim}{\bar{P}}_{\nu} = \underset{\sim}{\bar{R}} \quad , \quad \bar{M}_{\nu\nu} = \bar{K}_{\nu} \tag{5.2.28}$$

to be satisfied at smooth parts of $\bar{C}$ and the *concentrated force*

$$\{[\bar{M}_{t\nu}(\bar{s}_i + 0) - \bar{K}_t(\bar{s}_i + 0)] - [\bar{M}_{t\nu}(\bar{s}_i - 0) - \bar{K}_t(\bar{s}_i - 0)]\}\underset{\sim}{\bar{n}}(\bar{s}_i) \tag{5.2.29}$$

to be applied at each *corner* of $\bar{C}$.

These natural boundary conditions (5.2.28) and (5.2.29) are *exact* under K - L constraints.

5.3. Relation to three-dimensional stress state

The Eulerian stress and couple resultant tensors $\bar{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$ has been introduced in (5.1.3) *by definition*, as coefficients of strain measures in the invariant internal virtual work. Let us relate them now to the unknown stress distribution in a shell.

Let $\bar{C}$ be an *arbitrary smooth curve* at the deformed surface $\bar{M}$. According to K - L constraints, the curve $\bar{C}$ generates the cross-section surface $\partial\bar{P}$, which is orthogonal to $\bar{M}$, Fig. 17.

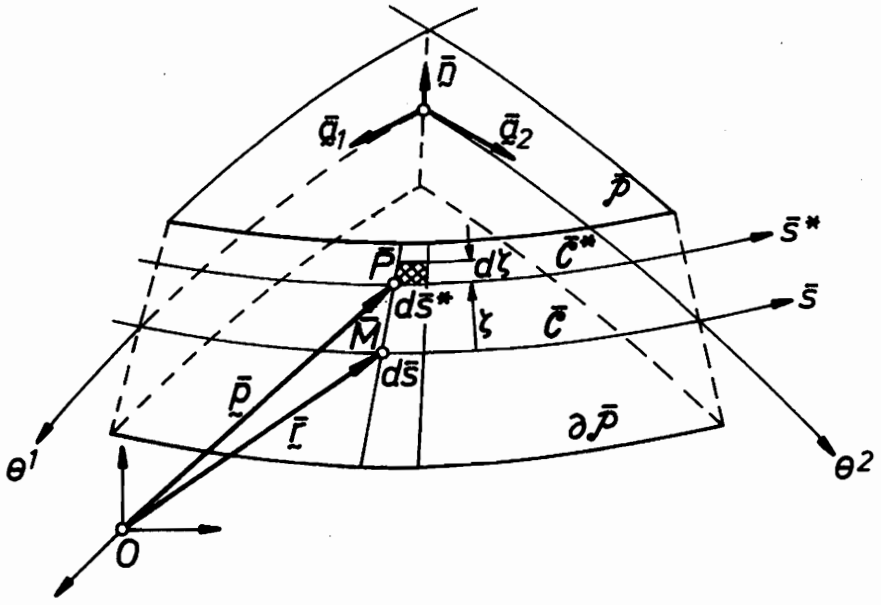

Fig. 17

Let $\bar{s}$ be the *length paramenter* of the curve $\bar{C}$ and $\underset{\sim}{\bar{\nu}}$, $\underset{\sim}{\bar{t}}$, $\underset{\sim}{\bar{n}}$ form the triad of orthonormal vectors associated with the point $\bar{M}$. Let us define the *second* curve $\bar{C}^*$, as the intersection of the surface $\partial\bar{P}$ and the surface ζ = const, with the length parameter $\bar{s}^*$ and with the orthonormal triad $\underset{\sim}{\bar{\nu}}^*, \underset{\sim}{\bar{t}}^*, \underset{\sim}{\bar{n}}^* \equiv \underset{\sim}{\bar{n}} = \underset{\sim}{\bar{g}}_3$. Recalling that, according to (3.1.1) to (3.1.4), for the normal coordinate system

$$\underset{\sim}{\bar{t}}^* = \frac{\partial \underset{\sim}{\bar{P}}}{\partial \bar{s}^*} = \bar{t}^{*\varphi} \underset{\sim}{\bar{g}}_\varphi = \bar{t}^{*\varphi} \bar{\mu}_\varphi^\alpha \underset{\sim}{\bar{a}}_\alpha = \bar{t}^\alpha \underset{\sim}{\bar{a}}_\alpha$$

$$\underset{\sim}{\bar{\nu}}^* = \bar{\nu}_\psi^* \underset{\sim}{\bar{g}}^\psi = \bar{\nu}_\psi^* \bar{\mu}_\beta^\psi \underset{\sim}{\bar{a}}^\beta = \bar{\nu}_\beta \underset{\sim}{\bar{a}}^\beta \tag{5.3.1}$$

$$= \underset{\sim}{\bar{t}}^* \times \underset{\sim}{\bar{n}}^* = \bar{t}^{*\varphi} \bar{\epsilon}_{\varphi 3 \psi} \underset{\sim}{\bar{g}}^\psi = \bar{\epsilon}_{\psi\varphi 3} \bar{t}^{*\varphi} \underset{\sim}{\bar{g}}^\psi$$

we obtain

$$\bar{t}^\alpha = \bar{\mu}_\varphi^\alpha \bar{t}^{*\varphi} \quad , \quad \bar{\nu}_\beta = \bar{\mu}_\beta^\psi \bar{\nu}_\psi^* \tag{5.3.2}$$

from which it follows that

$$\bar{t}^{*\varphi}\, d\bar{s}^* = \delta_\alpha^\varphi \bar{t}^\alpha\, d\bar{s} \tag{5.3.3}$$

$$\bar{\nu}_\psi^*\, d\bar{s}^* = \delta_\psi^\beta \bar{\mu}\, \bar{\nu}_\beta\, d\bar{s} \tag{5.3.4}$$

$$d\bar{s}^* = \bar{\mu}\, d\bar{s} \tag{5.3.5}$$

$$d\bar{A}^* = d\bar{s}^* d\zeta = \bar{\mu} d\bar{s} d\zeta \tag{5.3.6}$$

The *Cauchy stress vector*, acting across a surface element $d\bar{A}^*$ with the unit normal $\underset{\sim}{\bar{\nu}}^*$, is given [4,19] by

$$\underset{\sim}{\bar{T}}_{(\underset{\sim}{\bar{\nu}}^*)} = \underset{\sim}{\bar{T}} \underset{\sim}{\bar{\nu}}^* = \bar{T}^{i\psi} \underset{\sim}{\bar{g}}_i \bar{\nu}_\psi^* = \bar{T}^{\varphi j} \bar{\nu}_\varphi^* \underset{\sim}{\bar{g}}_j$$
$$= \bar{T}^{\varphi\psi} \underset{\sim}{\bar{g}}_\varphi \bar{\nu}_\psi^* + \bar{T}^{3\psi} \underset{\sim}{\bar{n}} \bar{\nu}_\psi^* \tag{5.3.7}$$

where $\underset{\sim}{\bar{T}} = \bar{T}^{ij} \underset{\sim}{\bar{g}}_i \otimes \underset{\sim}{\bar{g}}_j$ is the symmetric *Cauchy stress tensor* at the point $\bar{P} \in \partial\bar{P}$.

With respect ot an arbitrary point O in space, an action of the stress vector $\underset{\sim}{\bar{T}}_{(\underset{\sim}{\bar{\nu}}^*)}$ is statically equivallent to an action of the ***Eulerian resultant force and couple vectors*** $\underset{\sim}{\bar{N}}_{(\underset{\sim}{\bar{\nu}}^*)}$ and $\underset{\sim}{\bar{M}}_{(\underset{\sim}{\bar{\nu}}^*)}$, each acting on and measured per unit length of the curve $\bar{C}$, defined by

$$\int_{\bar{C}} \underset{\sim}{\bar{N}}_{(\bar{\nu})} d\bar{s} = \int_{\partial\bar{P}} \underset{\sim}{\bar{T}}_{(\bar{\nu}^*)} d\bar{A}^* \tag{5.3.8}$$

$$\int_{\bar{C}} (\underset{\sim}{\bar{r}} \times \underset{\sim}{\bar{N}}_{(\bar{\nu})} + \underset{\sim}{\bar{M}}_{(\bar{\nu})}) \, d\bar{s} = \int_{\partial\bar{P}} (\underset{\sim}{\bar{p}} \times \underset{\sim}{\bar{T}}_{(\bar{\nu}^*)}) d\bar{A}^* \tag{5.3.9}$$

Using (4.1.3) and (5.3.8) we have

$$\int_{\bar{C}} \underset{\sim}{\bar{M}}_{(\bar{\nu})} d\bar{s} = \int_{\partial\bar{P}} \zeta\underset{\sim}{\bar{n}} \times \underset{\sim}{\bar{T}}_{(\bar{\nu}^*)} \, d\bar{A}^* \tag{5.3.10}$$

Introducing (5.3.7) into (5.3.8) and (5.3.10) and using (5.3.6) and (5.3.4), after integration with respect to ζ, we obtain

$$\int_{\bar{C}} \underset{\sim}{\bar{N}}_{(\bar{\nu})} d\bar{s} = \int_{\bar{C}} \underset{\sim}{\bar{N}}^{\alpha} \bar{\nu}_{\alpha} d\bar{s} \, , \quad \int_{\bar{C}} \underset{\sim}{\bar{M}}_{(\bar{\nu})} d\bar{s} = \int_{\bar{C}} \underset{\sim}{\bar{M}}^{\alpha} \bar{\nu}_{\alpha} d\bar{s} \tag{5.3.11}$$

where

$$\underset{\sim}{\bar{N}}^{\beta} = \bar{Q}^{\alpha\beta} \underset{\sim}{\bar{a}}_{\alpha} + \bar{Q}^{\beta} \underset{\sim}{\bar{n}} \, , \quad \underset{\sim}{\bar{M}}^{\beta} = \bar{\epsilon}_{\alpha\lambda} \bar{R}^{\alpha\beta} \underset{\sim}{\bar{a}}^{\lambda} \tag{5.3.12}$$

$$\begin{aligned} \bar{Q}^{\alpha\beta} &= \int_{-h/2}^{h/2} \bar{\mu}\bar{\mu} \bar{T}^{\alpha}_{\varphi} \bar{T}^{\varphi\psi} \delta^{\beta}_{\psi} \, d\zeta \\ \bar{R}^{\alpha\beta} &= \int_{-h/2}^{h/2} \bar{\mu}\bar{\mu} \bar{T}^{\alpha}_{\varphi} \bar{T}^{\varphi\psi} \delta^{\beta}_{\psi} \, \zeta \, d\zeta \\ \bar{Q}^{\beta} &= \int_{-h/2}^{h/2} \bar{\mu} \bar{T}^{3\psi} \delta^{\beta}_{\psi} \, d\zeta \end{aligned} \tag{5.3.13}$$

Let us assume further that the middle surface load $\underset{\sim}{\bar{p}}$ defined in (5.2.2) is a *resultant* of the external forces, acted on the upper and lower surfaces of deformed shell, and the body forces in deformed shell space, resultant surface moment due to these forces being neglected. Then the *global force and moment equilibrium conditions* about the origin O of any part of the shell, cut out by a closed contour $\bar{C}$, take the form

$$\int_{\bar{C}} \underset{\sim}{\bar{N}}^{\beta}\bar{\nu}_{\beta}\, d\bar{s} + \int_{\bar{C}} \underset{\sim}{\bar{p}}\, d\bar{A} = \underset{\sim}{0} \tag{5.3.14}$$

$$\int_{\bar{C}} (\underset{\sim}{\bar{r}} \times \underset{\sim}{\bar{N}}^{\beta} + \underset{\sim}{\bar{M}}^{\beta})\bar{\nu}_{\beta} d\bar{s} + \int_{\bar{M}} \underset{\sim}{\bar{r}} \times \underset{\sim}{\bar{p}}\, d\bar{A} = \underset{\sim}{0} \tag{5.3.15}$$

By using Stockes theorem, these global equilibrium conditions can be put in the following *local* vector form

$$\begin{aligned} &\underset{\sim}{\bar{N}}^{\beta}\|_{\beta} + \underset{\sim}{\bar{p}} = \underset{\sim}{0} \\ &\underset{\sim}{\bar{M}}^{\beta}\|_{\beta} + \underset{\sim}{\bar{a}}_{\beta} \times \underset{\sim}{\bar{N}}^{\beta} = \underset{\sim}{0} \end{aligned} \tag{5.3.16}$$

or in components along deformed basis $\underset{\sim}{\bar{a}}_{\alpha}$, $\underset{\sim}{\bar{n}}$

$$\begin{aligned} &\bar{Q}^{\alpha\beta}\|_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{Q}^{\lambda} + \bar{p}^{\alpha} = 0 \\ &\bar{Q}^{\beta}\|_{\beta} + \bar{b}_{\alpha\beta}\bar{Q}^{\alpha\beta} + \bar{p} = 0 \end{aligned} \tag{5.3.17}$$

$$\bar{R}^{\alpha\beta}\|_{\beta} - \bar{Q}^{\alpha} = 0 \tag{5.3.18}$$

$$\bar{\epsilon}_{\beta\alpha}(\bar{Q}^{\alpha\beta} + \bar{b}^{\alpha}_{\lambda}\bar{R}^{\beta\lambda}) = 0 \tag{5.3.19}$$

Let us eliminate $\bar{Q}^{\alpha}$ from (5.3.17) by using (5.3.18) to obtain

$$\begin{aligned} &\bar{Q}^{\alpha\beta}\|_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{R}^{\lambda\beta}\|_{\beta} + \bar{p}^{\alpha} = 0 \\ &\bar{R}^{\alpha\beta}\|_{\beta\alpha} + \bar{b}_{\alpha\beta}\bar{Q}^{\alpha\beta} + \bar{p} = 0 \end{aligned} \tag{5.3.20}$$

The symmetry condition (5.3.19) can be satisfied by introducing the following new variables

$$\begin{aligned} &\bar{S}^{\alpha\beta} = \bar{Q}^{\alpha\beta} + \bar{b}^{\alpha}_{\lambda}\bar{R}^{\beta\lambda} = \bar{Q}^{(\alpha\beta)} + \frac{1}{2}(\bar{b}^{\alpha}_{\lambda}\bar{R}^{\beta\lambda} + \bar{b}^{\beta}_{\lambda}\bar{R}^{\alpha\lambda}) \\ &\bar{R}^{\alpha\beta} = \bar{R}^{(\alpha\beta)} + \bar{R}^{[\alpha\beta]} \end{aligned} \tag{5.3.21}$$

It is easy to see that (5.3.19) now becomes

$$\bar{\epsilon}_{\beta\alpha}\bar{S}^{\alpha\beta} = 0 \tag{5.3.22}$$

what means that $\bar{S}^{\alpha\beta}$ are components of a *symmetric* surface tensor.

Using (5.3.21) and (5.3.22) the *equilibrium equations* take now the form

$$\begin{aligned} &[\bar{S}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{R}^{(\lambda\beta)}]\|_{\beta} - \bar{b}^{\alpha}_{\lambda}\bar{R}^{(\lambda\beta)}\|_{\beta} + \bar{p}^{\alpha} = 0 \\ &\bar{R}^{(\alpha\beta)}\|_{\alpha\beta} + \bar{b}_{\alpha\beta}(\bar{S}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{R}^{(\lambda\beta)}) + \bar{p} = 0 \end{aligned} \tag{5.3.23}$$

Note that these equations do contain only *symmetric* quantities $\bar{S}^{\alpha\beta}$ and $\bar{R}^{(\alpha\beta)}$. In particular, the antisymmetric part of moments $\bar{R}^{[\alpha\beta]}$ has been eliminated. In fact, under K - L constraints $R^{[\alpha\beta]}$ are quantities of higher order. If we assume $\bar{R}^{[\alpha\beta]} = 0$ then under the following substitution

$$\bar{N}^{\alpha\beta} \Leftrightarrow \bar{S}^{\alpha\beta} \quad , \quad \bar{M}^{\alpha\beta} \Leftrightarrow \bar{R}^{(\alpha\beta)} \tag{5.3.24}$$

the equilibrium equations (5.3.23) become exactly the same in form as those derived in (5.2.3) from virtual work principle, and the vectors (5.3.12) become exactly the same as those defined by (5.2.5). The relations (5.3.24) give us the physical interpretation of the surface tensor components $\bar{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$ and their relation to a three-dimensional stress state in the shell.

5.4. Lagrangean quantities

Usually only the ***reference*** (undeformed) configuration is known in advance. It is desirable then to use those quantities which are defined in and/or referred to the *known geometry* of the *reference* middle surface M. Such shell quantities will be called *Lagrangean*.

Let us recall first, that according to (3.4.4) (2.5.11) and (2.5.12) we have the following transformation rules

$$\bar{\underset{\sim}{a}}_{\alpha} = \underset{\sim}{G}\underset{\sim}{a}_{\alpha} \quad , \quad \bar{\underset{\sim}{n}} = \underset{\sim}{G}\underset{\sim}{n} \tag{5.4.1}$$

$$(\sqrt{\frac{\bar{a}}{a}}\, v^{\beta})\|_{\beta} = \sqrt{\frac{\bar{a}}{a}}\, v^{\beta}|_{\beta} \tag{5.4.2}$$

$$(\sqrt{\frac{\bar{a}}{a}}\, T^{\alpha\beta})\|_{\beta} = \sqrt{\frac{\bar{a}}{a}}\,[T^{\alpha\beta}|_{\beta} + \bar{a}^{\alpha\kappa}(2\gamma_{\kappa\lambda}|_{\mu} - \gamma_{\lambda\mu}|_{\kappa})T^{\lambda\mu}] \tag{5.4.3}$$

These hold for any surface vector $\underset{\sim}{v} \in V$ and any symmetric surface tensor

$\underset{\sim}{T} \in T_2$, which components with respect to the reference and deformed configurations transform according to (2.2.5).

The transformation rule for an arc differential have been already obtained in (4.5.5) to be

$$d\bar{s} = \sqrt{1 + 2\gamma_{tt}}\, ds \tag{5.4.4}$$

and transformation formula for a surface differential can be obtained as follows

$$\begin{aligned} d\bar{A} &\equiv \underset{\sim}{\bar{n}} \cdot (\underset{\sim}{\bar{a}}_1 d\theta^1 \times \underset{\sim}{\bar{a}}_2 d\theta^2) = \bar{\epsilon}_{12} d\theta^1 d\theta^2 \\ &= \sqrt{\frac{\bar{a}}{a}}\, \epsilon_{12} d\theta^1 d\theta^2 = \sqrt{\frac{\bar{a}}{a}}\, dA \end{aligned} \tag{5.4.5}$$

Multiplying (4.1.4) and (4.1.5) by $d\bar{s}$ and using (5.4.4) the following transformation rules can be obtained as well

$$\bar{\nu}_\beta\, d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, \nu_\beta\, ds \tag{5.4.6}$$

$$\bar{t}_\beta\, d\bar{s} = (\delta^\alpha_\beta + 2\gamma^\alpha_\beta) t_\alpha\, ds \tag{5.4.7}$$

$$\bar{\nu}^\beta\, d\bar{s} = \sqrt{\frac{a}{\bar{a}}}\, (\delta^\beta_\alpha + 2\epsilon_{\alpha\lambda}\epsilon^{\beta\mu}\gamma^\lambda_\mu)\nu^\alpha\, ds \tag{5.4.8}$$

$$\bar{t}^\beta\, d\bar{s} = t^\beta\, ds \tag{5.4.9}$$

Note the simplicity of (5.4.6) and (5.4.9).

Consider an *arbitrary* smooth curve C at the middle surface M generating a normal surface ∂P. Let us introduce the *Piola-Kirchhoff stress vector* $\underset{\sim}{T}_{(\underset{\sim}{\nu}^*)}$, acting at a point $P \in \partial P$ across a surface element dA^* with the unit normal $\underset{\sim}{\nu}^*$, Fig. 18, and related to the Cauchy stress vector $\underset{\sim}{\bar{T}}_{(\underset{\sim}{\bar{\nu}}^*)}$ defined in (5.3.7) by the relation

$$\underset{\sim}{T}_{(\underset{\sim}{\nu}^*)}\, dA^* = \underset{\sim}{\bar{T}}_{(\underset{\sim}{\bar{\nu}}^*)}\, d\bar{A}^* \tag{5.4.10}$$

where

$$d\bar{A}^* = \sqrt{\frac{\bar{g}}{g}}\, dA^* \tag{5.4.11}$$

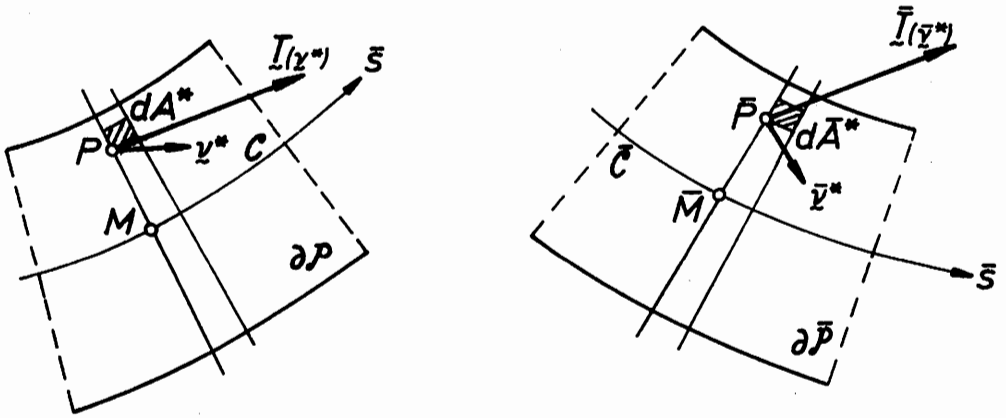

Fig. 18

The stress vector $\underset{\sim}{T}_{(\underset{\sim}{\nu}^*)}$ defines at $P \in \partial P$ the *first* and the *second Piola-Kirchhoff stress tensors* $\underset{\sim}{T}$ and $\underset{\sim}{S}$, respectively, by the following relations [19]

$$\underset{\sim}{T}_{(\underset{\sim}{\nu}^*)} = \underset{\sim}{T}\underset{\sim}{\nu}^* = \underset{\sim}{F}\underset{\sim}{S}\underset{\sim}{\nu}^* \tag{5.4.12}$$

where

$$\underset{\sim}{T} = T^{ij}\bar{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}_j = \sqrt{\frac{\bar{g}}{g}}\,\bar{\underset{\sim}{T}}(\underset{\sim}{F}^{-1})^T \tag{5.4.13}$$

$$\underset{\sim}{S} = S^{ij}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = \sqrt{\frac{\bar{g}}{g}}\,F^{-1}\bar{\underset{\sim}{T}}(\underset{\sim}{F}^{-1})^T \tag{5.4.14}$$

If we now use (5.4.10) to (5.3.8) and (5.3.10) then it is easily seen that it is possible to define the *Lagrangean resultant force and couple vectors* $\underset{\sim}{N}_{(\underset{\sim}{\nu})}$ and $\underset{\sim}{M}_{(\underset{\sim}{\nu})}$, each measured per unit length of C, such that

$$\int_C \underset{\sim}{N}_{(\underset{\sim}{\nu})}\,ds = \int_{\partial P} \underset{\sim}{T}_{(\underset{\sim}{\nu}^*)}\,dA^* = \int_C \underset{\sim}{G}\underset{\sim}{N}^\beta \nu_\beta\,ds \tag{5.4.15}$$

$$\int_C \underset{\sim}{M}_{(\underset{\sim}{\nu})}\,ds = \int_{\partial P} \zeta\bar{\underset{\sim}{n}} \times \underset{\sim}{T}_{(\underset{\sim}{\nu}^*)}\,dA^* = \int_C \underset{\sim}{G}\underset{\sim}{M}^\beta \nu_\beta\,ds \tag{5.4.16}$$

It follows now from (5.3.11), (5.4.1), (5.4.3) and (5.4.6) that

$$\bar{\underset{\sim}{N}}^\beta = \sqrt{\frac{a}{\bar{a}}}\,\underset{\sim}{G}\underset{\sim}{N}^\beta \quad , \quad \bar{\underset{\sim}{M}}^\beta = \sqrt{\frac{a}{\bar{a}}}\,(\underset{\sim}{G}^{-1})^T\underset{\sim}{M}^\beta \tag{5.4.17}$$

and

$$\underset{\sim}{N}^{\beta} = Q^{\alpha\beta}\underset{\sim}{a}_{\alpha} + Q^{\beta}\underset{\sim}{n} \quad , \quad \underset{\sim}{M}^{\beta} = \sqrt{\frac{\bar{a}}{a}}\,\epsilon_{\alpha\lambda}M^{\alpha\beta}\underset{\sim}{a}^{\lambda} \tag{5.4.18}$$

$$\begin{aligned} Q^{\alpha\beta} &= N^{\alpha\beta} - \bar{b}^{\alpha}_{\kappa}M^{\kappa\beta} \\ Q^{\beta} &= M^{\alpha\beta}\,|\,|_{\alpha} + \bar{a}^{\beta\kappa}(2\gamma_{\kappa\lambda}\,|_{\mu} - \gamma_{\kappa\mu}\,|_{\lambda})M^{\lambda\mu} \end{aligned} \tag{5.4.19}$$

The *Lagrangean stress and couple resultant tensors* defined by

$$\underset{\sim}{N} = N^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes a_{\beta} \quad , \quad \underset{\sim}{M} = M^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta} \tag{5.4.20}$$

are connected with the Eulerian tensors (5.1.2) by the *transformation formulae*

$$\underset{\sim}{N} = \sqrt{\frac{\bar{a}}{a}}\,\underset{\sim}{G}^{-1}\bar{\underset{\sim}{N}}(G^{-1})^{T} \quad , \quad \bar{\underset{\sim}{N}} = \sqrt{\frac{a}{\bar{a}}}\,\underset{\sim}{G}\underset{\sim}{N}\underset{\sim}{G}^{T} \tag{5.4.21}$$

$$\underset{\sim}{M} = \sqrt{\frac{\bar{a}}{a}}\,\underset{\sim}{G}^{-1}\bar{\underset{\sim}{M}}(\underset{\sim}{G}^{-1})^{T} \quad , \quad \bar{\underset{\sim}{M}} = \sqrt{\frac{a}{\bar{a}}}\,\underset{\sim}{G}\underset{\sim}{M}\underset{\sim}{G}^{T} \tag{5.4.22}$$

which follow from (5.4.17) and (5.4.1).

It is easy to recognize the same structure of transformations relating $\underset{\sim}{N}$, $\underset{\sim}{M}$ to $\bar{\underset{\sim}{N}}$, $\bar{\underset{\sim}{M}}$ and the one relating $\underset{\sim}{S}$ to $\bar{\underset{\sim}{T}}$ in continuum mechanics (5.4.14). The Lagrangean tensors $\underset{\sim}{N}$ and $\underset{\sim}{M}$ form a two-dimensional analogue of the second Piola-Kirchhoff stress tensor and may be called the *second Piola-Kirchhoff stress and couple resultant tensors*, respectively.

It follows from (5.1.2), (5.4.20) and (5.4.21), (5.4.22) that the components of the Eulerian and Lagrangean tensors in convected coordinate system are connected as follows

$$\bar{N}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,N^{\alpha\beta} \quad , \quad \bar{M}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,M^{\alpha\beta} \tag{5.4.23}$$

Despite the simple relations (5.4.23) the components appearing at both sides of (5.4.23) are referred to *different* tensor bases $\bar{\underset{\sim}{a}}_{\alpha} \otimes \bar{\underset{\sim}{a}}_{\beta}$ and $\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta}$, respectively. In many papers, written entirely in component notation with respect to convected coordinate system, this important geometrical difference between the Lagrangean and Eulerian tensors do not appear at all, which occasionally cause some confusions.

All formulae may be rewritten in terms of stress resultants and stress couples $n^{\alpha\beta}$ and $m^{\alpha\beta}$, related to those introduced in (5.1.12) by

$$\bar{n}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\, n^{\alpha\beta} \quad , \quad \bar{m}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\, m^{\alpha\beta} \tag{5.4.24}$$

Then (5.4.19) takes the form

$$\begin{aligned} Q^{\alpha\beta} &= n^{\alpha\beta} + \frac{1}{2}\,(\bar{b}^{\beta}_{\lambda} m^{\alpha\lambda} - \bar{b}^{\alpha}_{\lambda} m^{\lambda\beta}) \\ Q^{\beta} &= m^{\alpha\beta}\big|_{\alpha} + \bar{a}^{\beta\kappa}(2\gamma_{\kappa\lambda|\mu} - \gamma_{\lambda\mu|\kappa}) m^{\lambda\mu} \end{aligned} \tag{5.4.25}$$

and other relations become obvious as well.

5.5. Lagrangean shell equations

Let $\underset{\sim}{p}$ be the *surface force* measured per unit area of the *reference* middle surface. With the help of (5.4.17) and (5.4.2) we obtain from (5.2.10) and (5.2.8) the following *Lagrangean equilibrium equations* written in vector form

$$(\underset{\sim}{G}\underset{\sim}{N}^{\beta})\big|_{\beta} + \underset{\sim}{p} = \underset{\sim}{0} \tag{5.5.1}$$

and while identity (5.2.9) becomes

$$(\underset{\sim}{G}\underset{\sim}{M}^{\beta})\big|_{\beta} + \underset{\sim}{G}\underset{\sim}{a}_{\beta} \times \underset{\sim}{G}\underset{\sim}{N}^{\beta} = \underset{\sim}{0} \tag{5.5.2}$$

Keeping in mind that

$$\begin{aligned} \underset{\sim}{p} &= p^{\alpha}\underset{\sim}{a}_{\alpha} + p\,\underset{\sim}{n} = \\ &= \sqrt{\frac{\bar{a}}{a}}\,(\bar{p}^{\alpha}\underset{\sim}{\bar{a}}_{\alpha} + \bar{p}\,\underset{\sim}{\bar{n}}) \end{aligned} \tag{5.5.3}$$

the component form of the *Lagrangean equilibrium equations* with respect to the reference basis $\underset{\sim}{a}_{\alpha}$, $\underset{\sim}{n}$, follows from (5.5.1) to be

$$\begin{aligned} (1^{\alpha}_{\cdot\lambda} Q^{\lambda\beta} + n^{\alpha} Q^{\beta})\big|_{\beta} - b^{\alpha}_{\beta}(\varphi_{\lambda} Q^{\lambda\beta} + n Q^{\beta}) + p &= 0 \\ (\varphi_{\lambda} Q^{\lambda\beta} + n Q^{\beta})\big|_{\beta} - b_{\alpha\beta}(1^{\alpha}_{\cdot\lambda} Q^{\lambda\beta} + n^{\alpha} Q^{\beta}) + p &= 0 \end{aligned} \tag{5.5.4}$$

The latter result has been obtained by the author in [10,11].

Note that the Lagrangean equilibrium equations (5.5.4) contain only the quantities defined in an referred to the *known* geometry of the reference surface. The covariant differentiation is carried out with respect to the *known* geometry. Hence, these equations may be used directly

for numerical computation even at this stage of generality. However, the Lagrangean equations (5.5.4) are *much more complicated* than the Eulerian ones (5.2.3), as components of the surface deformation gradient tensor $\underset{\sim}{G}$ appear explicitly in (5.5.4). That is in complete analogy to equations of equilibrium [19] of continuum mechanics when the second Piola-Kirchhoff stress tensor is used. Because of the appearence of the non-linear displacement functions $1^{\alpha}_{.\lambda}$, φ_{λ}, n^{α}, n in (5.5.4), the Lagrangean equilibrium equations can be solved, in general case, only in terms of displacements u_{α}, w.

In order to derive Lagrangean natural boundary conditions, let us use the transformation rules (5.4.1) to (5.4.9) and rewrite the Eulerian virtual work principle (5.1.3) entirely in terms of Lagrangean quantities.

We note first, that (5.2.10) can easily be transformed to

$$-\iint_{M} (\underset{\sim}{G}\underset{\sim}{N}^{\beta})\big|_{\beta} \cdot \delta\underset{\sim}{u}\, dA + \int_{C} (\underset{\sim}{G}\underset{\sim}{N}^{\beta} \cdot \delta\underset{\sim}{u} - M^{\alpha\beta}\underset{\sim}{G}\underset{\sim}{n} \cdot \delta\underset{\sim}{u}_{,\alpha})\nu_{\beta}\, ds = \iint_{M} \underset{\sim}{p} \cdot \delta\underset{\sim}{u}\, dA + \int_{C} (\underset{\sim}{F} \cdot \delta\underset{\sim}{u} + \underset{\sim}{K} \cdot \delta\underset{\sim}{\Omega}_{t})\, ds \tag{5.5.5}$$

where now [10,11,12]

$$\delta\underset{\sim}{u} = \delta u_{\alpha}\underset{\sim}{a}^{\alpha} + \delta w\,\underset{\sim}{n} \tag{5.5.6}$$

$$\delta\underset{\sim}{\Omega}_{t} = \delta\underset{\sim}{\Omega} - \delta\gamma_{\alpha\beta}\left(\frac{1}{1+2\gamma_{tt}}\frac{\bar{a}}{a}\bar{a}^{\alpha\lambda}\bar{\epsilon}^{\beta\mu}\nu_{\lambda}\nu_{\mu}\right)\underset{\sim}{G}\underset{\sim}{n} \tag{5.5.7}$$

$$\delta\underset{\sim}{\Omega} = \frac{1}{2}\left[(\underset{\sim}{G}^{-1})^{T}\underset{\sim}{a}^{\alpha} \times \delta\underset{\sim}{u}_{,\alpha} + \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\underset{\sim}{G}\underset{\sim}{n} \times (\delta\underset{\sim}{u}_{,\alpha} \times \underset{\sim}{G}\underset{\sim}{a}_{\beta})\right] \tag{5.5.8}$$

The Lagrangean boundary force $\underset{\sim}{F}$ and couple $\underset{\sim}{K}$, per unit length of the boundary C, are related to $\bar{\underset{\sim}{F}}$ and $\bar{\underset{\sim}{K}}$ given in (5.2.22) and (5.2.23) by

$$\underset{\sim}{F} = \sqrt{1+2\gamma_{tt}}\,\bar{\underset{\sim}{F}} = F^{\alpha}\underset{\sim}{a}_{\alpha} + F\,\underset{\sim}{n} \tag{5.5.9}$$

$$\underset{\sim}{K} = \sqrt{1+2\gamma_{tt}}\,\bar{\underset{\sim}{K}} = \epsilon_{\alpha\lambda}K^{\alpha}\underset{\sim}{a}^{\lambda} + K\,\underset{\sim}{n} \tag{5.5.10}$$

In these relations we have used the following abbreviated notation

$$\underset{\sim}{G}\underset{\sim}{a}_{\alpha} = 1^{\lambda}_{.\alpha}\underset{\sim}{a}_{\lambda} + \varphi_{\alpha}\underset{\sim}{n}\ ,\ \underset{\sim}{G}\underset{\sim}{n} = n^{\lambda}\underset{\sim}{a}_{\lambda} + n\,\underset{\sim}{n} \tag{5.5.11}$$

The variations of the Lagrangean strain measures $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$ follow from (3.2.11), (2.2.4) and (2.2.23) to be

$$\begin{aligned}\delta\gamma_{\alpha\beta} &= \frac{1}{2}\,(1^{\lambda}_{\cdot\alpha}\delta l_{\lambda\beta} + 1^{\lambda}_{\cdot\beta}\delta l_{\lambda\alpha} + \varphi_{\alpha}\delta\varphi_{\beta} + \varphi_{\beta}\delta\varphi_{\alpha}) \\ &= \frac{1}{2}\delta(\underset{\sim}{G}\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{G}\underset{\sim}{a}_{\beta}) = \frac{1}{2}\,(\delta\underset{\sim}{u}_{,\alpha}\cdot\underset{\sim}{G}\underset{\sim}{a}_{\beta} + \underset{\sim}{G}\underset{\sim}{a}_{\alpha}\cdot\delta\underset{\sim}{u}_{,\beta})\end{aligned} \tag{5.5.12}$$

$$\begin{aligned}\delta\kappa_{\alpha\beta} &= -n\delta d_{\alpha\beta} - d_{\alpha\beta}\delta n - n^{\lambda}\delta d_{\lambda\alpha\beta} - d^{\lambda}_{\cdot\alpha\beta}\delta n_{\lambda} \\ &= -\delta[(\underset{\sim}{G}\underset{\sim}{a}_{\alpha})\,|_{\beta}\cdot\underset{\sim}{G}\underset{\sim}{n}] = -(\delta\underset{\sim}{u}_{|\alpha\beta} - \bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}\delta\underset{\sim}{u}_{,\kappa})\cdot\underset{\sim}{G}\underset{\sim}{n}\end{aligned} \tag{5.5.13}$$

Now with the help of (5.5.6) to (5.5.13) the *virtual work principle* (5.1.3) can be rewritten entirely in the *Lagrangean form* [10,11]

$$\begin{aligned}&\iint_{M}(N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})\,dA = \\ &= \iint_{M}\underset{\sim}{p}\cdot\delta\underset{\sim}{u}\,dA + \int_{C}(\underset{\sim}{F}\cdot\delta\underset{\sim}{u} + \underset{\sim}{K}\cdot\delta\underset{\sim}{\Omega}_{t})\,ds\end{aligned} \tag{5.5.14}$$

From (5.5.14) follow the Lagrangean equations of equilibrium (5.5.1) or (5.5.4) and the appropriate natural boundary conditions (see [10,11]), expressed entirely in terms of the Lagrangean quantities.

5.6. Modified equilibrium equations

The Eulerian equilibrium equations, written in component form (5.2.3) along deformed basis $\underset{\sim}{\bar{a}}_{\alpha}$, $\underset{\sim}{\bar{n}}$, can be expressed entirely in terms of the Lagrangean quantities.

According to (5.2.6), (5.2.7), (5.4.19), (5.4.21) and (5.4.23) we obtain the following relations

$$\bar{Q}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\,Q^{\alpha\beta} \quad , \quad \bar{Q}^{\beta} = \sqrt{\frac{a}{\bar{a}}}\,Q^{\beta} \tag{5.6.1}$$

Using transformation rules for covariant differentiation (5.4.2) and (5.4.3), the equilibrium equations (5.2.3) become

$$Q^{\alpha\beta}\,|_{\beta} + \bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}Q^{\lambda\beta} - \bar{b}^{\alpha}_{\beta}Q^{\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p}^{\alpha} = 0 \tag{5.6.2}$$

$$Q^{\beta}\,|_{\beta} + \bar{b}_{\alpha\beta}Q^{\alpha\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p} = 0 \tag{5.6.3}$$

or in terms of $N^{\alpha\beta}$ and $M^{\alpha\beta}$

$$(N^{\alpha\beta} - \bar{b}^{\alpha}_{\kappa} M^{\kappa\beta})\big|_{\beta} + \bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}(N^{\lambda\beta} - \bar{b}^{\lambda}_{\mu} M^{\mu\beta}) - \\ - \bar{b}^{\alpha}_{\beta}(M^{\lambda\beta}\big|_{\lambda} + \bar{a}^{\beta\kappa}\gamma_{\kappa\lambda\mu} M^{\lambda\mu}) + \sqrt{\frac{\bar{a}}{a}}\,\bar{p}^{\alpha} = 0 \tag{5.6.4}$$

$$M^{\alpha\beta}\big|_{\alpha\beta} + (\bar{a}^{\beta\kappa}\gamma_{\kappa\lambda\mu} M^{\lambda\mu})\big|_{\beta} - \bar{b}_{\alpha\beta}(N^{\alpha\beta} - \bar{b}^{\alpha}_{\kappa} M^{\kappa\beta}) + \sqrt{\frac{\bar{a}}{a}}\,\bar{p} = 0 \tag{5.6.5}$$

where (2.3.5) should be used for $\bar{a}^{\alpha\kappa}$, (2.2.1) for $\bar{b}_{\alpha\beta}$, (2.2.7) for $\sqrt{\frac{\bar{a}}{a}}$ and (2.3.4) for $\gamma_{\kappa\lambda\mu}$. This *mixed equilibrium equations*, obtained originally in terms of $n^{\alpha\beta}$ and $m^{\alpha\beta}$ by DANIELSON [27], are expressed entirely in terms of the Lagrangean quantities. Moreover, when the constitutive equations are used to express $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in terms of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, we obtain these equilibrium equations expressed entirely in terms of strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. Assuming that the boundary conditions are expressible in terms of strains only, this form of the equilibrium equations, together with the compatibility conditions (2.3.11), allow us to solve the shell problems directly in terms of strains. When the displacement field $\underset{\sim}{u}$ is needed as well, it can be obtained by additional independent integration of the strain-displacement relations (2.2.4) and (2.2.14).

The equations (5.6.4) and (5.6.5) may be used to obtain still another form of the *Lagrangean equilibrium equations*. Remembering that (5.6.2) and (5.6.3) are component forms of (5.2.8), let us transform in (5.2.8) the basis $\underset{\sim}{\bar{a}}_{\alpha}$, $\underset{\sim}{\bar{n}}$ as well according to (2.1.4) and (2.2.8) to obtain

$$1^{\alpha}_{.\lambda}(Q^{\lambda\beta}\big|_{\beta} + \bar{a}^{\lambda\kappa}\gamma_{\kappa\mu\beta}Q^{\mu\beta} - \bar{b}^{\lambda}_{\beta}Q^{\beta}) + \\ + n^{\alpha}(Q^{\beta}\big|_{\beta} + b_{\lambda\beta}Q^{\lambda\beta}) + p^{\alpha} = 0 \\ \varphi_{\lambda}(Q^{\lambda\beta}\big|_{\beta} + \bar{a}^{\lambda\kappa}\gamma_{\kappa\mu\beta}Q^{\mu\beta} - \bar{b}^{\lambda}_{\beta}Q^{\beta}) + \\ + n(Q^{\beta}\big|_{\beta} + b_{\lambda\beta}Q^{\lambda\beta}) + p = 0 \tag{5.6.6}$$

These equations (5.6.6) derived originally in [12] are equivallent to those of (5.5.4), but (5.6.6) do not contain the covariant derivatives of deformation gradient tensor components.

According to the polar decomposition of $\underset{\sim}{G}$, the transformation formulae (5.4.21) and (5.4.22) can be presented in the form

$$\bar{N} = \sqrt{\frac{a}{\bar{a}}}\, \underset{\sim}{R}\, \overset{v}{\underset{\sim}{N}}\, \underset{\sim}{R}^T \quad , \quad \bar{\underset{\sim}{M}} = \sqrt{\frac{a}{\bar{a}}}\, \underset{\sim}{R}\, \overset{v}{\underset{\sim}{M}}\, \underset{\sim}{R}^T \tag{5.6.7}$$

$$\begin{aligned} \overset{v}{\underset{\sim}{N}} &= \underset{\sim}{U}\, \underset{\sim}{N}\, \underset{\sim}{U} = N^{\alpha\beta} \overset{v}{\underset{\sim}{a}}_{\alpha} \otimes \overset{v}{\underset{\sim}{a}}_{\beta} \\ \overset{v}{\underset{\sim}{M}} &= \underset{\sim}{U}\, \underset{\sim}{M}\, \underset{\sim}{U} = \overset{v}{M}{}^{\alpha\beta} \overset{v}{\underset{\sim}{a}}_{\alpha} \otimes \overset{v}{\underset{\sim}{a}}_{\beta} \end{aligned} \tag{5.6.8}$$

Let us introduce the *vector resultant* with respect to the *intermediate* basis $\overset{v}{\underset{\sim}{a}}_{\alpha}$, $\underset{\sim}{n}$ by

$$\begin{aligned} \overset{v\beta}{\underset{\sim}{N}} &= \underset{\sim}{U}\, \underset{\sim}{N}^{\beta} = \sqrt{\frac{\bar{a}}{a}}\, \underset{\sim}{R}^T \bar{\underset{\sim}{N}}{}^{\beta} = \\ &= Q^{\alpha\beta} \overset{v}{\underset{\sim}{a}}_{\alpha} + Q^{\beta} \underset{\sim}{n} \end{aligned} \tag{5.6.9}$$

In terms of (5.6.9) the Lagrangean equilibrium equations (5.5.1) become

$$(\underset{\sim}{R}\, \overset{v\beta}{\underset{\sim}{N}})\,|_{\beta} + \underset{\sim}{p} = \underset{\sim}{0} \tag{5.6.10}$$

where

$$\underset{\sim}{R}\, \overset{v\beta}{\underset{\sim}{N}} = \overset{v\beta}{\underset{\sim}{N}} + \underset{\sim}{\Omega} \times \overset{v\beta}{\underset{\sim}{N}} + \frac{1}{2\cos^2 \omega/2}\, \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \overset{v\beta}{\underset{\sim}{N}}) \tag{5.6.11}$$

The component form of (5.6.10) with respect to $\overset{v}{\underset{\sim}{a}}_{\alpha}$, $\underset{\sim}{n}$ can be written by means of the quantities

$$\begin{aligned} T^{\alpha\beta} &= \overset{v\alpha}{\underset{\sim}{a}} \cdot \underset{\sim}{R}\, \overset{v\beta}{\underset{\sim}{N}} = (\bar{\underset{\sim}{a}}_{\lambda} \cdot \overset{v\alpha}{\underset{\sim}{a}}) Q^{\lambda\beta} + (\bar{\underset{\sim}{n}} \cdot \overset{v\alpha}{\underset{\sim}{a}}) Q^{\beta} \\ T^{\beta} &= \underset{\sim}{n} \cdot \underset{\sim}{R}\, \overset{v\beta}{\underset{\sim}{N}} = (\bar{\underset{\sim}{a}}_{\lambda} \cdot \underset{\sim}{n}) Q^{\lambda\beta} + (\bar{\underset{\sim}{n}} \cdot \underset{\sim}{n}) Q^{\beta} \end{aligned} \tag{5.6.12}$$

where (3.7.10) to (3.7.12) and (3.7.6) should be used to express these scalar products in terms of finite rotations. Then, using (3.8.7) and (3.8.5), after transformation we obtain [12]

$$\begin{aligned} &T^{\alpha\beta}\,|_{\beta} + \bar{a}^{\alpha\mu} (\delta^{\lambda}_{\mu} + \overset{v\lambda}{\gamma}_{\mu}) \overset{v}{\gamma}_{\lambda\kappa}\,|_{\beta} T^{\kappa\beta} - \\ &- \sqrt{\frac{a}{\bar{a}}}\, \epsilon^{\alpha\mu} \epsilon_{\lambda\beta} (\delta^{\beta}_{\mu} + \overset{v\beta}{\gamma}_{\mu}) b^{\lambda}_{\kappa} T^{\kappa} + \sqrt{\frac{a}{\bar{a}}}\, \epsilon^{\alpha\beta} \epsilon_{\kappa\mu} (\delta^{\mu}_{\beta} + \overset{v\mu}{\gamma}_{\beta}) p^{\kappa} = 0 \\ &T^{\beta}\,|_{\beta} + (\delta^{\lambda}_{\alpha} + \overset{v\lambda}{\gamma}_{\alpha}) b_{\lambda\beta} T^{\alpha\beta} + p = 0 \end{aligned} \tag{5.6.13}$$

Another equivallent representation of (5.6.10) along $\overset{\vee}{\underset{\sim}{a}}_{\alpha}$ may be obtained with the help of (5.6.2) and (5.6.3), together with (3.7.10) and (3.7.11), or (3.5.9), (2.1.4) and (2.2.8) which leads to

$$(\bar{\underset{\sim}{a}}_{\kappa} \cdot \underset{\sim}{a}^{\alpha})(Q^{\kappa\beta}|_{\beta} + \bar{a}^{\kappa\mu}\gamma_{\mu\lambda\beta}Q^{\lambda\beta} - \bar{b}^{\kappa}_{\beta}Q^{\beta}) +$$
$$+ (\bar{\underset{\sim}{n}} \cdot \overset{\vee}{\underset{\sim}{a}}{}^{\alpha})(Q^{\beta}|_{\beta} + \bar{b}_{\lambda\beta}Q^{\lambda\beta}) + \sqrt{\frac{a}{\bar{a}}}\epsilon^{\alpha\beta}\epsilon_{\kappa\mu}(\delta^{\mu}_{\beta} + \overset{\vee}{\gamma}{}^{\mu}_{\beta})p^{\kappa} = 0 \tag{5.6.14}$$

while the component of (5.6.10) along $\underset{\sim}{n}$ is the same as the second of (5.6.6). This representation has been noted in [12].

5.7. Constitutive equations of elastic shells

The system of equations of the non-linear theory of shells is not yet complete. Two shells with the same geometry and boundary conditions, under the same external loading may behave in different manner, because they are composed of different materials. Here we discuss some features of the two-dimensional constitutive equations for an *elastic* shell subject to K - L constraints.

Let $\sigma(\underset{\sim}{E})$ be a three-dimensional *strain energy function* (elastic potential) of the homogeneous *elastic solid* measured per *unit volume* of the *undeformed* shell configuration. It has been shown in [19,28] that for such a material the reduced form of the Lagrangean *constitutive equation* is

$$\underset{\sim}{S} = \frac{\partial\sigma(\underset{\sim}{E})}{\partial\underset{\sim}{E}} \tag{5.7.1}$$

The explicit form of $\sigma(\underset{\sim}{E})$ depends upon the assumed material symmetry conditions. If, for example, the solid is supposed to be *isotropic*, then $\sigma(\underset{\sim}{E})$ is an isotropic tensor function which, according to the representation theorem [19,28], has the form

$$\sigma = \sigma(I_{\underset{\sim}{E}}, II_{\underset{\sim}{E}}, III_{\underset{\sim}{E}}) \tag{5.7.2}$$

where the principal invariants of $\underset{\sim}{E} \in L_2$ are defined by

$$
\begin{aligned}
I_{\underset{\sim}{E}} &= \operatorname{tr}\underset{\sim}{E} = E^{i}_{i} \\
II_{\underset{\sim}{E}} &= \frac{1}{2}\left[(\operatorname{tr}\underset{\sim}{E})^{2} - \operatorname{tr}\underset{\sim}{E}^{2}\right] = \frac{1}{2}\left[(E^{i}_{i})^{2} - E^{i}_{j}E^{j}_{i}\right] \\
III_{\underset{\sim}{E}} &= \det\underset{\sim}{E} = \left|E^{i}_{j}\right| = \\
&= \frac{1}{6}\left[(\operatorname{tr}\underset{\sim}{E})^{3} - 3\operatorname{tr}\underset{\sim}{E}^{2}\operatorname{tr}\underset{\sim}{E} + 2\operatorname{tr}\underset{\sim}{E}^{3}\right]
\end{aligned}
\tag{5.7.3}
$$

Let us define a two-dimensional *shell strain energy function*, measured per *unit area* of the *undeformed* shell middle surface, by the following relation.

$$
\iint_{M} \Sigma \, dA = \iiint_{P} \sigma \, dA \tag{5.7.4}
$$

from which

$$
\Sigma = \int_{-h/2}^{h/2} \mu\sigma \, d\zeta \tag{5.7.5}
$$

Under K - L constraints $\underset{\sim}{E}$ is expressible in terms of two shell strain measures $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$ and normal coordinate ζ, according to (3.2.10) and (3.2.12), and $\mu = \mu(\zeta)$, as has been shown in (3.1.2). Thus performing the integration in (5.7.5) we obtain $\Sigma = \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$. As the work done by internal stress and couple resultants on virtual changes of strain measures should come from the virtual change of the shell strain energy, we obtain

$$
\iint_{M} \delta\Sigma \, dA = \iint_{M} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}) \, dA \tag{5.7.6}
$$

However

$$
\delta\Sigma = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}}\delta\gamma_{\alpha\beta} + \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}}\delta\kappa_{\alpha\beta} \tag{5.7.7}
$$

Moreover, the stresses and strains are symmetric. Therefore we obtain from (5.7.6) the following Lagrangean *constitutive equations* for an elastic shell

$$
N^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} + \frac{\partial\Sigma}{\partial\gamma_{\beta\alpha}}\right) \quad , \quad M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} + \frac{\partial\Sigma}{\partial\kappa_{\beta\alpha}}\right) \tag{5.7.8}
$$

The explicit form of (5.7.8) for an *isotropic* elastic shell under K — L constraints would follow directly from (5.7.5) with (5.7.2), (5.7.3), (3.2.10), (3.2.14) and (3.1.2).

Chapter 6

SHELL EQUATIONS UNDER SMALL ELASTIC STRAINS

The various shell relations discussed in chapters 2 to 5 have a purely geometrical character. They are exact under K - L constraints,are valid for unrestricted strains and rotations, and do not depend upon the shell material properties. We do not know, however,about the influence of K - L constraints on the unknown three-dimensional stresses and strains in a shell space.

In this chapter we shall discuss possible simplifications of the shell equations in the case of a thin shell composed of an isotropic elastic material. Here the strains are assumed to be small everywhere in the shell. The K - L constraints are initially rejected here. Series expansion technique combined with a priori estimates of stresses obtained by JOHN [31] make it possible to show that, within an error of the first approximation, the state of stress in the shell is approximately plane and parallel to the shell middle surface [32]. The change in shell thickness during deformation happens to be of primary importance in this case. However, the shell thickness changes may easily be taken into account only by proper modification of the approximate constitutive equations. Thus the exact results of chapters 2 to 5 may still serve here as a basis for various simplified shell relations.

We begin here by recalling the form of three-dimensional strain energy function of a linear elastic material. The displacements and strains are expanded into series with respect to the normal coordinate which also give the series form for the shell strain energy function. The order of magnitude of all terms in the strain energy function are then estimated for a thin shell under the assumption of small strains. The consistent first and second approximations to the shell strain energy are constructed, and the approximate constitutive equations of the first approximation theory of shells are derived. The consistently reduced canonical form of intrinsic shell euqations is derived, and simplified sets of shell equations valid for membrane, bending or inextensional bending shell problems are given.

6.1. Strain energy of a linear elastic material

For a *non-linear elastic material* the three-dimensional strain energy function $\sigma(\underset{\sim}{E})$ can be expanded into Taylor series

$$\sigma = \sigma(\underset{\sim}{O}) + \underset{\sim}{K} \cdot \underset{\sim}{E} + \frac{1}{2!} \underset{\sim}{L} \cdot (\underset{\sim}{E} \otimes \underset{\sim}{E}) + \frac{1}{3!} \underset{\sim}{M} \cdot (\underset{\sim}{E} \otimes \underset{\sim}{E} \otimes \underset{\sim}{E}) + \ldots \qquad (6.1.1)$$

where

$$\underset{\sim}{K} = \frac{\partial \sigma(\underset{\sim}{E})}{\sigma \underset{\sim}{E}} \Big|_{\underset{\sim}{E} = \underset{\sim}{O}}$$

$$\underset{\sim}{L} = \frac{\partial^2 \sigma(\underset{\sim}{E})}{\partial \underset{\sim}{E}\, \partial \underset{\sim}{E}} \Big|_{\underset{\sim}{E} = \underset{\sim}{O}} \qquad (6.1.2)$$

$$\underset{\sim}{M} = \frac{\partial^3 \sigma(\underset{\sim}{E})}{\partial \underset{\sim}{E}\, \partial \underset{\sim}{E}\, \partial \underset{\sim}{E}} \Big|_{\underset{\sim}{E} = \underset{\sim}{O}}$$

The structure of the *elasticity tensors* $\underset{\sim}{K} \in L_2$, $\underset{\sim}{L} \in L_4$, $\underset{\sim}{M} \in L_6$ for an isotropic material have been discussed in [29,30].

Let us assume that the reference shell configuration is free from internal stresses and the strain energy becomes zero in this configuration. Then

$$\sigma(\underset{\sim}{O}) = 0 \quad , \quad \underset{\sim}{K} = \underset{\sim}{O} \qquad (6.1.3)$$

For many engineering materials, such as steel or aluminium, the higher-order elastic constants appearing in $\underset{\sim}{M}$ are quite small. The yield strains for such materials are also very small. That makes the experimental determination of these constants virtually impossible and hardly necessary. However, these constants may be a factor in problems of shells composed of man-made (rubber-like) materials or in shell problems arising in biomechanics. Since we are interested here primarily in shell problems arising in engineering, it is reasonable to discuss only the *linear elastic materials* for which higher-order elasticities vanish by definition. In this case the strain energy function becomes

$$\sigma = \frac{1}{2} L^{ijkl} E_{ij} E_{kl} \qquad (6.1.4)$$

where the elasticity tensor has the following *symmetry conditions*

$$L^{ijkl} = L^{jikl} = L^{ijlk} = L^{klij} \qquad (6.1.5)$$

In normal coordinates of the reference shell configuration, the relation (6.1.4) takes the form

$$\sigma = \frac{1}{2}\,(L^{\varphi\psi\theta\sigma}E_{\varphi\psi}E_{\theta\sigma} + 2L^{\varphi\psi 33}E_{\varphi\psi}E_{33} + 4L^{3\psi 3\sigma}E_{3\psi}E_{3\sigma} + L^{3333}E_{33}E_{33}) \tag{6.1.6}$$

and the constitutive equations (5.7.1) may be expanded into

$$\begin{aligned} S^{\varphi\psi} &= L^{\varphi\psi\theta\sigma}E_{\theta\sigma} + L^{\varphi\psi 33}E_{33} \\ S^{3\psi} &= L^{3\psi 3\sigma}E_{3\sigma} + L^{3\psi\theta 3}E_{\theta 3} \\ S^{33} &= L^{33\theta\sigma}E_{\theta\sigma} + L^{3333}E_{33} \end{aligned} \tag{6.1.7}$$

When the material is *homogeneous* and *isotropic*, the elasticity tensor can be expressed only by means of two elastic constants

$$L^{ijkl} = \frac{1}{2(1+\nu)}\,(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{1-2\nu}\,g^{ij}g^{kl}) \tag{6.1.8}$$

where E and ν are the *Young's modulus* and the *Poisson's ratio* respectively.

6.2. Series expansions

Let $\chi : P \to \bar{P}$ be a *deformation function* of the shell regarded as a three-dimensional body, $\bar{P} = \chi(P)$, $\chi = \bar{\kappa} \circ \kappa^{-1}$. Under a general deformation, the material fibres which have been straight and normal to the reference surface M may become neither straight nor normal to the surface $\bar{M} = \chi(M)$, Fig. 19

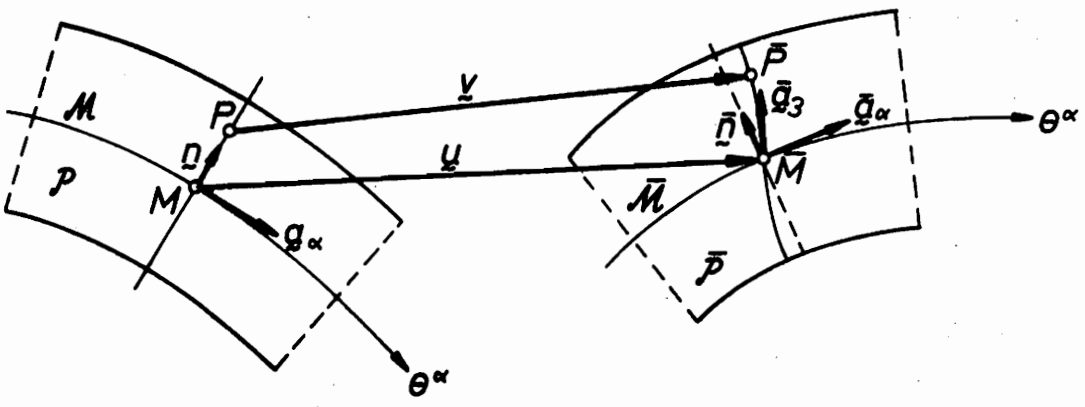

Fig. 19

The spatial displacement vector $\underset{\sim}{v} = \underset{\sim}{v}(P)$ defined in (3.2.1) can be expanded into series with respect to ζ [12,25] to obtain

$$\underset{\sim}{v} = \underset{\sim}{u} + \zeta\underset{\sim}{\beta} + \ldots \tag{6.2.1}$$

where, for *thin* shells, only the *linear* part of this expansion is indicated here. In the relation (6.2.1) the vector $\underset{\sim}{\beta}$ has the form

$$\underset{\sim}{\beta} = (\underset{\sim}{G} - \underset{\sim}{1})\underset{\sim}{n} = \underset{\sim}{\bar{a}}_3 - \underset{\sim}{n} = \beta^{\alpha}\underset{\sim}{a}_{\alpha} + \beta\,\underset{\sim}{n} \tag{6.2.2}$$

where

$$\underset{\sim}{G} = \underset{\sim}{F}(P)\Big|_{P = M} = \underset{\sim}{\bar{a}}_{\alpha} \otimes \underset{\sim}{a}^{\alpha} + \underset{\sim}{\bar{a}}_3 \otimes \underset{\sim}{n} \tag{6.2.3}$$

The vector $\underset{\sim}{\bar{a}}_3 = \underset{\sim}{\bar{a}}_3(\bar{M})$ defined by

$$\underset{\sim}{\bar{a}}_3 = \underset{\sim}{\bar{g}}_3(\bar{P})\Big|_{\bar{P} = \bar{M}} \tag{6.2.4}$$

is *tangent* at the surface $\bar{M}$ to this deformed material fibre, which before deformation has been normal to M. It is obvious that, in general, vector $\underset{\sim}{\bar{a}}_3$ may be *neither unit nor normal* to $\bar{M}$.

The *shell deformation gradient tensor* $\underset{\sim}{G} \in L_2$ defined by (6.2.3) describes *exactly* the deformation of the neighbourhood of the shell middle surface.

It follows from (6.2.1) to (6.2.4) that, when Kirchhoff-Love constraints are not used, deformation of the neighbourhood of the shell middle surface depends upon *two independent displacement parameters* $\underset{\sim}{u}$ and $\underset{\sim}{\beta}$.

For the components of *Green strain tensor* $\underset{\sim}{E}$ defined by (3.2.8) we obtain the following expansions [12]

$$\begin{aligned}
E_{\varphi\psi}(P) &= \frac{1}{2}\,(\underset{\sim}{\bar{g}}_{\varphi} \cdot \underset{\sim}{\bar{g}}_{\psi} - \underset{\sim}{g}_{\varphi} \cdot \underset{\sim}{g}_{\psi}) = \\
&= \delta^{\alpha}_{\varphi}\gamma^{\beta}_{\psi}[\gamma_{\alpha\beta} + \zeta\,\frac{1}{2}\,(\kappa_{\alpha\beta} + \kappa_{\beta\alpha}) + \zeta^2\nu_{\alpha\beta} + \ldots] \\
E_{3\psi}(P) &= \frac{1}{2}\,(\underset{\sim}{\bar{g}}_3 \cdot \underset{\sim}{\bar{g}}_{\psi} - \underset{\sim}{n} \cdot \underset{\sim}{g}_{\psi}) = \\
&= \delta^{\beta}_{\psi}(\gamma_{3\beta} + \zeta\,\frac{1}{2}\,\kappa_{3\beta} + \ldots) \\
E_{33}(P) &= \frac{1}{2}\,(\underset{\sim}{\bar{g}}_3 \cdot \underset{\sim}{\bar{g}}_3 - \underset{\sim}{n} \cdot \underset{\sim}{n}) = \gamma_{33} + \ldots
\end{aligned} \tag{6.2.5}$$

Here the components of the *shell strain measures* are defined by

$$
\begin{aligned}
2\gamma_{\alpha\beta} &= \varphi_{\alpha\beta} + \varphi_{\beta\alpha} + a^{\lambda\mu}\varphi_{\lambda\alpha}\varphi_{\mu\beta} + \varphi_{\alpha}\varphi_{\beta} \\
2\kappa_{(\alpha\beta)} &= \kappa_{\alpha\beta} + \kappa_{\beta\alpha} = \psi_{\alpha\beta} + \psi_{\beta\alpha} - b_{\alpha}^{\lambda}\varphi_{\lambda\beta} - b_{\beta}^{\lambda}\varphi_{\lambda\alpha} + \\
&\quad + a^{\lambda\mu}(\varphi_{\lambda\alpha}\psi_{\mu\beta} + \varphi_{\lambda\beta}\psi_{\mu\alpha}) + \varphi_{\alpha}\psi_{\beta} + \varphi_{\beta}\psi_{\alpha} \\
2\nu_{\alpha\beta} &= a^{\lambda\mu}\psi_{\lambda\alpha}\psi_{\mu\beta} - b_{\alpha}^{\lambda}\psi_{\lambda\beta} - b_{\beta}^{\lambda}\psi_{\lambda\alpha} + \psi_{\alpha}\psi_{\beta} \qquad (6.2.6) \\
2\gamma_{3\alpha} &= \varphi_{\alpha} + \beta_{\alpha} + a^{\lambda\mu}\varphi_{\lambda\alpha}\beta_{\mu} + \varphi_{\alpha}\beta \\
\kappa_{3\alpha} &= \psi_{\alpha} - b_{\alpha}^{\lambda}\beta_{\lambda} + a^{\lambda\mu}\psi_{\lambda\alpha}\beta_{\mu} + \psi_{\alpha}\beta \\
2\gamma_{33} &= 2\beta + a^{\lambda\mu}\beta_{\lambda}\beta_{\mu} + \beta^{2}
\end{aligned}
$$

where $\varphi_{\alpha\beta}$ and φ_{α} are given in (2.1.5) and

$$
\psi^{\lambda}_{\cdot\alpha} = \beta^{\lambda}\big|_{\alpha} - b_{\alpha}^{\lambda}\beta \ , \quad \psi_{\alpha} = \beta_{,\alpha} + b_{\alpha}^{\lambda}\beta_{\lambda} \qquad (6.2.7)
$$

The strain measures $\underset{\sim}{\gamma}$, $\underset{\sim}{\kappa}$ and $\underset{\sim}{\nu}$ appearing in (6.2.5) differ from those defined by (3.2.11) because here we do not impose the K - L constraints. Still in § 6.2 and § 6.3 we keep the same symbols for these generalized measures and show that within the first approximation they can be approximated by the measures used in (3.2.11).

Let us recall that in the normal coordinates we have, according to (3.1.2) and (3.1.3), the following expansions

$$
\begin{aligned}
g^{\varphi\psi} &= \delta_{\alpha}^{\varphi}\delta_{\beta}^{\psi}(a^{\alpha\beta} + 2\zeta b^{\alpha\beta} + 3\zeta^{2}b_{\lambda}^{\alpha}b^{\lambda\beta} + \ldots) \\
\mu &= 1 - 2\zeta H + \zeta^{2}K
\end{aligned}
\qquad (6.2.8)
$$

This allows us to express the components of elasticity tensor in series form as well

$$
L^{ijkl} = \delta_{a}^{i}\delta_{b}^{j}\delta_{c}^{k}\delta_{d}^{l}\,(L_{o}^{abcd} + \zeta L_{1}^{abcd} + \zeta^{2}L_{2}^{abcd} + \ldots) \qquad (6.2.9)
$$

In particular, for an *isotropic* material from (6.1.8) and (6.2.8) we obtain

$$L_o^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-2\nu} a^{\alpha\beta}a^{\lambda\mu}\right)$$

$$L_1^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left[2\left(a^{\alpha\lambda}b^{\beta\mu} + b^{\alpha\lambda}a^{\beta\mu}\right) + 2\left(a^{\alpha\mu}b^{\beta\lambda} + b^{\alpha\mu}a^{\beta\lambda}\right) + \right.$$

$$\left. + \frac{4\nu}{1-2\nu} \left(a^{\alpha\beta}b^{\lambda\mu} + b^{\alpha\beta}a^{\lambda\mu}\right)\right] \qquad (6.2.10)$$

$$L_2^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left[\left(3a^{\alpha\lambda}b_\kappa^\beta b^{\kappa\mu} + 4b^{\alpha\lambda}b^{\beta\mu} + 3b_\kappa^\alpha b^{\kappa\lambda}a^{\beta\mu}\right) + \right.$$

$$+ \left(3a^{\alpha\mu}b_\kappa^\beta b^{\kappa\lambda} + 4b^{\alpha\mu}b^{\beta\lambda} + 3b_\kappa^\alpha b^{\kappa\mu}a^{\beta\lambda}\right) +$$

$$\left. + \frac{2\nu}{1-2\nu} \left(3a^{\alpha\beta}b_\kappa^\lambda b^{\kappa\mu} + 4b^{\alpha\beta}b^{\lambda\mu} + 3b_\kappa^\alpha b^{\kappa\beta}a^{\lambda\mu}\right)\right]$$

$$L_o^{\alpha\beta33} = \frac{E}{2(1+\nu)} \cdot \frac{2\nu}{1-2\nu} a^{\alpha\beta}$$

$$L_1^{\alpha\beta33} = \frac{E}{2(1+\nu)} \cdot \frac{2\nu}{1-2\nu} 2b^{\alpha\beta} \qquad (6.2.11)$$

$$L_2^{\alpha\beta33} = \frac{E}{2(1+\nu)} \cdot \frac{2\nu}{1-2\nu} 3b_\kappa^\alpha b^{\kappa\beta}$$

$$L_o^{3\beta3\mu} = \frac{E}{2(1+\nu)} a^{\beta\mu}$$

$$L_1^{3\beta3\mu} = \frac{E}{2(1+\nu)} 2b^{\beta\mu} \qquad (6.2.12)$$

$$L_2^{3\beta3\mu} = \frac{E}{2(1+\nu)} 3b_\kappa^\beta b^{\kappa\mu}$$

$$L_o^{3333} = L^{3333} = \frac{E}{2(1+\nu)} \cdot \frac{2(1-\nu)}{1-2\nu} \qquad (6.2.13)$$

It follows now from (6.1.4), (6.2.5) and (6.2.9) that the three-dimensional strain energy function can be expressed as well in series form

$$\sigma = \sigma_o + \zeta\sigma_1 + \zeta_2\sigma_2 + \ldots \qquad (6.2.14)$$

where σ_o, σ_1 and σ_2 are expressible in terms of the quantities defined at the middle surface M by the relations [12]

$$\sigma_o = \frac{1}{2} L_o^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + L_o^{\alpha\beta 33}\gamma_{\alpha\beta}\gamma_{33} + 2L_o^{3\beta 3\mu}\gamma_{3\beta}\gamma_{3\mu} + \frac{1}{2} L_o^{3333}\gamma_{33}\gamma_{33} \tag{6.2.15}$$

$$\sigma_1 = L_o^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{(\lambda\mu)} + \frac{1}{2} L_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + L_o^{\alpha\beta 33}\kappa_{(\alpha\beta)}\gamma_{33} + L_1^{\alpha\beta 33}\gamma_{\alpha\beta}\gamma_{33} + 2L_o^{3\beta 3\mu}\gamma_{3\beta}\kappa_{3\mu} + 2L_1^{3\beta 3\mu}\gamma_{3\beta}\gamma_{3\mu} \tag{6.2.16}$$

$$\begin{aligned}\sigma_2 = {} & L_o^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\nu_{\lambda\mu} + \frac{1}{2} L_o^{\alpha\beta\lambda\mu}\kappa_{(\alpha\beta)}\kappa_{(\lambda\mu)} + L_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{(\lambda\mu)} + \\ & + \frac{1}{2} L_2^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + L_o^{\alpha\beta 33}\nu_{\alpha\beta}\gamma_{33} + L_1^{\alpha\beta 33}\kappa_{(\alpha\beta)}\gamma_{33} + \\ & + L_2^{\alpha\beta 33}\gamma_{\alpha\beta}\gamma_{33} + \frac{1}{2} L_o^{3\beta 3\mu}\kappa_{3\beta}\kappa_{3\mu} + 2L_1^{3\beta 3\mu}\gamma_{3\beta}\kappa_{3\mu} + \\ & + 2L_2^{3\beta 3\mu}\gamma_{3\beta}\gamma_{3\mu}\end{aligned} \tag{6.2.17}$$

In the case of an *isotropic* material the formulae (6.2.10) to (6.2.13) should be used in (6.2.15) to (6.2.17).

6.3. Approximate shell strain energy function

The constitutive equations (5.7.8) of an elastic shell have been expressed by means of a shell strain energy function defined by (5.7.5).

Let us now introduce the expansions (6.2.14) and (6.2.8) into (5.7.5) and perform the integration. As a result the following explicit formula for the *shell strain energy function* can be obtained

$$\Sigma = h[\sigma_o + \frac{h^2}{12}(\sigma_2 - 2H\sigma_1 + K\sigma_o) + \frac{h^4}{80}(\sigma_4 - 2H\sigma_3 + K\sigma_2) + \ldots] \tag{6.3.1}$$

It is seen that Σ is an *infinite series* with respect to the shell thickness h. In what follows we estimate the order of magnitude of all terms in (6.3.1) under the following assumptions:

a) The *shell is thin*, $h/R \ll 1$, where R is the *smallest principal radius of curvature* of the undeformed shell middle surface M.

b) The *shell deformation* is such that $(h/L)^2 \ll 1$, where L is the *smallest wavelength of deformation pattern* at the surface M.

c) The *strains are small* everywhere in the shell, $\max |E_r| = \eta \ll 1$, where E_r are three eigenvalues of the Green strain tensor $\underset{\sim}{E}$ of the shell.

Let the surface coordinate system be chosen in such a way that for the various components of metric and curvature tensors the following estimates are satisfied.

$$a_{\alpha\beta} \sim a^{\alpha\beta} = O(1) \quad , \quad b_{\alpha\beta} \sim b^{\alpha\beta} \sim b^{\alpha}_{\beta} = O(\frac{1}{R}) \tag{6.3.2}$$

Here $\sim$ means "is of the same order as" and $O(\ )$ means "of the order of".

The exact estimates of the stresses and their derivatives in an *interior* domain of a thin isotropic elastic shell under small strains have been discussed by JOHN [31]. He applied the standard energy integral method to the non-linear elliptic partial differential equations describing the behaviour of a thin three-dimensional body, subjected to the forces applied only at its lateral boundaries, and obtained the following *a priori estimates* for all stress components:

$$S^{\varphi\psi} = O(E\eta) \quad , \quad S^{\varphi 3} = O(E\eta\theta) \quad , \quad S^{33} = O(E\eta\theta^2) \tag{6.3.3}$$

The small parameter θ expresses the common measure of "smallness" of various quantities introduced in the assumptions a), b) and c) and has the form [31, 9, 19]

$$\theta = \max \left(\frac{h}{L} , \frac{h}{d} , \sqrt{\frac{h}{R}} , \sqrt{\eta} \right) \tag{6.3.4}$$

Here d is the distance of the point $P \in \mathcal{P}$ under consideration from the shell lateral boundary $\partial\mathcal{P}$.

The estiamtes (6.3.3) still remain unchanged [12] if we admit some smooth distributed loading, applied at the upper and/or the lower boundary of the shell space, which gives the resultant surface force

$$p^{\alpha} = O(E\eta\theta \ ) \quad , \quad p = O(E\eta\theta^2) \tag{6.3.5}$$

For the *bending* theory of shells the strains in a shell space caused by stretching and bending are of comparable order of magnitude. Thus it follows from (6.1.7) and (6.3.3) that

$$\gamma_{\alpha\beta} \sim h\kappa_{(\alpha\beta)} = O(\eta) \qquad \gamma_{3\beta} \sim h\kappa_{3\beta} = O(\eta\theta) \tag{6.3.6}$$

From (3.2.14), with the help of (6.3.6), (1.1.22) and (1.1.23), we obtain

$$h^2\nu_{\alpha\beta} = O(\nu\theta^2) \tag{6.3.7}$$

and the third of (6.1.7), together with (6.3.3), (6.3.6), (6.2.11) and (6.2.13), gives us as well

$$\gamma_{33} = -\frac{\nu}{1-\nu}\gamma_{\kappa}^{\kappa} + O(\eta\theta^2) = O(\nu\eta) \tag{6.3.8}$$

Note that the *transverse strains* γ_{33}, which describe change of the shell thickness during deformation, happen to be quite large although the corresponding *transverse stresses* S^{33} are very small, (6.3.3).

By using the estimates (6.3.6) to (6.3.8), we may now estimate the order of magnitude of the most important terms in (6.3.1).

According to (6.2.15) to (6.2.17), the formula (6.3.1) consists of the 4 terms proportional to h, the 20 terms proportional to $h^3/12$, an unspecified amount of terms proportional to $h^5/80$, and an unspecified amount of terms of the higher order.

The estimation of all terms proportional to $h^5/80$ may be omitted at once, since they are much smaller than the leading terms in (6.3.1). Careful estimation of the remaining 24 terms shows [12] that there are two leading terms $O(Eh\eta^2)$, one term $O(\nu Eh\eta^2)$, one term $O(\nu^2 Eh\eta^2)$ and five terms $O(Eh\eta^2\theta^4)$ or $O(\frac{\nu^2}{12} Eh\eta^2\theta^2)$. Thus, for the shell strain energy function we obtain the following approximation

$$\begin{aligned} \Sigma = {} & \frac{h}{2} L_o^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{(\alpha\beta)}\kappa_{(\lambda\mu)}) + hL_o^{\alpha\beta 33}\gamma_{\alpha\beta}\gamma_{33} + \\ & + \frac{h}{2} L_o^{3333}\gamma_{33}\gamma_{33} + \frac{h^3}{12} L_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{(\lambda\mu)} + 2hL_o^{3\beta 3\mu}(\gamma_{3\beta}\gamma_{3\mu} + \\ & + \frac{h^2}{48}\kappa_{3\beta}\kappa_{3\mu}) + \frac{h^3}{12} L_o^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\nu_{\lambda\mu} - 2H\kappa_{(\lambda\mu)}) + O(Eh\eta^2\theta^4) \end{aligned} \tag{6.3.9}$$

where the estimate $\frac{\nu^2}{12} \lesssim \theta^2$ has been used.

The relation (6.3.9) is the *consistent second approximation* to the shell strain energy function [12].

When a shell theory is based on (6.3.9), some additional higher-order stress and couple resultants, corresponding to the strain measures (6.2.6), should be introduced. In such shell theory the number of independent equilibrium equations and geometrical or natural boundary conditions increase to six, since both $\underset{\sim}{u}$ and $\underset{\sim}{\beta}$ should be used as independent displacemental variables. Such *consistent second approximation shell theory* has been constructed in the author's thesis [12], where all further details may be found.

Within the greater error the estimate for Σ becomes

$$\Sigma = \frac{h}{2} L_o^{\alpha\beta\lambda\mu} (\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \kappa_{(\alpha\beta)}\kappa_{(\lambda\mu)}) + hL_o^{\alpha\beta 33}\gamma_{\alpha\beta}\gamma_{33} + \\ + \frac{h}{2} L_o^{3333}\gamma_{33}\gamma_{33} + O(Eh\eta^2\theta^2) \tag{6.3.10}$$

The two terms with γ_{33} here may be transformed with the help of (6.3.8) and added to the first term to obtain

$$\Sigma = \frac{h}{2} (H^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} L_o^{\alpha\beta\lambda\mu}\kappa_{(\alpha\beta)}\kappa_{(\lambda\mu)}) + O(Eh\eta^2\theta^2) \tag{6.3.11}$$

where the *modified elasticity tensor* has the form

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} (a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu}) \tag{6.3.12}$$

The two elasticity tensors in (6.3.11) are not the same. The difference between them follows from (6.3.12) and (6.2.10) to be

$$H^{\alpha\beta\lambda\mu} - L_o^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \cdot \frac{2\nu^2}{(1-\nu)(1-2\nu)} \tag{6.3.13}$$

It means that by using $H^{\alpha\beta\lambda\mu}$ in place of $L_o^{\alpha\beta\lambda\mu}$ in (6.3.11) we would make an error $O(\frac{\nu^2}{24} Eh\eta^2)$. This is always permissible provided $\frac{\nu^2}{24} \lesssim O(\theta^2)$. Moreover, as the shear strains $\gamma_{3\beta}$ do not appear explicitly in (6.3.11), it is also convenient to express approximately $\kappa_{(\alpha\beta)}$ in terms of the surface strain measures. Note that, according to (6.2.5), (6.3.6) and

(6.3.8), the following estimate for the vector $\bar{\underset{\sim}{a}}_3$ may be used

$$\begin{aligned} \bar{\underset{\sim}{a}}_3 &= \sqrt{1+2\gamma_{33}}\ \bar{\underset{\sim}{n}}\ [1+O(\eta\theta)] = \\ &= (1 - \frac{\nu}{1-\nu}\gamma^{\kappa})\ \bar{\underset{\sim}{n}}\ [1 + O(\eta\theta)] \end{aligned} \tag{6.3.14}$$

This allow us to obtain

$$\begin{aligned} \kappa_{(\alpha\beta)} &\equiv \frac{1}{2}\,(\bar{\underset{\sim}{a}}_\alpha \cdot \bar{\underset{\sim}{a}}_{3,\beta} + \bar{\underset{\sim}{a}}_\beta \cdot \bar{\underset{\sim}{a}}_{3,\alpha} - 2\underset{\sim}{a}_\alpha \cdot \underset{\sim}{n}_{,\beta}) = \\ &= \kappa_{\alpha\beta} + \frac{\nu}{1-\nu}\,(b_{\alpha\beta} - \kappa_{\alpha\beta})\gamma^{\kappa} + O(\frac{\eta\theta}{R}, \frac{\eta^2\theta}{h}) \\ &= \kappa_{\alpha\beta} + O(\frac{\nu\eta}{R}, \frac{\nu\eta^2}{h}) \end{aligned} \tag{6.3.15}$$

Here $\kappa_{\alpha\beta}$ are the components of the tensor of change of curvature of the shell middle surface defined in (2.2.1).

Introducing (6.3.13) and (6.3.15) into (6.3.11), we obtain the final form of the *consistent first approximation* to the shell strain energy function

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}) + O(Eh\eta^2\theta^2) \tag{6.3.16}$$

The result had been originally obtained by KOITER [32] with an additional assumption that the *state of stress* in a shell in approximately *plane* and parallel to the middle surface. Here we have introduced only the assumptions a) to c) and then have taken into account the stress estimates (6.3.3), which have been obtained in [31] under the same assumptions. Thus, within the consistent first approximation to the strain energy, the state of stress in a shell is really *approximately plane*, which results only from the assumptions a) to c) and (6.3.5).

Note that the addition of terms of the type $b_\alpha^\lambda\gamma_{\lambda\beta}$ to the tensor $\kappa_{\alpha\beta}$ in (6.3.16) do not affect the accuracy of the first approximation, [32]. Thus, according to (2.2.2), the relation

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\rho_{\alpha\beta}\rho_{\lambda\mu}) + O(Eh\eta^2\theta^2) \tag{6.3.17}$$

is the *consistent first approximation* to the shell strain energy function as well.

The formula (6.3.16) expresses the known fact that, within the first approximation, the small strain energy of a thin elastic shell depends entirely upon the bending and stretching of its middle surface. The out-of-surface effect caused by a change of the shell thickness appears in (6.3.16) only in the modified elasticity tensor $H^{\alpha\beta\lambda\mu}$. This is the main reason why the results obtained in chapters 2 to 5 may serve as an adequate basis for deriving various approximate relations valid under the assumptions a) to c).

6.4. Constitutive equations

The *constitutive equations* of an isotropic elastic shell under small strains follow now from (5.7.8) and (6.3.16) to be

$$\begin{aligned} N^{\alpha\beta} &= H^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + O(Eh\eta\theta^2) \\ &= C\,[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}] + O(Eh\eta\theta^2) \\ M^{\alpha\beta} &= H^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} + O(Eh^2\eta\theta^2) \\ &= D\,[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^{\lambda}_{\lambda}] + O(Eh^2\eta\theta^2) \end{aligned} \tag{6.4.1}$$

Here C and D are the tensile-compressional and the flexural *rigidities* of the shell respectively, as defined by

$$C = \frac{Eh}{1-\nu^2} \quad , \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{6.4.2}$$

In terms of the "best" variables the *constitutive equations* have exactly the same form

$$\begin{aligned} n^{\alpha\beta} &= C[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}] + O(Eh\eta\theta^2) \\ m^{\alpha\beta} &= D[(1-\nu)\rho^{\alpha\beta} + \nu a^{\alpha\beta}\rho^{\lambda}_{\lambda}] + (Eh^2\eta\theta^2) \end{aligned} \tag{6.4.3}$$

which follows from (6.3.17).

Within the first approximation the constitutive equations (6.4.1) and (6.4.3) for a shell are exactly the same as the ones used in classical thin plate theory . They do not depend explicitly on the shell curvature, they are free from coupling terms, and their symmetric structure is also remarkable.

The constitutive equations (6.4.1) or (6.4.3) are linear in terms of the strain measures and can be inverted to obtain

$$\begin{aligned} \gamma_{\alpha\beta} &= A[(1+\nu)N_{\alpha\beta} - \nu a_{\alpha\beta}N^{\lambda}_{\lambda}] + O(\eta\theta^2) \\ \kappa_{\alpha\beta} &= B[(1+\nu)M_{\alpha\beta} - \nu a_{\alpha\beta}M^{\lambda}_{\lambda}] + O(\frac{\eta\theta^2}{h}) \end{aligned} \tag{6.4.4}$$

or

$$\begin{aligned} \gamma_{\alpha\beta} &= A[(1+\nu)n_{\alpha\beta} - \nu a_{\alpha\beta}n^{\lambda}_{\lambda}] + O(\eta\theta^2) \\ \rho_{\alpha\beta} &= B[(1+\nu)m_{\alpha\beta} - \nu a_{\alpha\beta}m^{\lambda}_{\lambda}] + O(\frac{\eta\theta^2}{h}) \end{aligned} \tag{6.4.5}$$

where

$$A = \frac{1}{Eh} = \frac{1}{C(1-\nu^2)} \quad , \quad B = \frac{12}{Eh^2} = \frac{1}{D(1-\nu^2)} \tag{6.4.6}$$

In the past 50 years many efforts have been made to derive "more exact" constitutive equations of the first approximation by taking into account various bending-stretching coupling terms. It is quite clear form (6.3.9) that, if only *some* selected terms $O(Eh\eta^2\theta^2)$ are taken into account, the resulting shell theory expressed only in terms of $N^{\alpha\beta}$ and $M^{\alpha\beta}$ *cannot* lead to the more accurate results. The accuracy of the strain energy function may be raised only when *all* terms $O(Eh\eta^2\theta^2)$ are taken into account. Thus the shell theory may be improved only within the *consistent second approximation* to the strain energy, in which the strain energy caused by components of shear $\gamma_{3\beta}$, change of normal due to shear $\kappa_{3\beta}$, higher-order strain $\nu_{\alpha\beta}$ and transverse strain γ_{33} are taken into account *together* with the strain energy caused by coupling between tensile-compressional and flexural strains. Then the formula (6.3.9) would provide us with the improved constitutive equations for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ *together* with the constitutive equations for some higher-order stress and couple resultants corresponding to the additional strain measures [12], which should be additionally introduced into the shell theory.

6.5. Reduced shell relations

Under the assumptions a) to c) many other shell relations discussed in chapters 2 to 5 may be simplified as well.

Under small strains we can use the following estimates

$$\frac{\bar{a}}{a} = 1 + 2\gamma_\lambda^\lambda + O(\eta^2) = 1 + O(\eta)$$

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta} = a_{\alpha\beta} + O(\eta) \qquad (6.5.1)$$

$$\bar{a}^{\alpha\beta} = a^{\alpha\beta} - 2\gamma^{\alpha\beta} + O(\eta^2) = a^{\alpha\beta} + O(\eta)$$

$$\bar{\Gamma}_{\lambda.\alpha\beta} = \Gamma_{\lambda.\alpha\beta} + O(\frac{\eta}{\lambda}) \quad , \quad \bar{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + O(\frac{\eta}{\lambda})$$

$$\bar{\epsilon}_{\alpha\beta} = \epsilon_{\alpha\beta} + O(\eta) \quad , \quad \bar{\epsilon}^{\alpha\beta} = \epsilon^{\alpha\beta} + O(\eta) \qquad (6.5.2)$$

where the parameter λ, used here to estimate the surface derivatives, is defined by [31, 9, 16]

$$\lambda = \frac{h}{\theta} = \min\ (L,\ d,\ \sqrt{hR},\ \frac{h}{\sqrt{\eta}}) \qquad (6.5.3)$$

The components of strain $\gamma_{\alpha\beta}$ are *quadratic polynomials* (2.2.4) in terms of displacements and their gradients. The components of change of curvature $\kappa_{\alpha\beta}$ follow from (2.2.14), where

$$n = [1 + \theta^\kappa_\kappa + \frac{1}{2}(\theta^\kappa_\kappa)^2 - \frac{1}{2}\theta^\kappa_\mu\theta^\mu_\kappa + \varphi^2][1 - \gamma^\lambda_\lambda + O(\eta^2)] \qquad (6.5.4)$$

$$n_\mu = [-(1 + \theta^\kappa_\kappa)\varphi_\mu + \varphi^\lambda(\theta_{\lambda\mu} - \omega_{\lambda\mu})][1 + O(\eta)] \qquad (6.5.5)$$

should be used. Note that in (6.5.4) we need to take into account the more accurate estimate for $\sqrt{\frac{a}{\bar{a}}}$, because the leading term $b_{\alpha\beta}$ of the product $nd_{\alpha\beta}$ is cancelled in (2.2.14) by the term $-b_{\alpha\beta}$.

From (2.2.4) we obtain

$$\gamma^\lambda_\lambda = \theta^\lambda_\lambda + \frac{1}{2}\theta^\lambda_\mu\theta^\mu_\lambda + \frac{1}{2}\varphi^\lambda\varphi_\lambda + \varphi^2 \qquad (6.5.6)$$

It is evident from (2.2.14) and (6.5.4) to (6.5.6) that, under small strains, the components of $\kappa_{\alpha\beta}$ become *polynomials* up to the *fifth degree* in displacements u_α, w and their gradients.

It follows from (3.5.7), (3.5.9) and (3.8.9) that for the quantities associated with intermediate basis $\overset{\mathrm{v}}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ we obtain the estimates

$$\overset{v}{\underset{\sim}{a}}_{\alpha} = \underset{\sim}{a}_{\alpha}[1 + O(\eta)] \ , \quad \overset{v\alpha}{\underset{\sim}{a}} = \underset{\sim}{a}^{\alpha}[1 + O(\eta)]$$

$$\overset{v}{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} + O(\eta^2) \ , \quad \overset{v}{\rho}_{\alpha\beta} = \rho_{\alpha\beta} + O(\frac{\eta^2}{R}) \tag{6.5.7}$$

The reduced formula for the finite rotation vector $\underset{\sim}{\Omega}$ follows from (3.7.17), (6.5.5), (2.1.5) and (6.1.4) to be

$$2\underset{\sim}{\Omega} = \{\epsilon^{\beta\alpha}[(2 + \theta^{\kappa}_{\kappa})\varphi_{\alpha} - \varphi^{\lambda}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})]\underset{\sim}{a}_{\beta} + 2\varphi\underset{\sim}{n}\}[1 + O(\eta)] \tag{6.5.8}$$

This relation is *quadratic* with respect to displacements and their gradients.

In a similar way many other geometrical results may be simplified as well [12]. In particular, by expanding the relations (4.7.7) to (4.7.10) in series with respect to the strain measures at the boundary, we obtain the following reduced components of the *vector of change of boundary curvature*

$$\begin{aligned} k_{tt} &\approx \kappa_{tt} + \sigma_t\gamma_{tt} \\ k_{\nu t} &\approx \kappa_{\nu t} + 2(\sigma_t - \underline{\kappa_{tt}})\gamma_{\nu t} - \tau_t\gamma_{\nu\nu} \\ k_{nt} &\approx 2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_{\nu}} + 2\kappa_{\nu}\gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) \end{aligned} \tag{6.5.9}$$

The simplified relations for physical compoments of the *effective internal force* follow from the reduction of (5.2.20) to give

$$\begin{aligned} P_{\nu\nu} &\approx N_{\nu\nu} - \sigma_{\nu}M_{\nu\nu} + 2\tau_t M_{t\nu} \\ P_{t\nu} &\approx N_{t\nu} + \tau_t M_{\nu\nu} - 2(\sigma_t - \underline{\kappa_{tt}})M_{t\nu} \\ P_{n\nu} &\approx \frac{dM_{\nu\nu}}{ds_{\nu}} + \frac{dM_{t\nu}}{ds} + \kappa_t(M_{\nu\nu} - M_{tt}) \end{aligned} \tag{6.5.10}$$

The relations (6.5.9) and (6.5.10) differ from those derived within the linear theory of shells by CHERNYKH [3] (in terms of different changes of curvatures and stress resultants) by the underlined terms only, which describe the boundary normal curvature changes during the shell deformation.

6.6. Bending shell equations

When the small strains in shell space caused by stretching and bending are of *comparable order* in the whole region, the error indicated in the constitutive equations (6.4.1) allows to make essentional reduction of equilibrium equations and compatibility conditions.

Within the same order of error, we obtain for $Q^{\alpha\beta}$ and Q^{β} in (5.4.19)

$$\begin{aligned} Q^{\alpha\beta} &= N^{\alpha\beta} + O(Eh\frac{2\eta}{R}\,,\ Eh\eta^2) \\ &= C[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}] + O(Eh\eta\theta^2) \end{aligned} \tag{6.6.1}$$

$$\begin{aligned} Q^{\beta} &= M^{\alpha\beta}|_{\alpha} + O(Eh\frac{2\eta^2}{\lambda}) \\ &= D[(1-\nu)\kappa^{\alpha\beta}|_{\alpha} + \nu a^{\alpha\beta}\kappa^{\lambda}_{\lambda}|_{\alpha}] + (Eh\frac{2\eta\theta^2}{\lambda}) \end{aligned} \tag{6.6.2}$$

Using (6.3.2), (6.3.6), (6.5.1) and (6.5.2) for some terms in the first of the compatibility conditions (2.3.11), we obtain the following estimates

$$\begin{aligned} \bar{a}^{\kappa\nu} b_{\kappa\lambda}\gamma_{\nu\beta\mu} &= O(\frac{\eta}{R\lambda}) = O(\frac{\eta\theta^2}{h\lambda}) \\ \bar{a}^{\kappa\nu} \kappa_{\kappa\lambda}\gamma_{\nu\beta\mu} &= O(\frac{\eta^2}{h\lambda}) = O(\frac{\nu\theta^2}{h\lambda}) \end{aligned} \tag{6.6.3}$$

and similarly for some terms in second of (2.3.11) we obtain

$$\begin{aligned} K\gamma^{\kappa}_{\kappa} &= O(\frac{\eta}{R^2}) = O(\frac{\eta\theta^2}{\lambda^2}) \\ \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda} &= O(\frac{\eta^2}{\lambda^2}) = O(\frac{\eta\theta^2}{\lambda^2}) \end{aligned} \tag{6.6.4}$$

The *compatibility conditions* (2.3.11) can now be reduced to the form

$$\kappa^{\beta}_{\alpha}|_{\beta} - \kappa^{\beta}_{\beta}|_{\alpha} = O(\frac{\eta\theta^2}{h\lambda}) \tag{6.6.5}$$

$$\gamma^{\beta}_{\alpha}|^{\alpha}_{\beta} - \gamma^{\alpha}_{\alpha}|^{\beta}_{\beta} - (b^{\beta}_{\alpha}\kappa^{\alpha}_{\beta} - b^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) + \frac{1}{2}(\kappa^{\beta}_{\alpha}\kappa^{\alpha}_{\beta} - \kappa^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) = O(\frac{\eta\theta^2}{\lambda^2}) \tag{6.6.6}$$

The equation (6.6.5) gives also the sharper estimate for Q^{β} to be

$$Q^{\beta} = D\kappa^{\lambda}_{\lambda}|^{\beta} + O(Eh^2\,\frac{\eta\theta^2}{\lambda}) = O(Eh^2\,\frac{\eta}{\lambda}) \tag{6.6.7}$$

Using the estimates (6.6.1) and (6.6.5), the *mixed equilibrium equations* (5.6.2) to (5.6.5) are reduced to

$$N_{\alpha}^{\beta}|_{\beta} + \bar{p}_{\alpha} = O(Eh\,\frac{\eta\theta^2}{\lambda}) \tag{6.6.8}$$

$$D\kappa_{\alpha}^{\alpha}|_{\beta}^{\beta} + (b_{\beta}^{\alpha} - \kappa_{\beta}^{\alpha})N_{\alpha}^{\beta} + \bar{p} = O(Eh^2\,\frac{\eta\theta^2}{\lambda^2}) \tag{6.6.9}$$

If the shell problem is going to be solved in terms of the strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ as independent variables, the constitutive equations (6.4.1) may be used to express the *equilibrium equations* in terms of the strain measures

$$\begin{aligned} &C[(1-\nu)\gamma_{\alpha}^{\beta}|_{\beta} + \nu\gamma_{\beta}^{\beta}|_{\alpha}] + \bar{p}_{\alpha} = O(Eh\,\frac{\eta\theta^2}{\lambda}) \\ &D\kappa_{\alpha}^{\alpha}|_{\beta}^{\beta} + C(b_{\beta}^{\alpha} - \kappa_{\beta}^{\alpha})[(1-\nu)\gamma_{\alpha}^{\beta} + \nu\delta_{\alpha}^{\beta}\gamma_{\lambda}^{\lambda}] + \bar{p} = O(Eh^2\,\frac{\eta\theta^2}{\lambda^2}) \end{aligned} \tag{6.6.10}$$

The relations (6.6.10), together with (6.6.5) and (6.6.6), form a *set of six equations* for six strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ to be solved. Four of the equations are *linear* and two are *quadratic* in the strain measures.

The *geometrical* boundary conditions may be expressed in terms of the strain measures through deformational variables $\underset{\sim}{k}_t$ and γ_{tt}, where (6.5.9) should be used. To the *natural* boundary conditions, expressed by $\underset{\sim}{P}_{\nu}$ and $\frac{d}{ds}M_{\nu\nu}$, we should apply (6.5.10) first. Then they can be rewritten in terms of the strain measures by using the constitutive equations (6.4.1). Thus, if the boundary conditions are expressible in terms of the strain measures and the surface forces $\underset{\sim}{\bar{p}}$ are given along the deformed basis, the geometrically nonlinear shell problem may be solved directly in terms of *strain measures* without first having had to calculate displacements. However, the displacement field may be calculated, if necessary, by additional integration of the strain-displacements relations (2.2.4) and (2.2.14), in which (6.5.4) to (6.5.6) should be used.

The geometrically non-linear bending shell problems may also be solved in terms of *displacements*. The consistently reduced set of three displacement equations follow easily from mixed equilibrium equations (6.6.10) by introducing the reduced strain-displacement relations (2.2.4), (2.2.14) and (6.5.4) to (6.5.6). Note that the first of (6.6.10) then becomes *quadratic* in displacements and their gradients, while the last one

contains displacement variables up to the *seventh* power.

If displacement and natural boundary conditions as well as the surface forces are known with respect to the *undeformed* geometry, the *Lagrangean* equilibrium equations (5.5.4) or (5.6.6) should be consistently reduced and expressed in terms of displacement variables. It is easy to see that, by substituting either (6.6.1) and (6.6.7) into the (5.5.4) or (6.6.10), (2.2.15) to (2.2.18) and (6.5.4) to (6.5.6) into (5.6.6.), we also obtain the consistently reduced sets of three equations in terms of displacements and their gradients. These equations are *more complex* than those following from the mixed equations (6.6.10), since in this case the components of the deformation gradient tensor appear explicitly in the equations.

6.7. Quasi-shallow shell equations

The simple structure of the relations (6.6.5) and (6.6.8) makes it possible to present the solution of bending shell equations by means of two scalar functions.

It is easy to see from (6.5.3) that here the wave length of deformation patterns have an *upper bound*

$$(\frac{h}{\lambda})^2 << 1 \quad \text{or} \quad |K|\lambda^2 << 1 \tag{6.7.1}$$

The *lower bound* has already been indicated as the assumption b) in § 6.3. Deformation patterns of the type appear frequently in bending shell problems [6].

According to (1.3.21) the double covariant derivative of any surface tensor may then be estimated as follows

$$T_{\alpha\beta|\lambda\mu} = T_{\alpha\beta|\mu\lambda} + O(T\frac{\theta^2}{\lambda^2}) \tag{6.7.2}$$

where $T = \max |T_r|$, and T_r are eigenvalues of $\underset{\sim}{T}$. Thus up to the error indicated in (6.7.2) the subsequent covariant differentiation is *interchangable*.

Let us introduce the *curvature function* W and the *stress function* F by the relations

$$\kappa_\alpha^\beta = W|_\alpha^\beta + O(\frac{\eta\theta^2}{h})$$
$$N_\alpha^\beta = \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}F|_\mu^\lambda + P_\alpha^\beta + O(Eh\eta\theta^2) \tag{6.7.3}$$

Both scalar functions should be three times continuously differentiable. With (6.7.3) the compatibility conditions (6.6.5) and equilibrium equations (6.6.8) are satisfied within the order of error already introduced into the equations, provided P_α^β is a particular solution of (6.5.7). The compatibility condition (6.6.6), with the help of (6.4.4) and (6.6.8), transforms to

$$AN_\alpha^\alpha|_\beta^\beta + (b_\beta^\alpha\kappa_\alpha^\beta - b_\alpha^\alpha\kappa_\beta^\beta) - \frac{1}{2}(\kappa_\beta^\alpha\kappa_\alpha^\beta - \kappa_\alpha^\alpha\kappa_\beta^\beta) +$$
$$+ A(1+\nu)\bar{p}^\alpha|_\alpha = O(\frac{\eta\theta^2}{\lambda^2}) \tag{6.7.4}$$

When (6.7.3) is introduced into (6.6.9) and (6.7.4), the following set of two equations for W and F should be satisfied [6, 12]

$$DW|_{\alpha\beta}^{\alpha\beta} + \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}(b_\beta^\alpha - W|_\beta^\alpha)F|_\mu^\lambda + (b_\beta^\alpha - W|_\beta^\alpha)P_\alpha^\beta + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2})$$
$$AF|_{\alpha\beta}^{\alpha\beta} - \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}(b_\beta^\alpha - \frac{1}{2}W|_\beta^\alpha)W|_\mu^\lambda + A[P_\alpha^\alpha|_\beta^\beta - (1+\nu)P_\beta^\alpha|_\alpha^\beta] = O(\frac{\eta\theta^2}{\lambda^2}) \tag{6.7.5}$$

These so-called *quasi-shallow shell equations* describe the behaviour of geometrically non-linear bending shell problems with "small" Gaussian curvature with respect to lengths of deformations patterns. In solving these equations, we do not need to refer to displacements, although some difficulties may arise with specification of P_α^β (due to the tangent surface load), as well as with the proper formulation of the boundary conditions in terms of W and F.

6.8. Canonical shell equations

The reduction of shell relations in § 6.3 to § 6.7 has been carried out under the assumption that the small strains in the shell space, caused by independent stretching and bending of the middle surface, are of comparable order in the *whole* shell region.

In many shell problems there are *some* regions in which the small strains caused by the membrane forces may happen to be of *essentially different*

order (higher or smaller by the factor θ^2) from those caused by the moments. Within these regions the reduced equations (6.6.5) and (6.6.8) may not be accurate enough, since they contain only the leading terms of one kind: changes of curvatures or membrane forces, respectively. Thus, in general, more accurately estimated equations are needed in order to describe properly the shell behaviour in these limitting cases as well.

The refinement of equations (6.6.5) and (6.6.8) may be carried out by selecting stress resultants $N^{\alpha\beta}$ and changes of curvatures $\kappa_{\alpha\beta}$ as two *independent variables* of the shell equations, [27]. Indeed, in this case we are able to obtain a much better estimate for $Q_{\alpha\beta}$ which, according to (5.4.19), (6.4.1), (6.5.1) and (2.2.1), becomes

$$Q^{\alpha\beta} = N^{\alpha\beta} - D(b^{\alpha}_{\kappa} - \kappa^{\alpha}_{\kappa})[(1-\nu)\kappa^{\kappa\beta} + \nu a^{\kappa\beta}\kappa^{\lambda}_{\lambda}] + O(Eh\eta\theta^4) \qquad (6.8.1)$$

This estimate introduces an error $O(Eh\,\frac{\eta\theta^4}{\lambda})$ into the equilibrium equations (5.6.2). Within the same error the second term of (5.6.2) can be reduced with the help of (6.5.1), (6.8.1), (6.4.4) and (6.6.8) as follows:

$$\begin{aligned}
\bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}Q^{\lambda\beta} &= (2\gamma^{\alpha}_{\lambda}|_{\beta} - \gamma_{\lambda\beta}|^{\alpha})N^{\lambda\beta} + O(Eh\,\frac{\eta\theta^4}{\lambda}) = \\
&= 2A[(1+\nu)N^{\alpha}_{\lambda} - \nu\delta^{\alpha}_{\lambda}N^{\kappa}_{\kappa}]|_{\beta}\,N^{\lambda\beta} - \\
&\quad - A[(1+\nu)N_{\lambda\beta} - \nu a_{\lambda\beta}N^{\kappa}_{\kappa}]|^{\alpha}N^{\lambda\beta} + O(Eh\,\frac{\eta\theta^4}{\lambda}) = \\
&= 2A[N^{\alpha}_{\lambda}N^{\lambda\beta} + \nu(N^{\alpha}_{\lambda}N^{\lambda\beta} - N^{\alpha\beta}N^{\lambda}_{\lambda})]|_{\beta} - \\
&\quad - \frac{1}{2}[(1+\nu)N_{\lambda\beta}N^{\lambda\beta} - \nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]|^{\alpha} + \\
&\quad + 2A[(1+\nu)N^{\alpha}_{\lambda}\bar{p}^{\lambda} - \nu N^{\lambda}_{\lambda}\bar{p}^{\alpha}] + O(Eh\,\frac{\eta\theta^4}{\lambda})
\end{aligned} \qquad (6.8.2)$$

But using the identities

$$\begin{aligned}
N^{\alpha}_{\lambda} &= \delta^{\alpha}_{\lambda}N^{\kappa}_{\kappa} - \epsilon^{\alpha\mu}\epsilon_{\lambda\nu}N^{\nu}_{\mu} \\
N^{\lambda\beta} &= a^{\beta\lambda}N^{\rho}_{\rho} - \epsilon^{\beta\gamma}\epsilon^{\lambda\kappa}N_{\gamma\kappa} \\
a^{\beta\lambda}\epsilon_{\lambda\nu} &= \epsilon^{\beta\lambda}a_{\lambda\nu}
\end{aligned} \qquad (6.8.3)$$

we obtain the relation

$$
\begin{aligned}
&(N_\lambda^\alpha N^{\lambda\beta} - N^{\alpha\beta}N_\lambda^\lambda)\,|_\beta = \\
&= a^{\alpha\beta}(N_{\lambda\kappa}N^{\lambda\kappa} - N_\lambda^\lambda N_\kappa^\kappa)\,|_\beta - (N_\lambda^\alpha N^{\lambda\beta} - N^{\alpha\beta}N_\lambda^\lambda)\,|_\beta = \\
&= \frac{1}{2}\,(N_{\lambda\beta}N^{\lambda\beta} - N_\lambda^\lambda N_\beta^\beta)\,|^\alpha
\end{aligned}
\tag{6.8.4}
$$

This relation is also an identity for any continuously differentiable *symmetric* surface tensor components.

With the help of (6.8.4) the relation (6.8.2) now takes a simpler form

$$
\begin{aligned}
\bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}Q^{\lambda\beta} = {} & 2A(N_\lambda^\alpha N^{\lambda\beta})\,|_\beta - \frac{1}{2}\,A[(1-\nu)N_{\lambda\beta}N^{\lambda\beta} + \nu N_\lambda^\lambda N_\beta^\beta]\,|^\alpha + \\
& + 2A[(1+\nu)N_\lambda^\alpha \bar{p}^\lambda - \nu N_\lambda^\lambda \bar{p}^\alpha] + O(Eh\,\frac{\eta\theta^4}{\lambda})
\end{aligned}
\tag{6.8.5}
$$

For the third term in (5.6.2), by taking into account (6.6.7) and (6.5.1), we obtain the following estimate

$$
-\bar{b}_\beta^\alpha Q^\beta = -D(b_\beta^\alpha - \kappa_\beta^\alpha)\kappa_\lambda^\lambda\,|_\beta + O(Eh\,\frac{\eta\theta^4}{\lambda}) \tag{6.8.6}
$$

In this way, all terms in equilibrium equations (5.6.2) have been consistently refined within an error $O(Eh\,\frac{\eta\theta^4}{\lambda})$, which is smaller by the factor θ^2 than the error used in the bending shell equations (6.6.8).

In exactly the same way, the reduction of the first of the compatibility conditions (2.3.11) may be carried out within a smaller error than in (6.6.5). Let us replace $\bar{a}^{\kappa\nu}$ by $a^{\kappa\nu}$ in the second term of (2.3.11). This, according to (6.5.1), introduces an error $O(\frac{\eta\theta^4}{h\lambda})$. Then using (2.3.4), (2.2.1) and (6.4.4) this term can be estimated as follows:

$$\epsilon_{\alpha\beta}\epsilon^{\lambda\mu}\bar{b}^{\kappa}_{\lambda}\gamma_{\kappa.\mu}{}^{\beta} = (b^{\kappa}_{\alpha} - \kappa^{\kappa}_{\alpha})(2\gamma^{\beta}_{\kappa}|_{\beta} - \gamma^{\beta}_{\beta}|_{\kappa}) -$$

$$- (b^{\kappa}_{\beta} - \kappa^{\kappa}_{\beta})\gamma^{\beta}_{\kappa}|_{\alpha} + O(\frac{\eta\theta^4}{h\lambda}) =$$

$$= A(b^{\kappa}_{\alpha} - \kappa^{\kappa}_{\alpha})[2(1+\nu)N^{\beta}_{\kappa}|_{\beta} - 2\nu N^{\lambda}_{\lambda}|_{\kappa} - (1-\nu)N^{\beta}_{\beta}|_{\kappa}] -$$

$$- A(b^{\kappa}_{\beta} - \kappa^{\kappa}_{\beta})[(1+\nu)N^{\beta}_{\kappa}|_{\alpha} - \nu\delta^{\beta}_{\kappa}N^{\lambda}_{\lambda}|_{\alpha}] + O(\frac{\eta\theta^4}{h\lambda}) = \tag{6.8.7}$$

$$= -2A(1+\nu)(b^{\beta}_{\alpha} - \kappa^{\beta}_{\alpha})\bar{p}_{\beta} + A\nu(b^{\beta}_{\beta} - \kappa^{\beta}_{\beta})N^{\lambda}_{\lambda}|_{\alpha} -$$

$$- A(1+\nu)[(b^{\kappa}_{\beta} - \kappa^{\kappa}_{\beta})N^{\beta}_{\kappa}|_{\alpha} + (b^{\kappa}_{\alpha} - \kappa^{\kappa}_{\alpha})N^{\lambda}_{\lambda}|_{\kappa}] + O(\frac{\eta\theta^4}{h\lambda})$$

Taking into account (6.6.9) together with (6.8.1), (6.8.5) and (6.8.6), the *refined equilibrium equations* take the following form

$$N^{\beta}_{\alpha}|_{\beta} - D[(1-\nu)b^{\lambda}_{\alpha}\kappa^{\beta}_{\lambda} + \nu b^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}]|_{\beta} - Db^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}|_{\beta} + D[(1-\nu)\kappa^{\lambda}_{\alpha}\kappa^{\beta}_{\lambda} + \nu\kappa^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}]|_{\beta} +$$

$$+ D\kappa^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}|_{\beta} + 2A(N^{\lambda}_{\alpha}N^{\beta}_{\lambda})|_{\beta} - \frac{1}{2}A[(1-\nu)N^{\beta}_{\lambda}N^{\lambda}_{\beta} + \nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]|_{\alpha} + \tag{6.8.8}$$

$$+ 2A[(1+\nu)N^{\lambda}_{\alpha}\bar{p}_{\lambda} - \nu N^{\lambda}_{\lambda}\bar{p}_{\alpha}] + (1+\gamma^{\lambda}_{\lambda})\bar{p}_{\alpha} = O(Eh\frac{\eta\theta^4}{\lambda})$$

$$D\kappa^{\alpha}_{\alpha}|^{\beta}_{\beta} + (b^{\alpha}_{\beta} - \kappa^{\alpha}_{\beta})N^{\beta}_{\alpha} + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2}) \tag{6.8.9}$$

Taking into account (6.7.4) together with (1.1.18) and (6.8.7), the *refined compatibility conditions* take the following form

$$\kappa^{\beta}_{\alpha}|_{\beta} - \kappa^{\beta}_{\beta}|_{\alpha} - A(1+\nu)[b^{\lambda}_{\beta}N^{\beta}_{\lambda}|_{\alpha} + b^{\beta}_{\alpha}N^{\lambda}_{\lambda}|_{\beta}] + A\nu b^{\beta}_{\beta}N^{\lambda}_{\lambda}|_{\alpha} +$$

$$+ A(1+\nu)[\kappa^{\lambda}_{\beta}N^{\beta}_{\lambda}|_{\alpha} + \kappa^{\beta}_{\alpha}N^{\lambda}_{\lambda}|_{\beta}] - A\nu\kappa^{\beta}_{\beta}N^{\lambda}_{\lambda}|_{\alpha} - \tag{6.8.10}$$

$$- 2A(1+\nu)b^{\beta}_{\alpha}\bar{p}_{\beta} + 2A(1+\nu)\kappa^{\beta}_{\alpha}\bar{p}_{\beta} = O(\frac{\eta\theta^4}{h\lambda})$$

$$AN^{\alpha}_{\alpha}|^{\beta}_{\beta} + (b^{\alpha}_{\beta}\kappa^{\beta}_{\alpha} - b^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) - \frac{1}{2}(\kappa^{\alpha}_{\beta}\kappa^{\beta}_{\alpha} - \kappa^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) +$$

$$+ A(1+\nu)\bar{p}^{\alpha}|_{\alpha} = O(\frac{\eta\theta^2}{\lambda^2}) \tag{6.8.11}$$

The relations (6.8.8) to (6.8.11) are the *canonical form of intrinsic equations* in the first-approximation non-linear theory of thin elastic shells [16]. They describe accurately the behaviour of a shell in the whole internal region independently of the ratio of bending and membrane strains.

The equations have been derived here in terms of symmetric variables $N^{\alpha\beta}$ and $\kappa_{\alpha\beta}$ and with all surface force components $\bar{p}_\alpha$, $\bar{p}$ taken into account. The refined shell equations of this kind have been originated by DANIELSON [27] in terms of (in our notation) $n^{\alpha\beta}$ and $-\kappa_{\alpha\beta}$ where only the normal component $\bar{p}$ of the surface force has been taken into account. KOITER and SIMMONDS [16] derived the equations in terms of $n^{\alpha\beta}$ and $-\rho_{\alpha\beta}$ in the absence of surface forces (cf. also [9]), while all the surface force components have been taken into account by the author [12, 33].

6.9. Membrane and inextensional bending equations

For some shell problems it is possible to predict in advance the type of solution in the whole internal shell region. The prediction may be used to reduce initially the canonical shell equations (6.8.8) to (6.8.11) within the error already introduced into the equations. The expected solution of the problem may then be obtained from a much simpler set of equations without any loss of accuracy.

Let γ and κ be the maximal values of the shell strain measures defined by

$$\gamma = \max |\gamma_\rho| \quad , \quad \kappa = \max |\kappa_\rho| \tag{6.9.1}$$

where γ_ρ and κ_ρ are eigenvalues of the tensors $\underset{\sim}{\gamma}$ and $\underset{\sim}{\kappa}$.

The ratio $\frac{\kappa h}{\gamma}$ describes at any $M \in M$ the estimate of the relation between the small strains in the shell caused by bending and stretching of the shell middle surface. In respect to the ratio it is convenient to introduce the following classification [12] of the shell theories:

1) $\frac{\kappa h}{\gamma} \leq O(\theta^2)$ - membrane theory

2) $\frac{\kappa h}{\gamma} = O(\theta)$ - small bending theory

3) $\frac{\kappa h}{\gamma} = O(1)$ - bending theory

4) $\frac{\kappa h}{\gamma} = O(\frac{1}{\theta})$ - large bending theory

5) $\frac{\kappa h}{\gamma} \geq O(\frac{1}{\theta^2})$ - inextensional bending theory

Let us use the estimates indicated in this classification to reduce the canonical shell equations (6.8.8) to (6.8.11).

For the *membrane* shell theory it is necessary to estimate in (6.8.8) to (6.8.11) all terms containing $\kappa_{\alpha\beta} = O(\frac{\eta\theta^2}{h})$. The terms of order of the error indicated in these equations may then be omitted. As a result we obtain the following reduced set of equations

$$N_\alpha^\beta|_\beta + 2A(N_\alpha^\lambda N_\lambda^\beta)|_\beta - \frac{1}{2}A[(1-\nu)N_\lambda^\beta N_\beta^\lambda + \nu N_\lambda^\lambda N_\beta^\beta]|_\alpha +$$
$$+ 2A[(1+\nu)N_\alpha^\lambda \bar{p}_\lambda - \nu N_\lambda^\lambda \bar{p}_\alpha] + (1+\gamma_\lambda^\lambda)\bar{p}_\alpha = O(Eh\frac{\eta\theta^4}{\lambda}) \qquad (6.9.2)$$
$$b_\beta^\alpha N_\alpha^\beta + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2})$$

$$\kappa_\alpha^\beta|_\beta - \kappa_\beta^\beta|_\alpha - A(1+\nu)[b_\beta^\lambda N_\lambda^\beta|_\alpha + b_\alpha^\beta N_\lambda^\lambda|_\beta] +$$
$$+ A\nu b_\beta^\beta N_\lambda^\lambda|_\alpha - 2A(1+\nu)b_\alpha^\beta \bar{p}_\beta = O(\frac{\eta\theta^4}{h\lambda}) \qquad (6.9.3)$$
$$AN_\alpha^\alpha|_\beta^\beta + A(1+\nu)\bar{p}^\alpha|_\alpha = O(\frac{\eta\theta^2}{\lambda^2})$$

It is easy to note that the non-linear (quadratic) equilibrium equations (6.9.2) can be solved with respect to $N^{\alpha\beta}$ *independently* of the state of strain in a shell, provided that two boundary conditions are expressible in terms of $N^{\alpha\beta}$. In this sense the membrane shell problems are *statically determined*. The additional conditions (6.9.3) for $N^{\alpha\beta}$ make the membrane shell equations with respect to $N^{\alpha\beta}$ *mathematically overdetermined*. This means that the membrane state of stress may occur in a shell only under *exceptional* circumstances.

Similarly, for the *inextensional bending* shell theory it is necessary to estimate in the canonical shell equations all terms containing $N^{\alpha\beta} = O(Eh\eta\theta^2)$. The terms of order of the error indicated in the equations (6.8.8) to (6.8.11) may then be omitted. As a result we obtain the following reduced set of equations

$$
\begin{aligned}
&N_{\alpha}^{\beta}|_{\beta} - D[(1-\nu)b_{\alpha}^{\lambda}\kappa_{\lambda}^{\beta} + \nu b_{\alpha}^{\beta}\kappa_{\lambda}^{\lambda}]|_{\beta} - Db_{\alpha}^{\beta}\kappa_{\lambda}^{\lambda}|_{\beta} + \\
&\quad + D[(1-\nu)\kappa_{\alpha}^{\lambda}\kappa_{\lambda}^{\beta} + \nu\kappa_{\alpha}^{\beta}\kappa_{\lambda}^{\lambda}]|_{\beta} + D\kappa_{\alpha}^{\beta}\kappa_{\lambda}^{\lambda}|_{\beta} + \\
&\quad + (1+\gamma_{\lambda}^{\lambda})\bar{p}_{\alpha} = O(Eh\frac{\eta\theta^4}{\lambda})
\end{aligned}
\tag{6.9.4}
$$

$$
\begin{aligned}
&D\kappa_{\alpha}^{\alpha}|_{\beta}^{\beta} + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2}) \\
&\kappa_{\alpha}^{\beta}|_{\beta} - \kappa_{\beta}^{\beta}|_{\alpha} = O(\frac{\eta\theta^4}{h\lambda}) \\
&b_{\beta}^{\alpha}\kappa_{\alpha}^{\beta} - b_{\alpha}^{\alpha}\kappa_{\beta}^{\beta} - \frac{1}{2}(\kappa_{\beta}^{\alpha}\kappa_{\alpha}^{\beta} - \kappa_{\alpha}^{\alpha}\kappa_{\beta}^{\beta}) = O(\frac{\eta\theta^2}{\lambda^2})
\end{aligned}
\tag{6.9.5}
$$

It is easy to note here that the non-linear (quadratic) compatibility conditions (6.9.5) can be solved with respect to $\kappa_{\alpha\beta}$ *independently* of the state of stress in a shell, provided that two boundary conditions are expressible in terms of $\kappa_{\alpha\beta}$. In this sense the inextensional bending shell problems are *geometrically determined*. The additional condition (6.9.4) for $\kappa_{\alpha\beta}$ makes the four equations with respect to $\kappa_{\alpha\beta}$ *mathematically over-determined*. This means that the inextensional bending state of strain may occur in a shell only under *exceptional* circumstances.

For the *small bending* {or *large bending*, respectively} shell theory we should estimate in (6.8.8) to (6.8.11) all terms containing $\kappa_{\alpha\beta} = O(\frac{\eta\theta}{h})$ {or $N_{\alpha\beta} = O(Eh\eta\theta)$}. It happens that within the error indicated in these equations only the terms quadratic in $\kappa_{\alpha\beta}$ {or quadratic in $N^{\alpha\beta}$} may be omitted here. However, with a possible small loss in accuracy, we may admit the greater error in the first of equilibrium equations and in the first of compatibility conditions. Then for both small and large bending shell problems the *bending shell euqations* of § 6.6 may be used.

Let us point out again that the advance simplification of shell equations is possible only when the unknown solution satisfies any of estimates indicated in 1) to 5) in the *whole* internal shell region. Still one should always check at the end of calculations to determine whether the solution obtained from simplified shell equations *indeed* represents a satisfactory and predicted type of solution.

Chapter 7

GEOMETRICALLY NON-LINEAR THEORY OF ELASTIC SHELLS UNDER RESTRICTED ROTATIONS

Polar decomposition theorem applied to displacement gradient tensor $\underset{\sim}{G}$ has made it possible to decompose in § 3.4 the shell deformation into the separate translation, strain and rotation. In the reduction of basic shell relations discussed in chapter 6 we have used the assumption that the strains are small everywhere in the shell. We have not made so far any restriction as to the magnitude of the *rotations* of the shell material elements.

For many engineering purposes it is hardly necessary to allow the rotations of any magnitude. Many shell structures would become unserviceable if really unrestricted rotations were permitted to occur. It is certainly worthwhile then to discuss the possible reduction of shell relations resulting from the consistently restricted rotations. The well-known classifications of the approximate shell equations have been proposed by MUSHTARI and GALIMOV [34] and KOITER [6]. In [34] the restrictions on the components φ_α and φ of the *linearized rotation vector* $\underset{\sim}{\phi}$ have been used to make a clear distinction between three approximate variants of shell equations with "small, medium or large *bending*". In [6] four approximate variants with "infinitesimal, small finite, moderate or large *deflections*" have been clearly defined by using various restrictions of *displacement gradients* and components of $\underset{\sim}{\phi}$. Note that in both classifications the name "rotation" is correctly avoided, since neither $\underset{\sim}{\phi}$ nor displacement gradients describe finite rotations of the shell material elements.

There are many papers in which various approximate variants of shell equations, obtained under similar to [6,34] assumptions, are called as to be valid under "infinitesimal, small, moderate, large etc. *rotations*". The names are given apparently on the intuitive basis, without defining precisely the meaning of the "rotations" which are supposed to be restricted in some sense. This sometimes may lead to confusions.

It is clear from chapter 3 that the consistent classification of the approximate shell relations, resulting from the appropriately restricted rotations, should be expressed in terms of either the finite rotation tensor $\underset{\sim}{R}$ or the finite rotation vector $\underset{\sim}{\Omega}$. We prefer to deal with the vector $\underset{\sim}{\Omega}$, since within small rotations it may be approximated by the linearized rotation vector $\underset{\sim}{\phi}$. However, when the rotations become not small the numerical values of $\underset{\sim}{\Omega}$ differ from those of $\underset{\sim}{\phi}$.

Here the consistent classification of shell relations is proposed in terms of the finite rotation vector $\underset{\sim}{\Omega}$. The case of small rotations coincides with the classical linear theory of shells. For moderate rotations various consistently reduced strain-displacement relations are discussed. The application of the Lagrangean virtual work principle leads then to the consistent sets of shell equations. The consistent strain-displacement relations are constructed also under the large rotations. At the end we discuss some relations for thin plates under small, moderate and large rotations.

7.1. Classification of rotations

The finite rotation vector $\underset{\sim}{\Omega}$ is uniquely defined in (3.4.1) by the *angle of rotation* ω around the *axis of rotation* described by the unit vector $\underset{\sim}{e}$. It is possible then to restrict either the magnitude of ω or the direction of $\underset{\sim}{e}$.

The angle ω appears in all the formulae through some trigonometric functions. Let us expand the functions into Taylor series at $\omega = 0$ to obtain

$$\begin{aligned} \sin\omega &= \omega - \frac{1}{3!}\omega^3 + \frac{1}{5!}\omega^5 - \dots \\ \cos\omega &= 1 - \frac{1}{2!}\omega^2 + \frac{1}{4!}\omega^4 - \dots \\ 2\cos^2\omega/2 &= 2 - \frac{1}{2!}\omega^2 + \frac{1}{4!}\omega^4 - \dots \end{aligned} \tag{7.1.1}$$

A substantial simplification of the shell relations may be obtained in the case when the functions (7.1.1) are approximated only by the *first terms* of the expansions. Approximation of the functions by first *two terms* results in only slight simplifications, which may happen to be useful in some particular shell problems.

Within geometrically non-linear theory of elastic shells the parameter θ defined in (6.3.4) has been used as a common measure of various small quantities. In terms of θ the following *classification* of magnitude of the *rotation angle* may be introduced:

1) $\omega \leq O(\theta^2)$ - small rotations
2) $\omega = O(\theta)$ - moderate rotations
3) $\omega = O(\sqrt{\theta})$ - large rotations
4) $\omega \geq O(1)$ - finite rotations

Within the *small rotation* shell theory $\omega << 1$ and (7.1.1) may be approximated only by the first terms as follows:

$$\begin{aligned} \sin\omega &= \omega + O(\theta^6) \\ \cos\omega &= 1 + O(\theta^4) \\ 2\cos^2\omega/2 &= 2 + O(\theta^4) \end{aligned} \tag{7.1.2}$$

For *moderate rotations* $\omega^2 << 1$, but (7.1.1) may still be approximated only by the first terms

$$\begin{aligned} \sin\omega &= \omega + O(\theta^3) \\ \cos\omega &= 1 + O(\theta^2) \\ 2\cos^2\omega/2 &= 2 + O(\theta^2) \end{aligned} \tag{7.1.3}$$

Within the definition of *large rotations* we have $\omega^4 << 1$. In this case approximation of the functions (7.1.1) only by the first terms would introduce a relative error $O(\theta)$ with respect to 1. Thus in this case the proper approximation should contain first two terms in the expansions

$$\begin{aligned} \sin\omega &= \omega - \frac{1}{3!}\omega^3 + O(\theta^2\sqrt{\theta}) \\ \cos\omega &= 1 - \frac{1}{2!}\omega^2 + O(\theta^2) \\ 2\cos^2\omega/2 &= 2 - \frac{1}{2!}\omega^2 + O(\theta^2) \end{aligned} \tag{7.1.4}$$

This classification restricts only the *magnitude* of $\underset{\sim}{\Omega}$ for $|\omega| < \pi/2$, since then $O(|\underset{\sim}{\Omega}|) = O(\sin\omega) = O(\omega)$. The *direction* of the rotation axis may still be arbitrary, Fig. 20 a. However, most of the shell structures are more flexible in the out-of-surface direction than in the direction tangent to the surface. It is recomended then to use *different restrictions*

on components of $\underset{\sim}{\Omega}$ in the direction tangent or normal to M. The name "small, moderate, large or finite rotation" may then be associated with the particular component $\underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha$ or $\underset{\sim}{\Omega} \cdot \underset{\sim}{n}$, Fig. 20 b and c.

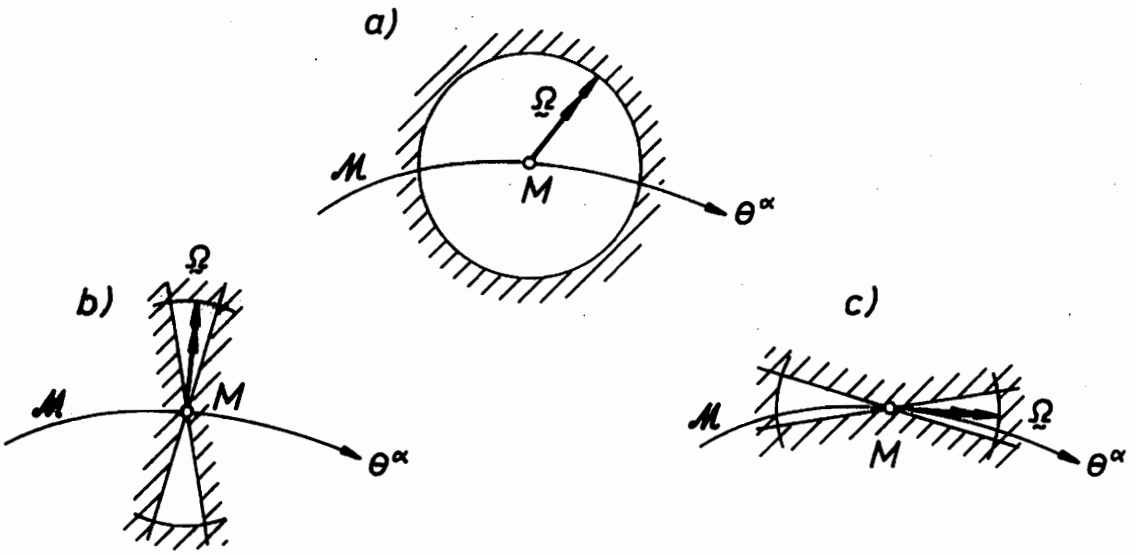

Fig. 20

7.2. Linear theory of shells

When all the rotations are assumed to be *small* then

$$|\underset{\sim}{\Omega}| = O(\theta^2) \quad , \quad \underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha = O(\theta^2) \quad , \quad \underset{\sim}{\Omega} \cdot \underset{\sim}{n} = O(\theta^2) \tag{7.2.1}$$

From (6.5.8) and (2.2.4) we obtain the following estimation of the linearized quantities

$$\varphi = O(\theta^2) \quad , \quad \varphi_\alpha = O(\theta^2) \quad , \quad \theta_{\alpha\beta} = O(\eta) \tag{7.2.2}$$

The appropriate reduction of (6.5.4), (6.5.5) and (2.1.4), (2.2.8) gives us

$$\begin{aligned} n &= 1 + O(\eta) & n_\mu &= -\varphi_\mu + O(\eta\theta^2) \\ \bar{\underset{\sim}{a}}_\alpha &= \underset{\sim}{a}_\alpha[1 + O(\theta^2)] \quad , & \bar{\underset{\sim}{n}} &= \underset{\sim}{n}[1 + O(\theta^2)] \end{aligned} \tag{7.2.3}$$

The *finite* rotation vector $\underset{\sim}{\Omega}$ given in (6.5.8) can be approximated by

$$\begin{aligned} \underset{\sim}{\Omega} &= \epsilon^{\beta\alpha}\varphi_\alpha \underset{\sim}{a}_\beta + \varphi \underset{\sim}{n} + O(\eta\theta^2) \\ &= \underset{\sim}{\phi} + O(\eta\theta^2) \end{aligned} \tag{7.2.4}$$

where $\underset{\sim}{\phi}$ is the *linearized* rotation vector defined in (2.1.9). Thus under small rotations both vectors $\underset{\sim}{\Omega}$ and $\underset{\sim}{\phi}$ coincide with accuracy to an error indicated in (7.2.4).

It follows from (2.2.4), (2.2.14), (6.5.4), (6.5.5) and (7.2.2) that under small roations the *strain-displacement* relations become linear in terms of displacements and take the form

$$\begin{aligned}\gamma_{\alpha\beta} &= \theta_{\alpha\beta} + O(\eta\theta^2) = \\ &= \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w + O(\eta\theta^2)\end{aligned} \tag{7.2.5}$$

$$\begin{aligned}\kappa_{\alpha\beta} &= -\frac{1}{2}[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b_\alpha^\lambda(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b_\beta^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] + O(\frac{\eta\theta^2}{\lambda}) = \\ &= -w|_{\alpha\beta} + b_\alpha^\lambda b_{\lambda\beta}w - b_\alpha^\lambda u_{\lambda|\beta} - b_\beta^\lambda u_{\lambda|\alpha} - b_{\alpha|\beta}^\lambda u_\lambda + O(\frac{\eta\theta^2}{\lambda})\end{aligned} \tag{7.2.6}$$

When the relations (7.2.4) to (7.2.6) are used in the Lagrangean virtual work principle (5.5.14), for the variations we obtain

$$\begin{aligned}\delta\gamma_{\alpha\beta} &= \frac{1}{2}\,(\delta u_{\alpha|\beta} + \delta u_{\beta|\alpha}) - b_{\alpha\beta}\delta w \\ \delta\kappa_{\alpha\beta} &= -\delta w|_{\alpha\beta} + b_\alpha^\lambda b_{\lambda\beta}\delta w - b_\alpha^\lambda \delta u_{\lambda|\beta} - b_\beta^\lambda \delta u_{\lambda|\alpha} - b_{\alpha|\beta}^\lambda \delta u_\lambda \\ \delta\underset{\sim}{\Omega}_t &= \delta\underset{\sim}{\phi} - (\delta\gamma_{\alpha\beta}\nu^\alpha t^\beta)\underset{\sim}{n} \\ \delta\underset{\sim}{\phi} &= \epsilon^{\beta\alpha}(\delta\varphi_\alpha \underset{\sim}{a}_\beta + \frac{1}{2}\,\delta\omega_{\beta\alpha}\underset{\sim}{n})\end{aligned} \tag{7.2.7}$$

Proceeding in exactly the same way as with the Eulerian virtual work principle in § 5.2, for the theory of small rotations we obtain the following *equilibrium equations*

$$\begin{aligned}(N^{\alpha\beta} - b_\kappa^\alpha M^{\kappa\beta})|_\beta - b_\kappa^\alpha M^{\kappa\beta}|_\beta + p^\alpha &= 0 \\ M^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta}(N^{\alpha\beta} - b_\kappa^\alpha M^{\kappa\beta}) + p &= 0\end{aligned} \tag{7.2.8}$$

to be satisfied within the internal region of M, the *natural boundary conditions*

$$\begin{aligned}P_{\nu\nu} &= (N^{\alpha\beta} - b_\kappa^\alpha M^{\kappa\beta})\nu_\alpha\nu_\beta + \tau_t M_{t\nu} = F_\nu + \tau_t K_t \equiv R_\nu \\ P_{t\nu} &= (N^{\alpha\beta} - b_\kappa^\alpha M^{\kappa\beta})t_\alpha\nu_\beta - \sigma_t M_{t\nu} = F_t - \sigma_t K_t \equiv R_t \\ P_{n\nu} &= M^{\alpha\beta}|_\alpha \nu_\beta - \frac{d}{ds}\,M_{t\nu} = F_n - \frac{d}{ds}\,K_t \equiv R_n \\ &\qquad M_{\nu\nu} = K_\nu\end{aligned} \tag{7.2.9}$$

to be satisfied at smooth parts of C, together with *concentrated force*

$$\{[M_{t\nu}(s_i+0) - K_t(s_i+0)] - [M_{t\nu}(s_i-0) - K_t(s_i-0)]\}\underset{\sim}{n}(s_i) \quad (7.2.10)$$

to be applied at each corner s_i of the contour C.

For the modified tensor of change of curvature $\rho_{\alpha\beta}$ we would obtain the relation

$$\rho_{\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b_\alpha^\lambda \omega_{\beta\lambda} + b_\beta^\lambda \omega_{\alpha\lambda}) + O(\frac{\eta\theta^2}{\lambda}) \quad (7.2.11)$$

If (7.2.5) and (7.2.11) were used in the Lagrangean virtual work principle (5.5.14) we would obtain the modified *equilibrium equations*

$$\begin{aligned} &[n^{\alpha\beta} + \frac{1}{2}(b_\kappa^\beta m^{\alpha\kappa} - b_\kappa^\alpha m^{\kappa\beta})]|_\beta - b_\kappa^\alpha m^{\kappa\beta}|_\beta + p^\alpha = 0 \\ &m^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} + p = 0 \end{aligned} \quad (7.2.12)$$

to be satisfied within the internal region of M, and in *natural boundary conditions* the components of effective internal force would take the form

$$\begin{aligned} P_{\nu\nu} &= [n^{\alpha\beta} + \frac{1}{2}(b_\kappa^\beta m^{\alpha\kappa} - b_\kappa^\alpha m^{\kappa\beta})]\nu_\alpha \nu_\beta + \tau_t m_{t\nu} \\ P_{t\nu} &= [n^{\alpha\beta} + \frac{1}{2}(b_\kappa^\beta m^{\alpha\kappa} - b_\kappa^\alpha m^{\kappa\beta})]t_\alpha \nu_\beta - \sigma_t m_{t\nu} \\ P_{n\nu} &= m^{\alpha\beta}|_\alpha \nu_\beta - \frac{d}{ds} m_{t\nu} \end{aligned} \quad (7.2.13)$$

Thus under small rotations the geometrically non-linear theory of shells reduces to the *classical linear theory of shells*. Various problems of the linear theory of shells have been discussed in detail in many papers and books, for example [2,3,4,32].

7.3. Moderate rotation shell theory

When all the rotations are assumed to be *moderate*, then

$$|\underset{\sim}{\Omega}| = O(\theta) \ , \ \underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha = O(\theta) \ , \ \underset{\sim}{\Omega} \cdot \underset{\sim}{n} = O(\theta) \quad (7.3.1)$$

From (6.5.8) and (2.2.4) we obtain the following estimates of the linearized quantities

$$\varphi = O(\theta) \ , \ \varphi_\alpha = O(\theta) \ , \ \theta_{\alpha\beta} = O(\theta^2) \tag{7.3.2}$$

It follows from (6.5.4) and (6.5.5) that now

$$\begin{aligned} n &= 1 - \frac{1}{2}\varphi^\lambda\varphi_\lambda + O(\eta\theta^2) \\ n^\lambda &= -\varphi^\lambda - \epsilon^{\mu\lambda}\varphi_\mu\varphi + O(\eta\theta) \end{aligned} \tag{7.3.3}$$

and for the basic vectors we obtain

$$\begin{aligned} \bar{\underset{\sim}{a}} &= (\underset{\sim}{a}_\alpha - \omega_{\lambda\alpha}\underset{\sim}{a}^\lambda + \varphi_\alpha\underset{\sim}{n})[1 + O(\theta^2)] \\ \bar{\underset{\sim}{n}} &= (-\varphi^\lambda\underset{\sim}{a}_\lambda + \underset{\sim}{n})[1 + O(\theta^2)] \end{aligned} \tag{7.3.4}$$

The finite rotation vector follows from the appropriate reduction of (6.5.8) to give

$$\begin{aligned} \underset{\sim}{\Omega} &= (\epsilon^{\beta\alpha}\varphi_\alpha + \frac{1}{2}\varphi^\beta\varphi)\underset{\sim}{a}_\beta + \varphi\underset{\sim}{n} + O(\eta\theta) \\ &= \underset{\sim}{\phi} + \frac{1}{2}\varphi^\beta\varphi\underset{\sim}{a}_\beta + O(\eta\theta) \end{aligned} \tag{7.3.5}$$

It is seen now that within moderate rotations the vector $\underset{\sim}{\Omega}$ *differs* from $\underset{\sim}{\phi}$ by one term quadratic in displacement gradients.

It follows now from (2.2.4), (2.2.14), (2.2.12), (2.2.17) and (7.3.3) that the *strain-displacement relations* can be approximated by

$$\begin{aligned} \gamma_{\alpha\beta} = \theta_{\alpha\beta} &- \frac{1}{2}(\theta^\lambda_\alpha\epsilon_{\lambda\beta} + \epsilon_{\lambda\alpha}\theta^\lambda_\beta)\varphi + \frac{1}{2}a_{\alpha\beta}\varphi^2 + \\ &+ \frac{1}{2}\varphi_\alpha\varphi_\beta + O(\eta\theta^2) \end{aligned} \tag{7.3.6}$$

$$\begin{aligned} \kappa_{\alpha\beta} = -\frac{1}{2}\{&\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha}) - \\ &- b_{\alpha\beta}\varphi^\lambda\varphi_\lambda - \frac{1}{2}\varphi^\lambda\varphi_\lambda(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) - \\ &- \varphi^\lambda[(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})|_\beta + (\theta_{\lambda\beta} - \omega_{\lambda\beta})|_\alpha - b_{\lambda\alpha}\varphi_\beta - b_{\lambda\beta}\varphi_\alpha] + \\ &+ \epsilon^{\mu\lambda}\varphi_\mu\varphi(\omega_{\lambda\alpha|\beta} + \omega_{\lambda\beta|\alpha})\} + O(\frac{\eta\theta^2}{\lambda}) \end{aligned} \tag{7.3.7}$$

The formula (7,3,7) may still be simplified. When we use (2.1.7), (1,3,16), (2.1.6) and (1,3.25) the term $\omega_{\lambda\alpha|\beta}$ can be transformed as follows

$$\omega_{\lambda\alpha|\beta} = \frac{1}{2}\,(u_{\alpha|\lambda\beta} - u_{\lambda|\alpha\beta}) =$$

$$= \frac{1}{2}\,(u_{\alpha|\beta\lambda} + R^{\kappa}_{\cdot\alpha\lambda\beta}u_{\kappa} - u_{\lambda|\beta\alpha} - R^{\kappa}_{\cdot\lambda\alpha\beta}u_{\kappa}) =$$

$$= \frac{1}{2}\,[(u_{\alpha|\beta} + u_{\beta|\alpha})|_{\lambda} - (u_{\lambda|\beta} + u_{\beta|\lambda})|_{\alpha} + (R^{\kappa}_{\cdot\beta\lambda\alpha} + R^{\kappa}_{\cdot\alpha\lambda\beta} - R^{\kappa}_{\cdot\lambda\alpha\beta})u_{\kappa}] =$$

$$= (\theta_{\alpha\beta} + b_{\alpha\beta}w)|_{\lambda} - (\theta_{\lambda\beta} + b_{\lambda\beta}w)|_{\alpha} + \frac{1}{2}\,(b^{\kappa}_{\lambda}b_{\beta\alpha} - b^{\kappa}_{\alpha}b_{\beta\lambda} +$$

$$+ b^{\kappa}_{\lambda}b_{\alpha\beta} - b^{\kappa}_{\beta}b_{\alpha\lambda} - b^{\kappa}_{\alpha}b_{\lambda\beta} + b^{\kappa}_{\beta}b_{\lambda\alpha})u_{\kappa} =$$

$$= \theta_{\alpha\beta|\lambda} - \theta_{\lambda\beta|\alpha} + b_{\alpha\beta}(w|_{\lambda} + b^{\kappa}_{\lambda}u_{\kappa}) - b_{\lambda\beta}(w|_{\alpha} + b^{\kappa}_{\alpha}u_{\kappa}) =$$

$$= \theta_{\alpha\beta|\lambda} - \theta_{\lambda\beta|\alpha} + b_{\alpha\beta}\varphi_{\lambda} - b_{\lambda\beta}\varphi_{\alpha} \qquad (7.3.8)$$

The result (7.3.8) allow us to omit also the last two terms in (7.3.7). Six terms enclosed within square brackets in (7.3.7) with the help of (7.3.8) can be transformed to

$$+ \frac{1}{2}\,\varphi^{\lambda}[\ldots] = \varphi^{\lambda}(\theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda} - b_{\alpha\beta}\varphi_{\lambda}) \qquad (7.3.9)$$

and the final expression for $\kappa_{\alpha\beta}$ takes the form

$$\kappa_{\alpha\beta} = -\frac{1}{2}\,[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] -$$

$$- \frac{1}{2}\,b_{\alpha\beta}\varphi^{\lambda}\varphi_{\lambda} + \underline{\frac{1}{4}\,\varphi^{\lambda}\varphi_{\lambda}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha})} + \qquad (7.3.10)$$

$$+ \varphi^{\lambda}(\theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda}) + O(\frac{\eta\theta^2}{\lambda})$$

This relation for $\kappa_{\alpha\beta}$ differs from that derived by KOITER [6] by the underlined term. According to our order-of-magnitude estimates this term is $O(\frac{\theta^3}{\lambda})$ and should be retained within an error $O(\frac{\eta\theta^2}{\lambda})$.

Thus within the consistently formulated geometrically non-linear shell theory, in which moderate rotations are allowed to occur, the strain tensor $\gamma_{\alpha\beta}$ is *quadratic* in displacements and their gradients, while the tensor of change of curvature $\kappa_{\alpha\beta}$ is a *third-order polynomial* in displacements and

their gradients.

In most of the engineering shell literature the formulae for the strain measures are approximated to within the greater error

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2} a_{\alpha\beta}\varphi^2 + \frac{1}{2}\varphi_\alpha\varphi_\beta + O(\eta\theta) \tag{7.3.11}$$

$$\kappa_{\alpha\beta} = -\frac{1}{2}[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] + O(\frac{\eta\theta}{\lambda}) \tag{7.3.12}$$

The linear approximation (7.3.12) for $\kappa_{\alpha\beta}$ can be justified keeping in mind the error already introduced into the approximate strain energy (6.3.16), since terms $O(\frac{\eta\theta}{\lambda})$ in $\kappa_{\alpha\beta}$ cannot affect the accuracy of the strain energy expression (6.3.16). The approximation (7.3.11) introduces into (6.3.16) an error $O(Eh\eta^2\theta)$ and occasionally may cause some small loss in accuracy of the solution.

When (7.3.11) and (7.3.12) is used in the Lagrangean virtual work principle (5.5.14) it can easily be transformed with the help of Stockes theorem as follows:

$$\begin{aligned}
IVW = \iint_M &\{N^{\alpha\beta}[\delta u_{\alpha|\beta} - b_{\alpha\beta}\delta w + a_{\alpha\beta}\varphi\tfrac{1}{2}\epsilon^{\lambda\mu}\delta u_{\mu|\lambda} + \varphi_\alpha(\delta w|_\beta + b^\lambda_\beta\delta u_\lambda)] + \\
&+ M^{\alpha\beta}[-\delta w|_{\alpha\beta} - 2b^\lambda_\alpha\delta u_{\lambda|\beta} - b^\lambda_{\alpha|\beta}\delta u_\lambda + b^\lambda_\alpha b_{\lambda\beta}\delta w]\}\, dA = \\
= \iint_M &[(N^{\alpha\beta}\delta u_\alpha)|_\beta - N^{\alpha\beta}|_\beta\delta u_\alpha - b_{\alpha\beta}N^{\alpha\beta}\delta w + (\tfrac{1}{2}\epsilon^{\beta\alpha}\varphi N^\lambda_\lambda\delta u_\alpha)|_\beta - \\
&- (\tfrac{1}{2}\epsilon^{\beta\alpha}\varphi N^\lambda_\lambda)|_\beta\delta u_\alpha + (\varphi_\alpha N^{\alpha\beta}\delta w)|_\beta - (\varphi_\alpha N^{\alpha\beta})|_\beta\delta w + (b^\alpha_\beta\varphi_\lambda N^{\lambda\beta})\delta u_\alpha - \\
&- (M^{\alpha\beta}\delta w|_\alpha)|_\beta + M^{\alpha\beta}|_\beta\delta w|_\alpha - 2(b^\alpha_\lambda M^{\lambda\beta}\delta u_\alpha)|_\beta + 2(b^\alpha_\lambda M^{\lambda\beta})|_\beta\delta u_\alpha - \\
&- (b^\alpha_\lambda M^{\lambda\beta})|_\beta\delta u_\alpha + b^\alpha_\lambda M^{\lambda\beta}|_\beta\delta u_\alpha + b^\lambda_\alpha b_{\lambda\beta}M^{\alpha\beta}\delta w]\, dA = \\
= -\iint_M &[(N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta} + \tfrac{1}{2}\epsilon^{\beta\alpha}\varphi N^\lambda_\lambda)|_\beta\delta u_\alpha - b^\alpha_\lambda(M^{\lambda\beta}|_\beta + \varphi_\beta N^{\beta\lambda})\delta u_\alpha + \\
&+ (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})|_\beta\delta w + b_{\alpha\beta}(N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta})\delta w]\, dA + \\
&+ \int_C [(N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta} + \tfrac{1}{2}\epsilon^{\beta\alpha}\varphi N^\lambda_\lambda)\delta u_\alpha + (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})\delta w + M^{\alpha\beta}\delta\varphi_\alpha]\nu_\beta\, ds
\end{aligned} \tag{7.3.13}$$

It follows from (4.3.3) and (5.2.14) that at the boundary C

$$\beta_\nu = (\underset{\sim}{\bar{n}} - \underset{\sim}{n}) \cdot \underset{\sim}{\bar{a}}_\alpha \nu^\alpha [1 + O(\eta)] = - \varphi_\alpha \nu^\alpha + O(\eta\theta)$$

$$\delta\beta_\nu = -\delta\varphi_\nu \tag{7.3.14}$$

$$\delta\varphi_t = (\delta w_{,\alpha} + b^\lambda_\alpha \delta u_\lambda) t^\alpha = \frac{d}{ds}(\delta w) + \sigma_t \delta u_t - \tau_t \delta u_\nu$$

and the last term in the line integral of (7.3.13) can be integrated by parts to obtain

$$\int_C (-M^{\alpha\beta}\delta\varphi_\alpha)\nu_\beta \, ds = \int_C (-M_{t\nu}\delta\varphi_t - M_{\nu\nu}\delta\varphi_\nu) \, ds =$$

$$= - \int_C (\tau_t M_{t\nu}\delta u_\nu - \sigma_t M_{t\nu}\delta u_t + \frac{dM_{t\nu}}{ds}\delta w + M_{\nu\nu}\delta\beta_\nu) ds + \tag{7.3.15}$$

$$+ \sum_{i=1}^{N} [M_{t\nu}(s_i + O) - M_{t\nu}(s_i - O)]\delta w(s_i)$$

According to the Lagrangean formulation (5.5.5) we can introduce the vector

$$\underset{\sim}{G}\underset{\sim}{N}^\beta = (N^{\alpha\beta} - b^\lambda_\lambda M^{\lambda\beta} + \underline{\frac{1}{2}\epsilon^{\beta\alpha}\varphi N^\lambda_\lambda})\underset{\sim}{a}_\alpha +$$

$$+ (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})\underset{\sim}{n} \tag{7.3.16}$$

in terms of which IVW becomes

$$IVW = -\iint_M (\underset{\sim}{G}\underset{\sim}{N}^\beta)|_\beta \, dA + \int_C \{[\underset{\sim}{G}\underset{\sim}{N}^\beta \nu_\beta + \frac{d}{ds}(M_{t\nu}\underset{\sim}{n})] \cdot \delta\underset{\sim}{u} + M_{\nu\nu}\delta\beta_\nu\} ds +$$

$$+ \sum_{i=1}^{N} [M_{t\nu}(s_i + O) - M_{t\nu}(s_i - O)]\delta w(s_i) \tag{7.3.17}$$

In the expression for EVW according to (5.5.5) we need to transform only the last line integral. Within the accuracy assumed already by taking only the linear terms in (7.3.12), we should take only the *linear terms* in (7.3.5) as well. Then after transformations we obtain

$$EVW = \iint_M \underset{\sim}{p} \cdot \delta\underset{\sim}{u}\, dA + \int_C \{[\underset{\sim}{F} + \frac{d}{ds}(K_t \underset{\sim}{n})] \cdot \delta\underset{\sim}{u} + K_\nu \delta\beta_\nu\}\, ds + \sum_{i=1}^{N} [K_t(s_i + 0) - K_t(s_i - 0)]\delta w(s_i) \tag{7.3.18}$$

It follows from the variational problem IVW = EVW that the *equilibrium equations* in vector form

$$\underset{\sim}{G}\underset{\sim}{N}^\beta|_\beta + \underset{\sim}{p} = \underset{\sim}{0} \tag{7.3.19}$$

or in component form

$$\begin{aligned} &(N^{\alpha\beta} - b_\lambda^\alpha M^{\lambda\beta} + \underline{\frac{1}{2}\epsilon^{\beta\alpha}\varphi N_\lambda^\lambda)|_\beta} - b_\beta^\alpha(M^{\lambda\beta}|_\lambda + \varphi_\lambda N^{\lambda\beta}) + p^\alpha = 0 \\ &(M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})|_\beta + b_{\alpha\beta}(N^{\alpha\beta} - b_\lambda^\alpha M^{\lambda\beta}) + p = 0 \end{aligned} \tag{7.3.20}$$

should be satisfied within the internal region of M, the *natural boundary conditions*

$$\begin{aligned} &\underset{\sim}{G}\underset{\sim}{N}^\beta \nu_\beta + \frac{d}{ds}(M_{t\nu}\underset{\sim}{n}) = \underset{\sim}{F} + \frac{d}{ds}(K_t\underset{\sim}{n}) \\ &M_{\nu\nu} = K_\nu \end{aligned} \tag{7.3.21}$$

should be satisfied at smooth parts of C, and the effective *concentrated force*

$$\{[M_{t\nu}(s_i + 0) - K_t(s_i + 0)] - [M_{t\nu}(S_i - 0) - K_t(s_i - 0)]\}\underset{\sim}{n}(s_i) \tag{7.3.22}$$

should be applied at each *corner* s_i of the boundary C.

In exactly the same way we could obtain all shell relations using the strain-displacement formula (7.3.6) in place of that (7.3.11). The resulting equilibrium equations and boundary conditions become then only a little more complicated but we shall not discuss them here.

A different type of simplification may be achieved if the *rotations around the normal* are supposed to be *small*, while the *rotations around tangent* to the surface are still *moderate*. Thus when

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{n} = O(\theta^2) \quad , \quad \underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha = O(\theta) \tag{7.3.23}$$

the estimates of linearized quantities become

$$\varphi = O(\theta^2) \quad , \quad \varphi_\alpha = O(\theta) \quad , \quad \theta_{\alpha\beta} = O(\theta^2) \tag{7.3.24}$$

Under (7.3.24) the surface strain tensor (7.3.6) can be approximated by

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + O(\eta\theta^2) \tag{7.3.25}$$

within *the same error* as in (7.3.6), while within an error $O(\frac{\eta\theta^2}{\lambda})$ there is no reduction of (7.3.7). Again, if the error $O(\frac{\eta\theta}{\lambda})$ is justified in (7.3.7) on the basis of an approximate strain energy function (6.3.16), the formula for $\kappa_{\alpha\beta}$ can be taken as in (7.3.12). When (7.3.25) and (7.3.12) are used in the Lagrangean virtual work principle, we obtain the equilibrium equations and boundary conditions discussed in (7.3.19) to (7.3.22) and (7.3.16), only underlined terms containing φ in (7.3.20) and (7.3.16) should be omitted.

Let us note,that under (7.3.23) four last linear terms in (7.3.12) are also $O(\frac{\eta\theta}{\lambda})$. If we seek the accuracy within the *small and moderate rotation* range, the omission of these terms might affect the accuracy of the solution within the small rotation range, since in this case the relative error would be only $O(\theta)$. However, if we are interested *only* within the *moderate rotation* range, the omission of these terms introduces relative error $O(\theta^2)$ and is permissible. Then using (7.3.25) and

$$\kappa_{\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + O(\frac{\eta\theta}{\lambda}) \tag{7.3.26}$$

from the Lagrangean virtual work principle we obtain the reduced *equilibrium equations*

$$\begin{aligned} &N^{\alpha\beta}{}_{|\beta} - b^\alpha_\beta(M^{\lambda\beta}{}_{|\lambda} + \varphi_\lambda N^{\lambda\beta}) + p^\alpha = 0 \\ &(M^{\alpha\beta}{}_{|\alpha} + \varphi_\alpha N^{\alpha\beta})_{|\beta} + b_{\alpha\beta}N^{\alpha\beta} + p = 0 \end{aligned} \tag{7.3.27}$$

and *natural boundary conditions* (7.3.21) where

$$\underset{\sim}{G}\underset{\sim}{N}^\beta = N^{\alpha\beta}\underset{\sim}{a}_\alpha + (M^{\alpha\beta}{}_{|\alpha} + \varphi_\alpha N^{\alpha\beta})\underset{\sim}{n} \tag{7.3.28}$$

The equations (7.3.27) are simple enough to be used in most engineering applications.

The *classical non-linear theory of shallow shells* represents the simplest possible version of the theory for moderate rotations. In this case usually the following *simultaneous* restrictions are supposed to be satisfied

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{a}_{\alpha} = O(\theta) \quad , \quad \underset{\sim}{\Omega} \cdot \underset{\sim}{n} = O(\theta^2) \tag{7.3.29}$$

$$|K| \leq O(\frac{\theta^2}{\lambda^2}) \tag{7.3.30}$$

$$u_{\alpha} \leq O(\theta^2) \tag{7.3.31}$$

The restrictions (7.3.29) allow only for *moderate out-of-surface rotations*. The condition (7.3.30) expresses the *shallowness* of the shell with respect to deformation patterns and allows to interchange the sequence of covariant differentiation. The conditions (7.3.31) allow for only *small tangent displacements*. Under the restrictions (7.3.29) to (7.3.31) the *strain-displacement relations* become extremely simple

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w + \frac{1}{2}\,w_{,\alpha}w_{,\beta} + O(\eta\theta^2) \tag{7.3.32}$$

$$\kappa_{\alpha\beta} = -w|_{\alpha\beta} + O(\frac{\eta\theta}{\lambda}) \tag{7.3.33}$$

and the *equilibrium equations* take the form

$$N_{\alpha}^{\beta}|_{\beta} + \underline{p_{\alpha}} = 0 \tag{7.3.34}$$

$$M_{\alpha}^{\beta}|_{\beta}^{\alpha} + (b_{\beta}^{\alpha} + w|_{\beta}^{\alpha})N_{\alpha}^{\beta} + p - \underline{w_{,\alpha}p^{\alpha}} = 0 \tag{7.3.35}$$

Under the restriction (7.3.30) the relation (7.3.33) already satisfies the compatibility conditions (6.6.5). The general solution of (7.3.34) may be given in terms of an Airy stress function

$$N_{\alpha}^{\beta} = \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}F|_{\mu}^{\lambda} + \underline{P_{\alpha}^{\beta}} \tag{7.3.36}$$

where here

$$P_{\alpha}^{\beta}|_{\beta} + p_{\alpha} = 0 \tag{7.3.37}$$

In the remaining equilibrium equation (7.3.35) we use (6.4.1), (7.3.33) and (7.3.36), and in the remaining compatibility condition (6.6.6) we use (6.4.4), (7.3.33), (7.3.34) and (7.3.36). As a result we obtain two *solving equations* of the geometrically non-linear theory of *shallow shells*

$$
\begin{aligned}
&Dw|^{\alpha\beta}_{\alpha\beta} - \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}(b^{\alpha}_{\beta} + w|^{\alpha}_{\beta})F|^{\lambda}_{\mu} - \underline{(b^{\alpha}_{\beta} + w|^{\alpha}_{\beta})P^{\beta}_{\alpha}} - p + \underline{w_{,\alpha}p^{\alpha}} = 0 \\
&AF|^{\alpha\beta}_{\alpha\beta} + \epsilon_{\alpha\lambda}\epsilon^{\beta\mu}(b^{\alpha}_{\beta} + \tfrac{1}{2}w|^{\alpha}_{\beta})w|^{\lambda}_{\mu} + \underline{\Lambda[P^{\alpha}_{\alpha}|^{\beta}_{\beta} - (1+\nu)P^{\alpha\ \beta}_{\beta\ \alpha}]} = 0
\end{aligned}
\tag{7.3.38}
$$

For many shell problems the small tangent displacements occur only under *small tangent surface load* such that

$$
p_{\alpha} \leq O(Eh\,\frac{\eta\theta^2}{\lambda}) \tag{7.3.39}
$$

This allow us to omit also the underlined terms in (7.3.34) to (7.3.38), [6]. Such classical equations of the non-linear theory of shallow shells have served many years as the basis in numerical solutions of various engineering shell problems.

7.4. Strain measures in the large rotation shell theory

If all the rotations are allowed to be *large*, then

$$
|\underset{\sim}{\Omega}| = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{a}_{\alpha} = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\sqrt{\theta}) \tag{7.4.1}
$$

From (6.5.8) and (2.2.4) we obtain the estimates for linearized quantities

$$
\varphi_{\alpha} = O(\sqrt{\theta}) \quad , \quad \varphi = O(\sqrt{\theta}) \tag{7.4.2}
$$

$$
\theta_{\alpha\beta} = -\frac{1}{2}(a_{\alpha\beta}\varphi^2 + \varphi_{\alpha}\varphi_{\beta}) + O(\eta) = O(\theta) \tag{7.4.3}
$$

Within an error $O(\eta\theta^2)$ the surface strain tensor $\gamma_{\alpha\beta}$ given by (2.2.4) *cannot be simplified.*

Introducing (6.5.6) into (6.5.4) under (7.4.2) we obtain

$$
n = 1 - \frac{1}{2}\varphi^{\lambda}\varphi_{\lambda} + \frac{1}{2}(\theta^{\lambda}_{\lambda})^2 - \theta^{\lambda}_{\mu}\theta^{\mu}_{\lambda} - \gamma^{\kappa}_{\kappa}(\theta^{\lambda}_{\lambda} + \varphi^2) + O(\eta\theta^2) \tag{7.4.4}
$$

$$
n^{\lambda} = -\varphi^{\lambda}(1 + \theta^{\kappa}_{\kappa}) + \varphi_{\mu}(\theta^{\mu\lambda} - \omega^{\mu\lambda}) + O(\eta\sqrt{\theta}) \tag{7.4.5}
$$

For the basic vectors we obtain

$$
\begin{aligned}
&\bar{\underset{\sim}{a}}_{\alpha} = \underset{\sim}{a}_{\alpha} + (\theta_{\lambda\alpha} - \omega_{\lambda\alpha})\underset{\sim}{a}^{\lambda} + \varphi_{\alpha}\underset{\sim}{n} \\
&\bar{\underset{\sim}{n}} = \{[-\varphi^{\lambda}(1 + \theta^{\kappa}_{\kappa}) + \varphi_{\mu}(\theta^{\mu\lambda} - \omega^{\mu\lambda})]\underset{\sim}{a}_{\lambda} + (1 - \tfrac{1}{2}\varphi^{\lambda}\varphi_{\lambda})\underset{\sim}{n}\}[1 + O(\theta^2)]
\end{aligned}
\tag{7.4.6}
$$

while the finite rotation vector remains as in (6.5.8).

When (2.2.17), (2.2.18), (7.4.4) and (7.4.5) are used in (2.2.14), the tensor of change of curvature can be reduced with an error $O(\frac{\eta\theta}{\lambda})$ to the relation

$$\kappa_{\alpha\beta} = -\frac{1}{2}[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b_\alpha^\lambda(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b_\beta^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] -$$

$$- \frac{1}{2} b_{\alpha\beta}\varphi^\lambda\varphi_\lambda + \frac{1}{4}\varphi^\lambda\varphi_\lambda(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} - b_\alpha^\lambda\omega_{\lambda\beta} - b_\beta^\lambda\omega_{\lambda\alpha}) + \qquad (7.4.7)$$

$$(\varphi^\lambda + \varphi_\mu\omega^{\mu\lambda} + \varphi^\lambda\theta_\kappa^\kappa - \varphi_\mu\theta^{\mu\lambda})(\theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda}) + O(\frac{\eta\theta}{\lambda})$$

The formula may still be simplified by using the estimate (7.4.3) to transform two terms in the last row of (7.4.7) as follows

$$\varphi^\lambda\theta_\kappa^\kappa - \varphi_\mu\theta^{\mu\lambda} = -\frac{1}{2}\varphi^\lambda\varphi^2 + O(\eta) \qquad (7.4.8)$$

If some small loss in accuracy of strain energy function is taken into account, we may admit even greater simplification

$$\kappa_{\alpha\beta} = -\frac{1}{2}[(1 - \frac{1}{2}\varphi^\lambda\varphi_\lambda)(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + b_\alpha^\lambda(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b_\beta^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] -$$

$$- \frac{1}{2} b_{\alpha\beta}\varphi^\lambda\varphi_\lambda + (\varphi^\lambda + \varphi_\mu\omega^{\mu\lambda})(\theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda}) + O(\frac{\eta\sqrt{\theta}}{\lambda}) \qquad (7.4.9)$$

Both expressions (7.4.7) and (7.4.9) for $\kappa_{\alpha\beta}$ contain displacements and their gradients up to *third* power.

In many engineering shell structures *rotations around normal* may be allowed to be *small* or *moderate* at most. Thus if

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega} \cdot \underset{\sim}{n} = O(\theta) \qquad (7.4.10)$$

then

$$\varphi = O(\theta) \quad , \quad \varphi_\alpha = O(\sqrt{\theta})$$

$$\theta_{\alpha\beta} = -\frac{1}{2}\varphi_\alpha\varphi_\beta + O(\eta) = O(\theta) \qquad (7.4.11)$$

The formula (2.2.4) for the strain tensor $\gamma_{\alpha\beta}$ is still not affected by (7.4.11), while the tensor of change of curvature reduces under (7.4.11) to

$$\kappa_{\alpha\beta} = -\frac{1}{2}\left[\left(1-\frac{1}{2}\varphi^{\lambda}\varphi_{\lambda}\right)\left(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha}\right)+b_{\alpha}^{\lambda}\left(\theta_{\lambda\beta}-\omega_{\lambda\beta}\right)+b_{\beta}^{\lambda}\left(\theta_{\lambda\alpha}-\omega_{\lambda\alpha}\right)\right]-$$
$$-\frac{1}{2}b_{\alpha\beta}\varphi^{\lambda}\varphi_{\lambda}+\left(\varphi^{\lambda}+\varphi_{\mu}\omega^{\mu\lambda}\right)\left(\theta_{\lambda\alpha|\beta}+\theta_{\lambda\beta|\alpha}-\theta_{\alpha\beta|\lambda}\right)+O\left(\frac{\eta\delta}{\lambda}\right) \tag{7.4.12}$$

Note that (7.4.12) and (7.4.9) have exactly the same forms, although the accuracies of both formulae are different.

Again, if some small loss in accuracy is taken into bargain, (7.4.12) may be reduced to

$$\kappa_{\alpha\beta} = -\frac{1}{2}\left[\left(1-\frac{1}{2}\varphi^{\lambda}\varphi_{\lambda}\right)\left(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha}\right)+b_{\alpha}^{\lambda}\left(\theta_{\lambda\beta}-\omega_{\lambda\beta}\right)+b_{\beta}^{\lambda}\left(\theta_{\lambda\alpha}-\omega_{\lambda\alpha}\right)\right]-$$
$$-\frac{1}{2}b_{\alpha\beta}\varphi^{\lambda}\varphi_{\lambda}+\varphi^{\lambda}\left(\theta_{\lambda\alpha|\beta}+\theta_{\lambda\beta|\alpha}-\theta_{\alpha\beta|\lambda}\right)+O\left(\frac{\eta\sqrt{\theta}}{\lambda}\right) \tag{7.4.13}$$

Exactly the same relation would follow from (7.4.9) when assumption about moderate rotations around normal were taken into account.

The exact formula (2.2.4) for $\gamma_{\alpha\beta}$ and any of the approximate relations (7.4.7), (7.4.9), (7.4.12) or (7.4.13) for $\kappa_{\alpha\beta}$ may be used in the Lagrangean virtual work principle (5.5.14) to obtain equilibrium equations and natural boundary conditions compatible with the assumption of large rotations.

7.5. Some relations of the non-linear theory of plates

The geometrically non-linear *theory of plates* may be considered as a particular case of the shell theory with specific reference (undeformed) configuration, in which the curvature tensor of the shell middle surface vanishes identically, $b_{\alpha\beta} = 0$. Under this condition many shell relations become much simpler. We found it worthwhile to sketch the reduction procedure and to discuss here at least some of the relations of the non-linear theory of thin elastic plates.

For the plate theory the *linearized quantities* discussed in § 2.1. reduce to

$$\begin{aligned} 1_{\alpha\beta} &= a_{\alpha\beta}+\varphi_{\alpha\beta} = a_{\alpha\beta}+\theta_{\alpha\beta}-\omega_{\alpha\beta} \\ \varphi_{\alpha\beta} &= u_{\alpha|\beta} \qquad \varphi_{\alpha} = w_{,\alpha} \\ \theta_{\alpha\beta} &= \frac{1}{2}\left(u_{\alpha|\beta}+u_{\beta|\alpha}\right) \\ \omega_{\alpha\beta} &= \frac{1}{2}\left(u_{\beta|\alpha}-u_{\alpha|\beta}\right) = \epsilon_{\alpha\beta}\varphi \end{aligned} \tag{7.5.1}$$

The exact formulae for the *plate strain measures* follow from (2.2.4) and (2.2.14) to be

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha} + u^{\lambda}{}_{|\alpha}u_{\lambda|\beta} + w_{,\alpha}w_{,\alpha}) \tag{7.5.2}$$

$$\kappa_{\alpha\beta} = -\,(nd_{\alpha\beta} + n_{\lambda}d^{\lambda}_{\cdot\alpha\beta}) \tag{7.5.3}$$

where

$$d_{\alpha\beta} = w|_{\alpha\beta} \quad , \quad d^{\lambda}_{\cdot\alpha\beta} = u_{\lambda|\alpha\beta} \tag{7.5.4}$$

$$n_{\lambda} = \sqrt{\frac{a}{\bar{a}}}\,(-w_{,\lambda} + w_{,\mu}u^{\mu}|_{\lambda} - w_{,\lambda}u^{\mu}|_{\mu}) \tag{7.5.5}$$

$$n = \sqrt{\frac{a}{\bar{a}}}\,(1 + u^{\lambda}|_{\lambda} + \frac{1}{2}\,u^{\lambda}|_{\lambda}u^{\mu}|_{\mu} - \frac{1}{2}\,u^{\lambda}|_{\mu}u^{\mu}|_{\lambda}) \tag{7.5.6}$$

Under *small strains* we obtain

$$n_{\lambda} = [-w_{,\lambda}(1 + u^{\mu}|_{\mu}) + w_{,\mu}u^{\mu}|_{\lambda}][1 + O(\eta)] \tag{7.5.7}$$

$$n = [1 + u^{\lambda}|_{\lambda} + \frac{1}{2}\,u^{\lambda}|_{\lambda}u^{\mu}|_{\mu} - \frac{1}{2}\,u^{\lambda}|_{\mu}u^{\mu}|_{\lambda}][1 + O(\eta)] \tag{7.5.8}$$

and $\kappa_{\alpha\beta}$ b ecomes *polynomial of the third order* in terms of displacement gradients

$$\begin{aligned}\kappa_{\alpha\beta} = \{&-w|_{\alpha\beta}[1 + u^{\lambda}|_{\lambda} + \frac{1}{2}\,(u^{\lambda}|_{\lambda})^{2} - \frac{1}{2}\,u^{\lambda}|_{\mu}u^{\mu}|_{\lambda}] + \\ &+ [w_{,\lambda}(1 + u^{\mu}|_{\mu}) - w_{,\mu}u^{\mu}|_{\lambda}]u^{\lambda}|_{\alpha\beta}\}[1 + O(\eta)]\end{aligned} \tag{7.5.9}$$

The finite rotation vector $\underset{\sim}{\Omega}$ for a plate can be obtained as follows

$$\begin{aligned}\underset{\sim}{\Omega} &= \frac{1}{2}\,(\underset{\sim}{a}_{\alpha} \times \bar{\underset{\sim}{a}}_{\beta}a^{\alpha\beta} + \underset{\sim}{n} \times \bar{\underset{\sim}{n}})[1 + O(\eta)] = \\ &= \frac{1}{2}\,\{\epsilon^{\beta\alpha}[(2 + u^{\lambda}|_{\lambda})w_{,\alpha} - w_{,\lambda}u^{\lambda}|_{\alpha}]\underset{\sim}{a}_{\beta} + 2\underset{\sim}{\varphi n}\}[1 + O(\eta)] = \\ &= \epsilon^{\beta\alpha}\{[w_{,\alpha} + \frac{1}{2}\,(w_{,\alpha}u^{\lambda}|_{\lambda} - w_{,\lambda}u^{\lambda}|_{\alpha})]\underset{\sim}{a}_{\beta} + \frac{1}{2}\,u_{\alpha|\beta}\underset{\sim}{n}\}[1 + O(\eta)]\end{aligned} \tag{7.5.10}$$

In plate theory the small parameter θ may also be defined by (6.3.4) and the parameter λ by (6.5.3), but R in this case is the *linear dimension* of the plate.

Restricting the magnitude of $\underset{\sim}{\Omega}$ given by (7.5.10) we may discuss various variants of plate equations under the small, moderate, large or finite rotations.

Under the *small rotations*

$$\bar{\underset{\sim}{a}}_\alpha = \underset{\sim}{a}_\alpha [1 + O(\theta^2)] \quad , \quad \bar{\underset{\sim}{n}} = \underset{\sim}{n}[1 + O(\theta^2)]$$
$$\underset{\sim}{\Omega} = \epsilon^{\beta\alpha}[w_{,\alpha}\underset{\sim}{a}_\beta + \frac{1}{2} u_{\alpha|\beta}\underset{\sim}{n}][1 + O(\theta^2)] \tag{7.5.11}$$

and for the strain measures we have

$$\gamma_{\alpha\beta} = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) + O(\eta\theta^2) \tag{7.5.12}$$

$$\kappa_{\alpha\beta} = - w|_{\alpha\beta} + O(\eta\theta^2) \tag{7.5.13}$$

The equilibrium equations (7.2.8) reduce here to a set of two uncoupled linear equations

$$N^{\alpha\beta}|_\beta + p^\alpha = 0 \tag{7.5.14}$$

$$M^{\alpha\beta}|_{\alpha\beta} + p = 0 \tag{7.5.15}$$

The equation (7.5.14) of in-plane equilibrium may be solved with the help of an Airy stress function (7.3.36) which, introduced into the compatibility condition, gives us the well-known biharmonic equation of the *plane elasticity*

$$F|^{\alpha\beta}_{\alpha\beta} + [P^{\alpha|\beta}_{\alpha|\beta} - (1 + \nu)P^{\alpha|\beta}_{\beta|\alpha}] = 0 \tag{7.5.16}$$

The equation (7.5.15) of out-of-plane equilibrium is solved directly by using the constitutive equations (6.4.1) and the relation (7.5.13) to obtain

$$Dw|^{\alpha\beta}_{\alpha\beta} - p = 0 \tag{7.5.17}$$

which is the well known biharmonic equation of the *classical linear theory of plates*.

Under the *moderate rotations* the estimates (7.3.2) are still valid, and from (7.3.3) and (7.3.4) we obtain

$$n_\lambda = - w_{,\lambda} + \frac{1}{2} w_{,\mu}(u^\mu|_\lambda - u_\lambda|^\mu) + O(\eta\theta)$$
$$n = 1 + O(\eta) \tag{7.5.18}$$

$$\bar{\underset{\sim}{a}}_\alpha = [\underset{\sim}{a}_\alpha + \frac{1}{2}(u^\lambda|_\alpha - u_\alpha|^\lambda)\underset{\sim}{a}_\lambda + w_{,\alpha}\underset{\sim}{n}][1 + O(\eta)] \tag{7.5.19}$$

$$\bar{\underset{\sim}{n}} = (-w_{,\lambda}\underset{\sim}{a}^\lambda + \underset{\sim}{n})[1 + (\eta)]$$

The finite rotation vector is reduced to

$$\underset{\sim}{\Omega} = [(\epsilon^{\beta\alpha}w_{,\alpha} + \frac{1}{4} w|^\beta \epsilon^{\lambda\mu} u_{\mu|\lambda})\underset{\sim}{a}_\beta + \frac{1}{2}\epsilon^{\beta\alpha}u_{\alpha|\beta}\underset{\sim}{n}][1 + O(\eta)] \tag{7.5.20}$$

It follows from (7.3.6) that in terms of displacement gradients the strain tensor $\gamma_{\alpha\beta}$ takes the form

$$\gamma_{\alpha\beta} = \frac{1}{2}[u_{\alpha|\beta} + u_{\beta|\alpha} + \frac{1}{4} a^{\lambda\mu}(u_{\lambda|\alpha} - u_{\alpha|\lambda})(u_{\mu|\beta} - u_{\beta|\mu}) + w_{,\alpha}w_{,\beta}] -$$
$$- \frac{1}{8} a^{\lambda\mu}[(u_{\lambda|\alpha} + u_{\alpha|\lambda})(u_{\mu|\beta} - u_{\beta|\mu}) + (u_{\lambda|\alpha} - u_{\alpha|\lambda})(u_{\mu|\beta} + u_{\beta|\mu})] + O(\eta\theta^2) \tag{7.5.21}$$

or with a greater error compatible with (7.3.11)

$$\gamma_{\alpha\beta} = \frac{1}{2}[u_{\alpha|\beta} + u_{\beta|\alpha} + \frac{1}{4} a^{\lambda\mu}(u_{\lambda|\alpha} - u_{\alpha|\lambda})(u_{\mu|\beta} - u_{\beta|\mu}) + w_{,\alpha}w_{,\beta}] + O(\eta\theta) \tag{7.5.22}$$

The appropriate relation for the tensor of change of curvature follows from (7.3.12) and (7.5.1) to be

$$\kappa_{\alpha\beta} = -w|_{\alpha\beta} + O(\frac{\eta\theta}{\lambda}) \tag{7.5.23}$$

Introducing (7.5.22) and (7.5.23) into the Lagrangean virtual work principle (5.5.14) we obtain the *equilibrium equations*

$$[N^{\alpha\beta} + \frac{1}{4}(u^\alpha|^\beta + u^\beta|^\alpha)N^\lambda_\lambda]|_\beta + p^\alpha = 0$$
$$(M^{\alpha\beta}|_\alpha + w_{,\alpha}N^{\alpha\beta})|_\beta + p = 0 \tag{7.5.24}$$

to be satisfied within the internal region of M, the *natural boundary conditions*

$$\underset{\sim}{G}\underset{\sim}{N}^\beta\nu_\beta + \underset{\sim}{n}\frac{d}{ds}M_{t\nu} = \underset{\sim}{F} + \underset{\sim}{n}\frac{d}{ds}K_t$$
$$M_{\nu\nu} = K_\nu \tag{7.5.25}$$

where

$$\underset{\sim}{G}\underset{\sim}{N}^{\beta}=[N^{\alpha\beta}+\frac{1}{4}\,(u^{\alpha}|_{\beta}-u^{\beta}|_{\alpha})N^{\lambda}_{\lambda}]\underset{\sim}{a}_{\alpha}+(M^{\alpha\beta}|_{\alpha}+w_{,\alpha}N^{\alpha\beta})\underset{\sim}{n} \tag{7.5.26}$$

to be satisfied at smooth parts of the plate boundary C, and the effective *concentrated force*

$$\{[M_{t\nu}(s_i+0)-K_t(s_i+0)]-[M_{t\nu}(s_i-0)-K_t(s_i-0)]\}\underset{\sim}{n} \tag{7.5.24}$$

to be applied at each corner s_i of the boundary C.

The equations (7.5.24) can be solved only in terms of *displacements*. The appropriate solving equations can easily be obtained by introducing the constitutive equations (6.4.1) and the strain displacement relations (7.5.22) and (7.5.23) into (7.5.24) to (7.5.27).

When the *rotations around normal* are supposed to be *small* while the *rotations around tangent* are still *moderate*, the estimates (7.3.24) allow to reduce further the strain tensor $\gamma_{\alpha\beta}$ to the form

$$\gamma_{\alpha\beta}=\frac{1}{2}\,(u_{\alpha|\beta}+u_{\beta|\alpha}+w_{,\alpha}w_{,\beta})+O(\eta\theta) \tag{7.5.28}$$

while $\kappa_{\alpha\beta}$ remains as in (7.5.23). In this case the appropriate *equilibrium equations* are

$$\begin{aligned} &N^{\alpha\beta}|_{\beta}+p^{\alpha}=0\\ &(M^{\alpha\beta}|_{\alpha}+w_{,\alpha}N^{\alpha\beta})|_{\beta}+p=0 \end{aligned} \tag{7.5.29}$$

and in the *natural boundary conditions* (7.5.25) we should use

$$\underset{\sim}{G}\underset{\sim}{N}^{\beta}=N^{\alpha\beta}\underset{\sim}{a}_{\alpha}+(M^{\alpha\beta}|_{\alpha}+w_{,\alpha}N^{\alpha\beta})\underset{\sim}{n} \tag{7.5.30}$$

The first of (7.5.29) can be satisfied with the help of an Airy stress function (7.3.36), and the compatibility conditions (6.6.5) are already satisfied by (7.5.23) because of the Euclidean geometry of the reference plate configuration. The remaining compatibility condition (6.7.4) and the equilibrium equation $(7.5.29)_2$ lead to the following set of two equations

$$\begin{aligned} &AF|^{\alpha\beta}_{\alpha\beta}+\frac{1}{2}\,\epsilon_{\alpha\lambda}\epsilon^{\beta\mu}w|^{\alpha}_{\beta}w|^{\lambda}_{\mu}+A[P^{\alpha}_{\alpha}|^{\beta}_{\beta}-(1+\nu)P^{\alpha}_{\beta}|^{\beta}_{\alpha}]=0\\ &Dw|^{\alpha\beta}_{\alpha\beta}-\epsilon_{\alpha\lambda}\epsilon^{\beta\mu}w|^{\alpha}_{\beta}F|^{\lambda}_{\mu}-w|^{\alpha}_{\beta}P^{\beta}_{\alpha}-p+w_{,\alpha}p^{\alpha}=0 \end{aligned} \tag{7.5.31}$$

which have been given without p^{α} forces by KARMÁN [45]. These classical equations of the non-linear theory of plates have served many years as a basis for numerical solutions of engineering plate problems.

Consider next the *large rotations*. It is resonable to assume here that only *out-of-plate rotations* are allowed to be *large*, while *in-plate rotations* are at most *moderate*. For this case we have estimates (7.4.11), under which the exact expression (7.5.2) for $\gamma_{\alpha\beta}$ *cannot be reduced*. Under (7.4.11) we obtain

$$n_{\mu} = -w|_{\mu}(1 + u^{\kappa}|_{\kappa}) + w|_{\lambda}u^{\lambda}|_{\mu} + O(\eta\sqrt{\theta}) \tag{7.5.32}$$

$$n = 1 + u^{\kappa}|_{\kappa} + \frac{1}{2}(u^{\lambda}|_{\lambda})^2 - \frac{1}{8}(u^{\lambda}|_{\mu} + u_{\mu}|^{\lambda})(u^{\mu}|_{\lambda} + u^{\lambda}|_{\mu}) + \\ + \frac{1}{4}(u^{\lambda}|_{\mu} - u_{\mu}|^{\lambda})(u_{\lambda}|^{\mu} - u^{\mu}|_{\lambda}) + O(\eta) \tag{7.5.33}$$

$$\theta_{\alpha\beta} = -\frac{1}{2}w|_{\alpha}w|_{\beta} + O(\eta) = O(\theta) \tag{7.5.34}$$

It follows from (7.4.7) that

$$\kappa_{\alpha\beta} = -(1 + u^{\kappa}|_{\kappa})w|_{\alpha\beta} + (1 + u^{\kappa}|_{\kappa})w|_{\mu}u^{\mu}|_{\alpha\beta} - \\ - w|_{\mu}u^{\mu}|_{\lambda}u^{\lambda}|_{\alpha\beta} + O(\frac{\eta\theta}{\lambda}) \tag{7.5.35}$$

But from (7.5.34) we can obtain the estimates

$$\theta^{\kappa}_{\kappa} \equiv u^{\kappa}|_{\kappa} = -\frac{1}{2}w|^{\kappa}w|_{\kappa} + O(\theta) \tag{7.5.36}$$

$$d_{\lambda\alpha\beta} \equiv \theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda} \\ = -w|_{\lambda}w|_{\alpha\beta} + O(\frac{\eta}{\lambda}) \tag{7.5.37}$$

with help of which the formula (7.5.35) may be put in an *alternative* form

$$\kappa_{\alpha\beta} = -w|_{\alpha\beta}[1 + \frac{1}{2}w|^{\kappa}w|_{\kappa} + \frac{1}{2}(w|^{\kappa}w|_{\kappa})^2 - \\ - w|_{\kappa}u^{\kappa}|_{\lambda}w|^{\lambda}] + O(\frac{\eta\theta}{\lambda}) \tag{7.5.38}$$

The formulae (7.5.35) and (7.5.38) for plates have their counterpart in shell theory in formula (7.4.7) from which they may be obtained by putting $b_{\alpha\beta} \equiv 0$.

If some small loss in accuracy is taken into bargain (7.5.35) and (7.5.38) may be replaced by

$$\begin{aligned} \kappa_{\alpha\beta} &= -(1 + u^{\kappa}|_{\kappa})w|_{\alpha\beta} - w|_{\mu}w|^{\mu}w|_{\alpha\beta} + O(\frac{\eta\sqrt{\theta}}{\lambda}) = \\ &= -(1 + \frac{1}{2}\, w|^{\kappa}w|_{\kappa})w|_{\alpha\beta} + O(\frac{\eta\sqrt{\theta}}{\lambda}) \end{aligned} \tag{7.5.39}$$

It is interesting to note, that within the error indicated in (7.5.39) $\kappa_{\alpha\beta}$ becomes polynomial of third order in the *normal displacement gradients* only.

The exact formula (7.5.2) for $\gamma_{\alpha\beta}$ together with any of derived relations (7.5.35), (7.5.38) or (7.5.39) for $\kappa_{\alpha\beta}$ may be used in the Lagrangean virtual work principle (5.5.14) to obtain equilibrium equations and natural boundary conditions, compatible with the assumption of large out-of-plane and moderate in-plane rotations of plate material elements.

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