RUHR-UNIVERSITÄT BOCHUM

Czeslaw Wozniak

Large Deformations of Elastic and Non-Elastic Plates, Shells and Rods

Heft Nr. 20

Mitteilungen aus dem Institut für Mechanik

Institut für Mechanik
RUHR-UNIVERSITÄT BOCHUM

Czesław Woźniak

Large Deformations of Elastic and Non-Elastic Plates, Shells and Rods

Mitteilungen aus dem Institut für Mechanik Nr. 20
März 1980

Editor:
Institut für Mechanik der Ruhr-Universität Bochum

These are the lecture notes prepared by the author during the Academic Year 1979/80 at the Ruhr-Universität Bochum. The lectures were delivered to a research staff of the Institut für Mechanik RUB. The author wish to express his sincere gratitude to all participants of the lectures for stimulating discussions. The special thanks are due to Prof. Dr.-Ing. H. Stumpf, Dr. sci. techn. F. K. Labisch and Dipl.-Ing. D. Weichert for many valuable remarks and to Frau Mönikes for her excellent work done in typewriting the notes.

ul. Miedzynarodowa 58 m. 63, 03-922 Warszawa, Poland.

CONTENTS

Page

PREFACE

The list of works on the non-linear mechanics of plates, shells and rods is extensive. However, the known monographs and treatises (for example, cf. [1,11,18,20-22,24-26,28,35,39]) are concerned mainly either with some special theories or with some kinds of materials and deformations. The aim of this treatise is to approach the non-linear mechanics of plates, shells and rods from more general point of view. Firstly, we provide a systematic treatment of the mechanics of large deformations of plates and shells made of an arbitrary material and of an aribrary thickness. We also extend the obtained results to describe the mechanics of rods. Secondly, we develop independently various approaches to the plate and shell theories and we derive the known formulations as the special cases. Thirdly, we discuss the wide class of the shell and rod problems which lead not to the equations but to the variational inequalities (non-classical or unilateral problems, [8]). To render the analysis more concise we shall confine ourselves only to the pure mechanical theories.

Within the general treatment presented in the work it is not possible to precise properly the regularity conditions for the fields under consideration in order to investigate successfully such topics as, for example, the existence, stability or convergence of solutions. Nevertheless, this treatment constitutes the basis for the analysis of more special problems and gives the new insight into the known shell and rod theories and the related problems.

The main topics of this contribution can be listed as follows:

1. The methods of formation of the general theories for plates and shells (Chapter A).
2. The derivation, modification and simplification of the special large deformation theories for unelastic shells (Chapter B).
3. The general formulation of the non-classical plate and shell boundary-value problems (Chapter C).

All results of the Chapter A and some from those of the Chapter C have been adopted to obtain the formulation of the general theories and non-classical boundary value problems for the large deformations of unelastic rods. Thus the analysis of rods is not treated separately but is a consequence of the results which have been obtained for plates and shells. This work deals mainly with plates and shells and only few sections are devoted to describe the mechanics of rods (Sec. 5.3. in the Chapter A, Sec. 3.5. in the Chapter B and Sec. 2.4. in the Chapter C).

The contents of the treatise is so arranged that, apart from the Prerequisites, the Sections 1,2,4 of the Chapter A can be read independently of each other. Sections 2,4,5 of the Chapter A are the keys to the Chapter B. Also Sections 1-3,7,9 of the Chapter B are independent each other as well as Sections 4,5,6 which have to be read after Section 3 of this Chapter. In the Chapter C the Sections 1,2 are independent. The reader is assumed to be familiar with the basic concepts of the contemporary non-linear continuum mechanics which can be found, for example, in [23,33,34,39].

Most from the presented results have not appeared in the literature. The list of references is restricted mainly to the works which are strictly related to the subject of this treatment. No attempt has been made to detail or even outline the history and the recent development on the plate, shell and rod theories(for the particulars the reader is referred to [26,1] and to the related papers). The fundamental concepts and the basic denotations used throughout the work are given below.

Bochum, March, 1980

Cz. Woźniak

PREREQUISITES

Shells and shell theories. Plates. Let E be the Euclidean 3-space referred to a fixed rectangular Carthesian coordinate system $Ox^1x^2x^3$ (the reference space). Throughout the lectures using the term "shell" we shall mean the parametrized shell, i.e., the pair (B, κ_R), where B is the body,[33], and κ_R is the smooth invertible mapping from B to the reference space E, $\kappa_R : B \to E$, such that $\kappa_R(B) = \Pi \times (h_-,h_+)$, where Π is the regular region on the plane Ox^1x^2 and (h_-,h_+) is the interval of the x^3-axis. The element B of the pair (B, κ_R) will be called the *shell-like body* and κ_R will be referred to as the *shell reference configuration*. It means that we confine ourselves only to shell reference configurations in which the shell-like body B has the from of the undeformed plate with the constant thickness $h_+ - h_-$. Such reference configurations may be never attained in the motion of the shell but their application in the non-linear shell treatment make it possible to obtain the relatively simple form of the basic relations (1). Every "shell structure" which is met in engineering can be represented by the finite system of parametrized shells.

Let $\underset{\sim}{\theta} \equiv (\theta^1,\theta^2)$ be an arbitrary point in $\bar{\Pi}$ and ξ an arbitrary number of the closed interval $\langle h_-,h_+\rangle$. The triples $\underset{\sim}{X} \equiv (\theta^1,\theta^2,\xi)$ will be used as the material coordinates of the shell-like body with its boundary. By the *shell theory* we shall mean the system of *equations of motion*, *kinetic boundary conditions* (laws of mechanics) and *constitutive relations* which describes the shell like body and in which all fields are *independent of the material coordinate* ξ.

The geometric meaning of a term "parametrized shell" which was introduced above is quite formal and includes the bodies which, roughly speaking, are neither "thin" nor have anything in common with the familiar in engineering concept of a shell. The applicability of the shell theories to the description of such bodies within continuum mechanics may be often impaired. On the other hand, the necessity of such general approach will be evident if we deal with arbitrary large deformations.

---

(1) The restriction imposed here on the choice of the shell reference configuration is not used in most of works on the shell theories.

In this case we have to take also into account configurations of the shell-like body which may not give any resemblance to the intuitive concept of a shell. We also do not make any suggestion that the shell is "thin", because under large deformations the shell-like body which is "thin" in one configuration may be not "thin" in another. To precise the terminology we shall *define the shell* within mechanics as the *shell-like body $\mathcal{B}$ governed by a certain shell theory.* Every such theory is related to the parametrized shell $(\mathcal{B}, \kappa_R)$.

If there exists the special shell configuration which can be interpreted as "undeformed" and if this configuration coincides with a certain shell reference configuration κ_R, then the shell will be called the *plate.* In such situation instead of the shell theory we can talk about the plate theory. Below, both terms will be used parallelly.

Rods and rod theories. Within the large deformation theory the geometric meaning of a term "rod" can be assumed in the identical form as that of a shell. It means that using throughout the work the term "rod" we shall understand the *parametrized rod,* i.e., the pair $(\mathcal{B},\kappa_R)$, where κ_R satisfies the conditions given above. We shall refer $\mathcal{B}$ to as the *rod-like body* and κ_R as the *rod reference configuration.* The triples $\underset{\sim}{X} \equiv (\theta^1,\theta^2,\xi)$ introduced above will be used as the material coordinates of the rod-like body with its boundary. By the *rod theory* we shall mean the system of *equations of motion, kinetic boundary conditions* (laws of mechanics) and *constitutive relations* which describes the rod-like body and in which all fields are *independent of the material coordinates* θ^1,θ^2. All comments on a geometric meaning of a term "parametrized shell", which were given above, also concern the geometric meaning of a term "parametrized rod". We shall *define the rod* within mechanics as the *rod-like body $\mathcal{B}$ governed by a certain rod theory.* Every rod theory is related to the parametrized rod $(\mathcal{B},\kappa_R)$.

General scheme of notation. The denotations used throughout this treatise are mainly based on those used in [26,33]. The time coordinate is denoted by t, $t \in I$, I being the known suitably chosen time interval, and the four-tuples (x^1,x^2,x^3,t) will be interpreted as the inertial coordinates in the Galilean space-time. We denote by $\underset{\sim}{x} = (x^1,x^2,x^3)$ the points of the reference space E and by $\underset{\sim}{X} = (\theta^1,\theta^2,\xi)$ the shell or rod material coordinates (cf. above). We refer $\underset{\sim}{x} = \underset{\sim}{p}(\underset{\sim}{X},t)$,

$\underset{\sim}{X} \in \overline{\kappa_R(\mathcal{B})}$, $t \in I$, to as the position vector, at the time instant t, of this point of the shell or rod (or their boundaries) which in the reference configuration κ_R occupies the place $\underset{\sim}{X}$. The symbol $\underset{\sim}{p} : \kappa_R(\mathcal{B})\times I \to E$ stands for the deformation function of the shell-like (or rod-like) body. All other denotations will be explained in the text of the work.

The tensor notation is used throughout the notes but the summation convention holds with respect to all kinds of indices (not only tensorial indices). The sub- and superscripts $\alpha,\beta,\gamma,\ldots$ $(K,L,M,\ldots)$ run over the sequence 1,2 (over 1,2,3) and are related to the material coordinates θ^1,θ^2 (to the material coordinates $\theta^1,\theta^2,\theta^3 \equiv \xi$). The subscript "R" informs us that the quantity under consideration is the density related to the regions $\kappa_R(\mathcal{B})$, (h_-,h_+), Π or their boundaries. The partial derivatives of an arbitrary function $f(X,t) \equiv f(\theta^1,\theta^2,\xi,t)$ (if they exist) are denoted by $f_{,\alpha} \equiv \partial f/\partial\theta^\alpha$, $f_{,3} \equiv \partial f/\partial\xi$, $\dot{f} \equiv \partial f/\partial t$ and $\nabla f \equiv (f_{,1},f_{,2},f_{,3})$, $\nabla\nabla f \equiv (\nabla f_{,1},\nabla f_{,2},\nabla f_{,3})$. The ordered sets of fields or numbers are denoted by $a_{(n)} \equiv \{a_a ; a = 1,\ldots,n\}$, $b^{(N)} \equiv \{b^A, A = 1,\ldots,N\}$, etc., and the indices a,b,... run always over the sequence 1,...,n. The vectors and the vector functions in the vector 3-space are underlined: $\underset{\sim}{p} \equiv (p_1,p_2,p_3)$, $\underset{\sim}{b} \equiv (b^1,b^2,b^3)$, etc..

CHAPTER A

FORMATION OF PLATE, SHELL AND ROD THEORIES

In this Chapter we are to give general formulations of some basic approaches to the theories of unelastic and elastic plates and shells. We are also to outline the characteristic features of these approaches and to trace the interrelations between them. To simplify the analysis for the time being we relate all fields exclusively to the reference configuration of the plate or shell in the sense defined in Prerequisites. The results obtained for shells we utilize to derive the general form of the rod theories.

1. DIRECT APPROACH

The term "direct approach" informs us, that the plate or shell theory (or some from its relations) is postulated independently of the governing equations of solid mechanics, [26]. Special kinds of the direct approach, based mainly on the concept of the Cosserat surface, are well known in the recent literature and can be found in [6,14,26,34]. Here we outline the direct approach in the more general form. It is based on the system of axioms and rules of interpretation of primary concepts in the terms of the classical solid mechanics. Apart from Secs. 3,5 of this Chapter, the direct approach, as a rule, will be not utilized in next sections of our treatise.

1.1. Deformations

The kinematics of plates and shells in the direct approach is based on the following Axiom.

AXIOM 1. For every shell [1] there exists the non-empty set Q of the ordered sets $q_{(n)} = \{q_a(\underset{\sim}{\theta},t),\ \underset{\sim}{\theta} \in \bar{\Pi},\ t \in I,\ a = 1,\dots,n\},\ n \geq 1$, of sufficiently regular real-valued functions $q_a(\cdot)$.

---

[1] For the convenience we often use the term "shell" instead of "plate and shell"; the shell has always to be understood in the sense given in the Prerequisites.

<u>Definition</u>. Every ordered set $q_{(n)}$, $q_{(n)} \in Q$, will be called the shell deformation function.

The term "shell deformation function" will be also used in other approaches to the shell theories and has not to be confused with the deformation function of the shell, which is defined on $\overline{\kappa_R(\mathcal{B})} \times I$, cf. the Prerequisites.

<u>Interpretation</u>. We shall assume that the shell deformation function $q_{(n)}$ is uniquely determined by the deformation function $\underset{\sim}{x} = \underset{\sim}{p}(\underset{\sim}{X},t)$, $\underset{\sim}{X} = (\theta^1,\theta^2,\xi) \in \bar{\Pi} \times \langle h_-,h_+\rangle$, of the shell like body. It means that there are known functionals $\varphi_a(\cdot)$, $a = 1,\dots,n$, defined on the set of all deformation functions $\underset{\sim}{p}$ of the shell-like body [1],

$$q_a(\underset{\sim}{\theta},t) = \varphi_a(\underset{\sim}{p}(\underset{\sim}{\theta},\cdot,t)) \ , \ a = (,\dots,n) \ , \tag{A1.1}$$

where $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$. However, in the direct approach developed here the relation inverse to (A1.1.) is not known, i.e., the deformation $\underset{\sim}{p}$ of the shell do not need to be uniquely determined in term of $q_{(n)}$.

<u>Example: the Cosserat surface</u>. Putting $n = 6$, $q_a = \delta^i_a r_i(\underset{\sim}{\theta},t) + \delta^i_{a-3} d_i(\underset{\sim}{\theta},t)$, $i = 1,2,3$, and $Q := \{q_{(6)} | [\underset{\sim}{a}_1,\underset{\sim}{a}_2,\underset{\sim}{d}] > 0\}$, where $\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha}$ we arrive at the known concept of the Cosserat surface i.e., the smooth surface given by $\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{\theta},t)$, with the field of directors $\underset{\sim}{d}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, such that $[\underset{\sim}{a}_1,\underset{\sim}{a}_2,\underset{\sim}{d}] > 0$ for every time instant $t, t \in I$. Eqs. (1.1), for example can be assumed as

$$q_i = p_i(\underset{\sim}{\theta},0,t) \ , \ q_{i+3} = d_i = \frac{\partial p_i(\underset{\sim}{\theta},0,t)}{\partial \xi} \ . \tag{A1.2}$$

In some formulations of the direct approach based on the Cosserat surface instead of Eqs. (A1.2.) stronger condition $\underset{\sim}{p} = \underset{\sim}{r} + \xi\underset{\sim}{d}$ is postulated, which implies Eqs. (A1.2.). This condition seems to be too strong because it eliminates alternative interpretations of the fields $\underset{\sim}{d},\underset{\sim}{r}$, which, for example, can be given by

---

[1] In the general case we can introduce the functionals $\varphi_a(\underset{\sim}{\theta},t,\cdot)$ defined on the space of all deformation functions $\underset{\sim}{p}$ of the shell-like body, but in all applications we deal only with the cases given by Eq. (A1.1.).

$$q_i = p_i(\underset{\sim}{\theta},0,t), \quad q_{i+3} = d_i = \int_{h_-}^{h_+} \frac{\partial p_i}{\partial \xi}\, d\xi = p_i(\underset{\sim}{\theta},h_+,t) - p_i(\underset{\sim}{\theta},h_-,t),$$

where $i = 1,2,3$.

Superposed rigid body motions. The rules of interpretation (A1.1.) yield the transformation formulae for the functions q_a, $a = 1,\ldots,n$, under an arbitrary rigid body motion $\underset{\sim}{p} \to \bar{\underset{\sim}{p}} = \underset{\sim}{Q}(t)\underset{\sim}{p} + \underset{\sim}{c}(t)$, where $\underset{\sim}{Q}(t)$ is, for every $t \in I$, an arbitrary rotation tensor (or an arbitrary orthogonal tensor if the reflections of E are assumed to be admissible) and $\underset{\sim}{c}(t)$, $t \in I$, is an arbitrary vector. If $q_{(n)}$ constitutes the ordered set of all components of a certain system of scalars or vectors then $p \to \underset{\sim}{Q}(t)\underset{\sim}{p} + \underset{\sim}{c}(t)$ imply

$$q_a \to A_a^b q_b + a_a, \quad a = 1,\ldots,n, \qquad \text{(A1.3.)}$$

where A_a^b are the known functions of $\underset{\sim}{Q}(t)$ and a_a are the known functions of $\underset{\sim}{Q}(t)$, $\underset{\sim}{c}(t)$. The more general rules of transformation can be also considered (cf. for example, [40]).

Strain measures and compatibility conditions. Let $E_{(r)} \equiv \{E_\rho,\ \rho = 1,\ldots,r\}$ be the ordered set of the differential operators acting on $q_{(n)}(\cdot,t)$, such that $e_\rho = E_\rho(q_{(n)})$, $\rho = 1,\ldots,r$ are invariants under arbitrary rigid body motion of the reference space (under group of transformation (A1.3.) in the special case mentioned above). If e_ρ are independent then $e_{(r)}$ will be called the ordered set of the strain measures, provided that r is the smallest number of independent invariants. The conditions imposed on $e_{(r)}$, which ensure the existence of the solution $q_{(n)}$ of the system of equations $e_{(r)} = E_{(r)}(q_{(n)})$, such that $q_{(n)} \in Q$, are called the compatibility conditions. The compatibility conditions are used in so-called intrinsic formulations of the shell theories, cf. [5], for example.

1.2. Forces

We shall introduce the forces via the concepts of the shell rate of work and the shell kinetic energy functions, which will be treated as the primary concepts.

AXIOM 2: To every regular subregion Π_o of Π at an arbitrary time instant $t, t \in I$, and for every shell deformation function $q_{(n)}$ there are assigned the values:

$$R_e(\Pi_o,t) \equiv \oint_{\partial\Pi_o} t^a_R \dot{q}_a dl_R + \int_{\Pi_o} f^a_R \dot{q}_a da_R \quad , \qquad \text{(A1.4)}$$

where $t^a_R = t^a_R(\underset{\sim}{\theta}, t, \underset{\sim}{n}_R)$, $f^a_R = f^a_R(\underset{\sim}{\theta}, t)$ and $\underset{\sim}{n}_R = (n_{R1}, n_{R2})$ is the outward unit normal vector on $\partial\Pi_o$ in the plane Ox^1x^2, and

$$R_i(\Pi_o,t) \equiv \int_{\Pi_o} (H^{a\alpha}_R \dot{q}_{a,\alpha} - h^a_R \dot{q}_a) da_R \quad , \qquad \text{(A1.5)}$$

where $H^{a\alpha}_R = H^{a\alpha}_R(\underset{\sim}{\theta}, t)$, $h^a_R = h^a_R(\underset{\sim}{\theta}, t)$.

Definitions. $R_e(\Pi_o,t)$, $R_i(\Pi_o,t)$ will be called the rates of work of external and internal forces in Π_o. Putting $p^a_R \equiv t^a_R(\underset{\sim}{\theta}, t, n_R)$ for $\underset{\sim}{\theta} \in \partial\Pi$ and for $\underset{\sim}{n}_R$ as the outward unit normal vector on $\partial\Pi$, we shall refer p^a_R, f^a_R as the shell external forces (boundary tractions and shell body forces, respectively). H^a_R, h^a_R will be called the shell internal forces (stresses).

Interpretation. The interpretation of the shell external and internal forces is provided by Eqs. (A.1.4), (A.1.5) and (A.1.1) in terms of their rate of work.

Remark. We have tacitly assumed that the introduced shell forces do the work only on $\dot{q}_{(n)}$ and $\nabla\dot{q}_{(n)}$. Such system of forces will be called of the first order or simple shell force system (related to the deformation function $q_{(n)}$). The second order shell force system with respect to $q_{(n)}$ will be introduced in Sec. 1.5.

AXIOM 3. To every $\underset{\sim}{\theta} \in \Pi$ is assigned the non-negative real valued differentiable function κ_R defined for every $q_{(n)}$ and $\dot{q}_{(n)}$

$$\kappa_R = \kappa_R(\underset{\sim}{\theta}, q_{(n)}, \dot{q}_{(n)}) \quad , \qquad \text{(A1.6)}$$

being the density related to Π and such that $\kappa_R = 0$ only if $\dot{q}_{(n)} = 0$.

Definition. The function κ_R will be called the shell simple kinetic energy function and the functions $-i_R^a$, $a = 1,\dots,n$, where

$$i_R^a \equiv \frac{d}{dt}\frac{\partial \kappa_R}{\partial \dot{q}_a} - \frac{\partial \kappa_R}{\partial q_a} \tag{A1.7}$$

will be named the shell inertia forces.

Let us observe that as a primitive concept we have not introduced the concept of mass but the concept of kinetic energy. The kinetic energy function (A1.6) will be called simple, being independent of $\nabla q_{(n)}$, $\nabla \dot{q}_{(n)}$ and higher derivatives.

Example. For the Cosserat surface $n = 6$, $q_{(6)} = (\underset{\sim}{r}, \underset{\sim}{d})$ and

$$R_e(\Pi,t) = \oint_{\partial\Pi} (\underset{\sim}{N}_R \cdot \dot{\underset{\sim}{r}} + \underset{\sim}{M}_R \cdot \dot{\underset{\sim}{d}})\, dl_R + \int_{\Pi} (\underset{\sim}{f}_R \cdot \dot{\underset{\sim}{r}} + \underset{\sim}{l}_R \cdot \dot{\underset{\sim}{d}})\, da_R$$

$$R_i(\Pi,t) = \int_{\Pi} (\underset{\sim}{N}_R^{\alpha} \cdot \dot{\underset{\sim}{r}}_{,\alpha} + \underset{\sim}{M}_R^{\alpha} \cdot \dot{\underset{\sim}{d}}_{,\alpha} + \underset{\sim}{m}_R \cdot \dot{\underset{\sim}{d}})\, da_R \ ,$$

$$\kappa_R = \frac{1}{2}\rho_R \dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} + \frac{1}{2}\alpha_R \dot{\underset{\sim}{d}} \cdot \dot{\underset{\sim}{d}} \ ,$$

where $t_R^{(6)} = (\underset{\sim}{N}_R, \underset{\sim}{M}_R)$, $f_R^{(6)} = (\underset{\sim}{f}_R, \underset{\sim}{l}_R)$, $H_R^{(6)\alpha} = (\underset{\sim}{N}_R^{\alpha}, \underset{\sim}{M}_R^{\alpha})$ and $h_R^{(6)} = (\underset{\sim}{0}, -\underset{\sim}{m}_R)$, components in brackets being the vectors of forces and moments, respectively, cf. [26].

1.3. Field equations

The interrelation between the shell force system and the shell deformation function due to the principles of mechanics will be obtained here from the following:

AXIOM 4: For every $t \in I$ and every function u_a, defined and continuous on $\bar{\Pi}$ and differentiable in Π the following relation holds

$$\oint_{\partial\Pi} p_R^a u_a \, dl_R + \int_{\Pi} (f_R^a - i_R^a) u_a \, da_R = \int_{\Pi} (H_R^{a\alpha} u_{a,\alpha} - h_R^a u_a)\, da_R \ . \tag{A1.8}$$

Eq. (1.1.8) represents the simplest case of the known principle of virtual work for the simple shell force system. From the foregoing condition we obtain the field equations of the shell theory. They include the equations of motion

$$H^{a\alpha}_{R,\alpha} + h^{a}_{R} + f^{a}_{R} = i^{a}_{R} \qquad \text{(A1.9)}$$

which have to hold in $\Pi \times I$, and the kinetic boundary conditions

$$H^{a\alpha}_{R} n_{R\alpha} = p^{a}_{R} \qquad \text{(A1.10)}$$

which have to hold nearly everywhere on $\partial\Pi \times I$, $n_R = (n_{r\alpha})$ being the outward unit normal to $\partial\Pi$ on the plane Ox^1x^2.

Example: For the Cosserat surface we obtain

$$\underset{\sim}{N}^{\alpha}_{R,\alpha} + \underset{\sim}{f}_R = \rho_R \ddot{\underset{\sim}{r}} \quad ,$$

$$\underset{\sim}{M}^{\alpha}_{R,\alpha} - \underset{\sim}{m}_R + \underset{\sim}{l}_R = \alpha_R \ddot{\underset{\sim}{d}} \quad ,$$

and

$$\underset{\sim}{N}^{\alpha}_{R} n_{R\alpha} = \underset{\sim}{N}_R \ , \qquad \underset{\sim}{M}^{\alpha}_{R} n_{R\alpha} = \underset{\sim}{M}_R \ .$$

Remark. From Eqs. (A1.8), (A1.3), by virtue of the known invariance conditions with respect to the translations in time and translations and rotations of the reference space, we can obtain the principles of conservation of the energy, of the momentum and that of the moment of momentum; cf., for example, [40].

1.4. Constitutive relations

Constitutive equations are the definitions of the different classes of materials. Thus no extra axioms are needed to introduce the constitutive equations into the direct approach to the shell theories.

Definition 1. The shell will be called hyperelastic if there exists the non-negative differentiable real valued function $\varepsilon_R = \varepsilon_R(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)})$, $\underset{\sim}{\theta} \in \Pi$, invariant with respect to the group of transformations $q_a \to A^{b}_{a} q_b + a_a$ (cf. Eq. (A1.3), and such that

$$H_R^{a\alpha} = \frac{\partial \varepsilon_R}{\partial q_{a,\alpha}} \quad , \qquad h_R^a = -\frac{\partial \varepsilon_R}{\partial q_a} \quad . \tag{A1.11}$$

The function ε_R (if it exists) will be called the shell strain energy function and Eqs. (A1.11) will be referred to as the constitutive equations of the hyperelastic shell.

Remark. For the hyperelastic shell the field equations and the constitutive equations can be derived from the following principle of the conservation of energy

$$\frac{d}{dt}\int_{\Pi} (\varepsilon_R + \kappa_R)\, da_R = \oint_{\partial\Pi} p_R^a \dot{q}_a\, dl_R + \int_{\Pi} f_R^a \dot{q}_a\, da_R \quad ,$$

which has to hold for every $\dot{q}_{(n)}$ and every $t \in I$.

From Eq. (A1.11) we conclude that under arbitrary rigid motion of the reference frame $\underset{\sim}{p} \to \underset{\sim}{Q}(t)\underset{\sim}{p} + \underset{\sim}{c}(t)$ we have $\Psi^a \to \bar{A}_b^a \Psi^b$, where $\bar{A}_b^a A_a^c = \delta_b^c$ and Ψ^a stands for an arbitrary element of the shell force system, provided that Eq. (A1.3) holds.

Definition 2. The shell will be called elastic if there exist the real valued functions $\tilde{H}_R^{a\alpha}(\underset{\sim}{\theta}, \nabla q_{(n)}, q_{(n)})$, $\tilde{h}_R^a(\underset{\sim}{\theta}, \nabla q_{(n)}, q_{(n)})$, $\underset{\sim}{\theta} \in \Pi$, such that

$$\begin{aligned} H_R^{a\alpha}(\underset{\sim}{\theta}, t) &= \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta}, \nabla q_{(n)}, q_{(n)}) \quad , \\ h_R^a(\underset{\sim}{\theta}, t) &= \tilde{h}_R^a(\underset{\sim}{\theta}, \nabla q_{(n)}, q_{(n)}) \quad . \end{aligned} \tag{A1.12}$$

Eqs. (A1.12) will be referred to as the constitutive equations of the elastic shell.

Definition 3. The shell will be called simple if there exist the functionals $\overset{\infty}{\underset{\sigma=0}{H}}{}_R^{a\alpha}(\underset{\sim}{\theta}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)})$, $\overset{\infty}{\underset{\sigma=0}{h}}{}_R^{a}(\underset{\sim}{\theta}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)})$, where $q_{(n)}^{(t)} \equiv q_{(n)}(\underset{\sim}{\theta}, t-\sigma)$, $\sigma \geq 0$, such that

$$\begin{aligned} H_R^{a\alpha}(\underset{\sim}{\theta}, t) &= \overset{\infty}{\underset{\sigma=0}{H}}{}_R^{a\alpha}(\underset{\sim}{\theta}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}) \quad , \\ h_R^a(\underset{\sim}{\theta}, t) &= \overset{\infty}{\underset{\sigma=0}{h}}{}_R^{a}(\underset{\sim}{\theta}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}) \quad . \end{aligned} \tag{A1.13}$$

Eqs. (A1.13) will be referred to as the constiutive equations of the simple shell.

We postulate that the constitutive equations have to be invariant under arbitrary rigid motion of the reference frame, i. e., for an arbitrary functional $\Psi^a(\underset{\sim}{\theta},\nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}$ on the RHS of the constitutive equations [1] and arbitrary $\underset{\sim}{Q}(\sigma)$, $\underset{\sim}{c}(\sigma)$, $\sigma \geq 0$, we have

$$\bar{A}^a_b \psi^b(\underset{\sim}{\theta}, q_{(n)}^{(t)}, \nabla q_{(n)}) = \psi^a(\underset{\sim}{\theta}, (A^b_{(n)} \nabla q_b)^{(t)}, (A^b_{(n)} q_b)^{(t)}),$$

where $A^b_{(n)} \equiv \{A^b_a, a = 1,\ldots,n\}$.

Example. For the elastic Cosserat surface we obtain

$$\underset{\sim}{N}^\alpha_R = \hat{\underset{\sim}{N}}^\alpha_R(\underset{\sim}{\theta}, \underset{\sim}{a}_\beta, \underset{\sim}{d}_{,\beta}, d), \qquad \underset{\sim}{a}_\beta \equiv \underset{\sim}{r}_{,\beta},$$

$$\underset{\sim}{M}^\alpha_R = \hat{\underset{\sim}{M}}^\alpha_R(\underset{\sim}{\theta}, \underset{\sim}{a}_\beta, \underset{\sim}{d}_{,\beta}, \underset{\sim}{d}),$$

$$\underset{\sim}{m}_R = \hat{\underset{\sim}{m}}_R(\underset{\sim}{\theta}, \underset{\sim}{a}_\beta, \underset{\sim}{d}_{,\beta}, \underset{\sim}{d}),$$

Remark 1. Every hyperelastic shell is elastic and every elastic shell is simple. The constitutive equations which define different classes of non-simple shells can be also included into the direct approach but we neglect them here (such shells will be analysed in the next Sections). It must be stressed that the definitions of different shells from the point of view of the constitutive equations (Definitions 1-3) have the sense only if related to the fixed set Q of the shell deformation functions $q_{(n)}$ and for the postulated system of the shell internal forces. Thus the definitions 1-3 are valid only for the first order system of shell internal forces (with respect to the shell deformation function $q_{(n)}$).

If the field equations (A1.9), (A1.10) (where i^a_R are defined by Eqs. (A1.7)) and the constitutive equation are known (i.e., the RHS of Eqs. (A1.6) and (A1.13) are known) then we say that a certain shell theory has been established.

---

[1] We treat Eqs. (A1.11), (A1.12) as the special cases of Eqs. (A1.13).

Remark 2. For the detailed analysis of the direct approach in the special case of the Cosserat surface and within the first order shell force systems the reader is referred to [26].

1.5. Generalization

Let us define in Axiom 2 the rates of work $R_e(\Pi_o,t)$, $R_i(\Pi_o,t)$ of external and internal forces, in the more general form

$$R_e(\Pi_o,t) \equiv \oint_{\partial\Pi_o} (t_R^a \dot{q}_a + t_R^{a\alpha}\dot{q}_{a,\alpha})dl_R + \int_{\Pi_o} (f_R^a \dot{q}_a + f_R^{a\alpha}\dot{q}_{a,\alpha})da_R,$$
$$R_i(\Pi_o,t) \equiv \int_{\Pi_o} (-H_R^{a\alpha\beta}\dot{q}_{a,\alpha\beta} + H_R^{a\alpha}\dot{q}_{a,\alpha} - h_R^a\dot{q}_a)da_R \tag{A1.14}$$

which holds for every regular subregion Π_o of Π and every $t \in I$. If $\Pi_o = \Pi$ then in Eqs. (A1.14) we replace t_R^a, $t_R^{a\alpha}$ by $p_R^a, p_R^{a\alpha}$, respectively. The ordered systems of the scalar, vector and tensor functions [1] $\{p_R^a, (p_R^{a\alpha}), f_R^a, (f_R^{a\alpha})\}$ and $\{h_R^a, (H_R^{a\alpha}), (H_R^{a\alpha\beta})\}$, where $a = 1,\ldots,n$, will be called the second-order shell external and internal force systems, respectively. Let us assume that the shell kinetic energy function κ_R in the Axiom 3 is given by

$$\kappa_R = \kappa_R(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \dot{q}_{(n)}, \nabla\dot{q}_{(n)}) \quad . \tag{A1.15}$$

Putting

$$i_R^a \equiv \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}_a} - \frac{\partial\kappa_R}{\partial q_a}, \quad i_R^{a\alpha} \equiv \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}_{a,\alpha}} - \frac{\partial\kappa_R}{q_{a,\alpha}} \quad , \tag{A1.16}$$

we shell refer the fields $i_R^a - i_{R,\alpha}^{a\alpha}$ to as the shell inertia forces The function (A1.15) will be called the shell second-order kinetic energy function. At least let us replace Eq. (A1.8) of Axiom 4 by the following one

---

[1] With respect to the group of transformation of the orthogonal Cartesian coordinates θ^α, $\alpha = 1,2$, in the region Π of the plane. For every fixed "a" symbols $(p_R^{a\alpha})$, $(f_R^{a\alpha})$, $(H_R^{a\alpha})$ denote vectors and $(H_R^{a\alpha\beta})$ denotes second order tensor.

$$\oint_{\partial\Pi}(p_R^a u_a + p_R^{a\alpha} u_{a,\alpha})dl_R + \int_{\Pi}[(f_R^a - i_R^a)u_a + (f_R^{a\alpha} - i_R^{a\alpha})u_{a,\alpha}]da_R =$$

$$= \int_{\Pi}(-H_R^{a\alpha\beta} u_{a,\alpha\beta} + H_R^{a\alpha} u_{a,\alpha} - h_R^a u_a)da_R \qquad (A1.17)$$

From (A1.17) we obtain the field equations for the second-order force system. They include the equations of motion

$$H_{R,\alpha\beta}^{a\alpha\beta} + H_{R,\alpha}^{a\alpha} + h_R^a + f_R^a - f_{R,\alpha}^{a\alpha} = i_R^a - i_{R,\alpha}^{a\alpha} \qquad (A1.18)$$

and the kinetic boundary conditions

$$(H_R^{a\alpha} + H_{R,\beta}^{a\beta\alpha})n_{R\alpha} + \frac{d}{dl_R}(H_R^{a\alpha\beta} n_{R\beta} t_{R\alpha}) = p_R^a - \frac{d}{dl_R}(p_R^{a\alpha} t_{R\alpha}) - (i_R^{a\alpha} - f_R^{a\alpha})n_{R\alpha},$$

$$H_R^{a\beta\alpha} n_{R\alpha} n_{R\beta} = -p_R^{a\alpha} n_{R\alpha} . \qquad (A1.19)$$

The Definition 1 of Sec. 1.4. has to be changed now by assuming $\varepsilon_R = \varepsilon_R(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)})$ and

$$H_R^{a\alpha\beta} = -\frac{\partial \varepsilon_R}{\partial q_{a,\alpha\beta}} , \quad H_R^{a\alpha} = \frac{\partial \varepsilon_R}{\partial q_{a,\alpha}} , \quad h_R^a = -\frac{\partial \varepsilon_R}{\partial q_a} . \qquad (A1.20)$$

The function ε_R in Eqs. (A1.20) will be referred to as the shell strain energy function [1]. Analogously, instead of Eqs. (A1.12), (A1.13) we have now

$$H_R^{a\alpha\beta}(\underset{\sim}{\theta}, t) = \tilde{H}_R^{a\alpha\beta}(\underset{\sim}{\theta}, \nabla\nabla q_{(n)}, \nabla q_{(n)}, q_{(n)}),$$

$$H_R^{a\alpha}(\underset{\sim}{\theta}, t) = \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta}, \nabla\nabla q_{(n)}, \nabla q_{(n)}, q_{(n)}) , \qquad (A1.21)$$

$$h_R^a(\underset{\sim}{\theta}, t) = \tilde{h}_R^a(\underset{\sim}{\theta}, \nabla\nabla q_{(n)}, \nabla q_{(n)}, q_{(n)}) ,$$

and

$$H_R^{a\alpha\beta}(\underset{\sim}{\theta}, t) = \underset{\sigma=0}{\overset{\infty}{H}}{}_R^{a\alpha\beta}(\underset{\sim}{\theta}, \nabla\nabla q_{(n)}^{(t)}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}) ,$$

---

[1] Mind, that Definition 1 of Sec. 1.4. is no longer valid, cf. Remark to Sec. 1.4.

$$H_R^{a\alpha}(\underset{\sim}{\theta},t) = \underset{\sigma=0}{\overset{\infty a\alpha}{H}}(\underset{\sim}{\theta},\nabla\nabla q_{(n)}^{(t)}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}), \qquad (A1.22)$$

$$h_R^{a}(\underset{\sim}{\theta},t) = \underset{\sigma=0}{\overset{\infty a}{h_R}}(\underset{\sim}{\theta},\nabla\nabla q_{(n)}^{(t)}, \nabla q_{(n)}^{(t)}, q_{(n)}^{(t)}),$$

respectively. Eqs. (A1.20) or (A1.21) or (A1.22) will be referred to as the constitutive equations (for the second order shell internal force system with respect to $q_{(n)}$) of elastic and simple shells, respectively.

The field equations (A1.18), (A1.19) with the definitions (A1.16) and the constitutive equations (A1.22) (which can be taken in the more special forms given by Eqs. (A1.21) or Eqs. (A1.20)) represent certain shell theory (with the second order shell force system and the second order kinetic energy function), provided that the RHS of Eqs. (A1.15), (A1.22) are known.

Remark 1. It can be observed that in the known Love-Kirchhoff shell theory we deal with the second-order shell force system.

Remark 2. The higher-order shell force systems can be also introduced but they will not be used in what follows.

2. APPROXIMATION OF THE SOLID MECHANICS RELATIONS

In this approach to the plate and shell theories no primary concept or axioms are needed apart from those which are included into the classsical solid mechanics relations. There exist many different schemes of approximating the solid mechanics relations of the shell-like body by the relations of the shell theory. The scheme we are going to present at this section seems to include, as the special cases, all known methods of derivation of the relations of the large deformation shell theories. We do not analyse, however, some from the approximate approaches used in the derivation of the linear or small deformation theories (cf. [12,27], for example). We start with the pure analytical description of the approximation procedure which is applied in this section.

2.0. Analytical preliminaries

Let X be the topological space of vector functions defined on a certain differntiable manifold (or on a manifold with its boundary), Y be the linear space and A be the known mapping from X to Y with the domain D(A) and the range R(A). Moreover, let Δ be the non-empty subset of R(A). Many problems of mechanics are described by the binary relations of the form: $A(x) = y$, $y \in \Delta$. If, for example, $\Delta = \{\theta\}$ then the problem will be described by the equation $A(x) = \theta$, if Δ is the set of all non-negative elements (with respect to a certain positive cone in X) then we shall obtain the inequality $A(x) \geq 0$ and if $\Delta = R(A)$ then the problem will be described by the mapping A. We shall denote domain of this binary relation by Ξ.

The problem we are to deal with will be referred to as the formal approximation of the binary relation $A(x) = y$, $y \in \Delta$. The procedure leading to this approximation will be realized in two following steps.

1. *The approximation relation.* We shall introduce in X the approximation relation $\sim$, putting $x \sim \tilde{x}$ iff $(x,\tilde{x}) \in \Xi \times \tilde{\Xi}$, where $\tilde{\Xi}$ is the known non-empty subset of X, such that $\tilde{\Xi} \subset D(A)$. The set $\tilde{\Xi}$ will be interpreted, roughly speaking, as a certain "approximation" of the domain Ξ of the relation we deal with.

<u>Definition 1</u>. An arbitrary field $\overset{\circ}{y} \equiv A(\tilde{x}) - A(x)$, where $x \sim \tilde{x}$, will be called the error field of the approximation $x \sim \tilde{x}$.

The error fields can be also represented by $\overset{\circ}{y} = A(\tilde{x}) - y$, $y \in \Delta$, and the set of all error fields (for the fixed approximation relation) will be denoted by $\overset{\circ}{Y}$; it is a subset in the linear space Y.

2. *The restriction of the error fields.* In the known approximation procedures we usually demand that all "errors" have to be, roughly speaking, "sufficiently small". In our formal approximation we shall only demand that not all $\overset{\circ}{y}$ belonging to $\overset{\circ}{Y}$ are admissible. To this aid we shall assign , to every $\tilde{x}$ with $\tilde{\Xi}$, the known non-empty subset $\overset{\circ}{Y}_{\tilde{x}}$ of $\overset{\circ}{Y}$. The condition $\overset{\circ}{y} \in \overset{\circ}{Y}_{\tilde{x}}$ will be called the restriction of the error fields due to the approximation relation $x \sim \tilde{x}$, where $\tilde{x}$ is an arbitrary but fixed element of $\tilde{\Xi}$. At the same time we shall postulate

that for every y, $y \in \Delta$, there exists at least one pair $(\tilde{x}, \overset{\circ}{y}) \in \tilde{\Xi} \times \overset{\circ}{Y}_{\tilde{x}}$, such that $A(\tilde{x}) - y = \overset{\circ}{y}$.

Definition 2. The relation: $A(\tilde{x}) - y \in \overset{\circ}{Y}_{\tilde{x}}$, $y \in \Delta$, $x \in \tilde{\Xi}$, will be called the formal approximation of the relation "$A(x) = y$, $y \in \Delta$", provided that the foregoing condition is satisfied.

Remark 1. In the formal approximation neither norms in the spaces X,Y nor a-priori estimations of the approximation procedure have been postulated. The general scheme outlined above has been established mainly in order to "approximate" the relations of the non-linear solid mechanics by the relations of the shell or rod theories. The known approximation methods of functional analysis are, as a rule, too restricitve to be applied to this aid in the case of large deformations of an arbitrary shell-like body.

In what follows we shall understand the concept of approximation exclusively in the sense of the formal approximation.

Remark 2. The formal approximation $A(\tilde{x}) - y \in \overset{\circ}{Y}_{\tilde{x}}$, $\tilde{x} \in \tilde{\Xi}$, $y \in \Delta$, can be usually treated as a "good approximation" (in a certain well defined sense) of the relation $A(x) = y$, $y \in \Delta$, not for all $x \in \Xi$ but only for $x \in \overset{\circ}{\Xi}$, where $\overset{\circ}{\Xi}$ is a certain subset of Ξ (as a rule $\tilde{\Xi}$ is the proper subset of $\overset{\circ}{\Xi}$). It means that the applicability of the formal approximation is usually restricted only to a certain class of problems.

Example 1. Let $y^*_\nu, y^*_\nu \in Y^*$, $\nu = 1,\dots,N$, be the known functionals the form of which can also depend on $\tilde{x}$, $\tilde{x} \in \tilde{\Xi}$. Putting $\overset{\circ}{Y}_{\tilde{x}} := \{\overset{\circ}{y} \mid y^*_\nu(\overset{\circ}{y}) = 0, \nu = 1,\dots,N\}$ we shall approximate the relation "$A(x) = y$, $y \in \Delta$" by the relation: $y^*_\nu(A(\tilde{x})) = y^*_\nu(y)$, $y \in \Delta$, $\nu = 1,\dots,N$.

Example 2. Let V be a certain linear functional space and $V_{\tilde{x}}$ the non-empty subset of V, known for every $\tilde{x} \in \tilde{\Xi}$. Putting $\overset{\circ}{Y}_{\tilde{x}} := \{\overset{\circ}{y} \mid \langle v, \overset{\circ}{y}^* \rangle = 0$ for every $v \in V_{\tilde{x}}\}$ we shall approximate the relation "$A(x) = y$, $y \in \Delta$" by the relation: $\langle v, A^*(\tilde{x}) \rangle = \langle v, y \rangle$ for every $v \in V_{\tilde{x}}$, $\tilde{x} \in \tilde{\Xi}$.

Now let V be the topological space and B the known mapping from $X \times V$ to Y (in applications B is, as a rule, certain bilinear form). We shall also deal with the problems of mechanics described by the relation: $B(x,v) = y$, $y \in \Delta$, for every v with V_x, $x \in \Xi$. To define the formal approxiamtion of this relation we shall introduce:

1. Two approximation relations $x \sim \tilde{x}$, $v \sim \tilde{v}$ in the spaces X,V, respectively (they are denoted by the same sign "$\sim$"), given by $(x,\tilde{x}) \in \Xi \times \tilde{\Xi}$, $(v,\tilde{v}) \in V_x \times \tilde{V}_{\tilde{x}}$, where $\tilde{\Xi}$, $\tilde{V}_{\tilde{x}}$ are the known subsets of X,V, respectively, such that $\tilde{\Xi} \times \tilde{V}_{\tilde{x}} \subset D(B)$ for every $\tilde{x} \in \tilde{\Xi}$. The error fields will be defined by $\overset{\circ}{y} \equiv B(\tilde{x},\tilde{v}) - B(x,v) = B(\tilde{x},\tilde{v}) - y$, $y \in \Delta$.

2. The restriction of the error fields, given by $\overset{\circ}{y} \in \overset{\circ}{Y}_{\tilde{x}}$, where $\overset{\circ}{Y}_{\tilde{x}}$ is, for every $\tilde{x} \in \tilde{\Xi}$, the known subset of Y. At the same time we demand that for every y, $y \in \Delta$, there exists at least one pair $(\tilde{x},\overset{\circ}{y}) \in \tilde{\Xi} \times \overset{\circ}{Y}_{\tilde{x}}$, such that: $B(\tilde{x},\tilde{v}) - y = \overset{\circ}{y}$ for every $\tilde{v} \in \tilde{V}_{\tilde{x}}$.

Definition 3. The relation "$B(\tilde{x},\tilde{v})$, $y \in \overset{\circ}{Y}_{\tilde{x}}$, $\tilde{x} \in \tilde{\Xi}$, $y \in \Delta$ for every $\tilde{v}$ with $\tilde{V}_{\tilde{x}}$", will be called the formal approximation of the relation: $B(x,v) = y$, $y \in \Delta$ for every v with V_x, $x \in \Xi$.

The difference between the formal approximation and a "good" approximation, described in the Remark 2, for the foregoing case still holds.

2.1. Solid mechanics relations

By the solid mechanics relations we shall mean the laws of motion and the constitutive relations. The laws of motion will be assumed here in the form of the field equations, which include the equations of motion and the kinetic (natural) boundary conditions. They will be related to the reference configuration κ_R of the shell-like body $\mathcal{B}$ (cf. the Prerequisites). Denoting by $\underset{\sim}{T}(\underset{\sim}{X},t)$, $\rho_R(\underset{\sim}{X})$, $\underset{\sim}{b}_R(\underset{\sim}{X},t)$, $\underset{\sim}{p}_R(\underset{\sim}{X},t)$, $\underset{\sim}{p}(\underset{\sim}{X},t)$ the second Piola-Kirchhoff stress tensor, the mass density, the body forces, the surface tractions and the deformation function, respectively, we shall assume the field equations in the well known form [1]

$$\rho_R \ddot{\underset{\sim}{p}} - \underset{\sim}{b}_R - \text{Div}(\nabla \underset{\sim}{p}\,\underset{\sim}{T}) = \underset{\sim}{0}, \quad \underset{\sim}{T} = \underset{\sim}{T}^T; \quad X \in \kappa_R(\mathcal{B}), t \in I,$$

$$(\nabla \underset{\sim}{p}\,\underset{\sim}{T})\,\underset{\sim}{n}_R - \underset{\sim}{p}_R = \underset{\sim}{0}; \quad \underset{\sim}{X} \in \partial\kappa_R(\mathcal{B}) \text{ almost everywhere, } t \in I, \tag{A2.1}$$

[1] Throughout the treatise the relations involving the derivatives may be understand also in the generalized sens, because the classical partial derivatives may not exist. Thus Eqs. (A2.1) may be interpreted as the laws of the conservation of momentum and that of moment of momentum for an arbitrary regular part of the region $\kappa_R(\mathcal{B})$ or its boundary $\partial\kappa_R(\mathcal{B})$.

where $\underset{\sim}{T}^T$ is the transpose of $\underset{\sim}{T}$ and $\underset{\sim}{n}_R$ is the unit normal outward to $\partial\kappa_R(\mathcal{B})$.

In order to write down the constitutive relations of solid mechanics we shall introduce the right Cauchy-Green deformation tensor $\underset{\sim}{C} \equiv (\nabla\underset{\sim}{p})^T\nabla\underset{\sim}{p}$ and an ordered set $\lambda^{(s)} = \{\lambda^\sigma, \sigma = 1,\dots,s\}$ of the real valued functions defined on $\Pi \times I$. The elements λ^σ represent all those fields of mechanics which do not appear in the field equations but are needed to describe the material properties of the body. To describe many different classes of materials we shall assume the constitutive equations in the general form [1]

$$f_\mu(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) = 0, \quad \mu = 1,\dots,m; \quad m = 6 + S ,$$

$$j(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) \le 0, \tag{A2.2}$$

$$\varphi(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)},\underset{\sim}{C}_o,\underset{\sim}{T}_o) \le 0 \text{ for every } \underset{\sim}{C}_o,\underset{\sim}{T}_o \text{ with } j(\underset{\sim}{X},\underset{\sim}{C}_o,\underset{\sim}{T}_o,\lambda^{(s)}) \le 0,$$

where $f_\mu(\underset{\sim}{X},\cdot)$ are, for every $\underset{\sim}{X}$ with $\kappa_R(\mathcal{B})$ independent functionals defined on the space of functions $(C(\underset{\sim}{X},\cdot),\underset{\sim}{T}(\underset{\sim}{X},\cdot),\lambda^{(s)}(\underset{\sim}{X},\cdot))$, $t \in I$ and $j(\underset{\sim}{X},\cdot)$, $\varphi(\underset{\sim}{X},\cdot)$ are, also for every $\underset{\sim}{X}$ with $\kappa_R(\mathcal{B})$, the real valued functions defined on $\mathcal{T}^2 \times \mathcal{T}^2 \times R^s$ and $(\mathcal{T}^2 \times \mathcal{T}^2)^2 \times R^s$, respectively, $\mathcal{T}^2$ is the space of all symmetric second-order tensors. The domain of $\mathcal{T}^2 \times \mathcal{T}^2 \times R^s$ where $j(\underset{\sim}{X},\cdot) \le 0$, is usually assumed to be the closed set, $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$. We shall also introduce the one-to-one cooorespondence $\underset{\sim}{S} : R^6 \ni \underset{\sim}{\sigma} = (\sigma^1,\dots,\sigma^6) \to \underset{\sim}{S}(\underset{\sim}{\sigma}) \in \mathcal{T}^2$, with the inverse $\underset{\sim}{\sigma} \equiv \underset{\sim}{S}^{-1}$, $\underset{\sim}{\sigma}(\underset{\sim}{T}) \in R^6$. We assume that the stress components $\sigma^\nu(\underset{\sim}{X},t)$ which can be expressed by Eqs. $(A2.2)_1$ exclusively in terms of $\underset{\sim}{C}(\underset{\sim}{X},t-\sigma)$, $\sigma \ge 0$, do not appear in Eqs. $(A2.2)_{2,3}$.

<u>EXAMPLES</u>

1. If $m = 6$, $j \equiv 0$, $\varphi \equiv 0$ and $f_\mu = 0$, $\mu = 1,\dots,6$, have the form

$$\underset{\sim}{T} = \underset{\sigma=0}{\overset{\infty}{\underset{\sim}{\mathcal{F}}}}(\underset{\sim}{X},\underset{\sim}{C}(\underset{\sim}{X},t-\sigma)) , \tag{$(A2.2)_1$}$$

---

[1] We assume that the fields $\lambda^1,\dots,\lambda^S$ represent the internal parameters and $\lambda^{S+1},\dots,\lambda^s$ characterize together with $\underset{\sim}{p}$, the kinematics of the medium (being, for example, the rates of plastic deformations); here S is the fixed integer, $0 \le S \le s$.

where $\underset{\sim}{F}$ is the response functional, then Eqs. $(A2.2)_1$ describe the simple material.

2. If $m = 6$, $j \equiv 0$, $\varphi \equiv 0$ and f_μ, $\mu = 1,\ldots,6$, are the known differential operators with respect to the time coordinate, then Eqs. (A2.2) describe the rate-type material, cf. [33]. If these operators are linear then we deal with the linear viso-elastic material.

3. Let $\underset{\sim}{S}$ be given by $(\sigma^1,\ldots\sigma^6) = (T^{11},T^{22},T^{33},T^{12},T^{13},T^{23})$. If $j(\underset{\sim}{X},\underset{\sim}{S}(\cdot))$ is the known continuous and convex real valued function for every $\underset{\sim}{X},\underset{\sim}{X} \in \kappa_R(\mathcal{B})$ (so that the domain of R^6, where $j(\underset{\sim}{X},\underset{\sim}{S}(\sigma)) \leq 0$, is a closed convex set) and Eqs. (A2.2) have the form

$$\dot{\underset{\sim}{C}} = \underset{\sim}{A}[\dot{\underset{\sim}{T}}] + \underset{\sim}{\Lambda}\,, \quad j(\underset{\sim}{X},\underset{\sim}{T}) \leq 0$$
$$\operatorname{tr} \underset{\sim}{\Lambda}(\underset{\sim}{T}_o - \underset{\sim}{T}) \leq 0 \text{ for every } \underset{\sim}{T}_o \text{ with } j(\underset{\sim}{X},\underset{\sim}{T}_o) \leq 0\,, \qquad (A2.2)_2$$

where $\underset{\sim}{\Lambda} = \underset{\sim}{S}(\lambda^{(6)}) \in T^2$ and $\underset{\sim}{A} = \underset{\sim}{A}(\underset{\sim}{X},\underset{\sim}{T},\underset{\sim}{C})$ is the known, for every $\underset{\sim}{X},\underset{\sim}{T},\underset{\sim}{C}$, linear mapping $T^2 \to T^2$ (here $m = s = 6$), then we shall deal with the elastic-perfectly plastic material. The condition $j(\underset{\sim}{X},\underset{\sim}{T}) = 0$ represents the yield condition, $\underset{\sim}{\Lambda}$ is the rate of the plastic deformations and Eq. $(A2.2)_3$ constitutes the principle of maximum plastic work, cf. [15,30].

4. Let $\underset{\sim}{\pi}$ be the orthogonal projection of R^6 on the closed convex set $K := \{\underset{\sim}{\sigma} | j(\underset{\sim}{S}(\underset{\sim}{\sigma})) \leq 0\} \subset R^6$. Introducing the denotation (cf. [8], p. 234)

$$J_\mu(\underset{\sim}{T}) \equiv \frac{1}{4\mu} \operatorname{tr}[\underset{\sim}{T} - \underset{\sim}{S}\,\underset{\sim}{\pi}\,\underset{\sim}{S}^{-1}(\underset{\sim}{T})][\underset{\sim}{T} - \underset{\sim}{S}\,\underset{\sim}{\pi}\,\underset{\sim}{S}^{-1}(\underset{\sim}{T})]\,,$$

where μ, $\mu \geq 0$, is the coefficient of the linear viscosity, and putting Eqs. (A2.2) in the form (here $m = 6$)

$$\dot{\underset{\sim}{C}} = \underset{\sim}{A}[\dot{\underset{\sim}{T}}] + \underset{\sim}{\Lambda}$$
$$\operatorname{tr} \underset{\sim}{\Lambda}(\underset{\sim}{T}_o - \underset{\sim}{T}) \leq J_\mu(\underset{\sim}{T}_o) - J_\mu(\underset{\sim}{T}) \quad \text{for every } \underset{\sim}{T}_o \in T^2, \qquad (A2.2)_3$$

we arrive at the elasto-visco plastic materials. It has been shown in [8] that if $\mu = 0$ then Eqs. $(A2.2)_3$ reduce to Eqs. $(A2.2)_2$

If $\mu > 0$ then Eq. (A2.4) will yield $\Lambda = \partial J_\mu / \partial T$ and if $\mu \geq 0$ then $\Lambda \in \partial J_\mu(T)$, where $\partial J_\mu(T)$ is the subdifferential of $J_\mu(\cdot)$ at T.

5. If $j(X,S(\cdot))$ is the known function as before, $E \equiv \frac{1}{2}(C - 1)$, and Eqs. (A2.2) have the form (with $m = 6$)

$$T = A[E] + \Lambda\ , \quad j(X,E) \leq 0$$
$$\text{tr}\ \Lambda(E_o - E) \leq 0 \text{ for every } E_o \text{ with } j(X,E_o) \leq 0\ , \tag{A2.2.)_4}$$

where $A = A(X)$ is the known linear transformation, $\Lambda = S(\lambda^{(6)}) \in \mathcal{T}^2$ and $j(X,0) \leq 0$, then we shall deal with the locking materials, [31].

6. If $m > 6$, and $S > 0$, then interpreting $\lambda^{S+1},\ldots,\lambda^{S+6}$ analogously as in Example 3 and ($\kappa = (\lambda^1,\ldots,\lambda^S)$ as the internal parameters describing the effect of the work hardening we shall postulate the equations of the form $(A2.2)_2$ with $j = j(X,T,\kappa)$ and the extra $m = 6$ scalar relations $\dot{\kappa} = K[\dot{T}]$, where $K = K(X,T,C)$ is the known linear mapping form $\mathcal{T}^2$ to $R^{m-6} = R^S$. This is the case of the elastic-plastic materials with the work hardening.

2.2. Simple approach

Now as the field x in the relations of Sec. 2.0. we shall take the triple $(p,T,\lambda^{(s)})$. The set of all $x = (p,T,\lambda^{(s)})$ which satisfy Eqs. (A2.1), (A2.2) for an aribtrary but fixed pair (p_R, b_R) will be denoted by Ξ. Thus the set Δ introduced in Sec. 2.0, stands for the set of all RHS of Eqs. (A2.1), (A2.2), i. e., of all 8 + m tuples $(\theta,\theta,\ 0,\ldots,0(\text{m-times}),\alpha,\beta)$ with α,β as arbitrary non-negative numbers. We have $\Xi = \Xi_1 \times \Xi_2 \times \Xi_3$, where $p \in \Xi_1$, $T \in \Xi_2$, $\lambda^{(s)} \in \Xi_3$. The approximation relation "$\sim$" of the Sec. 2.0. will be postulated by putting $\tilde{\Xi} = \tilde{\Xi}_1 \times \tilde{\Xi}_2 \times \tilde{\Xi}_3$. We shall denote $(p \sim \tilde{p},\ \tilde{T} \sim \tilde{T},\ \lambda^{(s)} \sim \tilde{\lambda}^{(s)} \equiv$ $\equiv (p,T,\lambda^{(s)}) \sim (\tilde{p},\tilde{T},\tilde{\lambda}^{(s)})$ where $p \sim \tilde{p}$ iff $(p,\tilde{p}) \in \Xi_1 \times \Xi_1$, $T \sim \tilde{T}$ iff $(T,\tilde{T}) \in \Xi_2 \times$ $\times \tilde{\Xi}_2$ and $\lambda^{(s)} \sim \tilde{\lambda}^{(s)}$ iff $(\lambda^{(s)},\tilde{\lambda}^{(s)} \in \Xi_3 \times \tilde{\Xi}_3$. The approximation $\tilde{\Xi}_1$ of the set Ξ_1 of the deformation functions $p(X,t) \equiv p(\theta,\xi,t)$ will be determined by the approximation relation

$$p(\theta,\xi,t) \sim \tilde{p}(\theta,\xi,t,\ q_{(n)}(\theta,t)) \text{ for some } q_{(n)} \in Q\ , \tag{A2.3}$$

where n is the fixed positive integer, $\tilde{\underset{\sim}{p}}(\cdot)$ is the known differentiable function and $q_{(n)} = \{q_a, a = 1,\ldots,n\}$ is an ordered set of arbitrary independent differentiable real valued functions q_a defined on $\bar{\Pi} \times I$. Moreover, the non-empty set Q is defined by $Q := \{q_{(n)} | \det \nabla\tilde{p} > 0$ for every $\underset{\sim}{X} \in \overline{\kappa_R(\mathcal{B})}\}$. An arbitrary ordered set $q_{(n)}$, $q_{(n)} \in Q$, will be referred to as the shell deformation function. All RHS of Eqs. (A2.3) represent the set $\tilde{\Xi}_1$.

Let $\underset{\sim}{S}$ be the one-to-one correspondence between R^6 and the space $\mathcal{T}^2$ of the symmetric second order tensors with components $T^{KL} = T^{LK}$ (1). It means that to every $\underset{\sim}{T}$ is uniquely assigned the ordered set $\underset{\sim}{\sigma} \equiv (\sigma^1,\ldots,\sigma^6)$. Let M, $0 \le M \le 6$, be the fixed integer. We shall assume that the mapping $\underset{\sim}{\sigma} \equiv \underset{\sim}{S}^{-1}$ for every $X \in \kappa_R(\mathcal{B})$ is such, that for every ν with $1 \le \nu \le M$, Eqs. $(A2.2.)_1$ here the form

$$\sigma^\nu(\underset{\sim}{T}) - \hat{\sigma}^\nu(\underset{\sim}{X},\underset{\sim}{C}(\underset{\sim}{X},t-s)) = 0 \quad , \quad \nu = 1,\ldots,M, \qquad \text{(A2.4)}$$

where $\hat{\sigma}^\nu(\underset{\sim}{X},\underset{\sim}{C}(\underset{\sim}{X},\cdot))$ are the known functionals. If M = 0 then there are no equations (A2.4) (2). For every μ with $M < \mu \le 6$ we assume that the stress components σ^μ are not uniquely determined by the history of deformation (i. e., they cannot be represent in the form analogous to that of given by Eqs. (A2.4). We seek to approximate the functions σ^μ by the representation

$$\sigma^\mu(\underset{\sim}{T}) \sim \tilde{\sigma}^\mu(\underset{\sim}{X},\tau^{(N)}(\underset{\sim}{\theta},t)), \; \mu = M+1,\ldots,6 \;, \qquad \text{(A2.5)}$$

where $\tau^{(N)} := \{\tau^A, A = 1,\ldots,N\}$ is an arbitrary ordered set of the sufficently regular real valued functions defined on $\Pi \times I$ and $\tilde{\sigma}^\mu(\underset{\sim}{X},\cdot)$ are the knonw for every $\underset{\sim}{X}$, $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$, regular real valued functions defined on R^N. We shall also assume that every $\tau^A = \tau^A(\underset{\sim}{\theta},t)$ is an invariant under arbitrary rigid motion of the reference space. Substituting into (A2.4) on the place of the Cauchy-Green deformtion tensor $\underset{\sim}{C}$, the function $\tilde{C} \equiv (\nabla\tilde{\underset{\sim}{p}})^T\nabla\tilde{\underset{\sim}{p}}$ (with $\tilde{p}$ from the RHS of Eqs. (A2.3), we shall postulate the following approximation of the second Piola Kirchhoff stress tensor

(1) Indices K,L.M,... run over 1,2,3 and are related to the coordinates $\theta^1,\theta^2,\theta^3 \equiv \xi$.

(2) For M = 6 we deal with the simple material; in this case in Eqs.(A2.2) $j \equiv 0$, $\varphi \equiv 0$, $S = 0$.

$$\underset{\sim}{T} \sim \underset{\sim}{\tilde{T}} = \underset{\sim}{S}(\underset{\sim}{\tilde{\sigma}}) \qquad \begin{aligned} \tilde{\sigma}^{\nu} &= \hat{\sigma}^{\nu}(\underset{\sim}{X},\underset{\sim}{C}(\underset{\sim}{X},t-s)), \quad \nu = 1,\dots,M, \\ \tilde{\sigma}^{\mu} &= \tilde{\sigma}^{\mu}(\underset{\sim}{X},\tau^{(N)}(\underset{\sim}{\theta},t)), \quad \mu = M+1,\dots 6 . \end{aligned} \tag{A2.6}$$

Some from the functions $\tilde{\sigma}^{\mu}$ may be assumed as identically equal to zero (or independent of $\tau^{(N)}$); it means that the suitable stress components σ^{μ} are neglected (or assumed to be known a-priori).

The functions λ^{σ}, $\sigma = 1,\dots,s$, defined on $\kappa_R(\mathcal{B}) \times I$, in the constitutive relations (A2.2) will be approximated by [1]

$$\lambda^{\sigma} \sim \tilde{\lambda}^{\sigma}(\underset{\sim}{X},\omega^{(p)}(\underset{\sim}{\theta},t)), \quad \sigma = 1,\dots,s , \tag{A2.7}$$

where $\omega^{(p)} := \{\omega^{\pi},\ \pi = 1,\dots,p\}$ is an arbitrary ordered set of the smooth real valued functions defined on $\Pi \times I$ and $\tilde{\lambda}^{\sigma}(\underset{\sim}{X},\cdot)$ are the known regular functions. We have stated before, that the fields $\lambda^1,\dots,\lambda^S$ represent the internal parameters (which have to be determined by the constitutive equations $f_{\mu} = 0$, $\mu = 6,\dots,6+S$) and $\lambda^{S+1},\dots,\lambda^{s}$ concern the kinematics of the body (they have to be determined by the relations $(A2.2)_{2,3}$). We shall now assume that ω^{π}, $\pi = 1,\dots,P$, are the arguments of the functions $\tilde{\lambda}^{\sigma}(\underset{\sim}{X},\cdot)$ for $\sigma = 1,\dots,S$ and that ω^{π}, $P+1,\dots,p$ are the arguments of the functions $\tilde{\lambda}^{\sigma}(\underset{\sim}{X},\cdot)$ for $\sigma = S+1,\dots,s$.

Putting $x = (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$, $\tilde{x} = (\underset{\sim}{\tilde{p}},\underset{\sim}{\tilde{T}},\tilde{\lambda}^{(s)})$, $\underset{\sim}{\tilde{T}} \equiv \underset{\sim}{S}(\underset{\sim}{\tilde{\sigma}})$, we have defined by Eqs. (A2.3), (A2.6), (A2.7), the relation $x \sim \tilde{x}$ introduced in Sec. 2.0. The set Ξ is the set of all x and $\tilde{\Xi}$ is the set of all $\tilde{x}$. It is the first step of the approximation procedure, which approximate the set Ξ by the set $\tilde{\Xi}$. If $M = 6$, i. e., if the material (for a fixed $\underset{\sim}{X}$, $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$) is simple, then we conclude that it is the constraint approximation of the basic field.

In what follows we shall use the concept of the shell strain measures. To this aid we denote by $E_{(r)} = \{E_{\rho},\ \rho = 1,\dots,r\}$ an ordered set of the differential independent operators acting on $q_{(n)}(\cdot,t)$, such that $e_{\rho} = E_{\rho}(q_{(n)})$ are invariants under arbitrary rigid motion

---

[1] The sign $\sim$ has always to be understand in the sense defined in Sec. 2.0. of this Chapter. The sets $\tilde{\Xi}_2$, $\tilde{\Xi}_3$ introduced above are the sets of all fields $\tilde{T},\tilde{\lambda}^{(s)}$ defined by the RHS of Eqs. (A2.6), (A2.7), respectively.

of the reference space and $(\nabla\tilde{\mathbf{p}})^T\nabla\tilde{\mathbf{p}} = \tilde{\mathbf{C}}(\mathbf{X}, e_{(r)})$, $e_{(r)} = \{e_\rho,\ \rho = 1,\dots,r\}$, $\tilde{\mathbf{C}}(\cdot)$ being the known function. The set $e_{(r)}$ will be called the shell strain measure; it is not uniquely determined by the foregoing conditions.

Now we shall define the error fields putting $\overset{o}{y} = (r_R,\ s_R\ ,\ a_\mu, a, \alpha)$, where

$$
\begin{aligned}
\mathbf{r}_R &\equiv \rho_R \ddot{\tilde{\mathbf{p}}} - \mathbf{b}_R - \mathrm{Div}\, \tilde{\mathbf{T}}_R \quad , \quad \tilde{\mathbf{T}}_R \equiv \nabla\tilde{\mathbf{p}}\tilde{\mathbf{T}}\, , \tilde{\mathbf{T}} = \tilde{\mathbf{T}}^T \\
\mathbf{s}_R &\equiv \tilde{\mathbf{T}}_R \mathbf{n}_R - \mathbf{p}_R\ , \\
a_\mu &\equiv \tilde{f}_\mu - f_\mu = \tilde{f}_\mu, \quad \mu = M+1,\dots,m\ , \qquad \text{(A2.8)} \\
a &\equiv \tilde{j} - j\ ,\ j \le 0\ , \\
\alpha &\equiv \tilde{\varphi} - \varphi\ ,\ \varphi \le 0\ ,
\end{aligned}
$$

and where we have denoted

$$
\begin{aligned}
\tilde{f}_\mu &= \tilde{f}_\mu(\mathbf{X}, e_{(r)}\tau^{(N)}, \omega^{(p)}) \equiv f_\mu(\mathbf{X}, \tilde{\mathbf{C}}, \tilde{\mathbf{T}}, \tilde{\lambda}^{(s)})\ , \\
\tilde{j} &= \tilde{j}(\mathbf{X}, e_{(r)}\tau^{(N)}, \omega^{(p)}) \equiv j\ (\mathbf{X}, \tilde{\mathbf{C}}, \tilde{\mathbf{T}}, \tilde{\lambda}^{(s)})\ , \\
\tilde{\varphi} &= \tilde{\varphi}(\mathbf{X}, e_{(r)}, \tau^{(N)}, \omega^{(p)}, e^{o}_{(r)}, \tau^{(N)}_o \equiv \varphi(\mathbf{X}, \mathbf{C}, \mathbf{T}, \lambda^{(s)}, \mathbf{C}_o, \mathbf{T}_o)\ , \qquad \text{(A2.9)} \\
\tilde{\mathbf{C}}_o &= \tilde{\mathbf{C}}(\mathbf{X}, e^{o}_{(r)})\ , \tilde{\mathbf{T}}_o = \mathbf{S}(\tilde{\boldsymbol{\sigma}})\ , \tilde{\sigma}^\nu = \hat{\sigma}^\nu(\mathbf{X}, \tilde{\mathbf{C}}_o)\ ,\ \tilde{\sigma}^\mu = \tilde{\sigma}^\mu(\mathbf{X}, \tau^{(N)}_o)\ , \\
\nu &= 1,\dots,M\ ,\quad \mu = M+1,\dots,6.
\end{aligned}
$$

We have not introduced the error fields $a_\nu = \tilde{f}_\nu$, $\nu = 1,\dots,M$, because the equations $f_\nu = 0$, $\nu = 1,\dots,M$, have the form given by Eqs. (A2.4) and can be eliminated from the system of relation by substituting σ^ν into $\mathbf{T} = \mathbf{S}(\boldsymbol{\sigma})$ in the field equations.

The second step of the approximation procedure is to restrict the error fields. To this aid we introduce the known ordered set.

$$
\Xi^\mu_{\bar{A}} = \Xi^\mu_{\bar{A}}(\theta, \xi,\ t, e_{(r)}, \tau^{(N)}, \omega^{(p)})\ ,
$$

defined on $\Pi \times (h_-, h_+) \times I \times R^{r+N+p}$, which are independent for every $\bar{A}$ and where $\mu = M+1,\dots,m = 6\ /\ S$, $\bar{A} = 1,\dots,\ N+P$ (1).

(1) In the special case we can assume that $\boldsymbol{\Phi}^a \equiv \partial\tilde{\mathbf{p}}/\partial q_a$, $a = 1,\dots,n$, and $\Xi^\mu_A \equiv \partial\sigma^\mu/\partial\tau_A$, $\mu = M+1,\dots,6, A=1,\dots,N$, $\Xi^\mu_{\bar{A}} \equiv \partial\lambda^\sigma/\partial\omega^\pi \delta^\mu_{\sigma+6}\delta^{\bar{A}-N}_\pi$, $\mu = 7,\dots, m + 6$; $\bar{A} = N+1,\dots,N+P$.

The restriction of the error $\overset{\circ}{y} = (\underset{\sim}{r}_R, \underset{\sim}{s}_R, a_{(m)}, a, \alpha)$ given by (A2.8) will be assumed in the form [1]

$$\int_{h_-}^{h_+} \underset{\sim}{r}_R \cdot \underset{\sim}{\Phi}^a \, d\xi = 0 \;, \quad \int_{h_-}^{h_+} \underset{\sim}{s}_R \cdot \underset{\sim}{\Phi}^a \, d\xi = 0 \;, \quad a = 1,\dots,n$$

$$\sum_{\mu=M+1}^{m} \int_{h_-}^{h_+} a_\mu \Xi^{\mu}_{\bar{A}} \, d\xi = 0 \;, \quad \bar{A} = 1,\dots, N+P \;, \tag{A2.10}$$

$$\int_{h_-}^{h_+} a \, d\xi = 0 \;, \quad \int_{h_-}^{h_+} \alpha \, d\xi = 0 \;.$$

It means that for every $\tilde{x} = (\tilde{\underset{\sim}{p}}, \tilde{\underset{\sim}{T}}, \tilde{\lambda}^{(s)}) \in \tilde{\Xi}$ the set $\overset{\circ}{Y}$ of all error fields $\overset{\circ}{y} = (\underset{\sim}{r}_R, \underset{\sim}{s}_R, a, \alpha)$ has been restricted to the subset $\overset{\circ}{Y}_{\tilde{x}}, \overset{\circ}{Y}_{\tilde{x}} \subset \overset{\circ}{Y}$, of the error fields satisfying Eqs. (A2.10). Mind, that for every $\mu = 1,\dots,M$ we have $a_\mu \equiv 0$.

Now by the direct calcualtions we shall prove that the approximation $\tilde{\Xi}$ of the set Ξ of triples $x = (\underset{\sim}{p}, \underset{\sim}{T}, \lambda^{(s)})$, given by Eqs. (A2.3), (A2.6), (A2.7), and the restriction of the error fields determined by Eqs. (A2.10) leads to the relations of the shell theory. Substituting the RHS of Eqs. $(A2.8)_{1,2}$ into $(A2.10)_{1,2}$, after the denotations

$$H_R^{a\alpha} \equiv \int_{h_-}^{h_+} \Phi^a_k \tilde{p}^k_{,K} \tilde{T}^{K\alpha} d\xi \qquad h^a_R \equiv -\int_{h_-}^{h_+} \Phi^a_{k,K} \tilde{p}^k_{,L} \tilde{T}^{LK} \, d\xi$$

$$f^a_R \equiv \int_{h_-}^{h_+} \Phi^a_k b^k_R \, d\xi + [\Phi^a_k]_{h_+} \overset{+}{p}{}^k_R + [\Phi^a_k]_{h_-} \overset{-}{p}{}^k_R \;, \tag{A2.11}$$

$$i^a_R \equiv \int_{h_-}^{h_+} \rho_R \Phi^a_k \ddot{\tilde{p}}{}^k \, d\xi \;; \quad \underset{\sim}{\theta} \in \Pi \;,$$

$$p^a_R \equiv \int_{h_-}^{h_+} \Phi^a_k p^k_R \, d\xi \;, \quad \theta \in \partial\Pi \text{ a.e.},$$

[1] The restriction of the form (A2.10) (but with the integral over Π has been used in [1] to obtain the equations of motion for the rods, cf. Sec. 5.3..

where $\overset{+}{\underset{\sim}{p}}_R$, $\overset{-}{\underset{\sim}{p}}_R$ are the surface tractions on the surfaces $\xi = h_+$, $\xi = h_-$, respectively, we obtain after simple calculations

$$H^{a\alpha}_{R,\alpha} + h^a_R + f^a_R = i^a_R \quad ; \quad \underset{\sim}{\theta} \in \Pi, \quad t \in I ,$$
$$H^{a\alpha}_R n_R = p^a_R \quad ; \quad \theta \in \partial\Pi \ \text{a.e.}, \quad t \in I . \tag{A2.12}$$

Let us also denote

$$g_A \equiv \int_{h_-}^{h_+} \tilde{f}_\mu \overset{-}{\Xi}{}^\mu_A \, d\xi \quad , \qquad \kappa \equiv \int_{h_-}^{h_+} \tilde{j} \, d\xi \quad , \qquad \psi \equiv \int_{h_-}^{h_+} \tilde{\varphi} \, d\xi . \tag{A2.13}$$

We observe that the integrands in Eqs. (A2.13) are the known explicit functions of $\xi, \xi \in (h_-, h_+)$. Thus the integrals in Eqs. (A2.13) can be calculated. Thus g_A are the known functionals and κ, ψ are the known functions of $\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(P)}$. In view of Eqs. (A2.2), (A2.10), (A2.13), we obtain

$$g_A(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(p)}) = 0, \quad A = 1,\ldots,N + P$$
$$\kappa(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(p)}) \leq 0,$$
$$\psi(\theta, e_{(r)}, \tau^{(N)}, \omega^{(p)}, \overset{o}{e}_{(r)} \tau^{(N)}_o) \leq 0 \text{ for every } \overset{o}{e}_{(r)}, \tau^{(N)}_o \tag{A2.14}$$
$$\text{with } \kappa(\underset{\sim}{\theta}, \overset{o}{e}_{(r)}, \tau^{(N)}_o, \omega^{(p)}) \leq 0.$$

At the same time substituting $\underset{\sim}{\tilde{T}} \equiv \underset{\sim}{\tilde{S}}(\underset{\sim}{\tilde{\sigma}})$, with $\underset{\sim}{\tilde{\sigma}}$ given by Eqs. (A2.6), into Eqs. $(A2.11)_{1,2}$ and denoting

$$\tilde{H}^{a\alpha}_R \equiv \int_{h_-}^{h_+} \Phi^a_k \tilde{p}^k_{,K} S^{K\alpha}(\tilde{\sigma}) \, d\xi \quad ,$$
$$\tilde{h}^a_R \equiv - \int_{h_-}^{h_+} \Phi^a_{k,K} \tilde{p}^k_{,L} S^{LK}(\tilde{\sigma}) \, d\xi \quad , \tag{A2.15}$$

we also obtain

$$H_R^{a\alpha}(\underset{\sim}{\theta},t) = \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta},q_{(n)},\nabla q_{(n)}\tau^{(N)}) ,$$
$$h_R^{a}(\underset{\sim}{\theta},t) = \tilde{h}_r^{a}(\underset{\sim}{\theta},q_{(n)}\nabla q_{(n)},\tau^{(N)}) . \tag{A2.16}$$

We can observe that the integrands in Eqs. (A2.15) are the known explicit functions of $\xi,\xi \in (h_-,h_+)$; it follows that the RHS of Eqs. (A2.16) are the known functionals (they are, at the same time, the functions with respect to $\tau^{(N)},\underset{\sim}{\theta}$).

The shell theory derived from the solid mechanics relations (A2.1), (A2.2) is represented by:

1. the field equations (A2.12)(the equations of motion and the kinetic boundary conditions),
2. the constitutive relations (A2.14), (A2.16). The example of the approach given above will be given in the Chapter B.

Remark. If $m = 6$, $M = 6$, $j \equiv 0$, $\varphi \equiv 0$, i.e., if we deal with the simple material, then there are no functions g_A and $\kappa \equiv 0$, $\psi \equiv 0$ (i.e. Eqs. (A2.14) are identities), the argument $\underset{\sim}{\tilde{\sigma}}$ in Eqs. (A2.15) is given by

$$\tilde{\sigma}^{\nu} = \tilde{\sigma}^{\nu}(\underset{\sim}{X},\underset{\sim}{\tilde{C}},(\underset{\sim}{X},t - \sigma)) , \quad \nu = 1,\ldots,6 ,$$

and the functions $\tilde{H}_R^{a\alpha}(\cdot)$, $\tilde{h}_R^{a}(\cdot)$ in Eqs. (A2.16) are independent of $\tau^{(N)}$:

$$H_R^{a\alpha} = \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta},q_{(n)},\nabla q_{(n)}), \quad h_R^{a} = \tilde{h}_R^{a}(\underset{\sim}{\theta},q_{(n)},\nabla q_{(n)}). \tag{A2.16$_1$}$$

Eqs. $(A2.16)_1$ represent the approximation of the constitutive equations of shells made of the simple materials.

Strain measures and compatibility conditions. By the shell strain measure we mean an arbitrary ordered set $e_{(r)}$ of the real valued functions e_ρ defined on $\Pi \times I$, such that $e_\rho = E_\rho(q_{(n)})$, $\rho = 1,\ldots,r$ (E_ρ are the differential operators acting on $q_{(n)}(\cdot,t)$), e_ρ are invariants under arbitrary rigid motion of the reference space and there exists the function $\underset{\sim}{\tilde{C}}(\underset{\sim}{X},e_{(r)})$ satisfying the condition $\underset{\sim}{\tilde{C}}(\underset{\sim}{X},e_{(r)}) = (\nabla\underset{\sim}{\tilde{p}})^T\nabla\underset{\sim}{\tilde{p}}$ for every $q_{(n)} \in Q$. It is obvious that the functions e_ρ have to be interrelated by certain conditions which will be referred to as the shell compatibility conditions. To obtain them let us denote by $\underset{\sim}{R}(\underset{\sim}{C})$ the Riemann-Christoffel tensor related to the metric tensor $\underset{\sim}{C}$. In the solid mechanics the compatibility condition has the well known form $\underset{\sim}{R}(\underset{\sim}{C}) = \underset{\sim}{O}$. Using the formal approximation procedure we shall introduce the approximation relation $\underset{\sim}{C} \sim \underset{\sim}{\tilde{C}}$, define the set $\overset{\circ}{Y}$ of error fields $\underset{\sim}{J}$ putting $\overset{\circ}{Y} := \{\underset{\sim}{J} | \underset{\sim}{J} = \underset{\sim}{R}(\underset{\sim}{\tilde{C}})\}$ and restrict the error fields by $\underset{\sim}{J} \in \overset{\circ}{Y}_{\underset{\sim}{\tilde{C}}}$, where $\overset{\circ}{Y}_{\underset{\sim}{\tilde{C}}}$ is the known, for every $\underset{\sim}{\tilde{C}} = \underset{\sim}{\tilde{C}}(\underset{\sim}{X}, e_{(r)})$, subset of $\overset{\circ}{Y}$. This restriction, for the time being, will be assumed in the form

$$\int_{h_-}^{h_+} J_{KLMN} G_\tau^{KLMN} \, d\xi = 0 \, , \quad \tau = 1,\dots,T,$$

where G_τ^{KLMN} are the known functions of $\underset{\sim}{X}, e_{(r)}$ and the derivatives of $e_{(r)}$ with respect to θ^α. Thus the "shell approximation" of the condition $\underset{\sim}{R}(\underset{\sim}{C}) = \underset{\sim}{O}$ will have the form

$$\int_{h_-}^{h_+} R_{KLMN}(C(X, e_{(r)})) G_\tau^{KLMN} \, d\xi = 0, \quad \tau = 1,\dots,T. \qquad \text{(A2.17)}$$

Eq. (A2.17) represents a system of interrelations imposed on $e_{(r)}$ which will be called the shell compatibility conditions. They are used in the intrinsic formulations of shell theories (in which as the unknowns we take the triples $(e_{(r)}, \tau^{(N)}, \omega^{(p)})$ togehter with the equilibrium equations and the constitutive relations. The number T of the shell compatibility conditions in the shell intrinsic formulations satsify the condition $T \geq r - n$.

2.3. Second order approach

The simple approach to the plate and shell theories leads to the equations of motion which are the differential equations of the first order with respect to θ^α. In the simple approach also the number of the kinetic boundary conditions is equal to the number of the equations of motion. The obtained systems of the shell external forces $\{p_R^a, f_R^a;$ $a = 1,\dots,n\}$ and the shell internal forces $\{h_R^a, (H_R^{a\alpha}, \alpha = 1,2);$ $a = 1,\dots,n\}$ will be called simple, i. e., they do the work on the fields $\dot{q}_a$ and $\dot{q}_a, \dot{q}_{a,\alpha}$, respectively. Now we are to develop an approach which leads to more general form of the field equations.

We seek to approximate the set Ξ_1 of all deformation function by the set $\tilde{\Xi}_1$, such that $(\underset{\sim}{p},\underset{\sim}{\tilde{p}}) \in \Xi_1 \times \tilde{\Xi}_1$ holds iff

$$\underset{\sim}{p}(\underset{\sim}{\theta},\xi,t) \sim \underset{\sim}{\tilde{p}}(\underset{\sim}{\theta},\xi,t,q_{(n)}(\underset{\sim}{\theta},t),\nabla q_{(n)}(\underset{\sim}{\theta},t)),q_{(n)} \in Q \,. \qquad \text{(A2.18)}$$

It is more general case than that given by Eq. (A2.3). At the same time we leave the approximations (A2.6), (A2.7) unchanged.

<u>Example</u>. The shell theory will be called the Cosserat surface theory if the approximation relation (A2.18) has the form

$$\underset{\sim}{p}(\underset{\sim}{\theta},\xi,t) \sim \underset{\sim}{r}(\underset{\sim}{\theta},t) + \xi \underset{\sim}{d}(\underset{\sim}{\theta},t) \,,$$

where $q_{(n)} = q_{(6)} = \{r_k, d_k, k = 1,2,3\}$ and $Q := \{q_{(6)} | [\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{d}] > 0\}$ where $\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha}$. It is a special case of Eqs. (A2.3), provided that the fields $\underset{\sim}{r},\underset{\sim}{d}$ are independent. For the known Love-Kirchhoff theory we postulate

$$\underset{\sim}{p}(\theta,\xi,t) \sim \underset{\sim}{r}(\theta,t) + \xi \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} \,, \quad \underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha} \,,$$

where $\{r_k, k = 1,2,3\} = q_{(3)}$ and $Q := \{q_{(3)} | \det a_{\alpha\beta} > 0 \,,\ a_{\alpha\beta} \equiv \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta\}$. It is a special case of Eq. (A2.18) but not of Eq. (A2.3).

The error fields are now also defined by means of Eqs. (A2.8), but the restriction of the error fields will be different ([1]). Putting

$$L_R \equiv \int_{h_-}^{h_+} \underset{\sim}{r}_R \cdot \underset{\sim}{\tilde{p}}\, d\xi + [\underset{\sim}{s}_R \cdot \underset{\sim}{\tilde{p}}]_{h_+} + [\underset{\sim}{s}_R \cdot \underset{\sim}{\tilde{p}}]_{h_-} \,, \quad \underset{\sim}{\theta} \in \Pi \,,$$

$$M_R \equiv \int_{h_-}^{h_-} \underset{\sim}{s}_R \cdot \underset{\sim}{\tilde{p}}\, d\xi \,, \quad \underset{\sim}{\theta} \in \partial\Pi \quad \text{a.e.} \,,$$

let us define the functionals

([1]) It may be shown that the restriction of the fields $\underset{\sim}{r}_R, \underset{\sim}{s}_R$, given by Eqs. $(A2.10)_{1,2}$, in the case of the approximation relation (A2.18) will lead to the uncorrect form of the shell field equations.

$$E_{\tau^{(N)}} = \int_{\Pi} L_R da_R + \oint_{\partial\Pi} M_R dl_R \quad . \tag{A2.19}$$

The functionals $E_{\tau^{(N)}}$ will be treated, for every $\tau^{(N)}$, as defined on the space Q of the shell deformation functions $q_{(n)}$. We shall restrict the error fields (A2.8) by postulating Eqs. $(A2.10)_{3-5}$ and replacing Eqs. $(A2.10)_{1,2}$ by the stationary condition

$$\delta E_{\tau^{(N)}} = 0 \tag{A2.20}$$

for an arbitrary but fixed $\tau^{(N)}$.

The simple calculations lead now to the following equations which hold in Π

$$\left(\frac{\partial L_R}{\partial q_{a,\alpha}}\right)_{,\alpha} - \frac{\partial L_R}{\partial q_a} = 0 \quad , \quad a = 1,\dots,n \quad , \quad \underset{\sim}{\theta} \in \Pi \quad , \quad t \in I \; , \tag{A2.21}$$

and to the conditions which have to be satisfied almoste everywhere on $\partial\Pi$

$$\frac{\partial M_R}{\partial q_a} - \frac{d}{dl_R}\left(\frac{\partial M_R}{\partial q_{a,\alpha}} t_{R\alpha}\right) + \frac{\partial L_R}{q_{a,\alpha}} n_{R\alpha} = 0 \quad ,$$

$$\frac{\partial M_R}{\partial q_{a,\alpha}} n_{R\alpha} = 0 \qquad a = 1,\dots,n \; , \; \underset{\sim}{\theta} \in \partial\Pi \; , \; t \in I \; , \tag{A2.22}$$

where $\underset{\sim}{t}_R$ is the unit vector tangent to $\partial\Pi$. Introducing the denotations

$$\underset{\sim}{\phi}^a \equiv \frac{\partial \underset{\sim}{\tilde{p}}}{\partial q_a} \quad , \quad \underset{\sim}{\psi}^{a\alpha} \equiv \frac{\partial \underset{\sim}{\tilde{p}}}{\partial q_{a,\alpha}} \quad , \tag{A2.23}$$

using the first Piola-Kirchhoff stress tensor $\underset{\sim}{\tilde{T}}_R \equiv \nabla \underset{\sim}{\tilde{p}} \underset{\sim}{\tilde{T}}$ and putting

$$H_R^{a\alpha\beta} \equiv - \int_{h_-}^{h_+} T_R^{k\beta} \psi_k^{a\alpha} d\xi \quad ,$$

$$H_R^{a\alpha} \equiv \int_{h_-}^{h_+} (\tilde{T}_R^{k\alpha}\phi_k^a + \tilde{T}_R^{kK}\psi_{k,K}^{a\alpha}) d\xi \quad , \qquad K = 1,2,3 \; ,$$

$$h_R^a \equiv -\int_{h_-}^{h_+} \tilde{T}_R^{kK} \Phi^a_{k,K} \, d\xi \quad ,$$

$$f_R^{a\alpha} \equiv \int_{h_-}^{h_+} b_R^k \Psi_k^{a\alpha} d\xi + [\Psi_k^{a\alpha} p_R^k]_{h_+} + [\Psi_k^{a\alpha} p_R^k]_{h_-} \quad ,$$

$$f_R^a \equiv \int_{h_-}^{h_+} b_R^k \Phi_k^a d\xi + [\Phi_k^a p_R^k]_{h_+} + [\Phi_k^a p_R^k]_{h_-} \quad ,$$

$$i_R^{a\alpha} \equiv \int_{h_-}^{h_+} \rho_R \ddot{\tilde{p}}^k \Psi_k^{a\alpha} d\xi = i_R^{a\alpha}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \dot{q}_{(n)}, \nabla\dot{q}_{(n)}, \ddot{q}_{(n)}, \nabla\ddot{q}_{(n)}) \, ,$$

$$i_R^a \equiv \int_{h_-}^{h_+} \rho_R \ddot{\tilde{p}}^k \Phi_k^a d\Xi = i_R^a(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \dot{q}_{(n)}, \nabla\dot{q}_{(n)}, \ddot{q}_{(n)}, \nabla\ddot{q}_{(n)}) \, , \qquad \text{(A2.24)}$$

for every $\underset{\sim}{\theta} \in \Pi$ as well as

$$p_R^{a\alpha} \equiv \int_{h_-}^{h_+} p_R^k \Psi_k^{a\alpha} d\xi \, , \qquad p_R^a \equiv \int_{h_-}^{h_+} p_R^k \Phi_k^a d\xi \qquad \text{(A2.25)}$$

for almost every $\underset{\sim}{\theta} \in \partial\Pi$ we obtain from Eqs. (A2.21), (A2.22), after many manipulations, the shell field equations. The *shell equations of motion* will be given by

$$H^{a\alpha\beta}_{R,\alpha\beta} + H^{a\alpha}_{R,\alpha} + h_R^a + f_R^a - f^{a\alpha}_{R,\alpha} = i_R^a - i^{a\alpha}_{R,\alpha} \qquad \text{(A2.26)}$$

and have to hold in Π (for every $t \in I$). The *shell kinetic boundary conditions* will have the form

$$H_R^{a\alpha} n_{R\alpha} + \frac{d}{dl_R}(H_R^{a\alpha\beta} n_{R\beta} t_{R\alpha}) + H^{a\beta\alpha}_{R,\beta} n_{R\alpha} = p^a_{OR} - (i_R^{a\alpha} - f_R^{a\alpha}) n_{R\alpha} \, ,$$

$$H_R^{a\alpha\beta} n_{R\alpha} n_{R\beta} = -p_{\underset{\sim}{R}}^{aN} \, , \qquad \text{(A2.27)}$$

and have to hold almoste everywhere on $\partial\Pi$ (for every $t \in I$); we have used here the extra denotations

$$p^a_{OR} \equiv p_R^a - \frac{d}{dl_R}(p_R^\alpha t_{R\alpha}) \, , \quad p_{\underset{\sim}{R}}^{aN} \equiv p_R^{a\alpha} n_{R\alpha} \, . \qquad \text{(A2.28)}$$

Let us substitute now to Eqs. (A2.24)$_{1,2,3}$ the RHS of $\tilde{\underset{\sim}{T}}_R = \nabla\tilde{\underset{\sim}{p}}\tilde{\underset{\sim}{T}}$, $\tilde{\underset{\sim}{T}} = \underset{\sim}{S}(\tilde{\underset{\sim}{\sigma}})$, where $\tilde{\underset{\sim}{\sigma}}$ is given by Eqs. (A2.6). Denoting

$$\tilde{H}_R^{a\alpha\beta} \equiv \tilde{H}_R^{a\alpha\beta}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}, \tau^{(N)}) \equiv -\int_{h_-}^{h_+} \tilde{p}^k_{,K} S^{K\beta}(\tilde{\underset{\sim}{\sigma}}) \Psi^{a\alpha}_k \, d\xi ,$$

$$\tilde{H}_R^{a\alpha} \equiv \tilde{H}_R^{a\alpha\beta}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)} \tau^{(N)}) \equiv \int_{h_-}^{h_+} (\tilde{p}^k_{,K} S^{K\alpha}(\tilde{\underset{\sim}{\sigma}}) \phi^a_k + \tilde{p}^k_{,L} S^{LK}(\tilde{\underset{\sim}{\sigma}}) \Psi^{a\alpha}_{k,K}) \, d\xi ,$$

$$H_R^a \equiv \tilde{h}_R^a(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)} \tau^{(N)}) \equiv -\int_{h_-}^{h_+} \tilde{p}^k_{,L} S^{LK}(\tilde{\underset{\sim}{\sigma}}) \; {}^a_{k,K} \, d\xi , \qquad \text{(A2.29)}$$

we obtain

$$\begin{aligned} H_R^{a\alpha\beta}(\underset{\sim}{\theta}, t) &= \tilde{H}_R^{a\alpha\beta}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}, \tau^{(N)}), \\ H_R^{a\alpha}(\underset{\sim}{\theta}, t) &= \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}, \tau^{(N)}), \qquad \text{(A2.30)} \\ h_R^a(\underset{\sim}{\theta}, t) &= h_R^a(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}, \tau^{(N)}), \end{aligned}$$

for every $\theta \in \Pi$, $t \in I$. Eqs. (A2.30) and (A2.14) will be referred to as the shell *constituitve relations*. IfM = 6 then the term $\tau^{(N)}$ will drop out form Eqs. (A2.30) and Eqs. (A2.14) will be identities (cf. Sec. 2.2).

The functions on the LHS of Eqs. (A2.26) are defined on $\bar{\Pi} \times I$ and the functions in Eqs. (A2.28) are defined almost everywhere on $\partial\Pi \times I$. On the RHS of Eqs. (A2.30) there are known constitutive functionals or functions and on the RHS of Eqs. (A2.26) there are known functions obtained from Eqs. (A2.24)$_{6,7}$.

The terms i_R^a, $i_R^{a\alpha}$ can be also obtained from the *shell kinetic energy function*, defined by

$$\kappa_R = \kappa_R(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \dot{q}_{(n)}, \nabla\dot{q}_{(n)}) \equiv \frac{1}{2}\int_{h_-}^{h_+} \rho_R \dot{\tilde{\underset{\sim}{p}}} \cdot \dot{\tilde{\underset{\sim}{p}}} \, d\xi , \qquad \text{(A2.2.31)}$$

by means of the relations

$$i_R^a = \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}_a} - \frac{\partial\kappa_R}{\partial q_a} \quad , \quad i_R^{a\alpha} = \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}_{a,\alpha}} - \frac{\partial\kappa_R}{q_{a,\alpha}} \quad . \tag{A2.32}$$

If the material of the shell-like body is hyperelastic, i. e., if Eqs. (A2.2) have the form

$$\underset{\sim}{T}_R = \sigma_R \frac{\partial\sigma}{\partial\nabla\underset{\sim}{p}} \quad ,$$

where $\sigma = \sigma(\underset{\sim}{X},\nabla\underset{\sim}{p})$, $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$, is the strain energy function, then introducing the *shell strain energy function*

$$\varepsilon_R = \varepsilon_R(\underset{\sim}{\theta},q_{(n)},\nabla q_{(n)},\nabla\nabla q_{(n)}) \equiv \int_{h_-}^{h_+} \rho_R \sigma(\underset{\sim}{X},\nabla\tilde{\underset{\sim}{p}})\, d\xi \tag{A2.33}$$

we obtain the constitutive equations (A2.30) in the form

$$H_R^{a\alpha\beta} = -\frac{\partial\varepsilon_R}{\partial q_{a,\alpha\beta}} \quad , \quad H_R^{a\alpha} = \frac{\partial\varepsilon_R}{q_{a,\alpha}} \quad , \quad h_R^a = -\frac{\partial\varepsilon_R}{\partial q_a} \quad , \tag{A2.34}$$

Eqs. (A2.34) can be treated as the definition of the hyperelastic shell provided that Eqs. (A2.23) hold.

<u>Remark 1</u>. The field equations of the shell theory (A2.26) (A2.27) as well as Eqs. (A2.24), (A2.25) can be also obtained by postulating that the relation

$$\oint_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{p}_R \cdot \delta\underset{\sim}{p}\, da_R + \int_{\kappa_R(\mathcal{B})} (\underset{\sim}{b}_R - \rho_R \ddot{\tilde{\underset{\sim}{p}}}) \cdot \delta\underset{\sim}{p}\, dv_R = \int_{\kappa_R(\mathcal{B})} \mathrm{tr}(\tilde{\underset{\sim}{T}}_R \nabla\delta\underset{\sim}{p})\, dv_R \tag{A2.35}$$

holds for every

$$\delta\underset{\sim}{p} = \frac{\partial\tilde{\underset{\sim}{p}}}{\partial q_a} q_a + \frac{\partial\tilde{\underset{\sim}{p}}}{\partial q_{a,\alpha}} \delta q_{a,\alpha}$$

where δq_a are arbitrary independent real-valued functions defined and continuous on $\bar{\Pi}$ and smooth in Π. In view of $\underset{\sim}{p} \sim \tilde{\underset{\sim}{p}}$, $\underset{\sim}{T}_R \sim \tilde{\underset{\sim}{T}}_R$, Eq. (A2.35) can be interpreted as certain "approximation" of the principle of virtual work for a shell-like body. The suitable restriction of the error fields $\underset{\sim}{s}_R$, $\underset{\sim}{r}_R$ can be derived from Eq. (A2.35) and from the definitions of these fields. It is represented by the condition

$$\oint_{\partial\Pi} \int_{h_-}^{h_+} \underset{\sim}{s}_R \cdot \delta\underset{\sim}{p}\, d\xi\, dl_R + \int_{\Pi} \Big(\int_{h_-}^{h_+} \underset{\sim}{r}_R \cdot \delta\underset{\sim}{p}\, d\xi + [\underset{\sim}{s}_R \cdot \delta\underset{\sim}{p}]_{h_+} + [\underset{\sim}{s}_R \cdot \delta p]_{h_-} \Big) da_R = 0 \qquad \text{(A2.36)}$$

which has to hold for every $\delta\underset{\sim}{p}$ defined above. The foregoing condition is equivalent to the condition (A2.20).

<u>Remark 2</u>. Now let us reject Eqs. (A2.23) and assume that $\underset{\sim}{\Phi}^a, \underset{\sim}{\Psi}^{a\alpha}$; $a = 1,\ldots,n$; $\alpha = 1,2$, are the known sufficiently regular and independent vector functions of arguments $\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)}$. The restriction of the error fields can be also postulated in more general form by assuming that the condition (A2.36) has to hold for every

$$\delta\underset{\sim}{p} = \underset{\sim}{\Phi}^a \delta q_a + \underset{\sim}{\Psi}^{a\alpha} \delta q_{a,\alpha} \quad , \qquad \text{(A2.37)}$$

where the functions δq_a have the same meaning as in the Remark 1. Under this assumption we shall obtain again the field equations (A2.26), (A2.27) with the denotations (A2.28) and the formulas (A2.24), (A2.25). If the conditions (A2.23) do not hold then Eqs. (A2.32) will not be valid; in this case also Eqs. (A2.34) will not hold even if the material of the shell-like body is hyperelastic.

<u>Remark 3</u>: Putting $\Psi^{a\alpha} = 0$ for $a = 1,\ldots,n$ and $\alpha = 1,2$ in the case described in the Remark 2 and rejecting the argument $\nabla q_{(n)}$ of the function $\underset{\sim}{\tilde{p}}$, we obtain the results of Sec. 2.2.. Thus the simple approach to the shell field equations can be treated as the special case of such second-order approach which is governed by the assumption that Eq. (A2.36) has to hold for every $\nabla\underset{\sim}{p}$ given by Eq. (A2.37).

2.4. <u>Generalization</u>

Within the procedure outlined above the approach to the shell theory is determined by the form of functions $\underset{\sim}{\tilde{p}}(\cdot), \tilde{\sigma}^{\mu}(\cdot), \tilde{\lambda}^{\sigma}(\cdot)$ in Eqs. (A2.3), (A2.6), (A2.7), respectively (they describe the approximation relation) and by the form of functions $\underset{\sim}{\Phi}^a(\cdot), \underset{\sim}{\Psi}^{a\alpha}(\cdot)$, $\Xi^{\mu}_{\bar{A}}(\cdot)$ in Eqs. (A2.24), (A2.25), (A2.29), $(A2.10)_3$, which characterize the restriction of the error fields. We shall see now that the restriction of the error fields can be assumed in the more general form than that postulated by Eqs. $(A2.10)_{3-5}$ and Eqs. (A2.36), (A2.37). To this aid, instead of the smooth or continuous functions $\underset{\sim}{\Phi}^a$, $\underset{\sim}{\Psi}^{a\alpha}$, $\Xi^{\mu}_{\bar{A}}$, let us introduce the functions $\bar{\underset{\sim}{\Phi}}^a$, $\bar{\underset{\sim}{\Psi}}^{a\alpha}$, $\bar{\Xi}_{\bar{A}}$, $\bar{b}$, $\bar{\beta}$ of the arguments $\underset{\sim}{\theta}, \xi, t$ and $q_{(n)}, \tau^{(N)}, \omega^{(p)}, \nabla q_{(n)}$. we assume that these functions can attain, for example, the non-zero values only on the finite subset of $\langle h_-, h_+\rangle$. Instead of the restriction of error fields $\underset{\sim}{s}_R$, $\underset{\sim}{r}_R$ given by Eqs. (A2.36), (A2.37), let us assume that the relation

$$\int_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{s}_R \cdot d(\delta \underset{\sim}{p}) + \int_{\kappa_R(\mathcal{B})} \underset{\sim}{r}_R \cdot d(\delta \underset{\sim}{p}) = 0 \tag{A2.38}$$

holds for every $d(\delta \underset{\sim}{p})$, such that

$$d(\delta \underset{\sim}{p}) = \begin{cases} (\delta q_a \bar{\underset{\sim}{\Phi}}^a + \delta q_{a,\alpha} \bar{\underset{\sim}{\Psi}}^{a\alpha}) da_R & \text{on } \Pi \times \{h_-\} \text{ and } \Pi \times \{h_+\}, \\ (\delta q_a d\bar{\underset{\sim}{\Phi}}^a + \delta q_{a,\alpha} d\bar{\underset{\sim}{\Psi}}^{a\alpha}) dl_R & \text{on } \partial\Pi \times (h_-, h_+) , \\ (\delta q_a d\bar{\underset{\sim}{\Phi}}^a + \delta q_{a,\alpha} d\bar{\underset{\sim}{\Psi}}^a) da_R & \text{on } \Pi \times (h_-, h_+) , \end{cases} \tag{A2.39}$$

where δq_a are arbitrary independent functions defined on and continuous $\bar{\Pi}$ and smooth in Π. Instead of the restrictions of the error fields $a_{(m)}$, a, α given by Eqs. $(A2.10)_{3-5}$, let us postulate the restrictions

$$\sum_{\bar{\mu}=M+1}^{m} \int_{h_-}^{h_+} a_\mu d\bar{\Xi}^{\mu}_{\bar{A}} = 0 \quad , \quad \bar{A} = 1,\dots,N+P \quad ,$$
$$\int_{h_-}^{h_+} a\, d\bar{b} = 0 \; , \quad \int_{h_-}^{h_+} \alpha\, d\bar{\beta} = 0 \; . \tag{A2.40}$$

Thus the functions $\bar{\underset{\sim}{\Phi}}^a, \dots, \bar{\beta}$ introduced above together will the error fields have to ensure the existence of the Stieltes integrals in Eqs. (A2.38), +A2.40). By the direct calculation we obtain again the shell field equations (A2.26), (A2.27) and the shell constitutive relations (A2.30), (A2.14), but instead of Eqs. $(A2.24)_{1-5}$, (A2.25) we arrive at the formulas with the Stieltes integrals [1]

(1) The terms $[\Psi^{a\alpha}_k]_{h_+}$, $[\Psi^{a\alpha}_k]_{h_-}$, in Eq. $(A2.24)_4$ have to be replaced by by $[\bar{\Psi}^{a\alpha}_k]_{h_+}$, $[\bar{\Psi}^{a\alpha}_k]_{h_-}$, respectively, cf. Eq. (A2.39)

$$H_R^{a\alpha\beta} \equiv -\int_{h_-}^{h_+} \tilde{T}_R^{k\beta} d\bar{\Psi}_k^{a\alpha}, \ldots, p_R^a \equiv \int_{h_-}^{h_+} p_R^k d\bar{\Phi}_k^a \quad , \tag{A2.41}$$

and instead of Eqs. (A2.13) we obtain

$$g_A \equiv \int_{h_-}^{h_+} \tilde{f}_\mu d\bar{\Xi}_A^\mu \quad , \quad \kappa \equiv \int_{h_-}^{h_+} \tilde{j}\, d\bar{b} \quad , \quad \psi \equiv \int_{h_-}^{h_+} \tilde{\varphi}\, d\,\bar{\beta} \quad . \tag{A2.42}$$

Analogously, instead of Eqs. (A2.29) for the response functionals we obtain the formulae

$$\begin{aligned} \tilde{H}_R^{a\alpha\beta} &\equiv -\int_{h_-}^{h_+} \tilde{p}^k_{,K} S^{K\beta}(\tilde{\underset{\sim}{\sigma}}) d\bar{\Psi}_k^{a\alpha} \\ \tilde{h}_R^{a\alpha} &\equiv \int_{h_-}^{h_+} \tilde{p}^k_{,K} S^{K\alpha}(\tilde{\underset{\sim}{\sigma}}) d\bar{\Phi}_k^a + \tilde{p}^k_{,L} S^{LK}(\tilde{\underset{\sim}{\sigma}}) d\bar{\Psi}_{k,K}^{a\alpha} \quad , \\ \tilde{h}_R^a &\equiv -\int_{h_-}^{h_+} \tilde{p}^k_{,L} S^{LK}(\tilde{\underset{\sim}{\sigma}}) d\bar{\Phi}_{k,K}^a \quad , \end{aligned} \tag{A2.43}$$

and Eqs. $(A2.24)_{6,7}$ for the inertia forces have now to be replaced by

$$i_R^a \equiv \int_{h_-}^{h_+} \rho_R \ddot{\tilde{p}}^k d\bar{\Phi}_k^a \quad , \quad i_R^{a\alpha} \equiv \int_{h_-}^{h_+} \rho_R \ddot{\tilde{p}}^k d\bar{\Psi}_k^{a\alpha} \quad . \tag{A2.44}$$

The analogous procedure can be also applied in order to obtain the more general form of the compatibility conditions, i. e., instead of Eqs. (A2.17) we obtain

$$\int_{h_-}^{h_+} R_{KLMN}(\tilde{\underset{\sim}{C}}(\underset{\sim}{X}, e_{(r)})\, d\bar{G}_\tau^{KLMN} = 0, \qquad \tau = 1,\ldots,T \;, \tag{A2.45}$$

where $\bar{G}_\tau^{KLMN}$ are the known functions.

If $d\bar{\Phi}^a = \Phi^a d\xi, \dots, d\bar{\Xi}^\mu_A = \Xi^\mu_A d\xi$, $d\bar{b} = d\xi$, $d\bar{\beta} = d\xi$, then we shall arrive at the formulas obtained in Secs. 2.2., 2.3.. The example of the equations of the shell theory obtained by the approach outlined above (which, at the same time, cannot be derived from the relations of Secs. 2.2., 2.3.) can be found in [21].

Summing up, the approximation of the solid mechanics relations (A2.1), (A2.2) leads to the shell theory determined by the field equations (A2.26), (A2.27) and by the constitutive relations (A2.30), (A2.14). The fields in the shell theory are related to the fields of the solid mechanics by means of Eqs. (A2.41), (A2.42), (A2.9) and by the approximation relations (A2.18), (A2.5), (A2.6), (A2.7) The shell response functionals in Eqs. (A2.30) are defined by Eqs. (A2.43) and the shell inertia forces are related to the distribution of the mass density ρ_R by Eqs. (A2.44). It must be stressed, however, that the relations (A2.26), (A2.27), (A2.30), (A2.14) of the shell theory constitute the "good" approximation of the solid mechanics relations (A2.1), (A2.2) only for a certain calss of problems (cf. Remark 2) in Sec. 2.0.). It means that the functions $\tilde{p}, \tilde{\underset{\sim}{\sigma}}, \tilde{\lambda}^{(s)}$ (cf. Eqs. (A2.3), (A2.6), (A2.7)) and the functions $\underset{\sim}{\bar{\Phi}}^a, \underset{\sim}{\bar{\Psi}}^{a\alpha}, \bar{\Xi}^\mu_A, \bar{b}, \bar{\beta}$ (cf. (A2.40)-(A2.44), which describe the analytical structure of the formal approximation, cannot be quite arbitrary, but have to lead to the shell theory which is a "good" approximation of the solid mechanics problems under consideration.

3. MIXED APPROACH

Mixed approach combines together the direct approach with the formal approximation of the solid mechanics relations. It means that some from the "shell" fields (i.e., the fields defined on $\bar{\Pi} \times I$, $\Pi \times I$ or almost everywhere on $\partial\Pi \times I$) are postulated *a priori* and others are derived from the relations of solid mechanics. Here we shall give only one example of the mixed approach. To this aid we shall assume that the material of the shell-like body is simple and that:

1. For every set $\delta q_{(n)}$ of real valued functions δq_a, defined and continuous in $\bar{\Pi}$ and smooth in Π, the following relation hold

$$\oint_{\partial\Pi} (p_R^a \delta q_a + p_R^{a\underset{\sim}{N}} \delta q_{a,\underset{\sim}{N}}) dl_R + \int_{\Pi} (f_R^a - i_R^a) \delta q_a da_R = \int_{\kappa_R(\mathcal{B})} \tilde{T}_R^{k\beta} \delta p_{k,\beta} dv_R \,, \tag{A3.1}$$

where

$$\delta q_{a,\underset{\sim}{N}} \equiv q_{a,\alpha} n_R^\alpha \,, \quad \delta p_k \equiv \Phi_k^a \delta q_a + \Psi_k^{a\alpha} \delta q_{a,\alpha}$$

$$\underset{\sim}{\tilde{T}}_R \equiv \nabla \underset{\sim}{\tilde{p}} \underset{\sim}{\tilde{T}} \,, \quad \underset{\sim}{\tilde{p}} = \underset{\sim}{\tilde{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)}),$$

$$\underset{\sim}{\tilde{T}} = \underset{\sim}{\tilde{T}}(\underset{\sim}{X}, \nabla \underset{\sim}{\tilde{p}}) \,,$$

and where $\underset{\sim}{\Phi}^a$, $\underset{\sim}{\Psi}^{a\alpha}$, $\underset{\sim}{\tilde{p}}$ are the known functions of $\underset{\sim}{X}$, $q_{(n)}$, $\nabla q_{(n)}$ and $\underset{\sim}{\tilde{T}}(\underset{\sim}{X}, \cdot)$ is the known response functional of the simple material.

2. The shell inertial forces are given by

$$i_R^a = \frac{d}{dt}\frac{\partial \kappa_R}{\partial \dot{q}_a} - \frac{\partial \kappa_R}{\partial q_a} \,, \quad \kappa_R \equiv \frac{1}{2}\int_{h_-}^{h_+} \rho_R \dot{\underset{\sim}{\tilde{p}}} \cdot \dot{\underset{\sim}{\tilde{p}}} \, d\xi \,.$$

3. The motion of the shell is approximated by the function $\underset{\sim}{x} = \underset{\sim}{\tilde{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)})$, $q_{(n)} \in Q$, where $Q := \{q_{(n)} \,|\, \det \nabla \underset{\sim}{\tilde{p}} > 0\}$.

From Eq. (A3.1) we obtain the shell equations of motion

$$H^{a\alpha\beta}_{R,\alpha\beta} + H^{a\alpha}_{R,\alpha} + h^{a}_{R} + f^{a}_{R} = i^{a}_{R} , \qquad \text{(A3.2)}$$

and the shell kinetic boundary conditions

$$\begin{aligned} &H^{a\alpha}_{R} n_{R\alpha} + \frac{d}{dl_R} (H^{a\alpha\beta}_{R} n_{R\beta} t_{R\alpha}) + H^{a\alpha\beta}_{R,\alpha} n_{R\beta} = p^{a}_{OR} , \\ &H^{a\alpha\beta}_{R} n_{R\alpha} n_{R\beta} = -p^{aN}_{R} , \end{aligned} \qquad \text{(A3.3)}$$

with the denotations (A2.28). We also obtain the constitutive relations

$$\begin{aligned} H^{a\alpha\beta}_{R} &= \tilde{H}^{a\alpha\beta}_{R} (\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}) \\ H^{a\alpha}_{R} &= \tilde{H}^{a\alpha}_{R} (\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}), \\ h^{a}_{R} &= \tilde{h}^{a}_{R} (\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}), \end{aligned} \qquad \text{(A3.4)}$$

with the RHS defined by the RHS of formulas $(A2.24)_{1-3}$ in which $\underset{\sim}{\tilde{T}} = \underset{\sim}{\tilde{T}}(\underset{\sim}{X}, \nabla\underset{\sim}{\tilde{p}})$ is the response functional

In the shell theory described by Eqs. (A3.2) - (A3.4), the shell external forces p^{a}_{R}, $p^{a\alpha}_{R}$, f^{a}_{R} have been introduced by the direct approach but the shell internal forces $H^{a\alpha\beta}_{R}$, $H^{a\alpha}_{R}$, h^{a}_{R} and the shell interia forces $-i^{a}_{R}$ are related to the "three-dimensional" description of the shell-like body, analogously as in the approximation approach.

4. CONSTRAINT APPROACH

In this section we are to detail the approach in which the plate or shell theory is included into the "three-dimensional" description of the shell-like body. Roughly speaking, we shall deal with the "three-dimensional" description of mechanics of the shell-like body in which all boundary value problems are "two-dimensional", i.e., they can be stated exclusively for the relations of the shell theory. The idea of such treatment of mechanics of the shell-like bodies in based on the concept of constraint and will bereferred to as the constraint approach.

4.0. Analytical preliminaries

The concept of constraints, which up to now has been used almost exclusively in mechanics (including some problems of thermo- and electromechanics), has more general meaning. In this subsection we are to introduce the concept of constraints independently of any problem of mechanics, i. e., as a certain analytical concept. Such approach makes it possible to use the concept of constraints in an arbitrary field theory. Moreover, the basic notions concerning constraints seem to be more clear when treated independently of the physical interpretations. It must be stressed, however, that such general approach to the concept of contraints is based on the hidden assumption that the relations we are to introduce are motivated or implied by certain physical situations.

Let X be the topological space of vector or scalar valued functions defined on a certain differentiable manifold Ω(or on its closure $\bar{\Omega}$) and let Y be a certain linear space [1]. Moreover, let A be the known mapping with the domain D(A) in X and with the range R(A) in the linear space Y. It means that for every $y, y \in R(A)$, there exists at least one element $x, x \in D(A)$, such that $A(x) = y$.

Now suppose that Δ is the known non-empty subset of R(A). The governing relations of many field theories, in which x,y are the fields under consideration, are given by

---

([1]) In the special cases, which will be referred to as strictly local, the manifold Ω reduces to the one point set.

$$A(x) = y \quad , \; y \in \Delta \; . \tag{A4.1}$$

Eq. (A4.1) represents certain binary relation with the domain $\Xi \equiv A^{-1}(\Delta)$ and the range Δ. If $\Delta = \{\theta\}$ then this relation will constitute the equation $A(x) = \theta$ and if $\Delta = R(A)$ then Eq. (A4.1) will represent the mapping. In what follows we shall assume that the relation given by Eq. (A4.1) is known, i.e., that there are known the operator A and the set Δ, $\Delta \subset R(A)$. We shall also interpret Eq. (A4.1) as the govering relation of a certain field theory.

Now suppose that within the field theory governed by Eq. (A4.1) we are to describe some special problem or some class of such problems. To this aid we have, roughly speaking, to restrict or to modify Eq. (A4.1) by taking into account certain "extra" data which characterize the class of problems under consideration. If, for example, Eq. (A4.1) represent the system of the differential equations then as the "extra" data we can take the suitable boundary or initial conditions and the "modification" of Eq. (A4.1) leads to a certain initial-boundary value problem. In the general case we shall say that certain constraints are imposed on Eq. (A4.1).

To formulate the formal definition of constraints (i.e., the definition independent of the physical structure of problems) let us firstly denot by $\overset{\circ}{Y}$ the subset of Y which is the range of the mapping $\Xi \times \Delta \to Y$ given by $\overset{\circ}{y} = A(x) - y$, $x \in \Xi$, $y \in \Delta$. The concept of the *constraints* (imposed on the relation given by Eq. (A4.1)) we shall obtain as a special case of more general concept of the *semiconstraints*.

<u>Definition 1</u>. We shall say that the semiconstraints are imposed on Eq. (A4.1) if there are known:

1. The non-empty subset $\tilde{\Xi}$ of $D(A)$,
2. The multifunction

$$m : \tilde{\Xi} \ni x \to \overset{\circ}{Y}_x \subset \overset{\circ}{Y} \quad ; \quad \overset{\circ}{Y}_x \neq \phi \quad , \tag{A4.2}$$

and if for every $y, y \in \Delta$, there exists at least one pair $(x, \overset{\circ}{y}) \in \tilde{\Xi} \times \overset{\circ}{Y}_x$, such that $A(x) = y + \overset{\circ}{y}$.

Definition 2. The semiconstraints (imposed on Eqa. (A4.1) will be called constraints if $\tilde{\Xi} \subset \Xi$; the foregoing inclusion will be called the constraint inclusion.

Generally speaking, in the case of constraints we restrict the domain of the binary relation given by Eq. (A4.1)
Let us observe, that for some $y, y \in \Delta$, it may happen that it does not exist $x, x \in \tilde{\Xi}$, such that $A(x) = y$. Thus the fild $\overset{\circ}{y}$, $\overset{\circ}{y} = A(x)-y$, can be interpreted as the field "maintaining" the semi constraints.

In what follows we shall use more general concept of semiconstraints instead that of constraints ([1]). If for every $x \in \operatorname{int} \tilde{\Xi}$ there is $(A(x) \in \Delta) \Rightarrow (\overset{\circ}{Y}_x = \{\theta\})$ then the semiconstraints will be called *correctly imposed* (on the relation given by Eq. (A4.1). If $(\overset{\circ}{y} \in \overset{\circ}{Y}_x) \Rightarrow (-\overset{\circ}{y} \in \overset{\circ}{Y}_x)$ holds for every $\overset{\circ}{y}$, $\overset{\circ}{y} \in \overset{\circ}{Y}$, then these semiconstraints will be called *bilateral* with respect to x, $x \in \tilde{\Xi}$, and if $(\overset{\circ}{y} \in \overset{\circ}{Y}_x) \Rightarrow (-\overset{\circ}{y} \notin \overset{\circ}{Y}_x)$ holds for every $\overset{\circ}{y}$ $\overset{\circ}{y} \in \overset{\circ}{Y} \setminus \{\theta\}$, then they will be called *unilateral* with respect to x, $x \in \tilde{\Xi}$. The multifunction (A4.2) will be referred to as the realization of semiconstraints. It must be stressed that in the problems of mechanics the realization of seminconstraints has to describe the physical character of the problems and the semiconstraints imposed on the relation (A4.1) have to lead to the resonable solutions of these problems.

Example. Let Eq. (A4.1) represents the mapping from the space $X(\Omega)$ of functions defined on the n-th demensional differentiable manifold to the linear space $Y(\Omega)$ of functions defined on the same manifold Ω. This is the case in which $\Delta = R(A)$. Let us also assume that $\Omega = \Pi \times \Gamma$, where Ω, Γ are the differentiable manifolds of the orders k, l respectively, $k + l = n$ ([2]). Our objective is to give an example of the constraints imposed on the mapping $A(x) = y$, $x = x(\xi,\eta)$, $y = y(\xi,\eta)$, $\xi \equiv (\xi^1,\ldots,\xi^k) \in \Pi$, $\eta \equiv (\eta^1,\ldots\eta^l) \in \Gamma$, which reduce the n-th dimensional problem described by the mapping A to a certain k-th dimensional problem (i.e., to the problem in which we deal exclusively with functions which are independent of $\underset{\sim}{\eta}$. To this aid we shall assume that the following objects are known, cf. [43]:

([1]) For the particulars the reader is referred to [44,45].

([2]) This global assumption can be replaced by more weak assumption that Ω is the fiber space over Π and Γ is the standard fiber.

1. The topological space $U(\Pi)$ of the vector functions q defined on Π,
2. The mapping Φ from $U(\Pi)$ to $X(\Omega)$: $x = \Phi(q)$, $q = q(\underset{\sim}{\xi})$, $\underset{\sim}{\xi} \in \Pi$.
3. The linear space $W(\Gamma)$ and the system of N linear independent functionals $w_A^* \in W^*(\Gamma)$, $A = 1,\dots,N$.

To impose the constraints on the mapping $A(x) = y$ we shall assume the subset $\tilde{\Xi}$ of $\Xi = D(A)$ in the form

$$\tilde{\Xi} := \{x \mid x = \Phi(q) \text{ for some } q \in Q\} \quad ,$$

where Q is the known non-empty subset of $D(\Phi)$, $D(\Phi) \subset U$. The set $\overset{\circ}{Y}$ is given by

$$\overset{\circ}{Y} := \{\overset{\circ}{y} \mid \overset{\circ}{y} = A(x) - y \text{ for some } x \in \Xi, y \in \Delta\}, \overset{\circ}{Y} \subset Y. \qquad \text{(A4.3)}$$

To define the realization of the constraints we have to introduce the non-empty subsets $\overset{\circ}{Y}_x$ of $\overset{\circ}{Y}$ for every $x \in \tilde{\Xi}$, i.e., for every $x = \Phi(q)$ $q \in Q$. These subsets will be assumed in the form ([1])

$$Y_x := \{\overset{\circ}{y} \mid \langle \overset{\circ}{y}(\underset{\sim}{\xi},\cdot), w_A^* \rangle = 0 \text{ for } A = 1,\dots,N \text{ and every } \underset{\sim}{\xi} \in \Pi\}. \qquad \text{(A4.4)}$$

Denoting

$$\tilde{A}_A(q,\underset{\sim}{\xi}) \equiv \langle A(\Phi(q))|_{\underset{\sim}{\xi}}, w_A^* \rangle , \qquad \tilde{A} \equiv \{\tilde{A}_A , \; A = 1,\dots,N\} ,$$

$$\tilde{y}_A(\underset{\sim}{\xi}) \equiv \langle y(\underset{\sim}{\xi},\cdot), w_A^* \rangle \quad , \qquad \tilde{y} \equiv \{\tilde{y}_A , \; A = 1,\dots,N\} , \; \underset{\sim}{\xi} \in \Pi ,$$

we obtain from Eqs.(A4.3), (A4.4)

$$\tilde{A}(q,\underset{\sim}{\xi}) = \tilde{y}(\underset{\sim}{\xi}) \quad , \; q \in Q , \; \underset{\sim}{\xi} \in \Pi . \qquad \text{(A4.5)}$$

Eq. (A4.5) has been obtained by imposing the special form of constraints on the mapping $A(x) = y$ and is independent of the variable $\underset{\sim}{\eta}$, $\underset{\sim}{\eta} \in \Gamma$. Thus the n-th dimensional problem described by $A(x) = y$ has been reduced to the k-th dimensional problem described by Eq. (A4.5)(in the sense explained above). It may be easily observed that the procedure outlined above can be also applied to the formation of the shell theories.

([1]) The functionals w_A^* in Eqa. (A4.4) can also depend on q,$q \in Q$.

Generalizations. Let V be the topological space of functions v and B be the known mapping form $X \times V$ to Y. The concept of the semiconstraints can be also introduced when we deal with the relation of the form

$$B(x,v) = y \ , \ y \in \Delta, \quad \text{for every } v \in V_x \ , \tag{A4.6}$$

where V_x are the known non-empty subsets of V and Δ is the known non-empty subset of R(B). We shall say that the semiconstraints are imposed on Eq. (A4.6) if there are known the non-empty subsets $\tilde{\Xi}$, $\tilde{V}_x$, $x \in \tilde{\Xi}$, of X,V, respectively, as well as the multifunction (A4.2) and if for every y, $y \in \Delta$, there exists at least one pair $(x,\overset{\circ}{y}) \in \tilde{\Xi} \times \overset{\circ}{Y}_x$ such that $B(x,v) = y + \overset{\circ}{y}$ for every $v \in V_x$. Here Y_x are the known subsets of $\overset{\circ}{Y}$, where

$$\overset{\circ}{Y} := \{\overset{\circ}{y} \mid \overset{\circ}{y} = B(x,v)-y \text{ for every } v \in V_x \text{ and some } (x,y) \in \Xi \times \Delta\}, \tag{A4.7}$$

and where we have denoted $\Xi := \{x | B(x,v) = y$ for every $v \in V_x$ and some $y \in \Delta\}$.

Now let F be the known mapping from $X \times Y$ to a certain linear space. Let us also denote by Ξ and Δ the domain and the range, respectively, of the binary relation given by $F(x,y) = \theta$. We shall say that the semiconstraints are imposed on this relation if there are known:

1. The non-empty subset $\tilde{\Xi}$ of X.
2. The multifunction
   $\tilde{\Xi} \ni x \to \overset{\circ}{Y}_x \subset \overset{\circ}{Y}$,

   where

$$\overset{\circ}{Y} := \{\overset{\circ}{y} | F(x,y + \overset{\circ}{y}) = \theta \text{ for some } x \in \Xi, \ y \quad \Delta\} \tag{A4.7$_1$}$$

   and if for every y,$y \in \Delta$, there exists at least one pair (x,y) $\in \tilde{\Xi} \times \overset{\circ}{Y}_x$, such that $F(x,y + \overset{\circ}{y}) = \theta$.

Remark. Let us observe that the concept of semiconstraints is closely related to the concept of formal approximation which has been introduced in Sec. 2.0. Generally speaking, the formal approximation can be expressed in terms of the semiconstraints. However, the concept of semiconstraints does not include any "approximation relation", i.e., the set $\tilde{\Xi}$ may not constitute any "approximation" of the set Ξ (which was the assumption of the procedure outlined in Sec. 2.0). Thus by imposing the semiconstraints (or constraints) on a certain binary relation (A4.1) we obtain the relation $A(x) - y \in Y_x$, $y \in \Delta$, which may not be interpreted as the approximation of the relation $A(x) = y$, $y \in \Delta$. Such situation will take place, for example, if the constraints imposed on the differential equations of mechanics lead to the initial-boundary value problems for these equations.

4.1. Examples and interpretation of constraints in solid mechanics

The concept of constraints is well known in mechanics, cf.[2,3,13, 33, 36-39, 41-45]. The aim of rather simple examples given below is to outline some special cases of constraints from the point of view of the general treatment developed in Sec. 2.0.

Example 1. Let Eq. (A4.1) stands for the mapping $\hat{\underset{\sim}{T}}(\underset{\sim}{C}) = \underset{\sim}{T}$ where $\hat{\underset{\sim}{T}}(\cdot)$ is the response functional, $\underset{\sim}{C} \in \Xi$, where Ξ is the set of all histories of the right Cauchy-Green deformation tensors and $\underset{\sim}{T} \in T^2$, where T^2 is the space of the symmetric second order tensors. Let

$$\overset{\circ}{Y} := \{\underset{\sim}{N} | \underset{\sim}{N} = \hat{\underset{\sim}{T}}(C) - \underset{\sim}{T} \text{ for some } \underset{\sim}{C} \in \Xi,\ \underset{\sim}{T} \in T^2\},$$

and assume that

$$\tilde{\Xi} := \{\underset{\sim}{C} | \underset{\sim}{C} \in \Xi,\ \alpha_\nu(\underset{\sim}{C}) = 0,\quad \nu = 1,\dots,N\ ;\ N \le 6\},$$

$$\overset{\circ}{Y}_{\underset{\sim}{C}} := \{\underset{\sim}{N} | t_r(\underset{\sim}{N}\underset{\sim}{D}) = 0 \text{ for every } \underset{\sim}{D} \in T^2 \text{ with } \operatorname{tr}(\frac{\partial \alpha_\nu}{\partial \underset{\sim}{C}} \underset{\sim}{D}) = 0\},\quad Y_{\underset{\sim}{C}} \subset T^2,\ \underset{\sim}{C} \in \tilde{\Xi}, \tag{A4.8}$$

where $\alpha_\nu(\cdot)$ are the known independent differentiable functions. Eqs. (A4.8) describe the constraints imposed on the constitutive relation $\hat{\underset{\sim}{T}}(\underset{\sim}{C}) = \underset{\sim}{T}$. We observe that these constraints lead to the new relation $\hat{\underset{\sim}{T}}(\underset{\sim}{C}) = \underset{\sim}{T} + \underset{\sim}{N}$ with $\underset{\sim}{N} = \lambda^\nu \partial\alpha_\nu / \partial\underset{\sim}{C}$, where λ^ν are arbitrary scalars. It is the well known case of the internal (material) constraints, [33]. Thus Eqs. (A4.8) represent certain constitutive hypothesis which can be incorporated into the constitutive relation, cf. [2]. On the other hand we also observe that Eqs. (A4.8) can be interpreted as a formal approximation of the constitutive equation $\hat{\underset{\sim}{T}}(\underset{\sim}{C}) = \underset{\sim}{T}$, which may constitute the good approximation for a certain class of motions. This is the class of motions satisfying the condition $\underset{\sim}{C} \in \overset{\circ}{\Xi}$, where $\overset{\circ}{\Xi}$ is an arbitrary subset of Ξ which can be "approximated" by the set $\tilde{\Xi}$ defined in Eqs. (A4.8). It means that the conditions given by Eqs. (A4.8) can be interpreted either as the constraints or as the formal approximation related to the constitutive relation $\hat{\underset{\sim}{T}}(\underset{\sim}{C}) = \underset{\sim}{T}$.

Remark. The semiconstraints which can be interpreted as a certain formal approximation (in the sense described in Sec. 2.0) will be called the *simplifying semiconstraints*. If such interpretation does not hold then we shall say that they are the *real semiconstraints*. This terminology is based on that introduced in [2]. In the special cases in which Ξ is a set of a certain kinematical fields (for example the set of deformation functions) and Y is the space of forces (for example the external forces acting at the fixed body) we can often interpret the condition $x \in \tilde{\Xi}$, where $\tilde{\Xi} \in \Xi$, as due to some of forces y, $y \in \overset{\circ}{Y}_x$. If such interpretation holds then we shall say that the constrains are reactive (this interpretation can be also extended on non-mechanical cases). Inversly, if the systems of forces y, $y \in \overset{\circ}{Y}_x$, are interpreted as due to the condition $x \in \tilde{\Xi}$, then we can say that the constraints are *material*. The terminology introduced above is based on that proposed in [2]. It must be stressed, however, that such terms as "real", "simplifying", "reative" or "material" applied to the concept of constraints or semiconstraints are not related to the formal structure of this concept but only to the interpretations of semicontstraints in different classes of problems.

Example 2. Now let the constitutive relation be given in the form $L_\mu(\underset{\sim}{C},\underset{\sim}{T}) = 0$, $\mu = 1,\dots,6$, where L_μ are the known differential operators with respect to the time coordinate (it is the constitutive relation of the rate-type material). Let on $L_\mu(\underset{\sim}{C},\underset{\sim}{T}) = 0$, $\mu = 1,\dots,6$, be imposed the constraints given again by Eqs. (A4.8). In this case we obtain (cf. Sec. 2.0, Generalizations) the relation $L_\mu(\underset{\sim}{C},\underset{\sim}{T} + \underset{\sim}{N}) = 0$, where $\underset{\sim}{N} = \lambda^\nu \partial\alpha_\nu/\partial\underset{\sim}{C}$.

Example 3. Let the relation given by Eq. (A4.1) represents the governing equations of the solid mechanics for the simple materials. It means that $A(x) = y$ stands for

$$- \text{Div}\, (\nabla \underset{\sim}{p}\, \hat{\underset{\sim}{T}}) + \rho_R \ddot{\underset{\sim}{p}} = \underset{\sim}{b}_R \,, \qquad (\nabla \underset{\sim}{p}\, \hat{\underset{\sim}{T}})\, \underset{\sim}{n}_R = \underset{\sim}{p}_R \,, \tag{A4.9}$$

where the values $\underset{\sim}{T}$ of the second Piola-Kirchhoff stress tensor have been determined by the suitable response functional $\hat{\underset{\sim}{T}} \equiv \hat{\underset{\sim}{T}}(\underset{\sim}{X}, \nabla \underset{\sim}{p}^{(t)})$. Here $x \equiv \underset{\sim}{p} \in \Xi \subset X, y \equiv (\underset{\sim}{b}_R, \underset{\sim}{p}_R) \in Y$, where Ξ is the set of all deformation functions (we assume that Ξ is a subset in a certain topological space X) and Y is the linear space of all external forces.

Let us denote $\underset{\sim}{p}_t \equiv \underset{\sim}{p}(\cdot, t)$, $t \in I$, and let Φ be the known mapping from the space of sufficiently regular functions p_t defined on $\overline{\kappa_R(\mathcal{B})}$ to a certain linear space, having for every $\underset{\sim}{p}_t$ the weak derivative $\Phi'(\underset{\sim}{p}_t)$, $t \in I$. Then the constraint inclusion $\tilde{\Xi} \subset \Xi$ can be determined by

$$\tilde{\Xi} := \{\underset{\sim}{p} \,|\, \Phi(\underset{\sim}{p}_t) = \theta, \quad t \in I\} \,. \tag{A4.10}$$

Putting $\tilde{\underset{\sim}{T}} \in \hat{\underset{\sim}{T}}(X, \nabla \underset{\sim}{p}^{(t)}), \underset{\sim}{p} \in \tilde{\Xi}$, we shall now define the set $\overset{\circ}{Y}$ by means of

$$\overset{\circ}{Y} := \{(\underset{\sim}{r}_R, \underset{\sim}{s}_R) \,|\, \underset{\sim}{r}_R = -\text{Div}(\nabla \underset{\sim}{p}\, \tilde{\underset{\sim}{T}}) + \rho_R \ddot{\underset{\sim}{p}} - \underset{\sim}{b}_R \,,\, \underset{\sim}{s}_R = (\nabla \underset{\sim}{p}\, \tilde{\underset{\sim}{T}})\underset{\sim}{n}_R - \underset{\sim}{p}_R \text{ for some } \underset{\sim}{p} \in \tilde{\Xi}\} \,. \tag{A4.11}$$

The external forces $(\underset{\sim}{r}_R, \underset{\sim}{s}_R)$ can be interpreted, for example as the forces "maintaining the constraints $\Phi(\underset{\sim}{p}_t) = \theta$, $t \in I$". Among many possible realizations of these constraints we can assume the multifunction $\tilde{\Xi} \ni \underset{\sim}{p} \to \overset{\circ}{Y}_{\underset{\sim}{p}} \subset \overset{\circ}{Y}$ determined by

$$\overset{\circ}{Y}_{\underset{\sim}{p}} := \{(\underset{\sim}{r}_R, \underset{\sim}{s}_R) \,|\, \oint_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{s}_R \cdot \underset{\sim}{v}\, da_R + \int_{\kappa_R(\mathcal{B})} \underset{\sim}{r}_R \cdot \underset{\sim}{v}\, dv_R = 0 \quad \text{for every } \underset{\sim}{v} \text{ with } \Phi'(\underset{\sim}{p}_t)\underset{\sim}{v} = \theta\} \,. \tag{A4.12}$$

In this case the "reaction forces" $\underset{\sim}{r}_R$, $\underset{\sim}{s}_R$ do not work on any "virtual displacement" $\underset{\sim}{v}$ defined above.

Summing up, we conclude that the constraints imposed on the mapping (A4.9), which are given by Eqs. (A4.10), (A4.12), lead to the system of relations

$$\begin{aligned} &\mathrm{Div}(\nabla \underset{\sim}{p}\, \underset{\sim}{T}) + \underset{\sim}{b}_R + \underset{\sim}{r}_R = \rho_R \ddot{\underset{\sim}{p}}\ , \\ &(\nabla \underset{\sim}{p}\, \underset{\sim}{T})\underset{\sim}{n}_R = \underset{\sim}{p}_R + \underset{\sim}{s}_R \\ &\Phi(\underset{\sim}{p}_t) = \underset{\sim}{\theta},\ t \in I\ , \\ &\oint_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{s}_R \cdot \underset{\sim}{v}\, da_R + \int_{\kappa_R(\mathcal{B})} \underset{\sim}{r}_R \cdot \underset{\sim}{v}\, dv_R = 0 \text{ for every } \underset{\sim}{v} \text{ with } \Phi'(\underset{\sim}{p}_t)\underset{\sim}{v} = \theta, \end{aligned} \tag{A4.13}$$

where $\tilde{\underset{\sim}{T}} \equiv \hat{\underset{\sim}{T}}(\underset{\sim}{X}, \nabla \underset{\sim}{p}^{(t)})$. The constraints under consideration are bilateral.

Example 4. Now let Φ be the known mapping from the space of sufficiently regular functions defined on $\partial\kappa_R(\mathcal{B})$ to a certain linear space, possessing the weak derivative $\Phi'(\gamma \underset{\sim}{p}_t)$ for every $\gamma \underset{\sim}{p}_t \equiv \underset{\sim}{p}(\cdot,t)|_{\partial\kappa_R(\mathcal{B})}$ [1]. Let the motions of the solid body be restricted by the kinematic boundary conditions of the form $\Phi(\gamma \underset{\sim}{p}_t) = \underset{\sim}{\theta}$, $t \in I$. It is a special case of constraints described in the foregoing example, which leads to the relations

$$\begin{aligned} &\mathrm{Div}(\nabla \underset{\sim}{p}\, \tilde{\underset{\sim}{T}}) + \underset{\sim}{b}_R = \rho_R \ddot{\underset{\sim}{p}}\ , \\ &(\nabla \underset{\sim}{p}\, \tilde{\underset{\sim}{T}})\underset{\sim}{n}_R = \underset{\sim}{p}_R + \underset{\sim}{s}_R\ , \\ &\Phi(\gamma \underset{\sim}{p}_t) = \theta, \\ &\oint_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{s}_R \cdot \underset{\sim}{v}\, da_R = 0 \text{ for every } v \text{ with } \Phi'(\gamma \underset{\sim}{p}_t)\underset{\sim}{v} = \theta\ , \end{aligned} \tag{A4.14}$$

where $\tilde{\underset{\sim}{T}}$ $\hat{\underset{\sim}{T}}(\underset{\sim}{X}, \nabla \underset{\sim}{p}^{(t)})$ as before. We deal here with the well known case of the solid body subjected to the kinematic bilateral boundary constraints.

(1) γf is here the trace of the function f on $\partial\kappa_R(\mathcal{B})$.

Example 5. Let us assume that the relation given by Eq. (A4.1) stands for the constitutive relation of the rate-type material $\underset{\sim}{L}(\underset{\sim}{T})[\dot{\underset{\sim}{T}}] = \dot{\underset{\sim}{C}}$, where $\underset{\sim}{L}(\underset{\sim}{T})[\cdot]$ is the linear mapping from the space $\mathcal{T}^2$ of all symmetric second order tensors to the space $\mathcal{T}^2$, defined for every $\underset{\sim}{T} \in \mathcal{T}^2$. Now we have $(\underset{\sim}{T},\dot{\underset{\sim}{T}}) \in \Xi \equiv \mathcal{T}^2 \times \mathcal{T}^2$ and $\dot{\underset{\sim}{C}} \in \Delta \equiv \mathcal{T}^2$. Let the subset $\tilde{\Xi}$ of Ξ be defined by

$$\tilde{\Xi} \equiv \Gamma \times \mathcal{T}^2, \quad \Gamma := \{\underset{\sim}{T} | j(\underset{\sim}{T}) \leq 0\} \tag{A4.15}$$

where $j(\cdot)$ is the known sufficiently regular mapping from $\mathcal{T}^2$ to R. Thus the set $\overset{\circ}{Y}$ will be determined by

$$\overset{\circ}{Y} := \{\underset{\sim}{D} | \underset{\sim}{D} = \underset{\sim}{L}(\underset{\sim}{T})[\dot{\underset{\sim}{T}}] - \dot{\underset{\sim}{C}} \text{ for some } \underset{\sim}{T} \in \Gamma, \dot{\underset{\sim}{T}} \in \mathcal{T}^2, \text{ and } \dot{C} \in \mathcal{T}^2\}.$$

Moreover, let the multifunction $\tilde{\Xi} \ni x \to \overset{\circ}{Y}_x \subset \overset{\circ}{Y}$, where now $x \equiv (\underset{\sim}{T},\dot{\underset{\sim}{T}})$, be assumed in the form

$$\overset{\circ}{Y}_x := \{\underset{\sim}{D} | \text{tr } \underset{\sim}{D}(\underset{\sim}{T}_o - \underset{\sim}{T}) \leq 0 \text{ for every } \underset{\sim}{T}_o \text{ with } j(\underset{\sim}{T}_o) \leq 0\}. \tag{A4.16}$$

Eqs. (A4.15), (A4.16) describe the constraints imposed on the constitutive relation $\underset{\sim}{L}(\underset{\sim}{T})[\dot{\underset{\sim}{T}}] = \dot{\underset{\sim}{C}}$. These constraints lead to

$$\begin{aligned} &\underset{\sim}{L}(\underset{\sim}{T})[\underset{\sim}{T}] = \underset{\sim}{C} + \underset{\sim}{D}, \\ &j(\underset{\sim}{T}) \leq 0, \\ &\text{tr}[\underset{\sim}{D}(\underset{\sim}{T}_o - \underset{\sim}{T})] \leq 0 \text{ for every } \underset{\sim}{T}_o \text{ with } j(\underset{\sim}{T}_o) \leq 0. \end{aligned} \tag{A4.17}$$

If $j(\underset{\sim}{T}) = 0$ is the yield condition (and $\underset{\sim}{L}$ is independent of $\underset{\sim}{T}$) then we can interpret Eq. (A4.17) as the constutive relation of the elastic-perfectly plastic material with $\underset{\sim}{D}$ as the rate of the plastic strain. Thus the elastic-perfectly plastic materials can be interpreted as the special rate-type materials with the constraints defined by Eqs.(A4.15), (A4.16).

The detailed analysis of the different special cases of constraints in solid mechanics can be found, for example, in [45].

4.2. Relations of solid mechanics with semiconstraints

The governing relations of solid mechanics have been described in Sec. 2 of this Chapter and their analytical from is given by Eqs. (A2.1), (A2.2). Now we are to interpret Eqs. (A2.1), (A2.2) as a certain binary relation. To this aid we shall introduce the linear space Y of five-tuples $y \equiv (\underset{\sim}{r}_R, \underset{\sim}{s}_R, a_{(m)}, a, \alpha)$, $a_{(m)} \equiv (a_1, \ldots, a_m)$, where $\underset{\sim}{s}_R$ are vector functions defined almost everywhere on $\partial\kappa_R(\ ) \times I$, $\underset{\sim}{r}_R$ are vector functions defined on $\kappa_R(\mathcal{B}) \times I$ and a_μ; $\mu = 1, \ldots, m$; a, α are real-valued functions defined on $\kappa_R(\mathcal{B}) \times I$. Moereover, let X be the topological space of the triples $x \equiv (\underset{\sim}{p}, \underset{\sim}{T}, \lambda^{(s)})$ of functions defined on $\kappa_R(\mathcal{B}) \times I$, such that $\underset{\sim}{T} = \underset{\sim}{T}^T$ and the local invertibility condition $\det \nabla\underset{\sim}{p} > 0$ holds for every $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$, $t \in I$ (i.e., the governing rellations of solid mechanics possess orientation-preserving solutions $\underset{\sim}{p}$). Let us assume, for the time being, that the external forces $(\underset{\sim}{p}_R, \underset{\sim}{b}_R)$, are arbitrary but fixed. Then Eqs. (A2.1), (A2.2) can be interpreted as the relation (for every fixed $(\underset{\sim}{p}_R, \underset{\sim}{b}_R)$) with the domain Ξ in the space X and with the range Δ which can be indentified with the zero element of the linear space Y.

Using the approach outlined in Sec. 4.0 we shall impose on the foregoing relation the semiconstraints in their general form. To this aid we have to specify the subset $\tilde{\Xi}$ of X and the multifunction $\tilde{\Xi} \ni x \to \overset{\circ}{Y}_x \subset \overset{\circ}{Y}$, where $\overset{\circ}{Y}$ is the subset of Y consisting of all five tuples $y = (\underset{\sim}{r}_R, \underset{\sim}{s}_R, a_{(m)}, a, \alpha)$ which are the values of the LHS of Eqs. (A2.1), (A2.2), respectively, for all $x = (\underset{\sim}{p}, \underset{\sim}{T}, \lambda^{(s)}) \in \Xi$.

Thus the governing relations of the solid mechanics with semiconstraints will be determined by:

1. *The field equations*, i.e., the *equations of motion*

$$\mathrm{Div}(\nabla\underset{\sim}{p}\,\underset{\sim}{T}) + \underset{\sim}{b}_R + \underset{\sim}{r}_R = \rho_R \ddot{\underset{\sim}{p}} \quad , \quad \underset{\sim}{T} = \underset{\sim}{T}^T , \tag{A4.18}$$

and the *kinetic boundary conditions*

$$(\nabla\underset{\sim}{p}\,\underset{\sim}{T})\underset{\sim}{n}_R = \underset{\sim}{p}_R + \underset{\sim}{s}_R , \tag{A4.19}$$

where every $(\underset{\sim}{p}_R, \underset{\sim}{b}_R)$ will be called the active external force and $(\underset{\sim}{s}_R, \underset{\sim}{r}_R)$ will be referred to as the constraint reaction force.

2. *The constitutive relations* given by

$$f_\mu(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) = a_\mu, \quad \mu = 1,\dots,m- \, ,$$

$$j(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) - a \le 0 \, , \tag{A4.20}$$

$$\varphi(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}, \underset{\sim}{C}_o,\underset{\sim}{T}_o) - \alpha \le 0 \text{ for every } (\underset{\sim}{C}_o,\underset{\sim}{T}_o) \text{ with}$$

$$j(\underset{\sim}{X},\underset{\sim}{C}_o,\underset{\sim}{T}_o,\lambda^{(s)}) - a \le 0$$

where, as usual $\underset{\sim}{C} \equiv (\nabla \underset{\sim}{p})^T \nabla \underset{\sim}{p}$ and where $(a_{(m)},a,\alpha)$ will be called the constraint constitutive reaction ([1]).

3. *The semiconstraint relation* given by

$$x \equiv (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)}) \in \tilde{\Xi} \, ,$$

$$\overset{o}{y} \equiv (\underset{\sim}{r}_R,\underset{\sim}{s}_R,a_{(m)},a,\alpha) \in \overset{o}{Y}_x \subset \overset{o}{Y}, \; x \in \tilde{\Xi} \tag{A4.21}$$

where $\tilde{\Xi}$ is the known subset of X (in the case of constraints $\tilde{\Xi} \in \Xi$) and Y_x are, for every $x \in \Xi$, the known subset of $\overset{o}{Y}$. The five-tuples $\overset{o}{y} \equiv (\underset{\sim}{r}_R,\underset{\sim}{s}_R,a_{(m)},a,\alpha)$, $\overset{o}{y} \in Y_x$, will be called the constraint reactions; they include the constraint reaction force $(\underset{\sim}{s}_R,\underset{\sim}{r}_R)$ and the constraint constitutive reaction $(a_{(m)},a,\alpha)$.

Remark. The semiconstraint relation (A4.21) is not an arbitrary relation but has to describe in the analytical form all "extra" informations about the class of problems we are to investigate, i.e., the informations which are not included in Eqs. (A2.1) and which can modify the form of Eqs. (A2.2). Generally speaking, the form of the semiconstraint relation (A4.21) and the constitutive relations (A4.20) has to ensure the existence of the physically reasonable solutions to the well stated problems of the solid mechanics. We shall assume that $\tilde{\Xi} = \Xi$ implies $\overset{o}{Y}_x = \{\theta\}$ for every $x \in \Xi$. It means that if there are no "extra" restrictions imposed on the set of triples $x = (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$ then there will be no reactions $\overset{o}{y}$, i.e., $\overset{o}{y} \equiv \theta$. We shall also assume that the condition given in the Definition 1 of Sec. 2.0 of this Chapter has to be satisfied for every active external force $(\underset{\sim}{p}_R,\underset{\sim}{b}_R)$.

([1]) The simple example of the constraint constitutive reaction is given by the field $\underset{\sim}{N}$ in the Example 1 of Sec. 4.1.

Interpretation. In the case of constraints the reactions $\overset{\circ}{y} \equiv (\underset{\sim}{r}_R, \underset{\sim}{s}_R, \underset{\sim}{a}_{(m)}, a, \alpha)$ can be interpreted either as "maintaning" the constraint inclusion $\tilde{\Xi} \subset \Xi$ or as "due" to that inclsuion(cf. the interpretation of constraints in solid mechanics given in Sec. 4.1. of this Chapter). The sets $\overset{\circ}{Y}_x$ have to be understood as the sets of all reactions $\overset{\circ}{y}$ which are able to"maintain" such deformations, stresses and constitutive parameters, described by the triple $x = (\underset{\sim}{p}, \underset{\sim}{T}, \lambda^{(s)})$, that always $x \in \tilde{\Xi}$.

4.3. Formation of shell theories

We shall obtain the shell theories by postulating the special form of the constraint relation (A4.21). Firstly, let us assume that the constitutive relations $(A2.2)_1$ for $\mu = 1,\dots,M$, where M is the fixed integer, $0 \le M \le 6$, have the form

$$\sigma^{\mu}(\underset{\sim}{T}) = \overset{\infty}{\underset{s=o}{\tilde{\sigma}^{\mu}}}(\underset{\sim}{X}, \underset{\sim}{C}(\underset{\sim}{X}, t-s)), \qquad \text{(A4.22)}$$

where $\underset{\sim}{\sigma} : T^2 \to R^6$ is the known one-to-one mapping with the inverse $\underset{\sim}{S} : R^6 \to T^2$. Simultaneously, we assume that the stress components $\sigma^{\mu}(\underset{\sim}{T})$, $\mu = M + 1,\dots,6$, are not uniquely determined by the history of motion, i.e., that for $\mu > M$ the relations of the form (A4.22) do not hold. If M = 6 then Eqs. (A2.2) reduce to the form given exclusively by Eq. (A4.22); in this case we assume that $m = 6$, $j \equiv 0$, $\varphi \equiv 0$ and the constitutive relations (A2.2) define the simple material. If M = 0 then there are no constitutive relations of the form (A4.22). Secondly, let us denote by $q_{(n)}$, $\tau^{(N)}$, $\omega^{(p)}$ the ordered sets of the sufficiently regular real valued functions defined on $\Pi \times I$ and let $\tilde{\underset{\sim}{p}}(X, q_{(n)}, \nabla q_{(n)})$, $\tilde{\sigma}^{\mu}(\underset{\sim}{X}, \tau^{(N)})$, $\mu = M + 1,\dots,6$, and $\tilde{\lambda}^{\sigma}(\underset{\sim}{X}, \omega^{(p)})$, $\sigma = 1,\dots,s$, be the known sufficiently regular functions. Let us also denote $\tilde{\underset{\sim}{C}} \equiv (\nabla\tilde{\underset{\sim}{p}})^T \nabla\tilde{\underset{\sim}{p}}$ and assume that the set $Q := \{q_{(n)} \,|\det \nabla\tilde{\underset{\sim}{p}} > 0\}$ is not empty. The relation $(A4.21)_1$ will be postulated now in the form given by:

$$\underset{\sim}{p} = \tilde{\underset{\sim}{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)}) \quad \text{for some} \quad q_{(n)} \in Q,$$

$$\underset{\sim}{T} = \tilde{\underset{\sim}{T}} \equiv \underset{\sim}{S}(\underset{\sim}{\sigma}) \text{ where } \sigma^{\mu} = \begin{cases} \overset{\infty}{\underset{s=0}{\hat{\sigma}^{\mu}}}(\underset{\sim}{X},\tilde{\underset{\sim}{C}}(\underset{\sim}{X},t-s)) \text{ for } \mu = 1,\ldots,M, \\ \tilde{\sigma}^{\mu}(\underset{\sim}{X},\tau^{(N)}) \text{ for } \mu = M+1,\ldots,6 \text{ and for some } \tau^{(N)}, \end{cases}$$

$$\lambda^{(s)} = \tilde{\lambda}^{(s)}(\underset{\sim}{X},\omega^{(p)}) \text{ for some } \omega^{(p)}. \qquad \text{(A4.23)}$$

It means that the set $\tilde{\Xi}$ in Eq. (A4.21) is the set of all triples $(\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$ which can be expressed in the form given by the RHS of Eqs. (A4.23). For the simple materials Eqs. (A4.23) reduce to the form given by Eq. $(A4.23)_1$ ([1]).

In order to specify the form of Eq. $(A4.21)_2$ we shall assume that there are known the functions $\bar{\Phi}^{a}$, $\bar{\Psi}^{a\alpha}$, $\bar{\Xi}^{\mu}_{A}$, $\bar{b}$, $\bar{\beta}$; $\mu = M+1,\ldots,m$; $A = 1,\ldots,N+P$, of the arguments $\underset{\sim}{X}$, $q_{(n)}$, $\nabla q_{(n)}$, $\tau^{(N)}$, $\omega^{(p)}$, such that there exist the Stieltes integrals in Eqs. (A2.38) (with the denotations (A2.39)) and in Eqs. (A2.40). The integer P, $0 \le P \le p$, is the number of the shell internal parameters ω^{π}, $\pi = 1,\ldots,P$. We postulate

$$\overset{o}{Y}_{x} := \{(\underset{\sim}{r}_{R},\underset{\sim}{s}_{R},a_{(m)},a,\alpha)\,|\,a_{\mu} = 0 \text{ for } \mu = 1,\ldots,M;$$

Eq. (A2.38) has to hold for every $\delta\underset{\sim}{p}$ given

by Eqs. (A2.39); Eqs. (A2.40) hold $\}$, (A4.24)

for every $x = (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)}) \in \tilde{\Xi}$. It means that the constraint reactions $\mathcal{Y} = (\underset{\sim}{r}_{R},\underset{\sim}{s}_{R},a_{(m)}a,\alpha)$ in the class of problems under consideration have to satisfy the conditions given by (A4.24).

Now by the direct calculations, which are analogous to those given in Secs. 2.2, 2.3, we shall obtain form (Eqs. (A4.18) - (A4.20), (A4.23), (A4.24) the system of relations of the shell theory. This system is determined by *shell eauations of motion* (A2.26), *the shell kinetic boundary conditions* (A2.27) and *the shell constitutive relations* (A2.30), (A2.14). At the same time we shall obtain the relations

([1]) We also assume that $\lambda^{\sigma} = \tilde{\lambda}^{\sigma}(\underset{\sim}{X},\omega^{1},\ldots,\omega^{P})$ for $\sigma = 1,\ldots,S$ and $\lambda^{\sigma} = \tilde{\lambda}^{\sigma}(\underset{\sim}{X},\omega^{P+1},\ldots,\omega^{p})$ for $\sigma = S+1,\ldots,s$, where $\lambda^{1},\ldots,\lambda^{S}$ are the "internal" constitutive parameters, i.e., described by Eqs. $(A4.20)_1$ for $\mu = 6+1,\ldots,6+S = m$, and $\lambda^{S+1},\ldots,\lambda^{s}$ are the "kinematical" constitutive parameters, characterizing certain strain incompatibilities (such as the rates of plastic strain), cf. Sec. 2.1. If $S=0$ or $S=s$ then $P=0$ or $P=p$, respectively. Analogously, $\omega^{1},\ldots,\omega^{P}$ and $\omega^{P+1},\ldots,\omega^{p}$, will be referred to as the shell internal and kinematical constitutive parameters, respectively.

of the form (A2.41) - (A2.44) with the denotations given by Eq. (A2.28). If $d\bar{\underset{\sim}{\Phi}}^a = \underset{\sim}{\Phi}^a d\xi$, $d\bar{\underset{\sim}{\Psi}}^{a\alpha} = \underset{\sim}{\Psi}^{a\alpha} d\xi$, $d\bar{\Xi}^{\mu}_{A} = \Xi^{\mu}_{A} d\xi$, $d\bar{b} = d\xi$, $d\bar{\beta} = d\xi$ and if the Stieltes integrals in Eqs. (A2.38), (A2.40) reduce to the Riemann integrals, then instead of Eqs. (A2.41) - (A2.44) we shall arrive at Eqs. (A2.24), (A2.25), (A2.29).

We can now observe that using the constraint approach outlined above we have obtained the relations analogous to those which have been obtained in Sec. 2 by the formal approximation approach to the shell theories. The interrelation between the concept of semiconstraints and that of the formal approximation procedure has been mentioned in Sec. 4.0. In the constraint approach to the shell theories we deal with the "simplifying" constraints (cf. the Remark to Sec. 4.1) which can be interpreted also in terms of the formal approximation procedure. That is why for the constraint reactions we have used here the same denotations $\underset{\sim}{r}_R, \underset{\sim}{s}_R, \underset{\sim}{a}_{(m)}, a, \alpha$ as for the error fields introduced in Sec. 2. On the other hand, the constraint approach to the shell theories does not involve any "approximations" and all resulting relations are consistent with the general relations of the solid mechanics with semiconstraints described in Sec. 4.2. Generally speaking, the semiconstraints or constraints imposed on certain binary relation lead to the new relation which may be not interpreted as the approximation of this binary relation. Thus the shell theory obtained via constraint approach can be treated independently of the classical solid mechanics relations as a certain new analytical "model" describing some class of problems for the shell-like bodies.

5. CONCLUSIONS

5.1. Relationship of results

It has been observed that the formal structure of the general shell and plate theories obtained as the result of the different approaches is similar. It consists of ([1]):

1. *The equations of motion* (cf. Eqs. (A1.18) and (A2.26))

$$H^{a\alpha\beta}_{R,\alpha\beta} + H^{a\alpha}_{R,\alpha} + h^{a}_{R} + f^{a}_{R} - f^{a\alpha}_{R,\alpha} = i^{a}_{R} - i^{a\alpha}_{R,\alpha} \tag{A5.1}$$

where

$$i^{a}_{R} = i^{a}_{R}(\underset{\sim}{\theta},t,q_{(n)},\underline{\nabla q_{(n)}},\dot{q}_{(n)},\underline{\nabla\dot{q}_{(n)}},\ddot{q}_{(n)},\underline{\nabla\ddot{q}_{(n)}}) \ ,$$

$$i^{a\alpha}_{R} = i^{a\alpha}_{R}(\underset{\sim}{\theta},t,q_{(n)},\nabla q_{(n)},\dot{q}_{(n)},\nabla\dot{q}_{(n)},\ddot{q}_{(n)},\nabla\ddot{q}_{(n)}) \ , \tag{A5.2}$$

are the known functions. The fields $H^{a\alpha\beta}_{R}, H^{a\alpha}_{R}\ h^{a}_{R}$ are called the shell internal forces, the fields f^{a}_{R}, $f^{a\alpha}_{R}$ represent the shell body forces and $-i^{a}_{R}$, $-i^{a\alpha}_{R}$ are referred to as the shell inertia forces. The ordered set $q_{(n)}$ is said to be the shell deformation function.

2. *The kinetic boundary conditions* (cf. Eqs. (A1.19) and (A2.27) with the denotations (A2.28))

$$H^{a\alpha}_{R}n_{R\alpha} + \underline{\frac{d}{dL_R}(H^{a\alpha\beta}_{R}t_{R\alpha}n_{R\beta})} + \underline{H^{a\beta\alpha}_{R,\beta}n_{R\alpha}} = p^{a}_{OR} \underline{-(i^{a\alpha}_{R} - f^{a\alpha}_{R})n_{R\alpha}}$$

$$\underline{H^{a\alpha\beta}_{R}n_{R\alpha}n_{R\beta}} = \underline{p^{a\underset{\sim}{N}}_{R}} \tag{A5.3}$$

where we have denoted

$$p^{a}_{OR} = p^{a}_{R} - \underline{\frac{d}{dL_R}(p^{a\alpha}_{R}t_{R\alpha})}\ , \qquad p^{a\underset{\sim}{N}}_{R} \equiv p^{a\alpha}_{R}n_{R\alpha}\ . \tag{A5.4}$$

([1]) For the simple shell force system all underlined terms drop out from the relations below; it is the case in which the function $\tilde{p}$ is independent of the argument $\nabla q_{(n)}$, cf. Eqs. (A2.18), $(A4.23)_1$.

The fields defined by Eqs. (A5.4) are called the shell surface tractions. The system of the shell surface tractions and the shell body forces will be referred to as the system of the shell external forces. Eqs. (A5.1), (A5.3) will be called the shell field equations. The derivatives in these equations may be also understand in the generalized sense. It means that the shell equations of motion can be also interpreted as obtained form Eqs. (A5.1) by the integration over an arbitrary regular part Π_o of Π and by the formal application of the divergence theorem. Analogously, the shell kinetic boundary conditions can be also interpreted as obtained form Eqs. (A5.3) by the integration over an arbitrary regular part L_o of $\partial\Pi$ and by formal application of the divergence theorem (with respect to the L_R-coordinate). Such situation will take place if, for example, the partial derivatives with respect to θ^α in Eqs. (A5.1), (A5.3) do not exist.

3. *The constitutive relations* (cf. Eqs. (A2.30), (A2.14))[1]:

$$\begin{aligned}
&\underline{H_R^{a\alpha\beta}} = \underline{\tilde{H}_R^{a\alpha\beta}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \nabla\nabla q_{(n)}, \tau^{(N)})},\\
&H_R^{a\alpha} = \tilde{H}_R^{a\alpha}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \underline{\nabla\nabla q_{(n)}}, \tau^{(N)}),\\
&h_R^{a} = \tilde{h}_R^{a}(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}, \underline{\nabla\nabla q_{(n)}}, \tau^{(N)}),
\end{aligned}\qquad \text{(A5.5)}$$

where $\tilde{H}_R^{a\alpha\beta}$, $\tilde{H}_R^{a\alpha}$, $\tilde{h}_R^{a}$ are the known response functionals and

$$\begin{aligned}
&g_A(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(p)}) = 0\,, \qquad A = 1,\dots,N+P\\
&\kappa(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(p)}) \le 0\,,\\
&\psi(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \dot{\omega}^{(p)}, \overset{o}{e}_{(r)}, \overset{(N)}{\underset{o}{\tau}}) \le 0 \text{ for every } \overset{o}{e}_{(r)}, \overset{(N)}{\underset{o}{\tau}} \text{ with } \kappa \le 0,
\end{aligned}\qquad \text{(A5.6)}$$

where g_A are the known functionals, κ, ψ are the known scalar functions and $e_{(r)} = E_{(r)}(q_{(n)})$ are the shell strain measures (E_ρ, $\rho = 1,\dots,r$, are the known differentials operators acting on

---

[1] Using the direct approach we have postulated only the special form of Eqs. (A2.30), (A2.14), given by Eqs. (A1.22). However, the form of the shell constitutive relations obtained in Secs. 2,4 can be also postulated within the direct approach.

$q_{(n)}(.,t))$. For the simple materials the term $\tau^{(N)}$ drops out from Eqs. (A5.6) and Eqs. (A5.6) become the identities (this is the special case which was postulated in Sec. 1). The ordered sets $\tau^{(N)}$, $\omega^{(p)}$ are referred to as the shell stress parameters and the shell constitutive parameters, respectively.

The fields in Eqs. (A5.1), (A5.2), (A5.5), (A5.6) are defined on $\Pi \times I$; analogously the fields in Eqs. (A5.3), (A5.4) are defined almost everywhere on $\partial\Pi \times I$. All these fields as well as the functionals and functions: i^a_R, $i^{a\alpha}_R$, $\tilde{H}^{a\alpha\beta}_R$, $\tilde{H}^{a\alpha}_R$, $\tilde{h}^a_R$, g_A, κ, ψ, have to satisfy the suitable regularity conditions. These regularity conditions have to ensure, roughly speaking, the existence of the physically reasonable solutions ot the well stated problems within the shell theory under consideration.

Now let us look at the mechanics of the shell like bodies from the point of view of the formal approximation approach and that of the constraint approach. Then the three sets of relations have to be satisfied, namely:

1. *The relations of the shell theory* (i.e., the relations independent of the material coordinate ξ, $\xi \in \langle h_-, h_+ \rangle$), given by Eqs. (A5.1), (A5.3), (A5.5), (A5.6) with the denotations (A5.2), (A5.4).

2. *The relations of the "three-dimensional" mechanics of the shell like body* (i.e., the relations dependent on all material coordinates θ^1, θ^2, ξ; $\underset{\sim}{\theta} \in \bar{\Pi}$, $\xi \in \langle h_-, h_+ \rangle$), given by the field equations (cf. Eqs. (A2.8) or A4.18),(A4.19))

$$\mathrm{Div}(\nabla\tilde{\underset{\sim}{p}}\,\tilde{\underset{\sim}{T}}) + \underset{\sim}{b}_R + \underset{\sim}{r}_R = \rho_R \ddot{\tilde{\underset{\sim}{p}}} \; ; \qquad \tilde{\underset{\sim}{T}} = \tilde{\underset{\sim}{T}}^T \; ,$$

$$(\nabla\tilde{\underset{\sim}{p}}\,\underset{\sim}{T})\underset{\sim}{n}_R = \underset{\sim}{p}_R + \underset{\sim}{s}_R \qquad \text{(A5.7)}$$

and by the constitutive relations (cf. Eqs. $(A2.8)_{3-5}$ or (A4.20))

$$f_\mu(\tilde{\underset{\sim}{X}},\tilde{\underset{\sim}{C}},\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)}) = a_\mu \; ; \qquad \mu = 1,\dots,m \; ,$$

$$j(\underset{\sim}{X},\tilde{\underset{\sim}{C}},\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)}) - a \leq 0 \; , \qquad \text{(A5.8)}$$

$$\varphi(\underset{\sim}{X},\tilde{\underset{\sim}{C}},\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)}, \tilde{\underset{\sim}{C}}_o,\tilde{\underset{\sim}{T}}_o) - \alpha \leq 0 \text{ for every } (\tilde{\underset{\sim}{C}}_o,\tilde{\underset{\sim}{T}}_o) \text{ with } j(\underset{\sim}{X},\tilde{\underset{\sim}{C}}_o,\tilde{\underset{\sim}{T}}_o,\tilde{\lambda}^{(s)}) \leq a \; .$$

where $\tilde{\mathbf{C}} \equiv (\nabla\tilde{\mathbf{p}})^T\nabla\mathbf{p}$.

3. *The interrelations between the fields in the shell theory and the fields in Eqs. (A5.7), (A5.8).* They are given by (cf. also Eqs. (A2.18), (A2.6), (A2.7) or Eqs. (A4.23)):

$$\tilde{\mathbf{p}} \equiv \tilde{\mathbf{p}}(\theta,\boldsymbol{\xi},t,q_{(n)},\nabla q_{(n)}) \quad ,$$

$$\tilde{\mathbf{T}} \equiv \mathbf{S}(\tilde{\boldsymbol{\sigma}}) \;, \tilde{\sigma}^{\mu} = \begin{cases} \hat{\sigma}^{\mu}(\mathbf{X},\mathbf{C}(\mathbf{X},t-s) & \text{for } \mu = 1,\dots,M \;, \\ \tilde{\sigma}^{\mu}(\mathbf{X},\tau^{(N)}) & \text{for } \mu = M+1,\dots,6, \end{cases}$$

$$\tilde{\lambda}^{(s)} \equiv \tilde{\lambda}^{(s)}(\mathbf{X},\omega^{(p)}) \quad , \qquad \text{(A5.9)}$$

and by Eqs. (A2.29), (A2.13), $(A2.24)_{4-7}$, (A2.25) with the denotations (A2.9).

Mind, that all three sets of relations mentioned above are exact from the point of view of the formal approximation approach as well as from that of the constraint approach. They give the general description of the mechanics of shell-like bodies. It must be stressed, however, that both procedures leading to the shell theories influence the form of the field equations of the classical solid mechanics (A2.1) by the presence of the "extra" forces $\mathbf{r}_R,\mathbf{s}_R$ in Eqs. (A5.7). These procedures also "modify" the material properties (A2.2) of the body by the presence of the "extra"fields $a_{(m)}$, a, α in Eqs. (A5.8). The form of the "extra" fields is strictly connected with the range of applications of the shell theory to the special problems (i.e., with the reliability of solutions) and will be detailed in Sec. 1.3 of the Chapter C. Eqs. (A5.7) - (A5.9) do not belong to the shell theory and are the basis for the determination of the "extra" fields $\mathbf{r}_R,\mathbf{s}_R,a_{(m)},a,\alpha$.

5.2. Features of different approaches

Apart from the relationship of results described in Sec. 5.1., every approach to the shell theories has also its own characteristic features. The main features of the direct approach are:

1. The relatively simple structure, based entirely on axioms, corresponding with that of the classical solid mechanics.

2. The approach does not involve any approximation procedure.

3. The approach does not contain full informations about the analytical structure of the kinetic energy function and the response functionals even in the simplest special cases.

Both the approximation and constraint approaches to the shell theories are based on the derivation of the shell theories from the "three-dimensional" solid mechanics.

The formal approximation of the solid mechanics relations, which leads to the plate and shell theoreis, is characterized by:

1. The possibility of obtaining many different forms of the shell governing equations, by introducing different functions $\tilde{\underset{\sim}{p}}, \underset{\sim}{\Phi}^{a}$, $\underset{\sim}{\psi}^{a\alpha}$, etc.

2. The difficulty in estimating the "error" involved in the approximation procedures (for the non-linear systems no general criteria of accuracy are known).

3. The well determined relation with the "three dimensional" theory and the possibility of extending the results on the non pure mechanical case (cf. [26] for example).

In the mixed approach the field equations of the shell theory are usually obtained by the direct approach and the shell constitutive relations via approximations of the "three-dimensional" constitutive relations. The structure of the shell relations, as a rule, is simpler than that obtained form the approximation approach because some from the underlined terms (in Eqs. (A5.1) - (A5.5) may be neglected. The mixed approach can also supply the interpretations of the fields in the shell equations in terms of the fields of the solid mechanics.

The main feature of the constraint approach is its consistency with the mechanics of shells treated as the three-dimensional bodies. However, the interrelation between the "three - dimensional" solutions of problems of the constraint approach and those obtained from the classical solid mechanics, remains open question as for as the non-linear theories are concerned. It has to be stressed that in the formation of the plate, shell and rod theories the constraints or semiconstraints are introduced to render the theory more tractable, [2], and the constraint reactions $\underset{\sim}{r}_R, \underset{\sim}{s}_R, a_{(m)}, a, \alpha$ can be interpreted as certain "imaginary" external fields acting at the shell like body.

5.3. Formation of rod theories

All obtained results concerning the plate and shell theories can be easily modified in order to obtain the rod theories (cf. the Prerequisites). Using, for example, the formal approximation approach, in the place of the shell deformation function $q_{(n)}(\underset{\sim}{\theta}, t)$ we have to introduce the rod deformation function $q_{(n)}(\xi, t)$, $\xi \in \langle h_-, h_+\rangle$, $t \in I$, cf. Eq. (A2.3). Analogously, instead of the "shell-type" functions $\tau^{(N)}(\underset{\sim}{\theta}, t)$, $\omega^{(p)}(\underset{\sim}{\theta}, t)$ (cf. Eqs. (A2.5), (A2.7) we have to introduce the "rod-type" functions $\tau^{(N)}(\xi, t)$, $\omega^{(p)}(\xi, t)$. Roughly speaking, in the formal passage from the shell theory to the rod theory in all relations in the place of $\underset{\sim}{\theta}$, ξ we have to substitute the material coordinates ξ, $\underset{\sim}{\theta}$, respectively, i.e., to make the interchange between the points $\underset{\sim}{\theta} \in \bar{\Pi}$ and $\xi \in \langle h_-, h_+\rangle$. Thus the integrals over (h_-, h_+) have to be replaced by the integrals over Π and inversely. Let us also observe that the integrals over $\partial\Pi$ have to be replaced by the sum of the values of the suitable integrands for $\xi = h_-$ and $\xi = h_+$. Analogously, the values of the fields for $\xi = h_-$ and $\xi = h_+$ have to be replaced by the integrals over $\partial\Pi$. This simple scheme, the geometric and physical sense of which is clear, makes it possible to obtain all needed relations of the rod theory (or relations which lead to the rod theory) directly from the suitable relations concerning the shell-like body. Thus by modifying the form of Eqs. (A5.1)-(A5.6) we obtain [1]

1. <u>Rod equations of motion</u> (for every $\xi \in (h_-, h_+)$ $t \in I$)

$$\underline{{}''H^a_{R,33}} + {}'H^a_{R,3} + h^a_R + f^a_R - \underline{{}'f^a_{R,3}} = i^a_R - \underline{{}'i^a_{R,3}} \qquad \text{(A5.10)}$$

---

[1] Mind that the indices $\alpha, \beta, \ldots$ (related to $\theta^\alpha, \theta^\beta, \ldots$) have to be replaced by the index 3 (related to $\xi \equiv \theta^3$), which in the following relations is omitted or represented by the "primes", i.e., $''H^a_R \equiv H^{a33}_R$, etc.

where

$$i^a_R = i^a_R(\xi,t,q_{(n)},q_{(n),3},\dot{q}_{(n)},\dot{q}_{(n),3},\ddot{q}_{(n)},\ddot{q}_{(n),3})$$

$$'i^a_R = {}'i^a_R(\xi,t,q_{(n)},q_{(n),3},\dot{q}_{(n)},\dot{q}_{(n),3},\ddot{q}_{(n)},\ddot{q}_{(n),3}) \tag{A5.11}$$

are the known functions and $q_{(n)} = q_{(n)}(\xi,t)$, $\xi \in (h_-,h_+)$, $t \in I$. The fields in Eqs. (A5.10) are related to the fields in solid mechanics relations by means of the formulae

$$"H^a_R \equiv -\int_\Pi \tilde{\underset{\sim}{T}}{}^{k3}_R d\bar{\Psi}^a_k , \qquad \tilde{\underset{\sim}{T}}_R \equiv \nabla\underset{\sim}{\tilde{p}}\underset{\sim}{\tilde{T}} ,$$

$$'H^a_R \equiv \int_\Pi (\tilde{T}^{k3}_R d\bar{\Phi}^a_k + \tilde{T}^{kK}_R d\bar{\Psi}^a_{k,K}) ,$$

$$h^a_R \equiv -\int_\Pi \tilde{T}^{kK}_R d\bar{\Phi}^a_{,K} ,$$

$$'f^a_R \equiv \int_\Pi b^k_R d\bar{\Psi}^a_k + \oint_{\partial\Pi} p^k_R d\bar{\Psi}^a_k , \tag{A5.12}$$

$$f^a_R \equiv \int_\Pi b^k_R d\bar{\Phi}^a_k + \oint_{\partial\Pi} p^k_R d\bar{\Phi}^a_k ,$$

$$'i^a_R \equiv \int_\Pi \rho_R \ddot{\tilde{p}}{}^k d\bar{\Psi}^a_k ,$$

$$i^a_R \equiv \int_\Pi \rho_R \ddot{\tilde{p}}{}^k d\bar{\Phi}^a_k ,$$

which can be deduced directly form Eqs. (A2.41), (A2.24). In more special cases in Eqs. (A5.12) we can assume $d\bar{\Phi}^a_k = \Phi^a_k da_R$ or $= \Phi^a_k dL_R$ and $d\bar{\Psi}^a_k = \Psi^a_k da_R$ or $= \Psi^a_k dL_R$, for $\underset{\sim}{\theta} \in \Pi$ or $\underset{\sim}{\theta} \in \partial\Pi$, respectively. Here $\bar{\Phi}^a_k$, $\bar{\Psi}^a_k$, Φ^a_k, Ψ^a_k are known functions of X, $q_{(n)}, q_{(n),3}$. The Eqs.(A2.22) have now their counterparts in the form $\underset{\sim}{\Phi}^a \equiv \partial\underset{\sim}{\tilde{p}}/\partial q_a$, $\underset{\sim}{\Psi}^a \equiv \partial\tilde{p}/\partial q_{a,3}$. We have also to assume that $\underset{\sim}{\tilde{p}} \equiv \underset{\sim}{\tilde{p}}(\underset{\sim}{\theta},\xi,q_{(n)},q_{(n),3})$ where $\tilde{p}$ (.) is the known function (cf. Secs. 2,4); if $\tilde{p}$ (.) is independent of the argument $q_{(n),3}$

then the underlined terms drop out from all relations. (Eqs. (A5.7) in this special case lead to those derived in [1]. The fields $"H^a_R$, $'H^a_R$, h^q_R, a = 1,...,n, and $-'i^a_R$, $-i^a_R$, a = 1,...,n, will be referred to as rod internal and inertia forces, respectively.

2. Rod kinetic boundary conditions (for $\xi = h_-$, $\xi = h_+$, $t \in I$)

$$\begin{aligned} &'H^a_R n + \underline{"H^a_{R,3} n} = p^a_R - \underline{('i^a_R - 'f^a_R)n} \\ &\underline{"H^a_R n} = \underline{'p^a_R} \end{aligned} \qquad (A5.13)$$

where n = +1 for $\xi = h_+$ and n = -1 for $\xi = h_-$. Using (A2.41), (A2.25) we obtain now

$$'p^a_R \equiv \int_\Pi p^k_R d\bar{\Psi}^a_k \ , \quad p^a_R \equiv \int_\Pi p^k_R d\bar{\Phi}^a_k \ . \qquad (A5.14)$$

The fields p^a_R, $'p^a_R$, f^a_R, $'f^a_R$, a = 1,...,n, will be referred to as the rod external forces. The form of Eqs. (A5.13) may be deduced from that of Eqs. (A5.3), (A5.4). Eqs. (A5.10, (A5.13) will be referred to as the *rod field equations*. They can be also interpreted in the integral generalized sense if the derivatives with respect ξ do not exist (cf. the comments to the shell field equations in Sec. (5.1)).

3. Rod constitutive relations (for $\xi \in (h_-,h_+)$, $t \in I$) can be deduced from Eqs. (A5.5), (A5.6). Modifying Eqs. (A5.5) we obtain

$$\underline{"H^a_R} = \underline{"\tilde{H}^a_R(\xi, q_{(n)}, q_{(n),3}, q_{(n),33}, \tau^{(N)})}$$

$$'H^a_R = '\tilde{H}^a_R(\xi, q_{(n)}, q_{(n),3}, \underline{q_{(n),33}}, \tau^{(N)})$$

$$h^a_R = \tilde{h}^a_R(\xi, q_{(n)}, q_{(n),3}, \underline{q_{(n),33}}, \tau^{(N)})$$

where the form of response functionals on RHS of Eqs. (A5.12) can be obtained from Eqs. (A5.9) and (A2.6) provided that $\tau^{(N)} \equiv \{\tau^A(\xi,t)$, $\xi \in (h_-,h_+), t \in I$, A = 1,...,N$\}$. Assuming that $\omega^{(p)} = \{\omega^\pi(\xi,t), \xi \in (h_-,h_+)$, $t \in I$, π =1,...,p$\}$ and that $e_\rho = E_\rho(q_{(n)})$, ρ = 1,...,r, where $E_\rho(.)$ are the ordinary differential operators with respect to ξ (such that $e_\rho = e_\rho(\xi,t)$ are, invariants under arbitrary rigid motions of the reference space), we also obtain

$$g_A(\xi,e_{(r)},\tau^{(N)},\omega^{(p)}) = 0, \quad A = 1,\dots,N+P \ ,$$

$$\kappa(\xi,e_{(r)},\tau^{(N)},\omega^{(p)}) \le 0,$$

$$\psi(\xi,e_{(r)},\tau^{(N)},\omega^{(p)},\overset{o}{e}_{(r)},\tau_o^{(N)}) \le 0 \text{ for every } \overset{o}{e}_{(r)},\tau^{N}_{(o)}$$

$$\text{with } \kappa(\xi,\overset{o}{e}_{(r)},\tau_o^{(N)},\omega^{(p)}) \le 0 \qquad \text{(A5.16)}$$

By virtue of Eqs. (A2.42) we have here

$$g_A \equiv \int_\Pi \tilde{f}_{\bar{\mu}} d\bar{\Xi}^{\bar{\mu}}_A \ , \quad \kappa \equiv \int_\Pi \tilde{j} d\bar{b}, \quad \psi \equiv \int_\Pi \tilde{\varphi} d\bar{\beta} \quad . \qquad \text{(A5.17)}$$

For the simple materials Eqs. (A5.16) are indentities and the arguments $\tau^{(N)}$ drops out from the RHS of Eqs. (A5.15).

All fields in Eqs. (A5.10, (A5.13), (A5.15), (A5.16) are independent of $\underset{\sim}{\theta} = (\theta^1,\theta^2)$; thus the mentioned above relations represent the general scheme of the rod theory. For every special rod theory the form of the functions and functionals i^a_R, $'i^a_R$, $''H^a_R$, $'H^a_R$, h^a_R, g_A,κ,ψ is assumed to be known and can be obtained from Eqs. $(A5.12)_{6,7}$, $(A5.12)_{1-3}$, (A5.17), provided that the functions $\bar{\Psi}^a_k$, $\bar{\Phi}^a_k,\bar{\Xi}^\mu_a,\bar{b},\bar{\beta}$(or functions Ψ^a_k,Φ^a_k, Ξ^μ_a) are known. The relations of the rod theories can be obtained by the direct, approximation, mixed or constraint approaches analogously as the plate and shell theories. All considerations concerning the plates or shells which have been carried on throughout this chapter hold also for rods provided that all relations are modified according to the remarks at the beginning of this section. That is why we do not detail separately in this treatise the problems of formation of the rod theories.

Let us observe, that the mechanics of the rod-like bodies, from the point of view of the formal approximation and constraint approaches, is described by:

1. The rod theory given by Eqs. (A5.10), (A5.13), (A5.15) and (A5.16).

2. The relations of the "three-dimensional" mechanics of the rod-like body, given by Eqs. (A5.7), (A5.8).

3. The interrelations between the fields in the rod theory and those in Eqs. (A5.7), (A5.8). They are given by Eqs. (A5.12) (A5.14), (A5.17) and by the equations of the form (A5.9) in which now $\nabla q_{(n)} \equiv q_{(n),3}$ and $q_{(n)}, \tau^{(N)}, \omega^{(p)}$ are the functions defined on $(h_-, h_+) \times I$.

CHAPTER B

SPECIAL PLATES AND SHELL THEORIES

In the Chapter A the general form of the governing relations of the plate, shell and rod theories has been obtained. Now we are to derive different plate and shell theories as the special cases of the general approaches which have been developed in the Chapter A. We confine ourselves mainly to the theories describing the mechanics of "thin" shell structures.

1. SCALAR PLATE THEORY

It is the plate theory in which only one real-valued function characterize the motion of the plate. The known applications of this theory are restricted to problems of the small deflections of thin plates. Nevertheless, the scalar plate theory constitutes a good illustration of the general relations of the Chapter A.

1.1. Governing relations

By the scalar plate theory we shall mean the plate theory in which the plate deformation function $q_{(n)}$ reduces to the simplest possible form $q_{(1)} = \{q\}$, where $q = q(\underset{\sim}{\theta}, t)$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$, is a sufficiently smooth scalar function. From Eqs. (A5.1) we shall obtain the equations of motion of the plate theory under consideration

$$H^{\alpha\beta}_{R,\alpha\beta} + H^{\alpha}_{R,\alpha} + h_R + f^{\alpha}_{R} - f^{\alpha}_{R,\alpha} = i_R - i^{\alpha}_{R,\alpha}. \tag{B1.1}$$

The kinetic boundary condition will be obtained as the special case of Eqs. (A5.3)

$$\begin{aligned} &H^{\alpha}_{R} n_{R\alpha} + \frac{d}{dl_R}(H^{\alpha\beta}_{R} t_{R\alpha} n_{R\beta}) + H^{\beta\alpha}_{R,\beta} n_{R\alpha} = p_{OR}, \\ &H^{\beta\alpha}_{R} n_{R\beta} n_{R\alpha} = -\underset{\sim}{p}^{N}_{R} \end{aligned} \tag{B1.2}$$

where

$$p_{OR} \equiv p_R - \frac{d}{dl_R}(p^{\alpha}_{R} t_{R\alpha}), \quad \underset{\sim}{p}^{N}_{R} \equiv p^{\alpha}_{R} n_{R\alpha}. \tag{B1.3}$$

The constitutive relations will be described by Eqs. (A 5.5), which now can be written down in the form

$$H_R^{\alpha\beta} = \widetilde{H}_R^{\alpha\beta}(\theta, q, \nabla q, \nabla\nabla q, \tau^{(N)}) ,$$
$$H_R^{\alpha} = \widetilde{H}_R^{\alpha} (\theta, q, \nabla q, \nabla\nabla q, \tau^{(N)}) , \tag{B1.4}$$
$$h_R = \widetilde{h}_R (\theta, q, \nabla q, \nabla\nabla q, \tau^{(N)}) ,$$

and by Eqs. (A5.6), in which $e_{(r)} = E_{(r)}(q)$ is the strain measure of the scalar plate theory. If there exists the plate kinetic energy function (cf. Sec.2.3. of the Chapter A) $\kappa_R = \kappa_R(\theta, q, \nabla q, \dot{q}, \nabla\dot{q})$ then

$$i_R = \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}} - \frac{\partial\kappa_R}{\partial q} , \qquad i_R^{\alpha} = \frac{d}{dt}\frac{\partial\kappa_R}{\partial\dot{q}_{,\alpha}} - \frac{\partial\kappa_R}{\partial q_{,\alpha}} . \tag{B1.5}$$

Thus the scalar plate theory is described by Eqs. (B1.1), (B1.2), (B1.4) and (A5.6). The relations of this theory will constitute the basis for the illustration of the special problems in the Chapter C. Here, form the foregoing relations we shall derive the well known linear theory of plates. To this aid we shall assume that the plate is hyperelastic, i. e., that there exists the plate strain energy function $\varepsilon_R = \varepsilon_R(\underset{\sim}{\theta}, \nabla q, \nabla\nabla q)$ such that the relations

$$H_R^{\alpha\beta} = -\frac{\partial\varepsilon_R}{\partial q_{,\alpha\beta}} , \quad H_R^{\alpha} = \frac{\partial\varepsilon_R}{\partial q_{,\alpha}} , \quad h_R = -\frac{\partial\varepsilon_R}{\partial q} = 0 \tag{B1.6}$$

hold for every $\underset{\sim}{\theta} \in \Pi$. Eqs. (B1.6) are the special case of Eqs. (B1.4), (A5.6).

<u>The classical plate theory</u>. Now suppose that the plate strain energy function ε_R has the form

$$\varepsilon_R = \frac{1}{2} C_R^{\alpha\beta\gamma\delta} q_{,\alpha\beta} q_{,\gamma\delta} ,$$

where $C_R^{\alpha\beta\gamma\delta}$ are the smooth functions defined on Π and constituting the strongly elliptic tensor for every $\underset{\sim}{\theta} \in \Pi$. Then Eqs. (B1.1) and (B1.6) yield

$$-(C_R^{\alpha\beta\gamma\delta} q_{,\gamma\delta})_{,\alpha\beta} + f_R - f_{R,\alpha}^{\alpha} = i_R - i_{R,\alpha}^{\alpha} .$$

If $C_R^{\alpha\beta\gamma\delta} = A_R\delta^{\alpha\beta}\delta^{\gamma\delta} + B_R\delta^{\alpha\gamma}\delta^{\beta\delta}$, where A_R, B_R are the scalar functions defined on Π ($A_R + B_R > 0$, $B_R > 0$), then the plate will be called isotropic. In this case

$$H_R^{\alpha\beta} = A_R\delta^{\alpha\beta}\delta^{\lambda\delta}q_{,\gamma\delta} + B_R\delta^{\alpha\gamma}\delta^{\beta\delta}q_{,\gamma\delta}. \tag{B1.7}$$

Moreover, if A_R, B_R are constants and $\kappa_R = \frac{1}{2}\alpha_R(\dot{q})^2$, where α_R is the positive number, then putting $D_R \equiv A_R + B_R$ and assuming that $f^{\alpha}_{R,\alpha} \equiv 0$ we shall obtain

$$-D_R\Delta\Delta q + f_R = \alpha_R\ddot{q}, \quad \Delta(\cdot) \equiv \delta^{\alpha\beta}(\cdot)_{,\alpha\beta} . \tag{B1.8}$$

The positive constant D_R is said to be the stiffness of the plate. Denoting $Q_R^{\alpha} \equiv H^{\beta\alpha}_{R,\beta}$ we also obtain from Eq. (B1.7) that

$$Q_R^{\alpha} = D_R\delta^{\alpha\beta}\Delta q_{,\beta} \tag{B1.9}$$

and from Eqs. (B1.2) the kinetic boundary conditions of the form

$$\begin{aligned} &Q_R^{\alpha}n_{R\alpha} + \frac{d}{dl_R}(H_R^{\alpha\beta}t_{R\alpha}n_{R\beta}) = p_{OR} , \\ &H_R^{\beta\alpha}n_{R\beta}n_{R\alpha} = -\underset{\sim}{p}^{N}_{R} . \end{aligned} \tag{B1.10}$$

Eqs. (B1.7) - (B1.10) represent the well known "classical" plate theory.

1.2. Interpretations

Let us assume, for the time being, that the argument $\tau^{(N)}$ drops out from Eqs. (B1.4) and that Eqs. (A5.6) are identities. Let us also interpret Eqs. (B1.1) - (B1.4) as obtained via the direct approach. Then the rule of interpretation of the function q should be known. Putting $h_+ + h_- = 0$ we can interpret q as the deflection of the midsurface of the plate. In this case (cf. Sec. 1.1. fo he Chapter A)

$$q(\underset{\sim}{\theta},t) = \underset{\sim}{p}(\underset{\sim}{\theta},0,t)\cdot\underset{\sim}{i}_3 , \tag{B1.11}$$

where $\underset{\sim}{i}_3$ is the versor of the x^3-axis in the reference space. In the direct approach all informations about the plate deformation are given by Eqs. (B1.11). It means that no kinematical hypothesis are needed when we approach the plate theory via the direct approach. On the other hand, this approach does not supply any informations for example, in the case of the "classical" plate theory, about the values of the constants A_R, B_R, α_R (apart from $D_R > 0$, $B_R > 0$, $\alpha_R > 0$, postulated in Sec. 1.1.). Thus we have obtained only the formal structure of the constitutive relations (B1.7), (B1.9) and of the inertial term $\alpha_R \ddot{q}$ in Eq. (B1.8). The interpretation of the plate forces $H_R^{\alpha\beta}$, f_R, p_{OR}, $p_R^{\underset{\sim}{N}}$ in Eqs. (B1.7), (B1.8), (B1.10) in the case of the direct approach is given by the formulae (cf. Sec. 1 of the Chapter A)

$$R_i = -\int_{\Pi} H_R^{\alpha\beta} \dot{q}_{\alpha\beta} da_R$$

$$R_e = \int_{\partial\Pi} (p_R \dot{q} + p_R^{\alpha} \dot{q}_{,\alpha}) dl_R + \int_{\Pi} f_R \dot{q}\, da_R = \int_{\Pi} f_R \dot{q}\, da_R + \tag{B1.12}$$

$$+ \int_{\partial\Pi} (p_{OR} \dot{q} + p_R^{\underset{\sim}{N}} \dot{q}_{,\underset{\sim}{N}}) dl_R \quad , \quad q_{,\underset{\sim}{N}} = q_{,\alpha} n_R^{\alpha} .$$

We conclude that if the derivatives $q_{,\alpha}$ are sufficiently small with respect to the unity (i.e., the $q_{,\alpha\beta}$ can be approximately treated as the components of the curvature tensor of the deflected midsurface of the plate) then H_R^{11}, H_R^{22} can be interpreted as the bending and H_R^{12} as the twisting couples. Analogously, $p_R^{\underset{\sim}{N}}$ will be the boundary bending couple, p_{OR} will be the boundary transverse force and f_R the plate body force. This interpretation is implied by the interpretation formula (B1.11) and by Eqs. (B1.12).

If Eqs. (B1.1) - (B1.4) are obtained from the approximation or constraint approaches then the form of the function $\underset{\sim}{\tilde{p}}(\cdot)$ in the approximation relation

$$\underset{\sim}{p}(\underset{\sim}{X}, t) \sim \underset{\sim}{\tilde{p}}(\underset{\sim}{X}, q, \Delta q)$$

or in the constraint relation

$$\underset{\sim}{p}(\underset{\sim}{X}, t) = \underset{\sim}{\tilde{p}}(X, q, \Delta q) \quad ,$$

respectively, have to be known. For example, the funcction $\tilde{\underset{\sim}{p}}(\cdot)$ can be assumed in the form

$$\underset{\sim}{p} = \theta^{\alpha}\underset{\sim}{i}_{\alpha} + q\underset{\sim}{i}_{3} + \xi\frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} \tag{B1.13}$$

where $\underset{\sim}{a}_{\alpha} = \underset{\sim}{i}_{\alpha} + q_{,\alpha}\underset{\sim}{i}_3$ and i_k is the versor of the x^k-axis. Eq. (B1.13) represents the known kinematic Kirchhoff hypothesis. Moreover, if the plate is hyperelastic then

$$\varepsilon_R = \int_{h_-}^{h_+} \rho_R \sigma(\underset{\sim}{X}, \nabla\tilde{\underset{\sim}{p}}(\underset{\sim}{X}, q, \nabla q))\, d\xi \, ,$$

and for the "classical" theory we shall obtain from Eqs. (B1.6), the formulae (B1.7) in which

$$A_R = \frac{\nu E h^3}{12(1-\nu^2)} \, , \quad B_R = \frac{E h^3}{12(1+\nu)} \, , \quad D_R = \frac{E h^3}{12(1-\nu^2)} \, ,$$

where $h \equiv h_+ - h_-$ and where E, ν are the known material constant.

It must be stressed that the results obtained in this section can be found in any elementary textbooks on the linear plate theory and have been derived only in order to provide the simple illustration of the general approaches developed in the Chapter A.

2. PSEUDO PLANE AND PLANE PROBLEMS

The plane problems are usual formulated independently of the plate theory as the plane strain or plane stress problems. Here we formulate the governing relations for more general class of such problems using the general approach of the Chapter A. By the pseudo-plane problems we understand the class of problems which, roughly speaking, are "inc- cluded" between plane stress and plane stress problems.

2.1. Pseudo-plane problems

Let us assume that the plate deformation function $q_{(n)}$ can be assumed as $q_{(n)} = q_{(3)} = \{q_\alpha, q;\ \alpha = 1,2\}$. Putting $h_- = -h$, $h_+ = h$, $h > 0$, and denoting by $\underset{\sim}{i}_k$, $k = 1,2,3$, the versors of the rectangular Carthesian coordinate system x^k in the reference space, we shall interpret $q_{(3)}$ by means of $q_\alpha = p_\alpha(\underset{\sim}{\theta},0,t)$, $q = \partial p_3(\underset{\sim}{\theta},0,t)/\partial\xi$ or $q_\alpha = \underset{\sim}{p}(\underset{\sim}{\theta},0,t)\cdot\underset{\sim}{i}_\alpha$, $q = [\partial\underset{\sim}{p}(\underset{\sim}{\theta},0,t)/\partial\xi]\cdot\underset{\sim}{i}_3$, when using the direct approach (cf. Eqs. (A1.1). Using the approximation approach, we assume Eq. (A2.3) in the special form

$$\underset{\sim}{p}(\underset{\sim}{\theta},\xi,t) \sim \underset{\sim}{\tilde{p}}(\underset{\sim}{\theta},\xi,t) \equiv \sum_{\alpha=1}^{2} q_\alpha \underset{\sim}{i}_\alpha + \xi q \underset{\sim}{i}_3 \quad , \qquad q_{(3)} \in Q \tag{B2.1}$$

where $Q := \{q_{(3)} | \det \nabla\tilde{p} > 0\}$. In the constraint approach the sign $\sim$ in Eq. (B2.1) has to be replaced by the equality.

Because the RHS of Eq. (B2.1) do not contain the derivatives of $q_{(3)}$ we can apply the simple approach (Sec. 2.1. of the Chapter A) or to use the first order shell force system in the direct approach. From Eqs. (A2.4) (replacing the sign $\sim$ by the equality) or from Eqs. (A1.9), (A1.10) we obtain the field equations ([1]) in the form

$$H^{\beta\alpha}_{R,\alpha} + h^\beta_R + f^\beta_R = i^\beta_R \quad ,$$

$$H^{3\alpha}_{R,\alpha} + h^3_R + f^3_R = i^3_R \quad ; \quad \underset{\sim}{\theta} \in \Pi \ , \quad t \in I \ ,$$

([1]) Eqs. (B2.2) can be also obtained from Eqs. (A2.36)-(A2.38) taking into account (A2.33)-(A2.35) and (B2.1). The same form of the equations we shall obtain from the constraint approach.

$$H_R^{\beta\alpha} n_{R\alpha} = p_R^{\beta} \quad ,$$

$$H_R^{3\alpha} n_{R\alpha} = p_R^{3} \quad ; \qquad \underset{\sim}{\theta} \in \partial\Pi, \text{ a.e. }, \ t \in I \ . \qquad \text{(B2.2)}$$

Using Eqs. (A2.5) with $\underset{\sim}{\phi}^a = \partial\underset{\sim}{\tilde{p}}/\partial q_a$, we arrive at [1]

$$H_R^{\beta\alpha} = \int_{-h}^{h} T_R^{\beta\alpha}\, d\xi, \quad H_R^{3\alpha} = \int_{-h}^{h} T_R^{3\alpha}\xi d\xi,$$

$$h_R^{\beta} = 0 \quad , \qquad h_R^{3} = -\int_{-h}^{h} T_R^{33}\, d\xi,$$

$$f_R^{\beta} = \int_{-h}^{h} b_R^{\beta}\, d\xi + \overset{+\beta}{p_R} + \overset{-\beta}{p_R} \quad , \quad f_R^{3} = \int_{-h}^{h} \xi b_R^{3}\, d\xi + h\overset{+3}{p_R} - h\overset{-3}{p_R} \ , \qquad \text{(B2.3)}$$

$$p_R^{\beta} = \int_{-h}^{h} \bar{p}_R^{\beta}\, d\xi, \qquad p_R^{3} = \int_{-h}^{h} \xi \bar{p}_R^{3}\, d\xi$$

$$i_R = \int_{-h}^{h} \rho_R\, d\xi\, \delta^{\beta\gamma}\ddot{q}_{\gamma} \ ,$$

$$i_R^{3} = \int_{-h}^{h} \rho_R \xi^2\, d\xi\, \ddot{q} \ .$$

Because of $\tilde{p}_k = q_\gamma \delta_k^\gamma + q\xi\delta_k^3$, in view of Eq. (A2.10) we obtain

$$\tilde{C}_{\alpha\beta} = \delta^{\gamma\delta} q_{\gamma,\alpha} q_{\delta,\beta} + \xi^2 q_{,\alpha} q_{,\beta} \quad ,$$

$$\tilde{C}_{\alpha 3} = \xi q\, q_{,\alpha} \quad , \quad \tilde{C}_{33} = q^2$$

It follows that $e_{(r)} = e_{(6)} = \{(\delta^{\gamma\delta} q_{\gamma,\alpha} q_{\delta,\beta}),\ q_{,\alpha}, q\}$. The constitutive relations we can assume in the general form (A2.15) where Eqs. (A2.6)-(A2.9) hold. For the hyperelastic shell they reduce to the equations

---

[1] We denote here $\underset{\sim}{p}_R = \overset{-k}{p_R} \underset{\sim}{i}_k$ in Eqs. (A2.1) to avoid the ambiguity.

$$H_R^{\beta\alpha} = \frac{\partial \varepsilon_R}{\partial q_{\beta,\alpha}} \quad , \quad h_R^{\beta} = - \frac{\partial \varepsilon_R}{\partial q_\beta} = 0 \quad ,$$

$$H_R^{3\alpha} = \frac{\partial \varepsilon_R}{\partial q_{,\alpha}} \quad , \quad h_R^{3} = - \frac{\partial \varepsilon_R}{\partial q} \; ; \; \varepsilon_R = \int_{-h}^{h} \rho_R \sigma(X,\widetilde{C})\, d\xi \; . \tag{B2.4}$$

If the plate is sufficiently thin and the external force resultant in x^3-direction can be neglected, then taking into account the mixed approach we shall postulate that the terms $H_R^{3\alpha}$, f_R^3, p_R^3, i_R^3 drop out from Eqs. (B2.2)-(B2.4). Thus the mixed approach, under foregoing conditions, yields

$$H_{R,\alpha}^{\beta\alpha} + f_R^{\beta} = \alpha_R \delta^{\beta\gamma} \ddot{q}_\gamma \quad , \quad \alpha_R \equiv \int_{-h}^{h} \rho_R \, d\xi \; ,$$

$$h_R^3 = 0 \; ; \; \underset{\sim}{\theta} \in \Pi \; , \; t \in I \; , \tag{B2.5}$$

$$H_R^{\beta\alpha} n_{R\alpha} = p_R^{\beta} \quad ; \quad \underset{\sim}{\theta} \in \partial\Pi \; , \; \text{a.e.}, \; t \in I \; .$$

For the hyperelastic matrials we shall assume here, using the mixed approach, that $\varepsilon_R = \varepsilon_R(\underset{\sim}{\theta}, (\delta^{\gamma\delta} q_{\gamma,\alpha} q_{\delta,\beta}), q)$. If, from $h_R^3 = -\partial\varepsilon_R/\partial q$ we can express q in terms of $q_{\gamma,\alpha}$, then Eqs. (B2.5) and Eqs. (B2.4) will lead to the system of two equations of motion for the two unknown functions $q_\gamma = q_\gamma(\underset{\sim}{\theta}, t)$. It is a well known plane stress of the theory of elasticity.

Now, using again the mixed approach, let us assume that the terms $H_R^{3\alpha}$, p_R^3, i_R^3 drop out from Eqs. (B2.2)-(B2.4) ([1]).
Then

$$H_{R,\alpha}^{\beta\alpha} + f_R^{\beta} = \alpha_R \delta^{\beta\gamma} \ddot{q}_\gamma \; ,$$

$$h_R^3 + f_R^3 = 0 \quad ; \quad \underset{\sim}{\theta} \in \Pi \; , \; t \in I \; , \tag{B2.6}$$

$$H_R^{\beta\alpha} n_{R\alpha} = p_R^{\beta} \; ; \; \underset{\sim}{\theta} \in \partial\Pi, \; \text{a.e.}, \; t \in I \; .$$

([1]) All assumptions postulated in the mixed approach or in the direct approach can be based on the physical premises.

Let us also assume that $q = 1$ is the solution of Eq. $(B2.6)_2$ for an arbitrary q_α, i.e., let $f^3_R = -h^3_R|_{q=1} = [\partial\varepsilon_R/\partial q]_{q=1}$. Then $\varepsilon_R = \varepsilon_R(\theta,(\delta^{\lambda\delta}q_{\gamma,\alpha}q_{\delta,\beta}))$ and Eqs. $(B2.6)_1$, $(B2.4)_1$ lead to the well known plane strain problem of the theory of elasticity.

The analysis of the pseudo plane problems can be found in [3].

2.2. Plane problems

Putting $q_{(n)} = q_{(2)} = \{q_\alpha, \ \alpha = 1,2\}$ and

$$\underset{\sim}{p}(\underset{\sim}{\theta},\xi,t) \sim \tilde{\underset{\sim}{p}}(\underset{\sim}{\theta},\varepsilon,t) \equiv \sum_{a=1}^{2} q_\alpha \underset{\sim}{i}_\alpha + \xi \underset{\sim}{i}_3 , \quad q_{(2)} \in Q , \tag{B2.7}$$

where $Q := \{q_{(2)} | \det \nabla\tilde{\underset{\sim}{p}} > 0\}$, we shall arrive at the plain problem. The field equations are

$$\begin{aligned} H^{\beta\alpha}_{R\ ,\alpha} + f^\beta_R &= \alpha_R \delta^{\beta\gamma}\ddot{q}_\gamma , \quad \underset{\sim}{\theta} \in \Pi , \quad t \in I , \\ H^{\beta\alpha}_R n_{R\alpha} &= p^\beta_R \quad , \quad \underset{\sim}{\theta} \in \partial\Pi \text{, a.e., } t \in I \end{aligned} \tag{B2.8}$$

with the denotations $(B2.3)_{1,5,7}$. For the hyperelastic material Eq. $(B2.4)_1$ holds with $\varepsilon_R = \varepsilon_R(\theta,(\delta^{\gamma\delta}q_{\gamma,\alpha}q_{\delta,\beta}))$:

$$H^{\beta\alpha}_R = \frac{\partial\varepsilon_R}{\partial q_{\beta,\alpha}} \quad . \tag{B2.9}$$

If the inertia forces in Eq. $(B2.8)_1$ can be neglected (via the mixed approach) then as the basic unknown can be taken $a_{\alpha\beta} \equiv \delta^{\gamma\delta}q_{\gamma,\alpha}q_{\delta,\beta}$ and

$$H^{\gamma\alpha} = \int_{-h}^{h} T^{\gamma\alpha} d\xi \ ,$$

where $T^{\gamma\alpha} = T^{\alpha\gamma}$ are the components of the second Piola-Kirchhoff stress tensor, $T^{\beta\alpha}_R = p^\beta_{,\gamma}T^{\gamma\alpha} = q^\beta_{,\gamma}T^{\gamma\alpha}$ [1]. Because of $H^{\beta\alpha}_R = q^\beta_{,\gamma}H^{\gamma\alpha}$ and denoting by a^δ_β the inverse matrix with respect to $q^\beta_{,\gamma}$, we can transform Eqs. (B2.8) to the form

$$\begin{aligned} H^{\delta\alpha}|_\alpha + f^\delta &= 0 \ , \\ H^{\delta\alpha} n_{R\alpha} &= p^\delta \quad , \end{aligned} \tag{B2.10}$$

---

(1)Here $q^\alpha = q_\alpha$ because q_α are related to the orthogonal Carthesian coordinate system x^k in the reference space.

where the inertia forces have been neglected, the vertial line denote the covariant derivative in the metric $a_{\alpha\beta}$ and where we have denoted

$$f^{\delta} \equiv a^{\delta}_{\beta} f^{\beta}_{R} \quad , \quad p^{\delta} \equiv a^{\delta}_{\beta} p^{\beta}_{R} \quad .$$

At the same time

$$H^{\delta\alpha} = 2 \frac{\partial \varepsilon_R}{\partial a_{\alpha\beta}} \quad , \quad \varepsilon_R = \varepsilon_R(\theta, (a_{\alpha\beta})) \quad . \tag{B2.11}$$

Let us assume that f^{δ}, p^{δ} are the known functions of $a_{\alpha\beta}$. Then Eqs. (B2.10), (B2.11) constitute the system of 5 equations for 6 unknown functions $H^{\delta\alpha}$, $a_{\alpha\beta}$ (because of $H^{[\delta\alpha]} = 0$, $a_{[\alpha\beta]} = 0$). To obtain the extra equations we shall take into account the compatibility conditions. Eq. (A2.20) can be now taken in the form

$$\int_{-h}^{h} R_{1212} \, d\xi = 0$$

where the component R_{1212} of the Riemann-Christoffel tensor is now the known function of $C_{\alpha\beta} = a_{\alpha\beta}$, $C_{\alpha 3} = 0$, $C_{33} = 1$. Thus the foregoing compatibility condition leads to the equation $R_{1212} = 0$ which together with Eqs. (B2.10), (B2.11) constitute the system of 6 equations for the 6 unknowns $a_{\alpha\beta}$, $H^{\delta\alpha}$. It is the well known intrinsic formulation ([1]) of the plane problem of elasticity. The analogous intrinsic formulations can be also applied to the pseudo plane elastic problems described by Eqs. (B2.5) or Eqs. (B2.5) or Eqs. (B2.6) and Eqs. $(B2.4)_{1,4,5}$, provided that the inertia forces can be neglected. The intrinsic formulations can be also applied to the non-elastic or even non-simple materials. In this case Eqs. (B2.11) has to be replaced by Eqs. (A2.15), in which $e_{(r)} = e_{(3)} = (a_{11}, a_{22}, a_{12})$ and by the equations

$$H^{\delta\alpha} = \int_{h}^{h} S^{\delta\alpha}(\underset{\sim}{\sigma}) \, d\xi$$

where $\underset{\sim}{\sigma}$ are determined by Eqs. (A2.6), (A2.7) (after replacing the sign $\sim$ in (A2.7) by the equality).

([1]) By the intrinsic formulation we mean such formulation of the governing relations in which the internal forces and the strain measures are the basic unknowns (instead of the deformation function).

3. EULERIAN AND LAGRANGIAN FORMULATIONS

In the Chapter A all densities have been related to the reference configuration of the shell. In this configuration the material coordinates θ^1, θ^2, ξ coincide with the Carthesian orthogonal coordinates x^1, x^2, x^3, respectively. The formulations of the plate shell or rod theories in which all fields are related either to a certain arbitrary but fixed configuration or to the actual configuration are termed Lagrangian or Eulerian formulations, respectively. If the fixed configuration (which can be called initial or undeformed) coincides with the reference configuration (¹) then the shell governing relations obtained or postulated in the Chapter A are expressed in the Lagrangian formulation. Such situation holds mainly in the plate theories. In the shell theories the initial configuration is usually assumed as different then the reference configuration.

3.0. Analytical preliminaries

Let Z^K, K = I, II, III, $\underset{\sim}{Z} = (Z^K) \in \Omega_R$, be the curvilinear coordinates in the region Ω of the space R^3, related to the Carthesian orthogonal coordinates x^k, k = 1,2,3, by means of $x^k = \varphi^k(\underset{\sim}{Z})$, $\underset{\sim}{Z} = (Z^K)$, det φ^k, K > O, $\varphi^k(.)$ being the known differentiable functions. We shall denote $g^k_M \equiv \varphi^k_{,M}$ (indices K.L,M,... run over I, II, III), $g_{MN} \equiv g^k_M g_{kN}$, $g \equiv \det g_{MN} = (\det \varphi^k_{,K})^2$. We shall use the well known formulae

$$\sqrt{g}_{,L} = \sqrt{g}\left\{{M \atop ML}\right\} , \quad g^k_{K,L} = \left\{{M \atop KL}\right\} g^k_M , \tag{B3.1}$$

where $\left\{{M \atop KL}\right\}$ are the Christoffel symbols.

Let $A^{k\cdot}_R$, where the dot stands for an arbitrary "dead" multiindex $M_1,\ldots,M_m$, be the field defined and smooth in Ω of the vector density related to the region Ω_R, where $\Omega_R = \varphi^{-1}(\Omega)$. Putting

$$\frac{1}{\sqrt{g}} A^{k\cdot}_R = A^{M\cdot} g^k_M , \tag{B3.2}$$

we obtain $A^{M\cdot}$ as the density related to the region Ω. Using (B3.1),

---

(¹) The term "reference configuration" has to be understood exclusively in the sense defined in the Prerequisites

(B3.2) we also obtain

$$\frac{1}{\sqrt{g}} A^{k\cdot}_{R,L} = A^{M\cdot}]_L g^k_M$$

where we have denoted

$$A^{M\cdot}]_L \equiv A^{M\cdot}_{\ ,L} + \{^{N}_{N\,L}\} A^{M\cdot} + \{^{M}_{K\,L}\} A^{K\cdot} \quad .$$

Mind, that $A^{ML}]_L = A^{ML}|_L$, where the vertical line denote the covariant derivative in the metric g_{MN}. Analogously

$$\frac{1}{\sqrt{g}} A^{k\cdot}_{R,LM} = (A^{N\cdot}]_{L,M} + \{^{P}_{PM}\} A^{N\cdot}{}_{L} + \{^{P}_{PN}\} A^{P\cdot}]_L) g^k_N \ ,$$

and $A^{NLM}]_L]_M = A^{NLM}|_{LM}$. Thus we have arrived at the relations

$$\frac{1}{\sqrt{g}} A^{kL}_{R,L} = A^{ML}|_L g^k_M \ , \quad \frac{1}{\sqrt{g}} A^{kLN}_{R,LN} = A^{MLN}|_{LN} g^k_M \quad . \qquad \text{(B3.3)}$$

Now let $\Omega_R = \Pi \times (h_-, h_+)$, Π being the regular region on the plane Ox^1x^2 and $h_- < 0$, $h_+ > 0$. Let $x^k = \psi^k(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Pi$, be the differentiable functions such that det $(\psi^k_{,\alpha}\psi_{k,\beta}) > 0$. It means that $x^k = \psi^k(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} = (\theta^\alpha) \in \Pi$, can be interpreted as the parametric representation of a certain surface π in R^3. The mapping $x^k = \varphi^k(\underset{\sim}{Z})$, $\underset{\sim}{Z} \in \Omega_R$, will be now assumed in the form $x^k = \psi^k(\underset{\sim}{\theta}) + N^k(\underset{\sim}{\theta})\,\xi$, where $Z^K = \delta^K_\alpha + \delta^K_3\,\xi$ (1), and $N^kN_k = 1$, $N^k\psi_{k,\alpha} = 0$ (i.e., $\underset{\sim}{N}$ is the unit vector normal to π). Then $g^k_M = \psi^k_{,\alpha}\,\delta^\alpha_M + N^k\delta^3_M + \xi N^k_{,\alpha}\delta^\alpha_M$, and on the coordinate surface $\xi = 0$ (i.e., on the surface π) we obtain

$$\sqrt{g} = \sqrt{a}, \quad a \equiv \det a_{\alpha\beta}, \ a_{\alpha\beta} = \psi^k_{,\alpha}\psi_{k,\beta}$$

$$\{^{3}_{\alpha\beta}\} = b_{\alpha\beta} \ , \ \{^{\beta}_{\alpha 3}\} = -b^\beta_\alpha \ , \ \{^{3}_{3\alpha}\} = \{^{\alpha}_{33}\} = \{^{3}_{33}\} = 0 \ ,$$

where $b_{\alpha\beta}$ are the components of the second metric tensor of the surface π.

Let the fields $A^{k\cdot}_R$, defined in Ω, are such $A^k_{R,3} \equiv 0$. Then also $A^{M\cdot}]_3 \equiv 0$ and from Eqs. (B3.2), (B3.3) we obtain on the surface π

---

(1) Mind, that Z^K are not the material coordinates of the shell-like body

$$\frac{1}{\sqrt{a}} A_R^{k\cdot} = A^{\alpha\cdot}\psi^k_{,\alpha} + A^{3\cdot}N^k ,$$

$$\frac{1}{\sqrt{a}} A^{k\alpha}_{R,\alpha} = A^{\gamma\alpha}|_\alpha \psi^k_{,\gamma} + A^{3\alpha}|_\alpha N^k , \qquad \text{(B3.4)}$$

$$\frac{1}{\sqrt{a}} A^{k\alpha\beta}_{,\alpha\beta} = A^{\gamma\alpha\beta}|_{\alpha\beta}\psi^k_{,\gamma} + A^{3\alpha\beta}|_{\alpha\beta}N^k ,$$

where the foregoing fields can be interpreted as defined on π. At the same time

$$A^{\alpha\beta}|_\beta = A^{\alpha\beta}||_\beta - b^\alpha_\beta A^{3\beta} ,$$

$$A^{3\beta}|_\beta = A^{3\beta}||_\beta + b_{\alpha\beta}A^{\alpha\beta}, \qquad \text{(B3.5)}$$

where the double vertical line denotes the covariant derivative on the surface π (the superscript "3" is the "dead" index in the differentiation on the surface).

Now let $A_R^{\cdot}$, where the dot stands for an arbitrary multiindex $K_1 \ldots K_m$, be the field in defined and smooth Ω of the scalar density related to the region Ω_R. Putting

$$\frac{1}{\sqrt{g}} A_R^{\cdot} = A^{\cdot} , \qquad \text{(B3.6)}$$

we obtain $A^{\cdot}$ as the density related to the region Ω. Using (B3.1), (B3.6) we arrive at

$$\frac{1}{\sqrt{g}} A^{\cdot}_{R,L} = A^{\cdot}[_L ,$$

where

$$A^{\cdot}[_L \equiv A^{\cdot}_{,L} + \{^{N}_{NL}\}A^{\cdot} .$$

Analogously, we obtain

$$\frac{1}{\sqrt{g}} A^{\cdot}_{R,LM} = A^{\cdot}[_{L,M} + \{^{P}_{PM}\}A^{\cdot}[_L = A^{\cdot}[_L[_M .$$

Let us observe that $A^{L}[_{L} = A^{L}|_{L}$, $A^{LM}[_{L}[_{M} = A^{LM}|_{LM}$. Thus we have arrived at the relations

$$\frac{1}{\sqrt{g}} A^{L}_{R,L} = A^{L}|_{L} \;, \quad \frac{1}{\sqrt{g}} A^{LM}_{R,LM} = A^{LM}|_{LM} \;. \tag{B3.7}$$

Let $\Omega_R = \Pi \times (h_-, h_+)$ and let $A^{\cdot}_{R,3} \equiv 0$. Then $A^{\cdot}[_{3} \equiv 0$ and on the surface π, $\pi = \underset{\sim}{\psi}(\Pi)$, we obtain

$$\frac{1}{\sqrt{a}} A^{\cdot}_{R} = A^{\cdot} \;, \quad \frac{1}{\sqrt{a}} A^{\alpha}_{R,\alpha} = A^{\alpha}|_{\alpha} \;, \quad \frac{1}{\sqrt{a}} A^{\alpha\beta}_{R,\alpha\beta} = A^{\alpha\beta}|_{\alpha\beta} \tag{B3.8}$$

where the covariant derivatives in Eqs. (B3.8) are related to the covariant derivatives on the surface π by means of Eqs. (B3.5).

Eqs. (B3.4), (B3.5), (B3.8) are the basis for the further analysis.

3.1. General formulation

Let $x^k = \psi^k(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Pi$, be the parametric representation of an arbitrary smooth surface π, $N^k = N^k(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Pi$, be the field of unit vectors normal to π, and $a_{\alpha\beta} = \psi^k_{,\alpha}\psi_{k,\beta}$, $a = \det a_{\alpha\beta}$. Let

$$A^{k\alpha\beta}_{R,\alpha\beta} + A^{k\alpha}_{R,\alpha} + a^{k}_{R} + f^{k}_{R} - f^{k\alpha}_{R,\alpha} = i^{k}_{R} - i^{k\alpha}_{R,\alpha} \tag{B3.9}$$

stands for a certain equation of motion of the shell theory. The suitable kinetic boundary conditions have the form

$$\begin{aligned} &A^{k\alpha}_{R} n_{R\alpha} + \frac{d}{dL_R}(A^{k\alpha\beta}_{R} n_{R\beta} t_{R\alpha}) + A^{k\beta\alpha}_{R,\beta} n_{R\alpha} = p^{k}_{oR} - (i^{k\alpha}_{R} - f^{k\alpha}_{R}) n_{R\alpha} \\ &A^{k\alpha\beta}_{R} n_{R\alpha} n_{R\beta} = -p^{k\underset{\sim}{N}}_{R} \;. \end{aligned} \tag{B3.10}$$

Multiplying Eq . (B3.9) by $(\sqrt{a})^{-1}$ and using Eqs. (B3.4) we obtain

$$A^{M\alpha\beta}|_{\alpha\beta} + A^{M\alpha}|_{\alpha} + a^{M} + f^{M} - f^{M\alpha}|_{\alpha} = i^{M} - i^{M\alpha}|_{\alpha} \;, \tag{B3.11}$$

where the covariant derivatives are determined by Eqs. (B3.5).

To transform the kinetic boundary conditions (B3.10) we shall use the known formulae

$$n_{R\alpha} = \sqrt{\frac{\lambda}{a}}\, n_\alpha , \quad \lambda \equiv \left(\frac{dl}{dl_R}\right)^2 , \quad t_{R\alpha} = \sqrt{\lambda}\, t_\alpha ,$$

$$\psi^k_{,\gamma\delta} = \{{}^{\mu}_{\gamma\delta}\}\psi^k_{,\mu} + b_{\gamma\delta}N^k , \quad N^k_{,\delta} = -b^\mu_\delta \psi^k_{,\mu} , \quad \frac{\sqrt{a}_{,\alpha}}{\sqrt{a}} = \{{}^{\beta}_{\beta\alpha}\}$$

where "l" is the arc parameter along the boundary $\partial\pi = \underset{\sim}{\psi}(\partial\Pi)$. Let us decompose the RHS of Eqs. (B3.10)

$$\frac{1}{\sqrt{\lambda}} p^k_{oR} = p^\gamma_o \psi^k_{,\gamma} + p^3_o N^k ,$$

$$\frac{1}{\sqrt{\lambda}} p^{k\underset{\sim}{N}}_R = p^{\gamma\underset{\sim}{N}} \psi^k_{,\gamma} + p^{3\underset{\sim}{N}} N^k , \qquad \text{(B3.12)}$$

and introduce the derivatives

$$A^{M\beta\alpha}]_\beta \equiv A^{M\beta\alpha}_{,\beta} + A^{M\beta\alpha}\{{}^{N}_{N\beta}\} + A^{N\beta\alpha}\{{}^{N}_{N\beta}\} , \qquad \text{(B3.13)}$$

where the Christoffel symbols are related to the system of curvilinear coordinates $z^K = \delta^K_\alpha \theta^\alpha + \delta^K_3 \xi$ introduced in Sec. 3.0 and are taken for $\xi = 0$. We shall also use the absolute derivatives along the boundary $\partial\pi$

$$\frac{D}{dl}\bar{A}^M = \frac{d}{dl}\bar{A}^M + \{{}^{N}_{N\delta}\}\bar{A}^N t^\delta \qquad \text{(B3.14)}$$

where $\underset{\sim}{t}$ is the unit vector tangent almost everywhere to $\partial\pi$. After some calcualtions we obtain the following alternative form of the shell kinetic boundary conditions (B3.10):

$$A^{M\alpha} n_\alpha + \frac{D}{dl}(\lambda A^{M\beta\alpha} n_\alpha t_\beta) + A^{M\beta\alpha}]_\beta n_\alpha = p^M_o - (i^{M\alpha} - f^{M\alpha}) n_\alpha$$

$$\lambda A^{M\alpha\beta} n_\alpha n_\beta = -ap^{M\underset{\sim}{N}} , \qquad \text{(B3.15)}$$

where M = 1,2,3 and where the denotations (B3.12) - (B3.14) hold.

At the same time the relations between the fields $A^{M\alpha\beta}$, $A^{M\alpha}$, a^{M} and the fields $A_R^{k\alpha\beta}$, $A_R^{k\alpha}$, a_R^{k} can be obtained from Eqs. $(B3.4)_1$ and have the form

$$A^{\alpha\cdot} = \frac{1}{\sqrt{a}} A_R^{k\cdot} \psi_{k,\beta} a^{\beta\alpha} \quad ,$$

$$A^{3\cdot} = \frac{1}{\sqrt{a}} A_R^{k\cdot} N_k \quad . \tag{B3.16}$$

The equations of motion (B3.11) and the kinetic boundary conditions (B3.15) constitute the system of the field equations of the plate or shell theory, related to an arbitrary smooth surface π, $\pi = \underset{\sim}{\psi}(\Pi)$, provided that in the reference configuration they are given by Eqs. (B3.9), (B3.10).

Remark. If both sides of the field equations are scalars (with respect to the group of transformation of the reference space), then they can be written down in the form

$$A_{R,\alpha\beta}^{\alpha\beta} + A_{R,\alpha}^{\alpha} + a_R + f_R - f_{R,\alpha}^{\alpha} = i_R - i_{R,\alpha}^{\alpha} \quad , \tag{B3.17}$$

and

$$A_R^{\alpha} n_{R\alpha} + \frac{d}{dl_R} (A_R^{\alpha\beta} n_{R\beta} t_{R\alpha}) + A_{R,\beta}^{\beta\alpha} n_{R\alpha} = p_{oR} - (i_R^{\alpha} - f_R^{\alpha}) n_{R\alpha} \quad ,$$

$$A_R^{\alpha\beta} n_{R\alpha} n_{R\beta} = -p_R^{\underset{\sim}{N}} \quad . \tag{B3.18}$$

Using the transforamtion formulae (B3.8) and

$$\frac{1}{\sqrt{\lambda}} p_{oR} = p_o \quad , \quad \frac{1}{\sqrt{\lambda}} p_R^{\underset{\sim}{N}} = p^{\underset{\sim}{N}} \quad , \tag{B3.19}$$

we obtain from (B3.17), (B3.18) the equation of motion.

$$A^{\alpha\beta}\big|_{\alpha\beta} + A^{\alpha}\big|_{\alpha} + a + f - f^{\alpha}\big|_{\alpha} = i - i^{\alpha}\big|_{\alpha} \quad , \tag{B3.20}$$

and the kinetic boundary conditions

$$A^{\alpha}n_{\alpha} + \frac{d}{dl}(\lambda A^{\beta\alpha}n_{\alpha}t_{\beta}) + A^{\beta\alpha}{}_{[\beta}n_{\alpha} = p_{o} - (i^{\alpha} - f^{\alpha})n_{\alpha} \; ,$$

$$\lambda A^{\alpha\beta}n_{\alpha}n_{\beta} = -a\overset{N}{\underset{\sim}{p}} \; , \qquad \text{(B3.21)}$$

where

$$A^{\beta\alpha}{}_{[\beta} = A^{\beta\alpha}{}_{,\beta} + A^{\beta\alpha}\{{}^{\gamma}_{\gamma\beta}\} \; .$$

All densities in Eqs. (B3.20), (B3.21) are related to the surface or to its boundary $\partial\pi$. Instead of Eqs. (B3.16) we obtain now

$$A^{\cdot} = \frac{1}{\sqrt{a}} A^{\cdot}_{R} \; . \qquad \text{(B3.22)}$$

3.2. Eulerian formulation of shell theories

Now let us assume that the parametric representation $x^k = \psi^k(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Pi$, of the surface π, is, for every time instant t, $t \in I$, given by $x^k = p^k(\underset{\sim}{\theta}, \xi_o, t)$, where ξ_o is an arbitrary but fixed value of ξ (the fixed number form $\langle h_-, h_+ \rangle$). Then we deal with the moving surface, i.e., with the family of surfaces π_t, $t \in I$. Putting $\pi = \pi_t$ for every t, $t \in I$, we shall refer Eqs. (B3.11), (B3.15) as well as Eqs. (B3.20), (B3.21), to as the Eulerian formulation of the shell field equations.

3.3. Lagrangian formulation of shell theories

If to every time instant t, $t \in I$, we assign one known surface π, given by $x^k = \psi^k(\theta)$, $\theta \in \Pi$, then we shall refer Eqs. (B3.11), (B3.15) as well as Eqs. (B3.20), (B3.21), to as the Lagrangian formulation of the shell field equation. If $\pi \equiv \Pi$ then the Lagrangian formulation will coincide with the formullation given by Eqs. (B3.9), (B3.10) or Eqs. (B3.17), (B3.18), i.e., with the formulation introduced in the Chapter A. If $x^k = \psi^k(\underset{\sim}{\theta}) = p^k(\underset{\sim}{\theta}, \xi_o, t_o)$, where t_o is the fixed "initial" time instant., then in the Lagrangian formulation all fields are related to the shell material surface $\xi = \xi_o$ in certain initial configuration. Moreover, if $\xi_o = 0{,}5(h_+ - h_-)$, this surface will be referred to as the shell midsurface in the reference configuration.

3.4. Interrealtions between Lagrangean and Eulerian quantities

Let all quantities related to the Eulerian formulation ("Eulerian" quantities) be distinguished by a dash. Let the covariant differentiation (B3.5) in this formulation be denoted by a stroke with a dash, i.e.

$$\bar{A}^{\alpha\beta}\bar{|}_{\beta} = \bar{A}^{\alpha\beta}\bar{||}_{\beta} - \bar{b}^{\alpha}_{\beta}\bar{A}^{3\beta} \ , \quad \bar{A}^{3\beta}\bar{|}_{\beta} = \bar{A}^{3\beta}\bar{||}_{\beta} + \bar{b}_{\alpha\beta}\bar{A}^{\alpha\beta} \tag{B3.23}$$

Then the formulae (B3.4) yield

$$\begin{aligned} \sqrt{\bar{a}}\,\bar{A}^{K\cdot}\bar{g}^{L}_{K} &= \sqrt{a}\,A^{L\cdot} \ , \\ \sqrt{\bar{a}}\,\bar{A}^{K\alpha}\bar{|}_{\alpha}\bar{g}^{L}_{K} &= \sqrt{a}\,A^{L\alpha}|_{\alpha} \ , \\ \sqrt{\bar{a}}\,\bar{A}^{K\alpha\beta}\bar{|}_{\alpha\beta}\bar{g}^{L}_{K} &= \sqrt{a}\,A^{L\alpha\beta}|_{\alpha\beta} \end{aligned} \tag{B3.24}$$

where $\bar{A}^{K\cdot}$, $A^{L\cdot}$ stand for $\bar{A}^{KM_1\ldots M_m}$, $A^{LM_1\ldots M_m}$, respectively (where $M_1\ldots M_m$ is an arbitrary "dead" multiindex [1]) and were

$$(\bar{g}^{L}_{K}) \equiv \begin{pmatrix} \bar{g}^{\beta}_{\alpha} & \bar{g}^{3}_{\alpha} \\ \bar{g}^{\beta}_{3} & \bar{g}^{3}_{3} \end{pmatrix} \equiv \begin{Bmatrix} \bar{\psi}^{k}_{,\alpha} \\ \bar{N}^{k} \end{Bmatrix} [N_k] \ . \tag{B3.25}$$

The quantities and the covariant derivatives which are not distinguished by a dash are assumed to be expressed in the Lagrangean formulation ("Lagrangian" quantities). From the formulae (B3.8) we obtain

$$\begin{aligned} \sqrt{\bar{a}}\,\bar{A}^{\cdot} &= \sqrt{a}\,A^{\cdot} \ , \\ \sqrt{\bar{a}}\,\bar{A}^{\alpha}\bar{|}_{\alpha} &= \sqrt{a}\,A^{\alpha}|_{\alpha} \ , \\ \sqrt{\bar{a}}\,\bar{A}^{\alpha\beta}\bar{|}_{\alpha\beta} &= \sqrt{a}\,A^{\alpha\beta}|_{\alpha\beta} \ . \end{aligned} \tag{B3.26}$$

The formulae dual to (B3.23) - (B3.25), i.e., obtained by interchanging the quantities and derivatives distinguished by a dash with those without

---

[1] If m = 0 then we assume $\bar{A}^{K\cdot} = \bar{A}^{K}$, $A^{L\cdot} = A^{L}$.

a dash, also hold. It must be remembered that the covariant derivatives are not the surface derivatives, i.e., that Eqs. (B3.5), (B3.23) hold.

Eqs. (B3.23) - (B3.25) and the corresponding dual equations describe the interrelation between the "Eulerian" quantities and the "Lagrangean" quantities. Many special cases of the foregoing formulae can be found in the literature on the shell theories, c.f. [28,29].

Remark. From Eqs. (B3.2), (B3.3), (B3.6), (B3.7) we obtain

$$\sqrt{\bar{g}}\,\bar{A}^{M\cdot}\,\bar{g}^{N}_{M} = \sqrt{g}\,A^{N\cdot} \quad ,$$

$$\sqrt{\bar{g}}\,\bar{A}^{ML}\big|_{L}\,\bar{g}^{N}_{M} = \sqrt{g}\,A^{NL}\big|_{L} \, ,$$

$$\sqrt{\bar{g}}\,\bar{A}^{MKL}\big|_{KL}\,\bar{g}^{N}_{M} = \sqrt{g}\,A^{NKL}\big|_{KL} \, ,$$

where $\bar{g}^{N}_{M} \equiv \bar{g}^{k}_{,M}\,g_{k,L}\,g^{LN}$ and

$$\sqrt{\bar{g}}\,\bar{A}^{\cdot} = \sqrt{g}\,A^{\cdot} \quad ,$$

$$\sqrt{\bar{g}}\,\bar{A}^{K}\big|_{K} = \sqrt{g}\,A^{K}\big|_{K} \quad ,$$

$$\sqrt{\bar{g}}\,\bar{A}^{KL}\big|_{KL} = \sqrt{g}\,A^{KL}\big|_{KL} \, .$$

Thus the Eqs. (B3.23) - (B3.25) constitute the special cases of the foregoing formulae.

3.5. Eulerian and Lagrangean formulations of rod theories

Let $x^k = \psi^k(\xi)$, $\xi \in (h_-,h_+)$, be the parametric representation of the smooth curve σ in the reference space. In the neighbourhood of σ we shall introduce the curvilinear coordinates $Z^M = \delta^M_\alpha \theta^\alpha + \delta^M_3 \xi$, $M = I, II, III$, putting $x^k = \psi^k(\xi) + N^k_a(\xi)\theta^\alpha$ where $\underset{\sim}{N}_\alpha$ are the unit vector fields defined on σ, such that $\underset{\sim}{N}_\alpha \cdot \underset{\sim}{N}_\beta = \delta_{\alpha\beta}$, $N^k_\alpha \psi_{k,3} = 0$. By $A^{k\cdot}_R(\xi)$, $\xi \in (h_-,h_+)$, we shall denote the smooth field of the vector density related to the stright line segment (h_-,h_+) (dot stands for an arbitrary multiindex $M_1 \ldots M_m$). Denoting $a = \psi^k_{,3}\,\psi_{k,3}$ we obtain the formulae

$$\frac{1}{\sqrt{a}} A^{k\cdot}_R = A^{\alpha\cdot} N^k_\alpha + A^{3\cdot} \psi^k_{,3} \tag{B3.27}$$

where $A^{M\cdot}$, $M = \alpha,3$, are the vector densitites of the field under consideration related to the unit length of the curve σ and expressed in the vector basis $\underset{\sim}{N}_\alpha, \underset{\sim}{\psi}_{,3}$. We obtain also

$$\frac{1}{\sqrt{a}} A^{k3}_{R,3} = A^{33}|_3 \psi^k_{,3} + A^{\alpha 3}|_3 N^k_\alpha \quad ,$$
$$\frac{1}{\sqrt{a}} A^{k33}_{,33} = A^{333}|_{33} \psi^k_{,3} + A^{\alpha 33}|_{33} N^k_\alpha \tag{B3.28}$$

where

$$A^{33}|_3 \equiv A^{33}_{,3} + \left\{ {3 \atop M3} \right\} A^{M3} + \left\{ {3 \atop M3} \right\} A^{3M} \quad ,$$
$$A^{\alpha 3}|_3 \equiv A^{\alpha 3}_{,3} + \left\{ {\alpha \atop M3} \right\} A^{M3} + \left\{ {3 \atop M3} \right\} A^{\alpha M} \quad , \tag{B3.29}$$

and where $\left\{ {N \atop M3} \right\}$ are the values of the Christoffel symbols for the curvilinear coordinate system Z^M, $M = I,II,III$, on the curve σ (i.e., for $\theta^\alpha = 0$). Analogously, for the field $A^{\cdot}_R = A^{\cdot}_R(\xi)$, $\xi \in (h_-,h_+)$, of the scalar density (defined on σ) related to the stright line segment (h_-,h_+), the following formulae hold

$$\frac{1}{\sqrt{a}} A^{\cdot}_R = A^{\cdot} \quad , \quad \frac{1}{\sqrt{a}} A^3_{R,3} = A^3|_3 \quad , \quad \frac{1}{\sqrt{a}} A^{33}_{R,33} = A^{33}|_{33} \tag{B3.30}$$

where $A^{\cdot}$ is the corresponding scalar density related to the curve σ.

Using Eqs. (B3.27) - (B3.30) we shall derive the Eulerian and Lagrangean form of the field equations of the rod theory (A5.10), (A5.13) provided that the terms in the field equations are either vectors or scalars (with respect to the group of transformations of the reference space). In this case the equations of motion (A5.10) have the form

$$''A^k_{R,33} + {}'A^k_{R,3} + a^k_R + b^k_R - {}'b^k_{R,3} = c^k_R - {}'c^k_{R,3} \quad ,$$
$$''A_{R,33} + {}'A_{R,3} + a_R + b_R - {}'b_{R,3} = c_R - {}'c_{R,3} \quad , \tag{B3.31}$$

where the symols $"A^k_R$, $'A^k_R, \ldots$ $'c^k_R$ and $"A_R$, $'A_R, \ldots,$ $'c_R$ stand for some from $"H^a_R$, $'H^a_R, \ldots,$ $'i^a_R$, respectively. Multiplying Eqs. (B3.31) by $(\sqrt{a})^{-1}$ and using Eqs. (B3.27) - (B3.30) we arrive at the relations

$$
\begin{aligned}
&"A^M|_{33} + 'A^M|_3 + a^M + b^M - 'b^M|_3 = c^M - 'c^M|_3 ; \quad M = \alpha, 3 , \\
&"A|_{33} + 'A|_3 + a + b - 'b|_3 = c - 'c|_3
\end{aligned}
\qquad \text{(B3.32)}
$$

where the derivatives are defined by Eqs. (B3.29). The kinetic boundary conditions (A5.13) in the case under consideration can be represented by

$$
\begin{aligned}
&'A^k_R n + "A^k_{R,3} n = e^k_R - ('c^k_R - 'b^k_R) n ; \qquad "A^k_R n = 'e^k_R , \\
&'A_R n + "A_{R,3} n = e_R - ('c_R - 'b_R) n ; \qquad "A_R n = 'e_R ,
\end{aligned}
\qquad \text{(B3.33)}
$$

where the symbols e^k_R, $'e^k_R$ as well as e_R, $'e_R$ stand for some from p^a_R, $'p^a_R$, respectively. After simple calculations (cf. also Secs. 3.0, 3.1) we obtain from (B3.33) the formulae

$$
\begin{aligned}
&'A^M n + "A]_3 n = e^M - ('c^M - 'b^M) n ; \qquad "A^M n = -a'e^M , \\
&'A\, n + "A[_3 n = e - ('c - 'b) n ; \qquad "A\, n = -a'e ,
\end{aligned}
\qquad \text{(B3.34)}
$$

where $M = \alpha, 3$ and where we have denoted

$$
\begin{aligned}
&A^M[_3 \equiv A^M_{,3} + \{^N_{N3}\} A^M + \{^N_{N3}\} A^N , \\
&A]_3 \equiv A_{,3} + \{^M_{M3}\} A ,
\end{aligned}
\qquad \text{(B3.35)}
$$

Eqs. (B3.32), (B3.34) represent the alternative form of the rod field equations, corresponding with Eqs. (B3.31), (B3.33), respectively. If $\psi^k(\xi) = p^k(\underset{\sim}{O}, \xi, t_o)$ then Eqs. (B3.32), (B3.34) constitute the Lagrangean formulation of the rod field equations. If $\psi^k(\xi) = p^k(\underset{\sim}{O}, \xi, t)$, where t is an arbitrary element of I(the curve σ is moving in the reference space), then Eqs. (B3.32), (B3.34) represent the Eulerian formulation of the rod field equations.

4. VECTOR THEORIES

By the vector plate or shell theory we shall mean the theory in which the shell deformation function $q_{(n)}$ has the form $q_{(3)} := \{r_i, i = 1,2,3\}$, where $\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{\theta}, t)$, $\underset{\sim}{\theta} \in \Pi$, represents, for every $t \in I$, the smooth surface in the reference space. The special examples of the vector plate and shell theories are known as the Kirchhoff plate approximation and the Love-Kirchhoff shell approximation. In this Section we shall derive the general form of the vector shell theory directly from the relations of Sec. 5 of the Chapter A. Such general form is not used in applications and is treated here as the illustration of the formulae given in the Chapter A and in Sec. 3 of this Chapter.

4.1. Governing relations

Putting $q_{(n)} = q_{(3)} = \underset{\sim}{r}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, $t \in I$, we shall assume that the function $\underset{\sim}{\tilde{p}}$ in Eqs. (A2.18) and $(4.23)_1$ depends on $\underset{\sim}{r}$ and $\nabla\underset{\sim}{r}$:

$$\underset{\sim}{\tilde{p}} \equiv \underset{\sim}{\tilde{p}}(\underset{\sim}{X},\underset{\sim}{r}(\underset{\sim}{\theta},t),\ \nabla\underset{\sim}{r}(\underset{\sim}{\theta},t)),\ \underset{\sim}{r} \in Q \quad , \tag{B4.1}$$

where $Q := \{\underset{\sim}{r} \,|\, \det \nabla\underset{\sim}{\tilde{p}} > 0\}$. From Eqs. (A5.1), (A5.3), (A5.4) we obtain the shell equations of motion

$$H^{k\alpha\beta}_{R,\alpha\beta} + H^{k\alpha}_{R,\alpha} + h^k_R + f^k_R - f^{k\alpha}_{R,\alpha} = i^k_R - i^{k\alpha}_{R,\alpha} \tag{B4.2}$$

and the shell kinetic boundary conditions

$$\begin{aligned} &H^{k\alpha}_R n_{R\alpha} + \frac{d}{dl_R}(H^{k\alpha\beta}_R n_{R\beta} t_{R\alpha}) + H^{k\beta\alpha}_{R,\beta} n_{R\alpha} = p^k_{oR} - (i^{k\alpha}_R - f^{k\alpha}_R)n_{R\alpha} \ , \\ &H^{k\alpha\beta}_R n_{R\alpha} n_{R\beta} = -p^{k\underset{\sim}{N}}_R \end{aligned} \tag{B4.3}$$

with the denotations

$$\begin{aligned} &p^k_{oR} \equiv p^k_R - \frac{d}{dl_R}(p^{k\alpha}_R t_{R\alpha}) \quad , \\ &p^{k\underset{\sim}{N}}_R \equiv p^{k\alpha}_R n_{R\alpha} \ . \end{aligned} \tag{B4.4}$$

Superscript "k", as usual, runs over the sequence 1,2,3. We deal here with three equations of motion and the six kinetic boundary conditions. The constitutive relations in their general can be derived from Eqs. (A5.5), (A5.6). The first set of these relations is given by ([1])

$$
\begin{aligned}
H_R^{k\alpha\beta} &= \tilde{H}_R^{k\alpha\beta}(\underset{\sim}{\theta},\nabla\underset{\sim}{r},\nabla^2\underset{\sim}{r},\tau^{(N)}) \\
H_R^{k\alpha} &= \tilde{H}_R^{k\alpha}(\underset{\sim}{\theta},\nabla\underset{\sim}{r},\nabla^2\underset{\sim}{r},\tau^{(N)}) \qquad \text{(B4.5)} \\
h_R^{k} &= \tilde{h}_R^{k}(\underset{\sim}{\theta},\nabla\underset{\sim}{r},\Delta^2\underset{\sim}{r},\tau^{(N)}) .
\end{aligned}
$$

Eqs. (B4.2), (B4.3) and (B4.5) constitute the most general form of the field equations (A5.1), (A5.3) and the constitute relations (A5.5), respectively, in the case of the vector shell theory. They are based exclusively on Eq. (B4.1) and no further assumptions (concerning; for example, the form of functions Φ_k^1, $\Psi_k^{1\alpha}$ in Eqs. (A2.24), (A2.25), (A2.29)) have been made. One from the consequences of such general approach is that Eqs. (B4.2), (B4.3), (B4.5) in the special case of the plane problems do not reduce to the well known form but involve also the terms with the couple-stresses. That is why we shall treat the general vector shell theory rather as an example of the general approaches developed in the Chapter A then as a starting point for further applications.

The form of the second set (A5.6) of the constitutive relations for a vector shell theory will be obtained by assuming that Eq. (B4.1) has the form $\tilde{\underset{\sim}{p}} = \underset{\sim}{r} + \xi\underset{\sim}{N}$, where $\underset{\sim}{N} = (\underset{\sim}{a}_1 \times \underset{\sim}{a}_2) / |\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|$, $\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha}$ ($\underset{\sim}{N}$ is the unit vector normal to the surface $\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, for an arbitrary but fixed t). It is a well known assumption used in the Love-Kirchhoff shell theories. The components of the metric tensor $\tilde{\underset{\sim}{C}} = (\nabla\tilde{\underset{\sim}{p}})^T\nabla\tilde{\underset{\sim}{p}}$ will be now given by

$$
\begin{aligned}
\tilde{C}_{\alpha\beta} &= a_{\alpha\beta} + 2\xi b_{\alpha\beta} + \xi^2 c_{\alpha\beta} \quad , \\
\tilde{C}_{\alpha 3} &= C_{3\alpha} = 0 \quad , \\
\tilde{C}_{33} &= 1 \quad ,
\end{aligned}
$$

([1]) The RHS of Eqs. (B4.5) are independent of $\underset{\sim}{r}$ because they have to be invariant with respect to the group of translations of the reference space.

where

$$a_{\alpha\beta} \equiv r^k_{,\alpha} r_{k,\beta} \quad , \quad b_{\alpha\beta} \equiv \frac{1}{2}(N^k_{,\alpha} r_{k,\beta} + r^k_{,\alpha} N_{k,\beta})$$

$$c_{\alpha\beta} \equiv N^k_{,\alpha} N_{k,\beta}$$

are the first, second and third fundamental tensors of the surface $\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$ (for every fixed t). At the same time $c_{\alpha\beta} = b_{\alpha\gamma} b_{\beta\delta} a^{\gamma\delta}$, where $a^{\gamma\delta} a_{\delta\alpha} = \delta^\gamma_\alpha$. It means that $\underset{\sim}{\tilde{C}}$ is now the function of arguments ξ, $a_{\alpha\beta}$, $b_{\alpha\beta}$. Because $a_{\alpha\beta}$, $b_{\alpha\beta}$ are invariants under arbitrary rigid motion of the reference space, then $e_{(r)} = e_{(6)} = (a_{\alpha\beta}, b_{\alpha\beta})$ are the shell strain measures. This rather trivial and well known result leads from Eqs. (A5.6) to the following general form of the constitutive relations of the vector plate and shell theories in which Eq. (B4.1) has the form $\underset{\sim}{\tilde{p}} = \underset{\sim}{r} + \xi \underset{\sim}{N}$:

$$q_A(\underset{\sim}{\theta}, (a_{\alpha\beta}), (b_{\alpha\beta}), \tau^{(N)}, \omega^{(P)}) = 0, \quad A = 1\ldots, N+P$$

$$\kappa(\underset{\sim}{\theta}, (a_{\alpha\beta}), (b_{\alpha\beta}), \tau^{(N)}, \omega^{(P)}) \le 0 \tag{B4.6}$$

$$\psi(\underset{\sim}{\theta}, (a_{\alpha\beta}), (b_{\alpha\beta}), \tau^{(N)}, \omega^{(P)}, (\overset{o}{a}_{\alpha\beta}), (\overset{o}{b}_{\alpha\beta}) \tau^{(N)}_o) \le 0 \text{ for every}$$

$$\overset{o}{a}_{\alpha\beta}, \overset{o}{b}_{\alpha\beta}, \tau^{(N)}_o \text{ with } \kappa(\theta, (\overset{o}{a}_{\alpha\beta}), (\overset{o}{b}_{\alpha\beta}), \tau^{(N)}_o, \omega^{(P)}) \le 0$$

If the material of the shell like body for a fixed $\underset{\sim}{\theta}$, $\underset{\sim}{\theta} \in \Pi$, is simple (i.e., it is simple in all points $\underset{\sim}{X} = (\underset{\sim}{\theta},\xi)$, $\xi \in (h_-,h_+)$, $\underset{\sim}{\theta}$ being fixed) then Eqs. (B4.6) are indentities and the argument $\tau^{(N)}$ drops out form Eqs. (B4.5) (cf. Sec. 2 of the Chapter A). If, moreover, the material for a fixed $\underset{\sim}{\theta}$, $\underset{\sim}{\theta} \in \Pi$, is elastic then the RHS of Eqs. (B4.5) are the known functions. If the material is hyperelastic and Eqs. (A2.33) hold then from Eqs. (A2.34) we obtain

$$H^{k\alpha\beta}_R = -\frac{\partial \varepsilon_R}{\partial r_{k,\alpha\beta}} \quad , \quad H^k_R = \frac{\partial \varepsilon_R}{\partial r_{k,\alpha}} \quad , \quad h^k_R = 0$$

where the shell strain energy function ε_R depends on $\underset{\sim}{\theta}$, $a_{\alpha\beta} = r^k_{,\alpha} r_{k,\beta}$, $b_{\alpha\beta} = 0.5(N^k_{,\alpha} r_{k,\beta} + r^k_{,\alpha} N_{k,\beta})$.

Remark. From the formal point of view we deal here with an arbitrary large deformation of the shell-like body. However, the vector shell theory in which $\tilde{p} = r + \xi N$ is not a good approximation in the case of the large deformations because the condition $C_{33} = 1$, as a rule, is never fulfilled with the sufficient accuracy (the thickness of the shell undergoes the remarkable changes during the large deformations).

4.2. Lagrangean and Eulerian formulations

Now we shall express the governing relations of the vector shell or plate theory in the Lagrangean and Eulerian formulation. Using Eqs. (B3.11) we obtain the equations of motion

$$\begin{aligned} H^{\gamma\alpha\beta}|_{\alpha\beta} + H^{\gamma\alpha}|_{\alpha} + f^{\gamma} - f^{\gamma\alpha}|_{\alpha} &= i^{\gamma} - i^{\gamma\alpha}|_{\alpha} \\ H^{3\alpha\beta}|_{\alpha\beta} + H^{3\alpha}|_{\alpha} + f^{3} - f^{3\alpha}|_{\alpha} &= i^{3} - i^{3\alpha}|_{\alpha} \end{aligned} \tag{B4.7}$$

where in view of Eqs. (B3.16)

$$\begin{aligned} H^{\gamma\alpha\beta} &= \frac{1}{\sqrt{a}} H_R^{k\alpha\beta} r_{k,\delta}\, a^{\delta\gamma} , & H^{\gamma\alpha} &= \frac{1}{\sqrt{a}} H_R^{k\alpha} r_{k,\delta} a^{\delta\gamma} , \\ H^{3\alpha\beta} &= \frac{1}{\sqrt{a}} H_R^{k\alpha\beta} N_k , & H^{3\alpha} &= \frac{1}{\sqrt{a}} H_R^{k\alpha} N_k , \\ f^{\gamma} &= \frac{1}{\sqrt{a}} f_R^k r_{k,\delta} a^{\delta\gamma} , & f^{3} &= \frac{1}{\sqrt{a}} f_R^k N_k \quad \text{etc. .} \end{aligned} \tag{B 4.8}$$

Analogously, from Eqs. (B3.15) we obtain the kinetic boundary conditions

$$\begin{aligned} & H^{\gamma\alpha} n_{\alpha} + \frac{D}{dL} (\lambda H^{\gamma\beta\alpha} n_{\alpha} t_{\beta}) + H^{\gamma\beta\alpha}]_{\beta} n_{\alpha} = p_o^{\gamma} - (i^{\gamma\alpha} - f^{\gamma\alpha}) n_{\alpha} , \\ & H^{3\alpha} n_{\alpha} + \frac{D}{dl} (\lambda H^{3\beta\alpha} n_{\alpha} t_{\beta}) + H^{3\beta\alpha}]_{\beta} n_{\alpha} = p_o^{3} - (i^{3\alpha} - f^{3\alpha}) n_{\alpha} , \\ & \lambda H^{\gamma\alpha\beta} n_{\alpha} n_{\beta} = - a p^{\gamma N} , \\ & \lambda H^{3\alpha\beta} n_{\alpha} n_{\beta} = - a p^{3N} . \end{aligned} \tag{B4.9}$$

where in view of Eqs. (B3.12)

$$p_o^{\gamma} = \frac{1}{\sqrt{\lambda}} p_{oR}^{k} \psi_{k,\delta} a^{\delta\gamma} , \qquad p_o^{3} = \frac{1}{\sqrt{\lambda}} p_{oR}^{k} N_k ,$$

$$p^{\gamma \underset{\sim}{N}} = \frac{1}{\sqrt{\lambda}} p_R^{k\underset{\sim}{N}} \psi_{k,\delta} a^{\delta\gamma} , \qquad p^{3\underset{\sim}{N}} = \frac{1}{\sqrt{\lambda}} p_R^{k\underset{\sim}{N}} N_k .$$

But the LHS of Eqs. (B4.8) are invariant under an arbitrary rigid motion of the reference space; it follows that the left hand sides of the suitable constitutive relations have also be invariant. Thus we conclude that

$$\begin{aligned} H^{\gamma\alpha\beta} &= \tilde{H}^{\gamma\alpha\beta}(\underset{\sim}{\theta}, (a_{\mu\nu}), (b_{\mu\nu}), \tau^{(N)}) , \\ H^{3\alpha\beta} &= \tilde{H}^{3\alpha\beta}(\underset{\sim}{\theta}, (a_{\mu\nu}), (b_{\mu\nu}), \tau^{(N)}) , \\ H^{\gamma\alpha} &= \tilde{H}^{\gamma\alpha}(\underset{\sim}{\theta}, (a_{\mu\nu}), (b_{\mu\nu}), \tau^{(N)}) , \\ H^{3\alpha} &= \tilde{H}^{3\alpha}(\underset{\sim}{\theta}, (a_{\mu\nu}), (b_{\mu\nu}), \tau^{(N)}) , \end{aligned} \qquad \text{(B4.10)}$$

where the RHS of Eqs. (B4.10) are known. The form of the constitutive relations (B4.6) remain unchanged.

Eqs. (B4.7) - (B4.10) can express either Lagrangean or Eulerian formulation of the vector shell theory (cf. Secs. 3.2. - 3.4. of this Chapter). In both cases the arguments $a_{\mu\nu}$, $b_{\mu\nu}$ of the functionals on the RHS of Eqs. (B4.10) are the components of the fundamental tensors of the midsurface of the shell during the motion, i.e., they are functions of the time coordinate.

Eqs. (B4.7), (B4.9), (B4.10) and (B4.6) are the general form of the vector shell theory. The different simplifications and special cases of this theory, mainly for the elastic materials, can be found in the recent literature. To the problems of the vector shell theories we shall come back in Sec. 9 of this Chapter.

Remark. Introducing the denotations

$$\underset{\sim}{Q}^{M\alpha} \equiv H^{M\alpha\beta}|_{\beta} + H^{M\alpha} ,$$

the equations of motion can be written down in more compact form

$$Q^{\gamma\alpha}||_{\alpha} - b^{\gamma}_{\beta}Q^{3\beta} + f^{\gamma} - f^{\gamma\alpha}|_{\alpha} = i^{\gamma} - i^{\gamma\alpha}|_{\alpha} ,$$

$$Q^{3\alpha}||_{\alpha} + b_{\alpha\beta}Q_{\alpha\beta} + f^{3} - f^{3\alpha}|_{\alpha} = i^{3} - i^{3\alpha}|_{\alpha} ,$$

where Eqs. (B3.5) have been taken into account.

4.3. Membrane theories

The analysis in Secs. 4.1, 4.2. was based on the derivation of the shell theories from the solid mechanics equations (By the approximation or by the constraint approach). Using the direct approach, we have to replace Eqa. (B4.1) by the weaker condition of the form $\underset{\sim}{r}(\underset{\sim}{\theta},t) = \underset{\sim}{p}(\underset{\sim}{\theta},\xi_{o},t)$, $\underset{\sim}{\theta} \in \Pi$, $t \in I$, where ξ_{o} is the fixed numer from $\langle h_{-},h_{+}\rangle$. Moreover, we can formulate the shell theory introducing the shell force system of the first order (cf. Sec. 1 of the Chapter A). Then the field equations (B4.7), (B4.9) will reduce to the form

$$H^{\gamma\alpha}|_{\alpha} + f^{\gamma} = i^{\gamma} \quad , \quad H^{3\alpha}|_{\alpha} + f^{3} = i^{3} ,$$

$$H^{\gamma\alpha}n_{\alpha} = p^{\gamma} \quad , \quad H^{3\alpha}n_{\alpha} = p^{3} .$$

Within the direct approach we shall also postulate that $i^{3} = 0$, $p^{3} = 0$, and $H^{k\beta}_{R}N_{k} = 0$ (i.e., that $H^{3\alpha} = 0$). Using Eqa. (B3.5) we obtain

$$H^{\gamma\alpha}||_{\alpha} + f^{\gamma} = i^{\gamma} , \quad b_{\alpha\beta}H^{\alpha\beta} + f^{3} = 0 \; ; \quad \theta \in \Pi,$$

$$H^{\gamma\alpha}n_{\alpha} = p^{\gamma} \; ; \; \theta \in \partial\Pi \quad \text{a.e.} \quad , \tag{B4.11}$$

At the same time let the material of the shell be simple; then the terms $\tau^{(N)}$ drops out from Eqs. (B4.10) and we can postulate the constitutive equations in the form

$$H^{\gamma\alpha} = \tilde{H}^{\gamma\alpha}(\theta,(a_{\mu\nu}) \quad . \tag{B4.12}$$

Eqs. (B4.11),(B4.12) represent the special case of the vector shell theories which are called the membrane theories (cf. also Sec. 9.3. of this Chapter).

5. MULTIVECTOR THEORIES

For the multilayered plates or shells we can introduce the constraints (or the formal approximations) in which the motion of the shell is uniquely determined (or approximated) by the motion of the limit surfaces of the layers. Using the simple shell force system (cf. Sec. 5 of the Chapter A) we can formulate then the plate and shell theories which are called multivector theories.

5.1. Governing relations

Let $h_1 = h_-$, $h_m = h_+$ and $h_\mu < h_{\mu+1}$ for $\mu = 1,\dots, m-1$, where h_μ, for $\mu = 1,\dots,m$, $m \geq 2$, are the known numbers or functions of $\underset{\sim}{\theta}$, $\underset{\sim}{\theta} \in \bar{\Pi}$. Let us split the plate or shell under consideration into $m-1$ disjointed (open) layers, assuming that $\Delta h_\mu \equiv h_{\mu+1} - h_\mu$, $\mu = 1,\dots, m-1$, is the thickness of an arbitrary layer in the reference configuration. Let the function $\underset{\sim}{\tilde{p}}$ in Eqs. (A2.3) and $(A4.23)_1$ be assumed in the form

$$\underset{\sim}{\tilde{p}} = \frac{h_{\mu+1} - \xi}{\Delta h_\mu} \underset{\sim}{r}_\mu(\underset{\sim}{\theta},t) + \frac{\xi - h_\mu}{\Delta h_\mu} \underset{\sim}{r}_{\mu+1}(\underset{\sim}{\theta},t) \quad , \quad \xi \in \langle h_\mu, h_{\mu+1} \rangle , \ \mu = 1,\dots,m-1 \tag{B5.1}$$

where $\underset{\sim}{x} = \underset{\sim}{r}_\mu(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, $\mu = 1,\dots,m$, represent for every t, $t \in I$, the "m" non-intersecting surfaces ([1]). It means that $q_{(n)} = q_{(3m)} = \{\underset{\sim}{r}_\mu, \mu = 1,\dots,m\}$ is the shell deformation function ([2]). In the direct approach instead of Eq. (B5.1) we only postulate that $\underset{\sim}{r}_\mu(\underset{\sim}{\theta},t) = \underset{\sim}{p}(\underset{\sim}{\theta},h_\mu, t)$, $\mu = 1,\dots,m$. Because the RHS of Eq. (B5.1) does not depend on the derivatives of $q_{(n)}$ then we shall assume that the system of the shell forces is simple. From Eqs. (A5.1) we obtain

$$\begin{aligned} &H^{(\mu)k\alpha}_{R,\alpha} + h^{(\mu)k}_R + f^{(\mu)k}_R = i^{(\mu)k}_R , \\ &H^{(\mu)k\alpha}_R n_{R\alpha} = p^{(\mu)k}_R \qquad , \ \mu = 1,\dots,m, \end{aligned} \tag{B5.2}$$

([1]) The set of all $r_{(m)} = \{\underset{\sim}{r}_\mu, \mu = 1,\dots,m\}$ satisfying this condition coincides with the set Q of all shell deformation functions.

([2]) The approach outlined in Secs. 5.1., 5.2. is based on the yet unpublished work of Z. Baczyński on the multilayered elastic bodies.

where in view of Eqs. (A2.11) and (B5.1) we have denoted

$$H_R^{(\mu)k\alpha} \equiv \int_{h_-}^{h_+} \Phi_1^{(\mu)k} \tilde{p}^{-1}_{,K} T^{K\alpha} d\xi \quad ,$$

$$h_R^{(\mu)k} \equiv -\int_{h_-}^{h_+} \Phi_{1,3}^{(\mu)k} \tilde{p}^{-1}_{,L} T^{L3} d\xi \quad .$$

$$f_R^{(\mu)k} \equiv \int_{h_-}^{h_+} \Phi_1^{(\mu)k} b_R^1 d\xi + [\Phi_1^{(\mu)k}]_{h_+} \overset{+1}{p}_R + [\Phi_1^{(\mu)k}]_{h_-} \overset{-1}{p}_R \quad ,$$

$$i_R^{(\mu)k} \equiv \int_{h_-}^{h_+} \rho_R \Phi_1^{(\mu)k} \ddot{\tilde{p}}^1 d\xi \quad ; \quad \underset{\sim}{\theta} \in \Pi \quad ,$$

$$p_R^{(\mu)k} \equiv \int_{h_-}^{h_+} \Phi_1^{(\mu)k} p_R^1 d\xi \quad ; \quad \theta \in \partial\Pi \text{ a.e.}, \quad \mu = 1,\ldots,m. \qquad \text{(B5.3)}$$

and where $\Phi_1^{(\mu)k} \equiv 0$ for $\xi \in (h_-, h_{\mu-1})$ and for $\xi \in (h_{\mu+1}, h_+)$, $\mu = 1,\ldots,m$. In the special case given by Eq. $(A2.23)_1$ we obtain

$$\Phi_1^{(\mu)k} = \begin{cases} 0 & \text{if } \xi \in \langle h_-, h_{\mu-1} \rangle \quad , \\ \dfrac{\xi - h_{\mu-1}}{\Delta h_{\mu-1}} \delta_1^k & \text{if } \xi \in (h_{\mu-1}, h_\mu) \quad , \\ \dfrac{h_{\mu+1} - \xi}{\Delta h_\mu} \delta_1^k & \text{if } \xi \in (h_\mu, h_{\mu+1}) \quad , \\ 0 & \text{if } \xi \in \langle h_{\mu+1}, h_+ \rangle \quad , \end{cases}$$

for $\mu = 1,\ldots,m$. Eqs. (B5.2) are the field equations of the multivector shell theory.

The general form of the constitutive relations will be derived from Eqs. (A5.5), (A5.6). Denoting $\underset{\sim}{r}_{(m)} \equiv \{\underset{\sim}{r}_\mu(\underset{\sim}{\theta},t),\ \mu = 1,\ldots,m\}$, $\underset{\sim}{d}_{(m-1)} \equiv$ $\equiv \{\underset{\sim}{r}_{\mu+1} - \underset{\sim}{r}_\mu,\ \mu = 1,\ldots,m-1\}$, we obtain

$$H_R^{(\mu)k\alpha} = \tilde{H}_R^{(\mu)k\alpha}(\underset{\sim}{\theta}, \nabla \underset{\sim}{r}_{(m)}, \underset{\sim}{d}_{(m-1)}, \tau^{(N)}),$$

$$h_R^{(\mu)k} = \tilde{h}_R^{(\mu)k}(\theta, \nabla r_{(m)}, d_{(m-1)}, \tau^{(N)}), \quad \mu = 1,\dots,m, \tag{B5.4}$$

where the RHS of Eqs. (B5.4) can be derived from Eqs. (A2.29):

$$\tilde{H}_R^{(\mu)k\alpha} \equiv \int_{h_-}^{h_+} \tilde{p}^{1}_{,K} S^{K\alpha}(\underset{\sim}{\tilde{\sigma}}) \phi_1^{(\mu)k} \, d\xi ,$$

$$\tilde{h}_R^{(\mu)k} \equiv -\int_{h_-}^{h_+} \tilde{p}^{1}_{,L} S^{L3}(\underset{\sim}{\tilde{\sigma}}) \phi_{1,3}^{(\mu)k} \, d\xi , \quad \mu = 1,\dots,m .$$

To obtain the second set of the constitutive relations we shall introduce the set $e_{(r)}$ of the shell strain measures. From $\underset{\sim}{\tilde{C}} = (\nabla \underset{\sim}{\tilde{p}})^T \nabla \underset{\sim}{\tilde{p}}$ we obtain

$$\tilde{C}_{\alpha\beta} = \left(\frac{h_{\mu-1} - \xi}{\Delta h_\mu}\right)^2 \underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{a}_{(\mu)\beta} + \frac{(h_{\mu+1} - \xi)(\xi - h_\mu)}{(\Delta h_\mu)^2} (a_{(\mu)\alpha} \cdot a_{(\mu+1)\beta} +$$

$$+ \underset{\sim}{a}_{(\mu)\beta} \cdot \underset{\sim}{a}_{(\mu+1)\alpha}) + \left(\frac{\xi - h_\mu}{\Delta h_\mu}\right)^2 \underset{\sim}{a}_{(\mu+1)\alpha} \cdot \underset{\sim}{a}_{(\mu+1)\beta} ,$$

$$\tilde{C}_{\alpha 3} = \frac{h_{\mu+1} - \xi}{(\Delta h_\mu)^2} \underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{d}_{(\mu)} + \frac{\xi - h_\mu}{(\Delta h_\mu)^2} \underset{\sim}{a}_{(\mu+1)\alpha} \cdot d_{(\mu)} ,$$

$$\tilde{C}_{33} = \frac{1}{(\Delta h_\mu)^2} \underset{\sim}{d}_{(\mu)} \cdot \underset{\sim}{d}_{(\mu)} \quad ; \quad \mu = 1,\dots, m-1 ,$$

where $\underset{\sim}{a}_{(\mu)\alpha} \equiv \underset{\sim}{r}_{(\mu),\alpha}$ and $\underset{\sim}{d}_{(\mu)} \equiv \underset{\sim}{r}_{(\mu+1)} - \underset{\sim}{r}_{(\mu)}$. Let us also denote $\underset{\sim}{d}_{(\mu)\alpha} \equiv \underset{\sim}{d}_{(\mu),\alpha}$. Then we conclude that as the shell measures of deformation we can take the scalar products $\underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{a}_{(\mu)\alpha}$, $\underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{d}_{(\mu)\beta} + \underset{\sim}{a}_{(\mu)\beta} \cdot \underset{\sim}{d}_{(\mu)\alpha}$, $\underset{\sim}{d}_{(\mu)\alpha} \cdot \underset{\sim}{d}_{(\mu)\beta}$, $\underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{d}_{(\mu)}$, $\underset{\sim}{d}_{(\mu)\alpha} \cdot \underset{\sim}{d}_{(\mu)}$, $\underset{\sim}{d}_{(\mu)} \cdot \underset{\sim}{d}_{(\mu)}$, $\mu = 1,\dots,m-1$ (they are not independent if $m > 2$). The ordered set of the foregoing $r = 14\,(m-1)$ scalar functions) will be denoted by $e_{(r)}$ and is the argument of the constitutive relations (A5.6).

Eqs. (B5.2), (B5.4) and (A5.6) (where $e_{(r)}$ has been defined above) constitute the governing relations of the multivector plate and shell theories.

Remark: The vector shell theory, outlined in the privious Section, is not the special case of the mulitvector shell theory, because it involves the second order shell force system.

5.2. Lagrangean and Eulerian formulations

To obtain the Lagrangean or Eulerian formulations of Eqs. (B5.2), (B5.4) we shall apply the results obtained in Sec. 3 of this Chapter. Using Eqs. (B3.11), (B3.16) we transform the equations of motion $(B5.2)_1$ to the form

$$
\begin{aligned}
&H^{(\mu)\gamma\alpha}|_{\alpha} + h^{(\mu)\gamma} + f^{(\mu)\gamma} = i^{(\mu)\gamma} , \\
&H^{(\mu)3\alpha}|_{\alpha} + h^{(\mu)3} + f^{(\mu)3} = i^{(\mu)^3} , \qquad \mu = 1,\dots,m ,
\end{aligned}
\tag{B5.6}
$$

where (cf. Eqs. (B3.5)) $H^{(\mu)\gamma\alpha}|_{\alpha} = H^{(\mu)\gamma\alpha}||_{\alpha} - b^{\gamma}_{\alpha}H^{(\mu)3\alpha}$, $H^{(\mu)3\alpha}|_{\alpha} = H^{(\mu)3\alpha}||_{\alpha} + b_{\alpha\beta}H^{(\mu)\alpha\beta}$ and

$$
\begin{aligned}
H^{(\mu)\gamma\alpha} &= \frac{1}{\sqrt{a_{(\mu)}}} H_R^{(\mu)k\alpha} r_{(\mu)k,\delta} a^{\delta\gamma}_{(\mu)} , \\
H^{(\mu)3\alpha} &= \frac{1}{\sqrt{a_{(\mu)}}} H_R^{(\mu)k\alpha} N_k , \\
h^{(\mu)\gamma} &= \frac{1}{\sqrt{a_{(\mu)}}} h_R^{(\mu)k} r_{(\mu)k,\delta} a^{\delta\gamma}_{(\mu)} , \\
h^{(\mu)3} &= \frac{1}{\sqrt{a_{(\mu)}}} h_R^{(\mu)k} N_k , \text{ etc., } \mu = 1,\dots,m,
\end{aligned}
\tag{B5.7}
$$

and where $a_{(\mu)\alpha\beta} = \underset{\sim}{a}_{(\mu)\alpha} \cdot \underset{\sim}{a}_{(\mu)\beta}$, $a_{(\mu)} = \det a_{(\mu)\alpha\beta}$ and $a^{\delta\gamma}_{(\mu)}$ are the components of the metric tensor inverse to the first fundamental tensor $a_{(\mu)\alpha\beta}$ of the surface $x_k = r_{(\mu)k}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, for an arbitrary but fixed t, $t \in I$. From Eqs. (B3.15) we obtain the transformed form of the kinetic boundary conditions $(B5.2)_2$:

$$H^{(\mu)\gamma\alpha} n_\alpha = p^{(\mu)\gamma} ,$$

$$H^{(\mu)3\alpha} n_\alpha = p^{(\mu)3} , \quad \mu = 1,\ldots,m , \tag{B5.8}$$

At the same time the constitutive equations (B5.4) will be transformed to the form

$$H^{(\mu)\gamma\alpha} = \tilde{H}^{(\mu)\gamma\alpha}(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}) ,$$

$$H^{(\mu)3\gamma} = \tilde{H}^{(\mu)3\gamma}(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}) ,$$

$$h^{(\mu)\gamma} = \tilde{h}^{(\mu)\gamma}(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}) ,$$

$$h^{(\mu)3} = \tilde{h}^{(\mu)3}(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}) , \quad \theta \in \Pi \quad \mu = 1,\ldots,m, \tag{B5.9}$$

where $e_{(r)}$, $r = 14(m-1)$, are the shell strain measures introduced in Sec. 5.1.. The LHS of Eqs. (B5.9) are invariant under an arbitrary rigid motion of the reference space.

Eqs. (B5.6), (B5.8), (B5.9) and (A5.6) express either Lagrangean or Eulerian formulation of the multivector shell theory (cf. Sec.3 of this Chapter).

5.3. Polynomial representation

The general form of the multivector plate or shell theories can be obtained by replacing Eq. (B5.1) by more general assumption

$$\underset{\sim}{\tilde{p}} = \sum_{\mu=1}^{m} \gamma^\mu(\underset{\sim}{\theta}, \xi) \underset{\sim}{r}_\mu(\underset{\sim}{\theta}, t) , \tag{B5.10}$$

where γ^μ are the known functions satisfying the conditions $\gamma^\mu(\underset{\sim}{\theta}, h_\nu) = \delta^\mu_\nu$ for every h_ν, $\nu = 1,\ldots,m$. It follows that the functions $\underset{\sim}{r}_\mu$, $\mu = 1,\ldots,m$, have the same sense as before, i.e., they represent, for every $t \in I$, the system of m non-intersecting surfaces. The governing relations in this case have again the form given by Eqs. (B5.2), (B5.4), (A5.6) (or by Eqs. (B5.6), (B5.8), (B5.9), (A5.6)) but the functions $\Phi_1^{(\mu)k}$ in Eqs. (B5.3) can be now assumed in the form $\Phi_1^{(\mu)k} = \gamma^\mu \delta_1^k$. If

$$\begin{aligned} &\gamma^\mu \equiv 0 \text{ for } \xi \in \langle h_-, h_{\mu-1}\rangle \text{ and for } \xi \in \langle h_{\mu+1}, h_+\rangle \\ &\gamma^\mu = \frac{\xi - h_{\mu-1}}{\Delta h_{\mu-1}} \text{ for } \xi \in (h_{\mu-1}, h_\mu), \quad \gamma^\mu = \frac{h_{\mu+1} - \xi}{\Delta h_\mu} \text{ for } \xi \in (h_\mu, h_{\mu+1}), \end{aligned} \tag{B5.11}$$

where $\mu = 1,\ldots,m$, then we arrive at the special case described in Secs. 5.1., 5.2.. If $\gamma^\mu(\underset{\sim}{\theta},.)$ is the known polynomial of the k-th order with respect to ξ, $\xi \in (h_-, h_+)$, $k \geq m - 1$, then we shall say that the polynomial representation of the multivector theory is given at $\underset{\sim}{\theta}$, $\underset{\sim}{\theta} \in \Pi$.

<u>Remark 1</u>. In the polynomial representation the RHS of Eqs. (B5.4) depend on all elements of the ordered sets $\nabla\underset{\sim}{r}_{(m)}$, $\underset{\sim}{d}_{(m-1)}$. In the representation given by Eqs. (B5.11) we obtain the recurrentive system of shell relations. In this system the RHS of Eqs. (B5.4) for $\mu = 1$ and $\mu = m$ depend only on $\nabla\underset{\sim}{r}_1$, $\nabla\underset{\sim}{r}_2, \nabla\underset{\sim}{r}_2 - \underset{\sim}{r}_1$ and on $\Delta\underset{\sim}{r}_{m-1}, \Delta\underset{\sim}{r}_m$, $\underset{\sim}{r}_m - \underset{\sim}{r}_{m-1}$, respectively, and for an arbitrary μ with $\{2,\ldots,m-1\}$ they depend only on $\nabla\underset{\sim}{r}_{\mu-1}, \nabla\underset{\sim}{r}_\mu, \nabla\underset{\sim}{r}_{\mu+1}$, $\underset{\sim}{r}_\mu - r_{\mu-1}$, $\underset{\sim}{r}_{\mu+1} - r_\mu$. Thus for the large m the representation given by Eqs. (B5.11) may be more effective in applications then the polynomial representation.

<u>Remark 2</u>. Let μ_o be an arbitrary but fixed positive integer $\mu_o \leq m$. The alternative form of the multivector shell theory can be obtained by assuming that $q_{(3m)} = \{\underset{\sim}{r}_{\mu o}, \underset{\sim}{d}_\mu ; \mu = 1,\ldots,\mu_o - 1, \mu_o + 1, \ldots m\}$ where $\underset{\sim}{d}_\mu \equiv \underset{\sim}{r}_\mu - \underset{\sim}{r}_{\mu o}$. The vectors $\underset{\sim}{d}_\mu$ are called the directors assigned to the surface given by $\underset{\sim}{r}_{\mu o} = \underset{\sim}{r}_{\mu o}(\underset{\sim}{\theta}, t)$, $\underset{\sim}{\theta} \in \Pi$, for every t. The motion of the shell is determined now by the motion of the surface with m - 1 directors. If m = 2 than we obtain the special case of the Cosserat surface shell theory, which will be detailed in the next Section.

6. COSSERAT SURFACE THEORIES

They are the shell theories in which the motion of the shell is approximated or determined (in the constraint approach) by the motion of a certain Cosserat surface, i.e., the surface with the smooth field of vectors defined on it.

6.1. Governing relations

By the Cosserat surface we mean the pair $(\pi_t, \underset{\sim}{d})$, where π_t is the surface in the reference space it will given by the parametric representation $\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \Pi$, for every t,t $\in$ I) and $\underset{\sim}{d}$ is the vector field on π_t, $\underset{\sim}{d} = \underset{\sim}{d}(\underset{\sim}{\theta},t)$, $\theta \in \Pi$, $t \in I$. Let us denote by Q the set of all six-tuples $(q_{(6)} = ((r_k,(d_k))$, such that $[\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{d}] > 0$, $\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha}$, for every time instant. Interpreting $\underset{\sim}{q}_{(6)}$ as the shell deformation function let us assume that the function $\underset{\sim}{\tilde{p}}$ in Eqs. (A2.3) or (A4.23) is given by

$$\underset{\sim}{\tilde{p}} = \underset{\sim}{r}(\theta,t) + \xi \underset{\sim}{d}(\theta,t) \qquad \text{(B6.1)}$$

for some $q_{(6)} = (\underset{\sim}{r}, \underset{\sim}{d}) \in Q$. The approximations $\underset{\sim}{p} \sim \underset{\sim}{\tilde{p}}$ or the constraints $\underset{\sim}{p} = \underset{\sim}{\tilde{p}}$ (cf. Secs.2 and 4 of the Chapter A) lead to the shell theory which will be called the Cosserat surface shell theory, provided that we shall deal with the simple shell force system (cf. Sec. 5.1. of the Chapter 1). In the direct approach, instead of Eq. (B6.1), we only postulate that $\underset{\sim}{r} = \underset{\sim}{p}(\underset{\sim}{\theta},\xi_o,t)$, $\underset{\sim}{d} = \underset{\sim}{p}(\underset{\sim}{\theta}, \xi_o,t) / \partial\xi$, where ξ_o is the fixed number with $\langle h_-, h_+ \rangle$.

Denoting $q_a = \delta^k_a r_k + \delta^k_{a-3} d_k$, $a = 1,\dots,6$, and

$$H^{a\alpha}_R = \delta^a_k H^{k\alpha}_R + \delta^{a-3}_k M^{k\alpha}_R \quad ,$$

$$h^a_R = \delta^a_k h^k_R - \delta^{a-3}_k m^k_R \quad ,$$

$$f^a_R = \delta^a_k f^k_R + \delta^{a-3}_k l^k_R \quad ,$$

$$p^a_R = \delta^a_k p^k_R + \delta^{a-3}_k s^k_R \quad ,$$

$$i^a_R = \delta^a_k i^k_R + \delta^{a-3}_k j^k_R \quad ,$$

we shall write Eqs. (A2.12) in the form

$$H^{k\alpha}_{R,\alpha} + f^k_R = i^k_R \ ,$$
$$M^{k\alpha}_{R,\alpha} - m^k_R + l^k_R = j^k_R \ , \tag{B6.2}$$

and

$$H^k_R n_R = e^k_R \ , \quad M^k_R n_R = s^k_R \tag{B6.3}$$

where, putting $\Phi^a_k = \partial\tilde{p}_k/\partial q_a$, $h_- = -h, h_+ = h$, we obtain from Eqs. (A2.11)

$$H^{ka}_R \equiv \int_{-h}^{h} [(r^k_{,\beta} + \xi d^k_{,\beta})\tilde{T}^{\beta\alpha} + d^k\tilde{T}^{3\alpha}] d\xi \ ,$$

$$M^{k\alpha}_R \equiv \int_{-h}^{h} [(r^k_{,\beta} + \xi d^k_{,\beta})\tilde{T}^{\beta\alpha} + d^k\tilde{T}^{3\alpha}]\xi\, d\xi \ ,$$

$$h^k_R \equiv 0, \quad m^k_R \equiv \int_{-h}^{h} [(r^k_{,\beta} + \xi d^k_{,\beta})\tilde{T}^{\beta 3} + d^k\tilde{T}^{33}]\, d\xi \ ,$$

$$f^k_R \equiv \int_{-h}^{h} b^k_R\, d\xi + \overset{+}{p}{}^k_R + \overset{-}{p}{}^k_R \ , \tag{B6.4}$$

$$l^k_R \equiv \int_{-h}^{h} b^k_R \xi\, d\xi + h(\overset{+}{p}{}^k_R - \overset{-}{p}{}^k_R) \ ,$$

$$e^k_R \equiv \int_{-h}^{h} p^k_R\, d\xi \ , \quad s^k_R \equiv \int p^k_R \xi\, d\xi \ ,$$

$$i^k_R \equiv \alpha_R \ddot{r}^k + \beta_R \ddot{d}^k \ , \quad j^k_R \equiv \beta_R \ddot{r}^k + \gamma_R \ddot{d}^k \ ,$$

$$\alpha_R \equiv \int_{-h}^{h} \rho_R\, d\xi \ , \quad \beta_R \equiv \int_{-h}^{h} \rho_R \xi\, d\xi \ , \quad \gamma_R \equiv \int_{-h}^{h} \rho_R \xi^2 d\xi \ .$$

It must be stressed that the field s^k_R introduced above has nothing in common with the field $\underset{\sim}{s}_R$ introduced in Secs. 2 and 4 of the Chapter A.

Eqs. (B6.2) represent the equations of motion and Eqs. (B6.3) are the kinetic boundary conditions of the Cosserat-surface shell theory.

Let us determine the set of strain measures for the Cosserat surface shell theory. From $\tilde{\underset{\sim}{C}} = (\nabla\tilde{\underset{\sim}{p}})^T\nabla\tilde{\underset{\sim}{p}}$ and Eqs. (B6.1) we obtain

$$\begin{aligned} \tilde{C}_{\alpha\beta} &= \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta + \xi(\underset{\sim}{a}_\alpha \cdot \underset{\sim}{d}_\beta + \underset{\sim}{a}_\beta \cdot \underset{\sim}{d}_\alpha) + \xi^2 \underset{\sim}{d}_\alpha \cdot \underset{\sim}{d}_\beta , \\ \tilde{C}_{\alpha 3} &= \underset{\sim}{a}_\alpha \cdot \underset{\sim}{d} + \xi \underset{\sim}{d}_\alpha \cdot \underset{\sim}{d} , \qquad\qquad (B6.5) \\ \tilde{C}_{33} &= \underset{\sim}{d} \cdot \underset{\sim}{d} , \end{aligned}$$

where $\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}_{,\alpha}$, $\underset{\sim}{d}_\alpha \equiv \underset{\sim}{d}_{,\alpha}$. Thus for $e_{(r)}$ we can take the ordered set of $r = 14$ functions $\underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta$, $(\underset{\sim}{a}_\alpha \cdot \underset{\sim}{d}_\beta + \underset{\sim}{a}_\beta \cdot \underset{\sim}{d}_\alpha)$, $\underset{\sim}{d}_\alpha \cdot \underset{\sim}{d}_\beta$, $\underset{\sim}{a}_\alpha \cdot \underset{\sim}{d}$, $\underset{\sim}{d}_\alpha \cdot \underset{\sim}{d}$, $\underset{\sim}{d} \cdot \underset{\sim}{d}$.

The first set of the constitutive relations can be derived from Eqs. (A2.16), (A2.15). For the Cosserat-surface shell theory we obtain [1]

$$\begin{aligned} H_R^{k\alpha} &= \tilde{H}_R^{k\alpha}(\theta, (\underset{\sim}{a}_\beta), (\underset{\sim}{d}_\beta), \underset{\sim}{d}, \tau^{(N)}) , \\ M_R^{k} &= \tilde{M}_R^{k\alpha}(\theta, (\underset{\sim}{a}_\beta), (\underset{\sim}{d}_\beta), \underset{\sim}{d}, \tau^{(N)}) , \qquad\qquad (B6.6) \\ m_R^{k} &= \tilde{m}_R^{k}(\theta, (\underset{\sim}{a}_\beta) , (\underset{\sim}{d}_\beta) , \underset{\sim}{d}, \tau^{(N)}) , \end{aligned}$$

where the RHS of Eqs. (B6.6) we obtain substituting $\tilde{T}^{KL} = S^{KL}(\tilde{\underset{\sim}{\sigma}})$ into Eqs. $(B6.4)_{1-4}$. The second set of the constitutive relations has the form (A2.14), where the set $e_{(r)} = e_{(14)}$ of the strain measures has been defined above.

The field equations (B6.2),(B6.3) and the constitutive relations (B6.6), (A2.14) are the governing relations of the Cosserat-surface shell theory. For the shells made of the simple materials the arguments $\tau^{(N)}$ drops out from Eqs. (B6.6) and we can also neglect Eqs. (A2.14) (cf. Secs. 2 and 5 of the Chapter A).

---

[1] We denote $(\underset{\sim}{a}_\beta) \equiv (\underset{\sim}{a}_1, a_2)$, $(\underset{\sim}{d}_\beta) \equiv (\underset{\sim}{d}_1, \underset{\sim}{d}_2)$.

6.2. Lagrangean and Eulerian formulations

The Lagrangean or Eulerian formulations of Eqs. (B6.2),(B6.3),(B6.6) can be easily obtained by means of the general transformation formulas (B3.11), (B3.15), (B3.16). Equations of motion (B6.2) will have the form

$$H^{M\alpha}|_{\alpha} + f^M = i^M \ , \quad M^{M\alpha}|_{\alpha} - m^M + l^M = j^M \ , \quad M = 1,2,3,$$

or, using the surface covariant derivatives

$$\begin{aligned} &H^{\gamma\alpha}||_{\alpha} - b^{\gamma}_{\alpha} H^{3\alpha} + f^{\gamma} = i^{\gamma} \ , \\ &H^{3\alpha}||_{\alpha} - b_{\alpha\beta} H^{\alpha\beta} + f^3 = i^3 \ , \\ &M^{\gamma\alpha}||_{\alpha} - b^{\gamma}_{\alpha} M^{3\alpha} - m^{\gamma} + l^{\gamma} = j^{\gamma} \ , \\ &M^{3\alpha}||_{\alpha} - b_{\alpha\beta} M^{\alpha\beta} - m^3 + l^3 = j^3 \ , \end{aligned} \tag{B6.7}$$

where

$$\begin{aligned} H^{\gamma\alpha} &= \frac{1}{\sqrt{a}} H^{k\alpha}_R r_{k,\beta} a^{\beta\gamma}, \qquad & H^{3\alpha} &= \frac{1}{\sqrt{a}} H^{k\alpha} N_k \ , \\ f^{\gamma} &= \frac{1}{\sqrt{a}} f^k_R r_{k,\beta} a^{\beta\gamma} \ , & f^3 &= \frac{1}{\sqrt{a}} f^k_R N_k \ , \ \text{etc.} \end{aligned} \tag{B6.8}$$

and where $a \equiv \det a_{\alpha\beta}$, $a_{\alpha\beta} \equiv \underset{\sim}{r}_{,\alpha} \cdot \underset{\sim}{r}_{,\beta}$ and $a^{\beta\gamma}$ are determined by $a^{\beta\gamma} a_{\gamma\alpha} = \delta^{\beta}_{\alpha}$ in terms of $a_{\alpha\beta}$. The transformed form of the kinetic boundary conditions (B6.3) will be derived from Eqs. (B3.15)

$$\begin{aligned} &H^{\gamma\alpha} n_{\alpha} = e^{\gamma} \ , \quad & &H^{3\alpha} n_{\alpha} = e^{3\alpha} \ , \\ &M^{\gamma\alpha} n_{\alpha} = s^{\gamma} \ , & &M^{3\alpha} n_{\alpha} = s^3 \ , \end{aligned} \tag{B6.9}$$

where e^M, s^M are related to e^k_R, s^k_R by the formulae of the form $(B3.12)_1$, i.e., by means of

$$e^{\gamma} = \frac{1}{\sqrt{\lambda}} p^k_R r_{k,\beta} a^{\beta\gamma}, \qquad e^3 = \frac{1}{\sqrt{\lambda}} p^k_R N_k \ ,$$

and analogously for s^{γ}, s^3.

Denoting now be $e_{(14)}$ the ordered set of the shell strain measures (cf. Eqs. (B6.5), we can transform the first set of the constitutive relations to the form

$$
\begin{aligned}
H^{\gamma\alpha} &= \tilde{H}^{\gamma\alpha}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),\\
H^{3\alpha} &= \tilde{H}^{3\alpha}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),\\
M^{\gamma\alpha} &= \tilde{M}^{\gamma\alpha}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),\\
M^{3\alpha} &= \tilde{M}^{3\alpha}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),\\
m^{\gamma} &= \tilde{m}^{\gamma}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),\\
m^{3} &= \tilde{m}^{3}(\underset{\sim}{\theta}, e_{(14)}, \tau^{(N)}),
\end{aligned}
\tag{B6.10}
$$

At the same time from $(B6.4)_{1-4}$ and from the equations of the form (B6.8) we obtain

$$
\begin{aligned}
\sqrt{a}\,H^{\gamma\alpha} &= \int_{-h}^{h} [(\delta^{\gamma}_{\beta} + \xi B^{\gamma}_{\beta})\tilde{T}^{\beta\alpha} + B^{\gamma}\tilde{T}^{3\alpha}]\,d\xi ,\\
\sqrt{a}\,H^{3\gamma} &= \int_{-h}^{h} (\xi B_{\beta}\tilde{T}^{\beta\gamma} + B\tilde{T}^{3\gamma})\,d\xi ,\\
\sqrt{a}\,M^{\gamma\alpha} &= \int_{-h}^{h} [(\delta^{\gamma}_{\beta} + \xi B^{\gamma}_{\beta})\tilde{T}^{\beta\alpha} + B^{\gamma}\tilde{T}^{3\alpha}]\,\xi\,d\xi ,\\
\sqrt{a}\,M^{3\alpha} &= \int_{-h}^{h} (\xi B_{\beta}\tilde{T}^{\beta\alpha} + B\tilde{T}^{3\alpha})\,\xi\,d\xi ,\\
\sqrt{a}\,m^{\gamma} &= \int_{-h}^{h} [(\delta^{\gamma}_{\beta} + \xi B^{\gamma}_{\beta})\tilde{T}^{\beta 3} + B^{\gamma}\tilde{T}^{33}]\,d\xi ,\\
\sqrt{a}\,m^{3} &= \int_{-h}^{h} (\xi B_{\beta}\tilde{T}^{\beta 3} + B\tilde{T}^{33})\,d\xi ,
\end{aligned}
\tag{B6.11}
$$

where we have denoted

$$
\begin{aligned}
B^{\gamma}_{\beta} &\equiv d^{k}_{,\beta} r_{k,\delta} a^{\delta\gamma}, \qquad & B^{\gamma} &\equiv d^{k} r_{k,\delta} a^{\delta\gamma} ,\\
B_{\gamma} &\equiv d^{k}_{,\beta} N_{k} , & B &\equiv d^{k} N_{k} .
\end{aligned}
\tag{B6.12}
$$

The RHS of Eqs. (B6.10) can be derived by substituting $\tilde{T}^{KL} = S^{KL}(\tilde{\sigma})$ into Eqs. (B6.11).

Eqs. (B6.7), (B6.9), (B6.10) and (A2.14) can be interpreted either as Lagrangean or as Eulerian formulation of the Cosserat-type theory of shells (cf. Sec. 3 of this Chapter).

6.3. Alternative form of governing relations

Now instead of vectors $M_R^{k\alpha}$, m_R^k, l_R^k, j_R^k, s_R^k we shall introduce the vectors $G_R^{k\alpha}$, g_R^k, k_R^k, e_R^k, t_R^k, respectively, defined by

$$
\begin{aligned}
G_R^{k\alpha} &= \varepsilon^{k}_{.lm} M_R^{m\alpha} d^l \quad , \\
g_R^k &= \varepsilon^{k}_{.lm} m_R^m d^l \quad , \\
k_R^k &= \varepsilon^{k}_{.lm} l_R^m d^l \quad , \\
a_R^k &= \varepsilon^{k}_{.lm} j_R^m d^l \quad , \\
t_R^k &= \varepsilon^{k}_{.lm} s_R^m d^l \quad ,
\end{aligned}
\qquad (B6.13)
$$

Substituting into RHS of Eqs. (B6.13) the RHS of Eqs. $(B6.4)_{2,4,6,8,10}$ we observe, that the fields introduced above represent the shell couple stresses and moments. Now we introduce the LHS of Eqs. (B6.13) into the governing equations of the shell theory. To this aid let us multiply Eqs. $(B6.2)_2$ by $\varepsilon_k^{.lm} d_l$. Let us also observe, that

$$
\varepsilon_k^{.lm}(M^{\alpha}_{Rm}\, d_{l,\alpha} + H^{\alpha}_{Rm} r_{l,\alpha} + m_{Rm} d_L) =
$$

$$
= \varepsilon_k^{.lm} \int_{-h}^{h} \tilde{p}_{m,M} \tilde{T}^{MN} \tilde{p}_{l,N}\, d\xi = 0 \quad ,
$$

which makes it possible to eliminate the functions $g_R^k = \varepsilon^{k}_{.lm} m_R^m d^l$. Thus Eqs. $(B6.2)_2$ can be transformed to the form

$$
G^{k\alpha}_{R,\alpha} + \varepsilon^{k}_{.lm} H_R^{m\alpha} r^l_{,\alpha} + k_R^k = a_R^k \quad .
$$

At the same time, using (B6.13), we shall transform $(B6.3)_2$, obtaining

$$G_R^{k\alpha} n_{R\alpha} = c_R^k .$$

Passing to the Lagrangean or Eulerian formulations, the two foregoing equations can be written down in the form

$$G^{\gamma\alpha}||_\alpha - b_\alpha^\gamma G^{3\alpha} + \epsilon^\gamma_{.\alpha} H^{3\alpha} + k^\gamma = a^\gamma ,$$

$$G^{3\alpha}||_\alpha + b_{\alpha\beta} G^{\alpha\beta} + \epsilon_{\alpha\beta} H^{\alpha\beta} + k^3 = a^3 ,$$

$$G^{\gamma\alpha} n_\alpha = c^\gamma, \quad G^{3\alpha} n_\alpha = c^3, \quad \epsilon^\gamma_{.\alpha} = a^{\gamma\beta}\epsilon_{\beta\alpha}, \quad \epsilon_{\beta\alpha} \equiv \varepsilon_{klm} r^k_{,\beta} r^l_{,\alpha} N^m .$$

It means that the equations of motion are now given by

$$\begin{aligned} &H^{\gamma\alpha}||_\alpha - b_\alpha^\gamma H^{3\alpha} + f^\gamma = i^\gamma , \\ &H^{3\alpha}||_\alpha + b_{\alpha\beta} H^{\alpha\beta} + f^3 = i^3 , \\ &G^{\gamma\alpha}||_\alpha - b_\alpha^\gamma G^{3\alpha} + a^{\gamma\beta}\epsilon_{\beta\alpha} H^{3\alpha} + k^\gamma = a^\gamma , \\ &G^{3\alpha}||_\alpha + b_{\alpha\beta} G^{\alpha\beta} + \epsilon_{\alpha\beta} H^{\alpha\beta} + k^3 = a^3 \end{aligned} \qquad (B6.14)$$

and the kinetic boundary conditions have the form

$$\begin{aligned} &H^{\gamma\alpha} n_\alpha = e^\gamma , \quad H^{3\alpha} n_\alpha = e^3 , \\ &G^{\gamma\alpha} n_\alpha = c^\gamma , \quad G^{3\alpha} n_\alpha = c^3 , \end{aligned} \qquad (B6.15)$$

Now let us represent the vector field $\underset{\sim}{d}$ in the form

$$d_k = d^\alpha r_{k,\alpha} + dN_k, \quad d = d_k N^k, \quad d^\alpha = a^{\alpha\beta} d_k r^k_{,\beta}$$

and let us also denote

$$\begin{aligned} &R^{\alpha\beta} \equiv [\delta^\alpha_\gamma + (d^\alpha||_\gamma - b^\alpha_\gamma d)\xi]\tilde{T}^{\gamma\beta} + d^\alpha \tilde{T}^{3\beta}, \\ &R^\alpha \equiv (d_{,\gamma} + b_{\gamma\beta} d^\beta)\tilde{T}^{\gamma\alpha} + d\tilde{T}^{3\alpha} . \end{aligned} \qquad (B6.16)$$

After simple calculations we arrive at the formulae

$$H^{\alpha\beta} = \frac{1}{\sqrt{a}}\int_{-h}^{h} R^{\alpha\beta} d\,\xi\,, \quad H^{3\beta} = \frac{1}{\sqrt{a}}\int_{-h}^{h} R^{\beta} d\,\xi$$

$$G^{\alpha\beta} = \frac{1}{\sqrt{a}}\int_{-h}^{h} (R^{\gamma\alpha} d - R^{\alpha} d^{\gamma})\,\xi\, d\xi\, a^{\alpha\beta}\epsilon_{\delta\gamma}\,, \tag{B6.17}$$

$$G^{3\alpha} = \frac{1}{\sqrt{a}}\int_{-h}^{h} R^{\gamma\alpha} d^{\beta}\,\xi\, d\,\xi\,\epsilon_{\beta\gamma}\,.$$

We can also introduce the new shell strain measures. Denoting

$$D_{\alpha\beta} \equiv d_{\alpha}||_{\beta} - b_{\alpha\beta} d\,, \quad D_{\alpha} \equiv d_{,\alpha} + b_{\alpha\beta} d^{\beta} \tag{B6.18}$$

we obtain at equations

$$\tilde{C}_{\alpha\beta} = a_{\alpha\beta} + 2D_{(\alpha\beta)}\xi + D_{\alpha\gamma} D_{\beta\delta} a^{\gamma\delta}\xi^{2}\,,$$

$$\tilde{C}_{\alpha 3} = a_{\alpha\beta} d^{\beta} + (D_{\beta\alpha} d^{\beta} + D_{\alpha} d)\,\xi\,, \tag{B6.19}$$

$$\tilde{C}_{33} = d^{\alpha} d^{\beta} a_{\alpha\beta} + (d)^{2}\,,$$

where $a_{\alpha\beta} \equiv \underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_{\beta}$. Now for $e_{(r)}$ we can take the ordered set of $r = 15$ functions $a_{\alpha\beta}$, $D_{(\alpha\beta)}$, $D_{\alpha\gamma} D_{\alpha\delta} a^{\gamma\delta}$, $a_{\alpha\beta} d^{\beta}$, $D_{\beta\alpha} d^{\beta} + D_{\alpha} d$, $d^{\alpha} d^{\beta} a_{\alpha\beta}$, $(d)^{2}$. The first set of the constitutive relations will be now assumed in the form

$$H^{M\beta} = \tilde{H}^{M\beta}(\underset{\sim}{\theta}, e_{(15)}, \tau^{(N)})\,,$$

$$G^{M\beta} = \tilde{G}^{M\beta}(\underset{\sim}{\theta}, e_{(15)}, \tau^{(N)})\,, \tag{B6.20}$$

where the RHS of Eqs. (B6.20) can be obtained by substituting $\underset{\sim}{T} = \underset{\sim}{S}(\tilde{\underset{\sim}{\sigma}})$ into Eqs. (B6.16) and the RHS of the resulting equations into Eqs. (B6.17).

The equations of motion (B6.14), the kinetic boundary conditions (B6.15) and the constitutive relations (B6.20), (A2.24) constitute the alternative form of the governing relations of the Cosserat-surface shell theory.

Remark. After simple calculations we shall obtain the following interrelations between the fields in the equations (B6.14), (B6.15) and in the equations (B6.7), (B6.9):

$$G^{\beta\gamma} = d\,\epsilon_{\alpha\cdot}^{\ \beta}\, M^{\alpha\gamma} + d\,\epsilon^{\beta}_{\cdot\alpha}\, M^{3\gamma} \ ,$$

$$G^{3\gamma} = d^{\alpha}\epsilon_{\alpha\beta}M^{\beta\gamma},$$

$$k^{\beta} = d\,\epsilon_{\alpha\cdot}^{\ \beta}\, l^{\alpha} + d^{\alpha}\epsilon^{\beta}_{\cdot\alpha}\, l^{3} \ ,$$

$$k^{3} = d^{\alpha}\epsilon_{\alpha\beta}l^{\beta}$$

and analogously for: a^{γ}, a^{3} and j^{γ}, j^{3}, c^{γ}, c^{3} and s^{γ}, s^{3}, where $d \equiv d^{k}N_{k}$, $d^{\alpha} \equiv d^{k}r_{k,\beta}a^{\beta\alpha}$.

6.4. Elastic-perfectly plastic shells

Let the constitutive relations for an elastic-perfectly plastic material be assumed in the form (cf. also Sec. 2.1. of the Chapter A)

$$\dot{\underset{\sim}{C}} - \underset{\sim}{A}[\dot{\underset{\sim}{T}}] - \underset{\sim}{\Lambda} = 0 \ ,$$

$$j(\underset{\sim}{X},\underset{\sim}{T}) \le 0 \ , \qquad \text{(B6.21)}$$

$$\mathrm{tr}\underset{\sim}{\Lambda}(\underset{\sim}{T}_0 - \underset{\sim}{T}) \le 0 \text{ for every } \underset{\sim}{T}_0 \text{ with } j(\underset{\sim}{X},\underset{\sim}{T}_0) \le 0 \ ,$$

where $\underset{\sim}{\Lambda} = (\Lambda_{\alpha\beta})$, $\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}^{T}$, is the rate of the plastic strain, $\underset{\sim}{A}[\dot{\underset{\sim}{T}}]$ is the elastic part of the strain rate and $j(\underset{\sim}{X},\underset{\sim}{T}) = 0$ is the yield condition ([1]). Eqs. (B6.21) are the special form of the general constitutive relations (A2.2).

To obtain the constiutive relations for the elastic-perfectly plastic shells within the Cosserat-surface shell theory we have firstly to introduce the approximation relations (A2.5) - (A2.7). To this aid we introduce the mapping $\underset{\sim}{S}$ (cf. Sec. 2 of the Chapter A) assuming

---

([1]) The form of the linear transformation A and that of the function j(.) can also depend on the deformation gradient $\nabla\underset{\sim}{p}$.

that $\underset{\sim}{T} = \underset{\sim}{S}(\underset{\sim}{\sigma})$ has the trivial form $(T^{11}, T^{22}, T^{33}, T^{12}, T^{13}, T^{23}) = (\sigma^1,\ldots,\sigma^6)$. It means that the stress components σ^μ, $\mu = 1,\ldots,6$, simply coincide with the suitable components of the second Piola-Kirchhoff stress tensor T^{KL}, $K,L = 1,2,3$ ([1]). We shall approximate all stress components, assuming (A2.5), (A2.6) in the form

$$\underset{\sim}{T}^{KL}(\underset{\sim}{X},t) \sim \tilde{\underset{\sim}{T}}^{KL} = \tilde{S}^{KL}_A(\underset{\sim}{X})\tau^A(\underset{\sim}{\theta},t) \text{ or } \tilde{\underset{\sim}{T}} = \tilde{\underset{\sim}{S}}[\tau^{(N)}], \tag{B6.22}$$

where $A = 1,\ldots,N$ (summation convention holds), $\underset{\sim}{X} \equiv (\underset{\sim}{\theta},\xi)$ and $\tilde{S}^{KL}$ are the known functions ([2]). We also have $\underset{\sim}{\Lambda} = \underset{\sim}{S}(\lambda_{(6)})$, i.e., $(\Lambda_{11}, \Lambda_{22}, \Lambda_{33}, \Lambda_{12}, \Lambda_{13}, \Lambda_{23}) = (\lambda_1,\ldots,\lambda_6)$, and we shall postulate the approximation (A2.7) in the form which corresponds to that given by Eq. (B6.5), i.e., in the form

$$\begin{aligned}
\Lambda_{\alpha\beta}(\underset{\sim}{X},t) &\sim \tilde{\Lambda}_{\alpha\beta} = \overset{o}{\omega}_{\alpha\beta}(\underset{\sim}{\theta},t) + \xi\overset{1}{\omega}_{\alpha\beta}(\underset{\sim}{\theta},t) + \xi^2\overset{2}{\omega}_{\alpha\beta}(\underset{\sim}{\theta},t) \ ,\\
\Lambda_{\alpha 3}(\underset{\sim}{X},t) &\sim \tilde{\Lambda}_{\alpha 3} = \overset{o}{\omega}_{\alpha}(\underset{\sim}{\theta},t) + \xi\overset{1}{\omega}_{\alpha}(\underset{\sim}{\theta},t) \ ,\\
\Lambda_{33}(\underset{\sim}{X},t) &\sim \tilde{\Lambda}_{33} = \overset{o}{\omega}(\underset{\sim}{\theta},t) \ ,
\end{aligned} \tag{B6.23}$$

where, as usual, $\underset{\sim}{X} = (\underset{\sim}{\theta},t)$, and the ordered set of arbitrary regular functions $\overset{o}{\omega}_{\alpha\beta}$, $\overset{1}{\omega}_{\alpha\beta}$, $\overset{2}{\omega}_{\alpha\beta}$, $\overset{o}{\omega}_{\alpha}$, $\overset{1}{\omega}_{\alpha}$, $\overset{o}{\omega}$ plays the role of the set $\omega^{(p)}$, $p = 14$, in Eq. (A2.7). The strain componenets C_{KL} (the components of the right Cauchy-Green deformation tensor) are approximated by $\tilde{C}_{KL}$, i.e., $(C_{KL} \sim \tilde{C}_{KL})$, where $\tilde{C}_{KL}$ for the Cosserat type shell theory have to be assumed in the form (B6.5).

Now we can introduce the "error" fields a_μ, $\mu = 1,\ldots,6$, a,α, defined by Eqs. (A2.8). Putting $(E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}) = (a_1,\ldots,a_6)$, we obtain from Eqs. (A2.8), (A2.9)

$$\begin{aligned}
\underset{\sim}{E} &\equiv \dot{\tilde{\underset{\sim}{C}}} - \underset{\sim}{A}[\tilde{\underset{\sim}{S}}[\dot{\tau}^{(N)}]] - \tilde{\underset{\sim}{\Lambda}} \ ,\\
a &\equiv j(\underset{\sim}{X},\tilde{\underset{\sim}{S}}[\tau^{(N)}]) - j(\underset{\sim}{X},\underset{\sim}{T}) \ ,\\
\alpha &\equiv \mathrm{tr}\tilde{\underset{\sim}{\Lambda}}\tilde{\underset{\sim}{S}}[\tau_o^{(N)} - \tau^{(N)}] - \mathrm{tr}\underset{\sim}{\Lambda}(\underset{\sim}{T}_o - \underset{\sim}{T}) \ .
\end{aligned}$$

The foregoing formulae have been introduced here only in order to give an example of Eqs. $(A2.8)_{3-5}$, (A2.9). In the second step of the

---

([1])We have tacitly assumed here, that neither from the stress components is determined by the history of motion, i.e., we have assumed $M = 0$ in Eqs. (A2.6).

([2]) Mapping $\underset{\sim}{S} : R^6 \to \mathcal{T}^2$ has nothing in common with the linear transformation $\tilde{\underset{\sim}{S}}$.

formal approximation procedure we have to restrict the error fields by means of Eqs. $(A2.10)_{3-5}$. To this aid we shall assume the functions Ξ^{μ}_{A}, $\mu = 1,\dots,6$, $A = 1,\dots,N$ (cf. Eqs. (A2.10), where now $P = 0$, $M = 0$, $m = 6$) in the form

$$
(\Xi^{1}_{A},\dots,\Xi^{6}_{A}) = (\Xi^{11}_{A},\ \Xi^{22}_{A},\ \Xi^{33}_{A},\ \Xi^{12}_{A},\ \Xi^{13}_{A},\ \Xi^{23}_{A}) \ ,
$$

$$
\Xi^{KL}_{A} = \frac{\partial \tilde{T}^{KL}}{\partial \tau^{A}} = \tilde{S}^{KL}_{A}(\underset{\sim}{X}) \quad . \tag{B6.24}
$$

From Eqs.(A2.13) we obtain

$$
g_A \equiv \int_{-h}^{h} (\tilde{\tilde{C}}_{KL} - A_{KLMN}\tilde{S}^{MN}_{B}\dot{\tau}^{B} - \tilde{\Lambda}_{KL})\tilde{S}^{KL}_{A}\, d\xi \ ,
$$

$$
\kappa \equiv \int_{-h}^{h} j(\underset{\sim}{X},\underset{\sim}{\tilde{S}}[\tau^{(N)}])\, d\xi \ , \tag{B6.25}
$$

$$
\psi \equiv \int_{-h}^{h} \tilde{\Lambda}_{KL}\tilde{S}^{KL}_{A}\, d\xi\,(\tau^{A}_{o} - \tau^{A}) \quad .
$$

But all the integrands above are the known functions of ξ and all integrals in Eqs. (B6.25) can be calculated. Denoting by $e_{(14)}$ the ordered set of the strain measures in Eqs. (B6.5) and using Eqs. (B6.25) we shall obtain the constitutive relations (A2.14) in which $r = 14$, $p = 14$, $P = 0$. If A_{KLMN} depend only on $\underset{\sim}{X}$ then denoting

$$
\overset{o}{G}{}^{KL}_{A} = \frac{1}{\sqrt{a}}\int_{-h}^{h} \tilde{S}^{KL}_{A}\, d\xi,\quad \overset{1}{G}{}^{\alpha L}_{A} \equiv \frac{1}{\sqrt{a}}\int_{-h}^{h} \xi\tilde{S}^{\alpha L}_{A}\, d\xi,\quad \overset{2}{G}{}^{\alpha\beta}_{A} \equiv \frac{1}{\sqrt{a}}\int_{-h}^{h} \xi^{2}\tilde{S}^{\alpha\beta}_{A}\, d\xi
$$

and taking into account (B6.5), (B6.23), we obtain from Eqs. (B6.25)

$$
\begin{aligned}
&(\overline{\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{a}_{\beta}}^{\,\boldsymbol{\cdot}} - \overset{o}{\omega}_{\alpha\beta})\overset{o}{G}{}^{\alpha\beta}_{A} + (\overline{\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{d}_{\beta} + \underset{\sim}{a}_{\beta}\cdot\underset{\sim}{d}_{\alpha}}^{\,\boldsymbol{\cdot}} - \overset{1}{\omega}_{\alpha\beta})\overset{1}{G}{}^{\alpha\beta}_{A} + \\
&+ (\overline{\underset{\sim}{d}_{\alpha}\cdot\underset{\sim}{d}_{\beta}}^{\,\boldsymbol{\cdot}} - \overset{2}{\omega}_{\alpha\beta})\overset{2}{G}{}^{\alpha\beta}_{A} + (\overline{\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{d}}^{\,\boldsymbol{\cdot}} - \overset{o}{\omega}_{\alpha})\overset{o}{G}{}^{\alpha 3}_{A} + \\
&+2(\overline{\underset{\sim}{d}_{\alpha}\cdot\underset{\sim}{d}}^{\,\boldsymbol{\cdot}} - \overset{1}{\omega}_{\alpha})\overset{1}{G}{}^{\alpha 3}_{A} + (\overline{\underset{\sim}{d}\cdot\underset{\sim}{d}}^{\,\boldsymbol{\cdot}} - \overset{o}{\omega})\overset{o}{G}{}^{33}_{A} - D_{AB}\dot{\tau}^{B} = 0 \ ,
\end{aligned}
$$

$$\kappa(\underset{\sim}{\theta},\tau^{(N)}) \leq 0 \quad ,$$

$$(\tau_o^A - \tau^A)(\overset{o}{\omega}_{\alpha\beta}\overset{o\alpha\beta}{G}_A + \overset{1}{\omega}_{\alpha\beta}\overset{1\alpha\beta}{G}_A + \overset{2}{\omega}_{\alpha\beta}\overset{2\alpha\beta}{G}_A +$$

$$+\overset{o}{\omega}_{\alpha}\overset{o\alpha 3}{G}_A + 2\overset{1}{\omega}_{\alpha}\overset{1\alpha 3}{G}_A + \overset{o}{\omega}\overset{o33}{G}_A) \leq 0 \quad \text{for every } \tau_o^{(N)} \text{ with } \kappa(\theta,\tau_o^{(N)}) \leq 0, \tag{B6.26}$$

where

$$D_{AB} \equiv \frac{1}{\sqrt{a}}\int_{h_-}^{h_+} A_{KLMN}\tilde{S}_B^{MN}\tilde{S}_A^{KL}\, d\xi \ , \quad \underset{\sim}{a}_{\alpha} \equiv \underset{\sim}{r}_{,\alpha} \ , \quad \underset{\sim}{d}_{\alpha} \equiv \underset{\sim}{d}_{,\alpha} \quad ,$$

and D_{AB} have to constitute the positive definite N × N matrix. At the same time form Eqs. (B6.11), (B6.22) we obtain

$$\begin{aligned}
H^{\gamma\alpha} &= (\overset{o\gamma\alpha}{G}_A + B^{\gamma}_{\beta}\overset{1\beta\alpha}{G}_A + B^{\gamma}\overset{o3\alpha}{G}_A)\tau^A \quad , \\
H^{3\gamma} &= (B_{\beta}\overset{1\beta\gamma}{G}_A + B\overset{o3\gamma}{G}_A)\tau^A \quad , \\
M^{\gamma\alpha} &= (\overset{1\gamma\alpha}{G}_A + B^{\gamma}_{\beta}\overset{2\beta\alpha}{G}_A + B^{\gamma}\overset{13\alpha}{G}_A)\tau^A \quad , \\
m^{\gamma} &= (\overset{o\gamma 3}{G}_A + B^{\gamma}_{\beta}\overset{1\beta 3}{G}_A + B^{\gamma}\overset{o33}{G}_A)\tau^A \quad , \\
m^{3} &= (B_{\beta}\overset{1\beta 3}{G}_A + B\overset{o33}{G}_A)\tau^A \quad ,
\end{aligned} \tag{B6.27}$$

where the functions B^{γ}_{β}, B^{γ}, B_{γ}, B are the strain shell measures defined by Eqs. (B6.12).

Eqs. (B6.26), (B6.27) are the constitutive relations of the elastic-perfectly plastic shells described by the Cosserat-surface shell theory ([1]). Together with the field equations (B6.7), (B6.9) they constitute the shell governing relations of the Cosserat-type shell theory. The basic unknowns are:

1. the shell deformation function $q_{(6)} = (\underset{\sim}{r}(\underset{\sim}{\theta},t), \underset{\sim}{d}(\underset{\sim}{\theta},t))$, which approximate (or describes in constraint approach) the motion of the shell by means of Eqs. (B6.1),

---

([1]) Eqs. (B6.26), (B6.27), which have been obtained here form the formal approximation approach, can be also derived by the constraint approach.

2. the function $\tau^{(N)} = \{\tau^A(\underset{\sim}{\theta},t),\ A = 1,\ldots,N\}$ which approximate (or describes) the stresses in shell by means of Eqs. (B6.22),

3. the function $\omega^{(p)} = ((\overset{o}{\omega}_{\alpha\beta}),\ (\overset{1}{\omega}_{\alpha\beta}),\ (\overset{2}{\omega}_{\alpha\beta}),\ (\overset{o}{\omega}_{\alpha}),\ (\overset{1}{\omega}_{\alpha}),\ \overset{o}{\omega})$, where $p = 14$ and $(\overset{k}{\omega}_{\alpha\beta}) \equiv (\overset{k}{\omega}_{11}),\ \overset{k}{\omega}_{22},\ \overset{k}{\omega}_{12}),\ k = 0,1,2$, which approximates (or describes) the rate of the plastic strain by means of Eqs. (B6.23).

If instead of the field equations (B6.7), (B6.9) we use the field equations (B6.14), (B6.15) then instead of Eqs. (B2.27) we have to take into account the constitutive equations obtained from Eqs. (B6.16), (B6.17), (B6.22). They will have the form

$$
\begin{aligned}
H^{\alpha\beta} &= [\overset{o\alpha\beta}{\underset{A}{G}} + (d^{\alpha}\|_{\gamma} - b^{\alpha}_{\gamma}d)\underset{A}{G}^{\gamma\beta} + d^{\alpha}\overset{o3\beta}{\underset{A}{G}}]\tau^{A} \quad ,\\
H^{3\beta} &= [(d_{,\gamma} + b_{\gamma\beta}d^{\beta})\overset{o\gamma\beta}{\underset{A}{G}} + d\underset{A}{G}^{3\alpha}]\tau^{A} \quad ,\\
G^{\alpha\beta} &= \{d[\overset{1\gamma\alpha}{\underset{A}{G}} + (d^{\gamma}\|_{\delta} - b^{\gamma}_{\delta}d)\overset{2\delta\alpha}{\underset{A}{G}} + d^{\gamma}\overset{13\alpha}{\underset{A}{G}}] - \\
&\quad - d^{\gamma}[(d_{,\delta} + b_{\delta\varepsilon}d^{\varepsilon})\overset{1\delta\alpha}{\underset{A}{G}} + d\overset{13\alpha}{\underset{A}{G}}]\}a^{\beta\delta}\epsilon_{\delta\gamma}\tau^{A} \quad ,\\
G^{3\alpha} &= d^{\beta}[\overset{1\gamma\alpha}{\underset{A}{G}} + (d^{\gamma}\|_{\delta} - b^{\gamma}_{\delta}d)\overset{2\delta\alpha}{\underset{A}{G}} + d^{\gamma}\overset{13\alpha}{\underset{A}{G}}]\epsilon_{\beta\gamma}\tau^{A} \quad ,
\end{aligned}
\tag{B6.28}
$$

where $d = d_k N^k$, $d^{\alpha} = a^{\alpha\beta}d_k r^k_{,\beta}$ are the strain measures. Eqs. (B6.14), (B6.15), (B6.26), (B6.28) constitute the alternative form of governing relations of the elastic-perfectly plastic shells described by the Cosserat-surface shell theory.

7. MODIFICATION OF THE SHELL THEORIES. GENERAL CASE.

The problem we are going now to detail can be stated as follows: how from the special plate or shell theory one can derive new plate or shell theories? In this Section we shall try to give rather general answer to that questions using the analytical methods described in Secs. 2.0. and 4.0. of the Chapter A.

7.0. Preliminaries

Let be known certain shell theory described by the Eqs. (A2.12), (A2.14), (A2.16). Let us substitute the RHS of Eqs. (A2.16) into Eqs. (A2.12). Then we obtain the equations

$$\begin{aligned} &\tilde{H}^{a\alpha}_{R,\alpha} + \tilde{h}^{a}_{R} + f^{a}_{R} - i^{a}_{R} = 0 \; ; \; \underset{\sim}{\theta} \in \Pi, \; t \in I \\ &\tilde{H}^{a\alpha}_{R} n_{R\alpha} - p^{a}_{R} = 0 \; ; \; \underset{\sim}{\theta} \in \partial\Pi \text{ a.e.}, \; t \in I \; , \end{aligned} \tag{B7.1}$$

which together with Eqs. (A2.14) describe the shell theory under consideration. Putting $x \equiv (q_{(n)} \tau^{(N)})$, let us write down Eqs. (B7.1) in the form $A(x) = \theta$, where A is the operator with the domain D(A) in the space X of the sufficiently regular fields x defined on $\Pi \times I$ and with the range R(A) in a certain linear space Y. By Ξ, $\Xi \subset D(A)$ we shall denote the set KerA, i.e., the set of all solutions of $A(x) = \theta$.

In the Sec. 7.1. we shall introduce the formal approximation of the equation $A(x) = \theta$ (cf. Sec. 2.0. of the Chapter A) and in Sec. 7.2. we shall impose the semiconstraints on this equation (cf. Sec. 4.0. of the Chapter A). In both cases we obtain certain new "modified" shell theory.

7.1. Modification by approximation

To construct the formal approximation the equation $A(x) = \theta$ we have:

1. To introduce the set $\tilde{\Xi}$, $\tilde{\Xi} \subset D(A)$, which "approximates"the set Ξ. Then the formal approximation relation $\sim$ will be given by

$$x \sim \tilde{x} \Rightarrow (x,\tilde{x}) \in \Xi \times \tilde{\Xi} \; .$$

After that we can define the set $\overset{\circ}{Y}$, $\overset{\circ}{Y} \subset Y$, of the error fields

$\overset{\circ}{y}$, $\overset{\circ}{y} \in \overset{\circ}{Y}$, putting $Y = A(\tilde{\Xi})$, i.e., $\overset{\circ}{y} = A(\tilde{x})$, $\tilde{x} \in \tilde{\Xi}$. In view of Eqs. (B7.1) we shall write $\overset{\circ}{y} = (r_R^{(n)}, s_R^{(n)})$, where $r_R^{(n)} = \{r_R^a(\underset{\sim}{\theta}, t),$ $a = 1,\ldots,n\}$, $s_R^{(n)} = \{s_R^a(\underset{\sim}{\theta}, t)$, $a = 1,\ldots,n\}$ and

$$
\begin{aligned}
r_R^a &= -\tilde{H}_{R,\alpha}^{a\alpha} - h_R^a - f_R^a + i_R^a \quad , \\
s_R^a &= \tilde{H}_R^{a\alpha} n_{R\alpha} - p_R^a \quad , \quad a = 1,\ldots,n \ .
\end{aligned}
\tag{B7.2}
$$

2. To restrict the error fields by introducing the multifunction $\tilde{\Xi} \ni \tilde{x} \to \overset{\circ}{Y}_{\tilde{x}} \subset \overset{\circ}{Y}$ and putting $\overset{\circ}{y} \in \overset{\circ}{Y}_{\tilde{x}}$, $\tilde{x} \in \tilde{\Xi}$ (1).

The general scheme of the formal approximation of Eqs. (B7.1) then leads to the relation

$$
A(\tilde{x}) \in \overset{\circ}{Y}_{\tilde{x}} \ , \quad \tilde{x} \in \tilde{\Xi} \ , \tag{B7.3}
$$

or, in the explicit form, to the relations

$$
\begin{aligned}
&\tilde{H}_{R,\alpha}^{a\alpha} + \tilde{h}_R^a + f_R^a + r_R^a = i_R^a \quad , \\
&\tilde{H}_R^{a\alpha} n_{R\alpha} = p_R^a + s_R^a \quad , \\
&(r_R^{(n)}, s_R^{(n)}) \equiv \overset{\circ}{y} \in \overset{\circ}{Y}_x \ , \quad \tilde{x} \equiv (q_{(n)}, \tau^{(N)}) \in \tilde{\Xi} \ .
\end{aligned}
\tag{B7.4}
$$

Eqs. (B7.4) together with (A2.24) represent the new shell theory obtained from the shell theory described by Eqs. (B7.1), (A2.14), by the formal approximation approach (2).

<u>Example 1</u>. Let V be the set of the fields $v_{(n)} = \{v_a, a = 1,\ldots,n\}$, where v_a are the sufficiently regular real valued functions defined on $\bar{\Pi}$. Let $V_{\tilde{x}}$ be for every $\tilde{x}$, $\tilde{x} \in \tilde{\Xi}$, the known non-empty subset of V. Moreover let $\langle v, y\rangle_t$ be the rate of work of the forces $\overset{\circ}{y} = (r_R^{(n)}, s_R^{(n)})$ on the field v. Then the restriction of the error vields can be assumed in the form

$$
\overset{\circ}{Y}_{\tilde{x}} := \{\overset{\circ}{y} \mid \langle v, \overset{\circ}{y}\rangle_t = 0 \quad \text{for every } v \text{ with } V_{\tilde{x}}\}, \ \tilde{x} \in \tilde{\Xi} \ .
$$

The relation (B7.3) is given now by

(1) At the same time the given conditions in Sec. 2.0 of the Chapter A has to be satisfied

(2) We tacitly assume that there should exist the physically reasonable solutions of the correctly stated problems for Eqs. (B7.4), (A2.24).

$$<v,A(\tilde{x})>_t = 0 \quad \text{for every} \quad v \text{ with } V_{\tilde{x}}, \quad \tilde{x} \in \tilde{\Xi} . \qquad \text{(B7.5)}$$

Example 2. Let Λ be the known set of the sufficiently regular vector valued functions $\lambda = (\lambda^1,\dots,\lambda^M)$ defined on $\bar{\Pi} \times I$ and $L_{\tilde{x}}$ be, for every $\tilde{x} \in \tilde{\Xi}$, the known linear operator acting from $\Lambda = D(L_{\tilde{x}})$ to the space Y. The restriction of the error fields can be assumed in the form

$$\overset{\circ}{Y}_{\tilde{x}} := \{\overset{\circ}{y} | \overset{\circ}{y} = L_{\tilde{x}}[\underset{\sim}{\lambda}] \text{ for some } \underset{\sim}{\lambda} \in \Lambda\}, \ \tilde{x} \in \tilde{\Xi} .$$

The relation (B7.3) is now given by

$$A(\tilde{x}) - L_{\tilde{x}}(\underset{\sim}{\lambda}) = \theta, \quad \tilde{x} \in \tilde{\Xi} . \qquad \text{(B7.6)}$$

Remark. The modification by the formal approximation outlined above can be also applied to the field equations of solid mechanics; then $x = (\underset{\sim}{p},\underset{\sim}{T})$, $\overset{\circ}{y} = (\underset{\sim}{r}_R,\underset{\sim}{s}_R)$ and A is given by the LHS of Eqs. (A2.1).

7.2. Modification by semiconstraints

Let us impose on the equation $A(x) = \theta$ given by Eqs. (B7.1) the semiconstraints, introducing the set $\tilde{\Xi}$, $\tilde{\Xi} \subset D(A)$, and the multifunction $m : \tilde{\Xi} \ni x \to \overset{\circ}{Y}_x \subset \overset{\circ}{Y}$, $\overset{\circ}{Y}_x \neq \phi$. Then we shall arrive at the relation

$$A(x) \in \overset{\circ}{Y}_x , \ x \in \tilde{\Xi} , \qquad \text{(B7.7)}$$

which, in the explicit form, coincides with the relation (B7.4) provided that $\tilde{x}$ is now replaced by x (the field x in Eq. (B7.7) is not treated now as the approximation). The elements $\overset{\circ}{y}$ of $\overset{\circ}{Y}$ are called the reaction fields. The multifunctions m in the special cases have the forms analogous to those givenin Examples 1,2 of the Sec. 7.1.. We see that the formal structure of the relation obtaining as the result of the modification by the approximation is the same as that obtained by the semiconstraints. However, the interpretations of both approaches are different, because not all "semiconstraint approaches" can be interpreted as certain kinds of the formal approximations (cf. Sec. 4.2. of the Chapter A).

7.3. Passage to the second order shell force systems

The shell equations of motion $(A2.12)_1$ involve the simple shell internal force system $\{(H^{a\alpha}_R),\ h^a_R,\ a = 1,\dots,n\}$,i.e., they are the first order differential equations with respect to θ^α (cf. Sec. 2.2. of the Chapter A). Now we are to show that by the suitable modification of Eqs. (A2.12) we can arrive at the shell field equations with the second order shell internal force system,i.e., at the differential equations of motion of the second order with respect to θ^α. To this aid we shall denote by $\bar{Q}$ the set of the ordered sets $q_{(\bar{n})} = \{q_{\bar{a}},\ \bar{a} = 1,\dots,\bar{n}\}$ of the sufficiently regular real-valued functions $q_{\bar{a}}$ defined on $\bar{\Pi} \times I$, such that the function $q_{(n)}$, given by $q_a = \varphi_a(\underset{\sim}{\theta}, q_{(\bar{n})}, \nabla q_{(\bar{n})})$, $a = 1,\dots,n$, $\bar{n} < n$, satisfies the condition $q_{(n)} \in Q$, $\varphi_{(n)}$ being the known ordered set of the independent differentiable functions defined on $\Pi \times R^{3\bar{n}}$. Moreover let $\Phi^{\bar{a}}_a$, $\Psi^{\bar{a}\alpha}_a$ be the known functions which can be assumed in the form [1]

$$\Phi^{\bar{a}}_a = \frac{\partial\varphi_a}{\partial q_{\bar{a}}}\ ,\quad \Psi^{\bar{a}\alpha}_a = \frac{\partial\varphi_a}{\partial q_{\bar{a},\alpha}}\ .$$

Let the sets $\tilde{\Xi}$, $\overset{\circ}{Y}_x$ (or $\overset{\circ}{Y}_{\tilde{x}}$) in Secs. 7.1., 7.2., be given by

$$\begin{aligned}\tilde{\Xi} := \{x \equiv (q_{(n)}, \tau^{(N)}) \mid q_a = \varphi_a(\underset{\sim}{\theta}, q_{(\bar{n})}, \nabla q_{(\bar{n})}),\ a = 1,\dots,n \text{ for}\\ \text{some } q_{(\bar{n})} \in \bar{Q}\}\end{aligned} \tag{B7.8}$$

$$\overset{\circ}{Y}_x := \{\overset{\circ}{y} \equiv (r^{(n)}_R, s^{(n)}_R) \mid \int_{\partial\Pi} s^a_R h_a dl_R + \int_{\Pi} r^a_R h_a da_R = 0 \text{ for every}$$

$$h_a = \Phi^{\bar{a}}_a h_{\bar{a}} + \Psi^{\bar{a}\alpha}_a h_{\bar{a},\alpha} \text{ and every } h_{(\bar{n})} \in H\}$$

where H is the set of all ordered sets $(h_1,\dots,h_{\bar{n}})$ of regular real valued functions defined on $\bar{\Pi}$ (continuous in $\bar{\Pi}$ and differentiable in Π). Now substituting to the integrands in Eqs. $(B7.8)_2$ in the place of s^a_R, r^a_R the LHS of Eqs. $(B7.2)_{1,2}$, respectively, after simple calculations we obtian

$$H^{\bar{a}\alpha\beta}_{R,\alpha\beta} + H^{\bar{a}\alpha}_{R,\alpha} + h^{\bar{a}}_R + f^{\bar{a}}_R - f^{\bar{a}\alpha}_{R,\alpha} = i^{\bar{a}}_R - i^{\bar{a}\alpha}_{R,\alpha}\ ,$$

---

[1] The indicies $a, \bar{a}$ run over $1,\dots,n$ and $\bar{1},\dots,\bar{n}$, respectively, where $n > \bar{n}$.

$$h_R^{\bar a} \equiv h_R^a \Phi_a^{\bar a} - H_R^{a\beta} \Phi_{a,\beta}^{\bar a} ,$$

$$f_R^{\bar a} \equiv f_R^a \Phi_a^{\bar a} \quad , \quad f_R^{\bar a \alpha} \equiv f_R^a \Psi_a^{\bar a \alpha} ,$$

$$p_R^{\bar a} \equiv p_R^a \Phi_a^{\bar a} \quad , \quad p_R^{\bar a \alpha} \equiv p_R^a \Psi_a^{\bar a \alpha} \quad , \quad \text{etc.}$$

It means that by the modification of Eqs. (B7.1) we obtain Eqs. (B7.9) in which we deal with the second order shell force system. Eqs. (B7.9) can be treated either as the approximation of Eqs. (B7.1) (in this case $\varphi_{(n)}$ is the approximation of $q_{(n)}$, $q_{(n)} \sim \varphi_{(n)}$) or as the result of the semiconstraints.

7.4. Application of the constraint functions

In the foregoing Sections the real valued functions q_a , $a = 1,\dots,n$, have been treated as independent. Now let us assume that they are interrelated by the equations $h_\mu(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}) = 0$, $\mu = 1,\dots,M$, where h_μ are the known independent differentiable functions. Let $\tilde{Q}$, $\tilde{Q} \subset Q$, be the set of all regular solutions of $h_\mu = 0$, $\mu = 1,\dots,M$; we assume that $\tilde{Q} \neq \phi$. Moreover, let Λ be the set of all sufficiently regular vector valued functions $\underset{\sim}{\lambda}_R = (\lambda_R^1,\dots,\lambda_R^M)$ defined on $\bar{\Pi} \times I$ and M set of regular functions $\underset{\sim}{\mu}_R = (\mu_R^1,\dots,\mu_R^M)$ defined on $\partial\bar{\Pi} \times I$, a.e.. On Eqs. (B7.1) we now impsoe the semiconstraints, putting

$$\tilde{\Xi} := \{x \equiv (q_{(n)}, \tau^{(N)}) | q_{(n)} \in \tilde{Q}\}$$

$$\overset{o}{Y}_x := \{\overset{o}{y} \equiv (r_R^{(n)}, s_R^{(n)}) | r_R^a = -\lambda_R^\mu \frac{\partial h_\mu}{\partial q_a} + (\lambda_R^\mu \frac{\partial h_\mu}{\partial q_{a,\alpha}})_{,\alpha} \quad ,$$

$$s_R^a = -\lambda_R^\mu \frac{\partial h_\mu}{\partial q_{a,\alpha}} n_{R\alpha} + \mu_R^\mu \frac{\partial \bar{h}_\mu}{\partial q_a} - \frac{d}{dl_R} (\mu_R^\mu \frac{\partial \bar{h}_\mu}{\partial (dq_a/dl_R)}) ,$$

$$\text{for some } \underset{\sim}{\lambda}_R \in \Lambda \text{ and some } \underset{\sim}{\mu}_R \in M\}, \quad x \in \tilde{\Xi} \quad , \tag{B7.10}$$

where

$$\bar{h}_\mu \equiv 0 \text{ if } \left. \frac{\partial h_\mu(\underset{\sim}{\theta},\cdot)}{\partial q_{a,\underset{\sim}{N}}} \right|_{\underset{\sim}{\theta} \in \partial\Pi} \neq 0 \quad , \quad q_{a,\underset{\sim}{N}} = q_{a,\alpha} n_R^\alpha \quad ,$$

$$\bar{h}_\mu \equiv h_\mu(\underset{\sim}{\theta}\cdot)|_{\underset{\sim}{\theta} \in \partial\Pi} \quad \text{if} \quad \frac{\partial h_\mu(\underset{\sim}{\theta},\cdot)}{\partial q_{a,\underset{\sim}{N}}} \equiv 0 \quad . \tag{B7.11}$$

It means that $\bar{h}_\mu$ are either the boundary values of the functions h_μ (when h_μ do not depend on $q_{a,\underset{\sim}{N}}$) or they are identically equal to zero. The vector functions $\underset{\sim}{\lambda}_R, \underset{\sim}{\mu}_R$ will be called the shell constraint functions.

Subsituting to Eqs. (B7.4) in the places of r^a_R, s^a_R the expressions introduced by Eq. $(B7.10)_2$ and taking also into account $h_\mu(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}) = 0$, $\mu = 1,\ldots,M$, we arrive at the system of equations:

$$
\begin{aligned}
&(\tilde{H}^{a\alpha}_R + \lambda^\mu_R \frac{\partial h_\mu}{\partial q_{a,\alpha}})_{,\alpha} + \tilde{h}^a_R - \lambda^\mu_R \frac{\partial h_\mu}{\partial q_a} + f^a_R = i^a_R \quad , \\
&h_\mu(\underset{\sim}{\theta}, q_{(n)}, \nabla q_{(n)}) = 0 \quad , \quad \mu = 1,\ldots,M \ ; \quad \underset{\sim}{\theta} \in \Pi, \\
&(\tilde{H}^{a\alpha}_R + \lambda^\mu_R \frac{\partial h_\mu}{\partial q_{a,\alpha}}) n_{R\alpha} = \mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial q_a} - \frac{d}{dl_R}(\mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial (dl_R/\partial q_a)}) + p^a_R \ , \\
&\bar{h}_\mu(\underset{\sim}{\theta}, q_{(n)}, q_{(n),\alpha}) = 0 \ ; \ \underset{\sim}{\theta} \in \partial\Pi.
\end{aligned}
\qquad (B7.12)
$$

Eqs. (B7.12) have been obtained by imposing semiconstraints on Eqs. (B7.1).

8. MODIFICATION OF THE COSSERAT-SURFACE SHELL THEORY

Now we are to show some applications of the general procedure outlined in Sec. 7 to the Cosserat surface shell theory.

8.1. General case

Substituting the RHS of Eqs. (B6.6) into Eqs. (B6.2), we obtain a special case of Eqs. (B7.1). Then using the approach developed in Sec. 7.4. we shall derive the special case of Eqs. (B7.12), related to the Cosserat-surface shell theory. Passing to the Lagrangean or Eulerian formulations (cf. Sec. 6.2. of this Chapter) we obtain the equations

$$\begin{aligned}
&h_\mu(\underset{\sim}{\theta},\underset{\sim}{r},\underset{\sim}{d},\nabla\underset{\sim}{r},\nabla\underset{\sim}{d}) = 0, \quad \mu = 1,\dots,M \quad , \\
&\tilde{H}^{\gamma\alpha}\|_\alpha - b^\gamma_\alpha \tilde{H}^{3\alpha} + f^\gamma + \overset{\circ}{f}{}^\gamma = i^\gamma \\
&\tilde{H}^{3\alpha}\|_\alpha + b_{\alpha\beta}\tilde{H}^{\alpha\beta} + f^3 + \overset{\circ}{f}{}^3 = i^3 \quad , \qquad\qquad (B8.1) \\
&\tilde{M}^{\gamma\alpha}\|_\alpha - b^\gamma_\alpha \tilde{M}^{3\alpha} - \tilde{m}^\gamma + l^\gamma + \overset{\circ}{l}{}^\gamma = j^\gamma \quad , \\
&\tilde{M}^{3\alpha}\|_\alpha + b_{\alpha\beta}\tilde{M}^{\alpha\beta} - \tilde{m}^3 + l^3 + \overset{\circ}{l}{}^3 = j^3 \quad ,
\end{aligned}$$

where

$$\begin{aligned}
&\sqrt{a}\,\overset{\circ}{f}{}^\gamma = [-\lambda^\mu_R \frac{\partial h_\mu}{\partial r_k} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial r_{k,\alpha}})_{,\alpha}] r_{k,\beta} a^{\beta\gamma} \quad , \\
&\sqrt{a}\,\overset{\circ}{f}{}^3 = [-\lambda^\mu_R \frac{\partial h_\mu}{\partial r_k} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial r_{k,\alpha}})_{,\alpha}] N_k \quad , \qquad\qquad (B8.2) \\
&\sqrt{a}\,\overset{\circ}{l}{}^\gamma = [-\lambda^\mu_R \frac{\partial h_\mu}{\partial d_k} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{k,\alpha}})_{,\alpha}] r_{k,\beta} a^{\beta\gamma} \quad , \\
&\sqrt{a}\,\overset{\circ}{l}{}^3 = [-\lambda^\mu_R \frac{\partial h_\mu}{\partial d_k} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{k,\alpha}})_{,\alpha}] N_k \quad ,
\end{aligned}$$

which here to be satisfied in $\Pi \times I$. Analogously, we obtain the boundary conditions

$$\tilde{H}^{\gamma\alpha} n_\alpha = e^\gamma + \overset{o}{e}{}^\gamma , \qquad \tilde{M}^{\gamma\alpha} n_\alpha = s^\gamma + \overset{o}{s}{}^\gamma ,$$
$$\tilde{H}^{3\alpha} n_\alpha = e^3 + \overset{o}{e}{}^3 , \qquad \tilde{M}^{3\alpha} n_\alpha = s^3 + \overset{o}{s}{}^3 , \tag{B8.3}$$

where

$$\sqrt{a}\, \overset{o}{e}{}^\gamma = [\lambda^\mu_R \frac{\partial h_\mu}{\partial r_{k,\alpha}} n_{R\alpha} + \mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial r_k} - \frac{d}{dl_R} (\mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial (dr_k/dl_R)})]\, r_{k,\beta} a^{\beta\gamma} ,$$

$$\sqrt{\lambda}\, \overset{o}{e}{}^3 = [\lambda^\mu_R \frac{\partial h_\mu}{\partial r_{k,\alpha}} n_{R\alpha} + \mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial r_k} - \frac{d}{dl_R} (\mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial (dr_k/dl_R)})]\, N_k ,$$

$$\sqrt{\lambda}\, \overset{o}{s} = [\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{k,\alpha}} n_{R\alpha} + \mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial d_k} - \frac{d}{dl_R} (\mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial (dd_k/dl_R)})]\, r_{k,\alpha} a^{\beta\gamma} , \tag{B8.4}$$

$$\sqrt{\lambda}\, \overset{o}{s}{}^3 = [\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{k,\alpha}} n_{R\alpha} + \mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial d_k} - \frac{d}{dl_R} (\mu^\mu_R \frac{\partial \bar{h}_\mu}{\partial (dd_k/dl_R)})] N_k ,$$

which have to be satisfied almost everywhere on $\partial\Pi \times I$. Eqs. (B8.1) - (B8.4) represent, together with Eqs. (A2.14), the modifications of the Cosserat-surface shell theory, due to the presence of the constraint relation $(B8.1)_1$.

If we take into account the alternative form of the governing relations of the Cosserat-surface shell theory (cf. Sec. 6.3. of this Chapter), then Eqs. $(B8.1)_{4,5}$ have to be replaced by

$$\tilde{G}^{\gamma\alpha}\|_\alpha - b^\gamma_\alpha \tilde{G}^{3\alpha} + a^{\gamma\beta}\epsilon_{\beta\alpha}\tilde{H}^{3\alpha} + k^\gamma + \overset{o}{k}{}^\gamma = e^\gamma ,$$
$$\tilde{G}^{3\alpha}\|_\alpha + b_{\alpha\beta}\tilde{G}^{\alpha\beta} + \epsilon_{\alpha\beta}\tilde{H}^{\alpha\beta} + k^3 + \overset{o}{k}{}^3 = e^3 , \tag{B8.5}$$

where

$$\sqrt{a}\, \overset{o}{k}{}^\gamma \equiv \varepsilon^k_{\cdot lm} d^l [-\lambda^\mu_R \frac{\partial h_\mu}{\partial d_m} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{m,\alpha}})_{,\alpha}]\, r_{k,\beta} a^{\beta\gamma}$$
$$\sqrt{a}\, \overset{o}{k}{}^3 \equiv \varepsilon^k_{\cdot lm} d^l [-\lambda^\mu_R \frac{\partial h_\mu}{\partial d_m} + (\lambda^\mu_R \frac{\partial h_\mu}{\partial d_{m,\alpha}})_{,\alpha}]\, N_k \tag{B8.6}$$

and Eqs. $(B8.3)_{2,4}$ by

$$\tilde{G}^{\gamma\alpha} n_\alpha = c^\gamma + \overset{o}{c}{}^\gamma , \quad \tilde{G}^{3\alpha} n_\alpha = c^3 + \overset{o}{c}{}^3 , \tag{B8.7}$$

where

$$\sqrt{\lambda}\,\overset{o}{c}{}^{\gamma} \equiv \varepsilon^{k}_{\cdot lm} d^{l}[\lambda^{\mu}_{R}\frac{\partial h_{\mu}}{\partial d_{m,\alpha}} n_{R\alpha} + \mu^{\mu}_{R}\frac{\partial \bar{h}_{\mu}}{\partial d_{m}} - \frac{d}{dl_{R}}(\mu^{\mu}_{R}\frac{\partial \bar{h}_{\mu}}{\partial (dd_{m}/dl_{R})})]\, r_{k,\beta} a^{\beta\gamma},$$

$$\sqrt{\lambda}\,\overset{o}{c}{}^{3} \equiv \varepsilon^{k}_{\cdot lm} d^{l}[\lambda^{\mu}_{R}\frac{\partial h_{\mu}}{\partial d_{m,\alpha}} n_{R\alpha} + \mu^{\mu}_{R}\frac{\partial \bar{h}_{\mu}}{\partial d_{m}} - \frac{d}{dl_{R}}(\mu^{\mu}_{R}\frac{\partial \bar{h}_{\mu}}{\partial (dd_{m}/dl_{R})})]\, N_{k}. \qquad (B8.8)$$

Postulating different special forms of Eqs. $(B8.1)_1$ we obtain different modifications of the Cosserat-surface shell theory. The two special cases of such modifications will be presented in the next subsections.

8.2. Love-Kirchhoff shell theory

It is the well known theory which can be obtained from Eqs. (B8.1)-(B8.4) by assuming Eqs. $(B8.1)_1$ in the form $d^k r_{k,\alpha} = 0$, $d^k d_k - 1 = 0$. They are the Love-Kirchhoff constraints, which can be also given by $d_k = N_k, \underset{\sim}{N} = \underset{\sim}{N}(\underset{\sim}{\theta},t)$, being the field of the unit vectors normal to the surface π_t for every $t \in I$. In Eqs. $(B8.1)_2$ there is now $M = 3$ and we shall deal with the three constraint real-valued functions $\lambda^{\alpha}_{R}, \lambda^{3}_{R}$. Using Eqs. $(B8.2)_{1,2}$ and $(B8.4)_{1,2}$ we obtain

$$\begin{aligned}
\overset{o}{f}{}^{\gamma} &= \frac{1}{\sqrt{a}}(\lambda^{\alpha}_{R} d_k)_{,\alpha} r^{k}_{,\beta} a^{\beta\gamma} = -b^{\gamma}_{\alpha}\lambda^{\alpha}, \qquad \lambda^{\alpha} \equiv \lambda^{\alpha}_{R}/\sqrt{a}\\
\overset{o}{f}{}^{3} &= \frac{1}{\sqrt{a}}(\lambda^{\alpha}_{R} d_k)_{,\alpha} N^{k} = \lambda^{\alpha}\|_{\alpha},\\
\overset{o}{k}{}^{\gamma} &= \frac{1}{\sqrt{a}}\,\varepsilon_{k}^{\cdot lm}(\lambda^{3}_{R} d_m + \lambda^{\alpha}_{R} r_{m,\alpha}) d_l r^{k}_{,\beta} a^{\beta\gamma} = a^{\gamma\beta}\epsilon_{\beta\alpha}\lambda^{\alpha},\\
\overset{o}{k}{}^{3} &= \frac{1}{\sqrt{a}}\,\varepsilon_{k}^{\cdot lm}(\lambda^{3}_{R} d_m + \lambda^{\alpha}_{R} r_{m,\alpha}) d_l N^{k} = 0 .
\end{aligned} \qquad (B8.9)$$

In view of the results of Sec. 6.3. we can observe that if $d_k = \alpha N_k$ (for an arbitrary $\alpha \neq 0$) then $G^{3\alpha} = 0$, $k^3 = 0$, $a^3 = 0$, $c^3 = 0$ (cf. Remark to SEc. 6.3.). Denoting $Q^{\alpha} \equiv H^{3\alpha} + \lambda^{\alpha}$, we derive from Eqs. (B8.1), (B8.5), (B8.9) the following form of the equation of motion of the Love-Kirchhoff theory

$$
\begin{aligned}
&H^{\gamma\alpha}\|_{\alpha} - b^{\gamma}_{\alpha}Q^{\alpha} + f^{\gamma} = i^{\gamma} \quad , \\
&Q^{\alpha}\|_{\alpha} + b_{\alpha\beta}H^{\alpha\beta} + f^{3} = i^{3} \quad , \\
&G^{\gamma\alpha}\|_{\alpha} - a^{\gamma\beta}\epsilon_{\beta\alpha}Q^{\alpha} + k^{\gamma} = a^{\gamma} \, , \\
&b_{\alpha\beta}G^{\alpha\beta} + \epsilon_{\alpha\beta}H^{\alpha\beta} = 0 \quad .
\end{aligned}
\tag{B8.10}
$$

At the same time from Eqs. (B6.16), (B6.17) we obtain the constitutive relations

$$
\begin{aligned}
&H^{\alpha\gamma} = \tilde{H}^{\alpha\gamma} \equiv \frac{1}{\sqrt{a}} \int_{-h}^{h} (\delta^{\alpha}_{\beta} - b^{\alpha}_{\beta}\xi) S^{\beta\gamma}(\tilde{\underset{\sim}{\sigma}})\, d\,\xi \quad , \\
&G^{\alpha\gamma} = \tilde{G}^{\alpha\gamma} \equiv \frac{1}{\sqrt{a}} \int_{-h}^{h} (\delta^{\delta}_{\varepsilon} - b^{\delta}_{\varepsilon}\xi)\xi S^{\varepsilon\gamma}(\tilde{\underset{\sim}{\sigma}})\, d\xi\, a^{\alpha\beta}\epsilon_{\delta\beta} \quad .
\end{aligned}
\tag{B8.11}
$$

Substituting the RHS of Eqs. (B8.11) into $(B8.10)_4$ we arrive at the identity. Eqs. (B8.10) are the well known equations of motion of the Love-Kirchhoff shell theory which have been obtained here by the modification of the Cosserat-surface shell theory.

Passing to the kinetic boundary conditions of the Love-Kirchhoff shell theory we obtain from Eqs. $(B8.4)_{1,2}$ and (B8.8)

$$
\begin{aligned}
&\overset{o}{e}{}^{\gamma} = \mu b^{\gamma}_{\delta} t^{\delta} \; , \quad \overset{o}{e}{}^{3} = \lambda^{\alpha} n_{\alpha} - \frac{d\mu}{dl} \quad , \\
&\overset{o}{c}{}^{\gamma} = \mu\epsilon_{\delta\beta} t^{\delta} a^{\beta\gamma} \quad , \quad \overset{o}{c}{}^{3} = \bar{\mu} \quad .
\end{aligned}
\tag{B8.12}
$$

where $\mu, \bar{\mu}$ are the constraint function on the boundary (mind, that $\bar{h}_{\mu} = 0$ is given by $d^{k}(r_{k,\alpha} t^{\alpha}_{R}) = 0$, $d^{k}d_{k} - 1 = 0$. Thus the kinetic boundary conditions will be written down in the form

$$
\begin{aligned}
&\tilde{H}^{\gamma\alpha} n_{\alpha} = e^{\gamma} + \mu b^{\gamma}_{\delta} t^{\delta} \; , \\
&Q^{\alpha} n_{\alpha} = e^{3} - \frac{d\mu}{dl} \quad , \\
&\tilde{G}^{\gamma\alpha} n_{\alpha} = c^{\gamma} - \mu\,\epsilon_{\alpha\beta} t^{\delta} a^{\beta\gamma} \; , \\
&\bar{\mu} = 0 \quad .
\end{aligned}
\tag{B8.13}
$$

Let us also observe that the constraint functions Q^{α},μ can be easily eliminated from Eqs. (B8.10) and (B8.13).

Remark. Using Eq. $(B.8.1)_5$ instead of Eqs. $(B8.5)_2$, in view of $\overset{o}{l}{}^3 = -\lambda^3$, we shall obtain in the place of the identity $(B8.10)_4$ (provided that Eqs. (B8.11) hold) the equation for the constraint function λ^3. Then from Eqs. $(B8.1)_{2-5}$ it follows that

$$H^{\gamma\alpha}|_{\alpha} + f^{\gamma} - b^{\gamma}_{\alpha}\lambda^{\alpha} = i^{\gamma} ,$$

$$H^{3\alpha}|_{\alpha} + f^{3} + \lambda^{\gamma}||_{\gamma} = i^{3} ,$$

$$M^{K\alpha}|_{\alpha} + m^{K} + l^{K} - \lambda^{K} = j^{K} ,$$

which is another form of the equations of motion in the Love-Kirchhoff shell theory.

8.3. Generalized Love-Kirchhoff shell theory

The Love-Kirchhoff shell theory, even for very thin shells, cannot be applied successfully to the problems of large deformations. From the restriction $d^k d_k = 1$ it follows that the thickness of the shell (measured in the direction of vector $\underset{\sim}{d}$) is constant during the deformation. This condition is not satisfied (even as the approximation) when the large deformations are concerned. Putting asside the condition $d^k d_k = 1$, let us assume Eqs. $(B8.1)_1$ in the form $d^k r_{k,\alpha} = 0$. Now M = 2 and we shall deal with the two sreal-valued constraint functions λ^{α}_{R}. Using the formulas of Sec. 8.1. we obtain again Eqs. $(B8.10)_{1-3}$. The remaining equation we obtain from Eq. (B8.1), in which now $\overset{o}{l}{}^3 \equiv 0$:

$$M^{3\alpha}||_{\alpha} + b_{\alpha\beta}M^{\alpha\beta} - m^3 + l^3 = j^3 \quad . \tag{B8.14}$$

At the same time Eqs. $(B6.11)_{3,4,5}$ yield

$$M^{\alpha\beta} = \tilde{M}^{\alpha\beta} \equiv \frac{1}{\sqrt{a}}\int_{-h}^{h}(\delta^{\alpha}_{\gamma} - db^{\alpha}_{\gamma}\xi)\,\xi\, S^{\gamma\beta}(\underset{\sim}{\tilde{\sigma}})\, d\xi ,$$

$$M^{3\alpha} = \tilde{M}^{3\alpha} \equiv \frac{1}{\sqrt{a}}\int_{-h}^{h}(d_{,\gamma}S^{\gamma\alpha}(\underset{\sim}{\tilde{\sigma}}) + dS^{3\alpha}(\underset{\sim}{\tilde{\sigma}}))\,\xi\, d\xi , \tag{B8.15}$$

$$m^3 = \tilde{m}^3 \equiv \frac{1}{\sqrt{a}} \int_{-h}^{h} (ds^{33}(\tilde{\underset{\sim}{\sigma}}) + d_{,\alpha} s^{\alpha 3}(\tilde{\underset{\sim}{\sigma}}))\, d\xi \quad ,$$

where $d = d(\underset{\sim}{\theta}, t)$ is the function defined by $d \equiv d_k N^k$. For the simple materials Eqs. $(B8.10)_{1-3}$, (B8.14), after substituting RHS of Eqs. (B8.11), (8.15), constitute the system of six equations for six basic unknown r_k, d, Q^α. Passing to the kinetic boundary conditions we obtain from Eqs. (B8.4)

$$\begin{aligned} \overset{o}{e}{}^\gamma &= \mu b^\gamma_\delta t^\delta \;, \qquad \overset{o}{e}{}^3 = -\frac{d\mu}{dl} + \lambda^\alpha n_\alpha \;, \\ \overset{o}{s}{}^\gamma &= \mu t^\gamma \;, \qquad \overset{o}{s}{}^3 = 0 \;, \end{aligned} \qquad \text{(B8.16)}$$

and then from Eqs. (B8.3)

$$\begin{aligned} \tilde{H}^{\gamma\alpha} n_\alpha &= e^\gamma + \mu b^\gamma_\delta t^\delta \;, \qquad Q^\alpha n_\alpha = e^3 - \frac{d\mu}{dl} \;, \\ \tilde{M}^{\gamma\alpha} n_\alpha &= s^\gamma + \mu t^\gamma \;, \qquad \tilde{M}^{3\alpha} n_\alpha = s^3 \;. \end{aligned} \qquad \text{(B8.17)}$$

Remark. Instead of Eqs. $(B8.5)_1$ we can use Eqs. $(B8.1)_4$ in which $\overset{o}{l}{}^\gamma = -\lambda^\gamma$. Then we have also to take into account the interrelation $Q^\gamma - \lambda^\gamma = H^{3\alpha}$ (cf. Sec. 8.2.) which allows to eliminate either Q^α or λ^α from the shell field equations.

9. SIMPLIFIED SHELL THEORIES

To describe some special classes of problems within the plate or shell theory we often try to make this theory more tractable by introducing certain postulated *a priori* simplifications into the shell governing relations. As the known examples of such simplification we can mention the procedures leading to the small deformation or linear theories, to the theories of shallow shells, to the membrane theories, [13] , and to the different variants of the small deformation theories in which we restrict the values of the strains, [5,24], deflections, [18], deformation gradients, [7], or rotations, [28,29]. Other examples can be given by simplified descriptions of certain non-homogeneous shells with the aid of the theory of homogeneous shells and by simplified descriptions of some "perforated" shells by the theory of simply connected shells. The plate and shell theories obtained by such simplications will be called the *simplified theories*. The objective of this section is to show that the simplified plate and shell theories can be interpreted as obtained either by imposing the special form of semiconstraints on a certain plate or shell theory or by applying special kind of formal approximation. Such semiconstraints or approximations will be referred to as the *effective semiconstraints* or the *effective formal approximations*, respectively.

9.1. Effective semiconstraints and the shell theories

We shall start with some pure analytical concepts. Let X be a certain topological functional space, Y be the linear space and $A, \bar{A}$ be the mappings form X to Y such that $D(\bar{A}) \subset D(A)$. Let us assume that the class of problems under consideration is governed by the binary relation $A(x) = y$, $y \in \Delta$; with the domain Ξ. Let us also define the sets

$$\tilde{\Xi} := \{x \,|\, \bar{A}(x) = y \quad \text{for some } y \in \bar{\Delta}\};$$
$$\overset{\circ}{Y} := \{\overset{\circ}{y} \,|\, \overset{\circ}{y} = A(x) - \bar{A}(x) \text{ for some } x \in \tilde{\Xi}\}\ , \qquad \text{(B9.1)}$$

where $\bar{\Delta} \equiv R(\bar{A}) \cap \Delta \neq \phi$.

We shall say that the *semiconstraints* imposed on the relation $A(x) = y$, $y \in \Delta$ *are effective* if there are known the operator $\bar{A}$ and the multifunction

$$D(\bar{A}) \supset \tilde{\Xi} \ni x \to \overset{o}{Y}_x \subset \overset{o}{Y} \tag{B9.2}$$

such that for every $y \in \Delta$ there exists the pair $(x, \overset{o}{y}) \in \tilde{\Xi} \times \overset{o}{Y}_x$ satisfying the condition $A(x) - y = \overset{o}{y}$.

Thus the effective semiconstraints imposed on the binary relation $A(x) = y$, $y \in \Delta$, lead to the binary relation $\bar{A}(x) = y$, $y \in \bar{\Delta}$ and to the conditions $A(x) - \bar{A}(x) \in \overset{o}{Y}_x$, $x \in \tilde{\Xi}$.

Remark. All foregoing statements can be also expressed in terms of the formal approximation procedure (cf. Sec. 2.0. of the Chapter A). Thus the fields $\overset{o}{y}$, $\overset{o}{y} \in \overset{o}{Y}$, can be called the error fields of the approximation $x \sim \tilde{x}$, where $x \sim \tilde{x}$ iff $(x, \tilde{x}) \in \Xi \times \tilde{\Xi}$ and $\overset{o}{y} = A(\tilde{x}) - A(x)$ (since $\bar{A}(\tilde{x}) = y = A(x)$). The conditions $\overset{o}{y} \in \overset{o}{Y}_{\tilde{x}} \subset \overset{o}{Y}$ can be treated as the restrictions of the error fields and the approach can be called the *effective formal approximation approach.* Thus the effective formal approximation of a certain binary relation always leads to another binary relation and to the independent condition which describes the range of its applicability.

Up to now the whole procedure has been quite formal. In the applications we look for such operator $\bar{A}$ that the relation $\bar{A}(x) = y$, $y \in \bar{\Delta}$, can be treated as a certain effective simplification of the relation $A(x) = y$, $y \in \Delta$. At the same time we demand that the "error fields" $\overset{o}{y}$ due to this simplification should be, roughly speaking, "sufficiently small". Thus the obtained relations $\bar{A}(x) = y$, $y \in \bar{\Delta}$, will constitute the "effective" tool in the further analysis only if the field $x, x \in \tilde{\Xi}$, satisfies the condition $A(x) - \bar{A}(x) \in \overset{o}{Y}_x$.

Now assume that the relation "$A(x) = y$, $y \in \Delta$" coincides with a certain relation of the shell theory. The formal approach outlined above can be successfully applied to obtain the simplified relations "$\bar{A}(x) = y$, $y \in \bar{\Delta}$" in the cases in which we deal with the phenomena or quantities of the "different order" of magnitude. In such situations we introduce the operator $\bar{A}$ putting $A(x) = \bar{A}(x) + \overset{o}{y}$, cf. Eq. $(B9.1)_2$, where $\bar{A}$ characterizes the phenomena or quantities of the "first order" and $\overset{o}{y}$ are the quantities of the "higher order" which in the "effective" description of the problems, given by $\bar{A}(x) = y$, $y \in \bar{\Delta}$, cf. Eq. $(B9.1)_1$, can be neglected. Thus $\overset{o}{Y}_x$ has to be interpreted as the set of such "higher order" terms which are "sufficiently small" to be neglected in the effective description of the problem. However, from the point of view of the simplifying procedures known in the mechanics of the shell-like bodies, the sets $\overset{\circ}{Y}_x$ are often not defined but rather intuitively described by the conditions of the form $\overset{\circ}{Y}_x := \{\overset{\circ}{y} \mid \|\overset{\circ}{y}\| \ll \|\bar{A}(x)\|\}$ (where $\|\cdot\|$ is the suitable norm in the linear space Y). If to every $x, x \in \Xi$, we assign certain "small" parameter v, and if the asymptotic procedure are used then we usually assume that $\overset{\circ}{Y}_x = O(v)$, $v = v(x)$. Such procedures have been used, for example, in [5,7,12,14,16-19,27-29], where different simplified theories of the "small deflections", "moderate deflections" "small, large,moderate rotations", etc., have been introduced.

Example. Let $y = \sin x$ stands for the mapping $y = A(x)$. In the large rotation theory (cf. [29], p.91, x will be here interpreted as the value of the rotation vector of the shell element) for $A(x) \equiv \sin x$ we assume that $\bar{A}(x) = x - \frac{1}{3!}x^3$. Thus the relation $y = \sin x$ is simplified to the form $\bar{A}(x) = y$ given by $y = x - \frac{1}{3!}x^3$, cf. Eqs. $(B9.1)_1$. At the same time it is assumed that $\overset{\circ}{y} \equiv A(x) - y = \sin x - (x - \frac{1}{3!}x^3) \in Y_x = O(v^{5/2})$, cf. $(B9.1)_2$, where v is the small parameter of the shell, [20]; here sets $\overset{\circ}{Y}_x$ are independent of x. In the theory of moderate rotations, [29], we assume that if $A(x) = \sin x$ then $\bar{A}(x) = x$ and $\overset{\circ}{y} = \sin x - x \in \overset{\circ}{Y}_x = O(v^3)$.

9.2. Special cases

Let us assume that a certain "fundamental" motion of the shell is described by the ordered set of functions $\overset{\circ}{q}_{(n)}(\underset{\sim}{\theta},t) = \{\overset{\circ}{q}_a(\underset{\sim}{\theta},t), a = 1,\dots,n\}$ and let us take into account the second motion determind by $q_{(n)} = \overset{\circ}{q}_{(n)} + \varepsilon w_{(n)}$, $w_{(n)} = \{w_a(\underset{\sim}{\theta},t), a = 1,\dots,n\}$, where ε, $\varepsilon \in \langle 0,1\rangle$, is a certain scalar parameter (1). Let us also assume that the shell is elastic and that the mapping $A(x) = y$, where $y = (f_R^{(n)}, p_R^{(n)})$, $x = w_{(n)}$, stands for the system of shell governing relations (obtained by substituting RHS of Eqs. (A2.26) into Eqs. (A2.12); the ordered set $\tau^{(N)}$ drops out from the quations for the elastic shell). The operator A now depends on the parameter ε, $A = A_\varepsilon$. Putting $A_\varepsilon = \bar{A} + o(\varepsilon)$, we assign to the operator A_ε certain linear (with respect to $w_{(n)}$) operator $\bar{A}$. Putting also $y \in \Delta \subset R(A)$, $\bar{\Delta} = R(\bar{A}) \cap \Delta$, we obtain the relation $\bar{A}(w_{(n)}) = y$, $y \in \bar{\Delta}$, which describes the theory of small elastic deformations superimposed on the finite deformation $\overset{\circ}{q}_{(n)}$. If

(1) We assume here that $\tilde{p} = \tilde{p}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)})$, i.e., that the shell theory under consideration is obtained by the constraint approach.

$\overset{\circ}{q}_{(n)}$ is time independent and characterizes certain natural shell configuration, then $\bar{A}(w_{(n)}) = y$, $y \in \bar{\Delta}$, represents the linear elastic shell theory. At the same time the condition $\overset{\circ}{y} \equiv A(w_{(n)}) - y \in \overset{\circ}{Y}_{w_{(n)}}$ has to ensure that the values of the "reaction field" $\overset{\circ}{y}$ are sufficiently small. In this cas Y has to be the normed space and the multifunction (B9.1) can be assumed in the form $\overset{\circ}{Y}_{w_{(n)}} := \{\overset{\circ}{y} \mid \|\overset{\circ}{y}\| \leq \eta \|A(w_{(n)})\|\}$ where η is a certain positive number, $\eta << 1$.

If we assume that $q_{(n)} = \overset{\circ}{q}_{(n)}(\underset{\sim}{\theta}) + \varepsilon^{\kappa} w_{\kappa(n)}(\underset{\sim}{\theta})$, $\kappa = 1,2,\ldots$, then putting $A_{\varepsilon} = \bar{A}^{\kappa} + o(\varepsilon^{\kappa})$ we obtain, under the conditions introduced above, the successive approximations in the statics of the elastic shells, given by $\bar{A}^{\kappa}(w_{\kappa(n)}) = y$, $\kappa = 1,2,\ldots$.

The different simplified shell theories play an important role in the development of the foundations of shell theories. In the next subsection we shall give an example of such theory, using the concept of the effective semiconstraints.

9.3. Example: membrane shell theory

The membrane shell theory (which has been introduced via direct approach in Sec. 4.3. of this Chapter, cf. also [13]) will be derived below from the generalized Love-Kirchhoff shell theory by imposing on the relations of this theory the special form of the effective semiconstraints. To this aid we shall treat the thickness 2h of the shell in the reference configuration as the small parameter, putting

$$\int_{-h}^{h} x(\xi)\, d\xi = 2hx(0) + o(h) \ , \qquad \text{(B9.3)}$$

for an arbitrary smooth function x defined on (-h,h), $x \in X$.

To use the results of Sec. 9.1. we shall assume that

$$A(x) = \int_{-h}^{h} x\, d\xi, \quad \bar{A}(x) = 2hx(0) \equiv \int_{-h}^{h} x(0)\, d\xi, \quad \overset{\circ}{Y} = o(h) \ ,$$

for every $x \in X$. We shall also use the denotation $\bar{A} \equiv \pi(A)$; it can be easily seen that now $\pi(\bar{A}) = \bar{A}$ and $\pi(o(h)) = 0$, where o(h) is interpreted here as the integral operator defined by Eq. (B9.3). Thus π can be treated as the projection defined by

$$\pi\Big(\int_{-h}^{h} x(\xi)\, d\xi\Big) = 2hx(0), \quad x \in X . \tag{B9.4}$$

Thus from Eqs. (B6.4) we obtain

$$\begin{aligned}
&\pi(H_R^{k\alpha}) = 2h(r^k_{,\beta}\tilde{T}^{\beta\alpha} + d^k\tilde{T}^{3\alpha})\big|_{\xi=0} , \\
&\pi(M_R^{k\alpha}) = 0 , \\
&\pi(m_R^k) = 2h(r^k_{,\beta}\tilde{T}^{\beta 3} + d^k\tilde{T}^{33})\big|_{\xi=0} , \\
&\pi(f_R^k) = 2hb_R^k\big|_{\xi=0} + \overset{+}{p}{}_R^k + \overset{-}{p}{}_R^k , \\
&\pi(l_R^k) = 0 , \quad \pi(e_R^k) = 2hp_R^k\big|_{\xi=0} , \quad \pi(s_R^k) = 0 , \quad \pi(j_R^k) = 0 , \\
&\pi(i_R^k) = 2h\rho_R\big|_{\xi=0}\, \ddot{r}^k ,
\end{aligned} \tag{B9.5}$$

where we have assumed $\overset{+}{p}{}_R^k , \overset{-}{p}{}_R^k = O(h)$. Then Eqs. (B6.8), (B6.17), (B6.16) yield

$$\begin{aligned}
&\pi(i^\gamma) = \pi(j^\gamma) = \pi(G^{\alpha\beta}) = \pi(G^{3\alpha}) = 0 , \\
&\pi(H^{\alpha\beta}) = \frac{2h}{\sqrt{a}}\tilde{T}^{\alpha\beta}\big|_{\xi=0} ,
\end{aligned} \tag{B9.6}$$

because of the relations $d = d^k r_{k,\beta} a^{\beta\alpha} = 0$ which hold in the generalized Love-Kircchoff theory. Now from Eqs. $(B8.6)_{1-3}$, (B8.10), (B8.11) we obtain $Q^\alpha = o(h)$ and then

$$\begin{aligned}
&H^{\gamma\alpha}\|_\alpha + f^\gamma = i^\gamma + o(h) , \\
&b_{\alpha\beta}H^{\alpha\beta} + f^3 = i^3 + o(h) , \\
&\tilde{m}^3 = o(h)
\end{aligned} \tag{B9.7}$$

where

$$\tilde{m}^3 = \frac{2h}{\sqrt{a}}(dS^{33}(\tilde{\sigma}) + d_{,\alpha}S^{\alpha 3}(\tilde{\sigma}))\big|_{\xi=0} . \tag{B9.8}$$

where, by virtue of Eqs. 6B6.8), (B9.5) - (B9.8), we have denoted

$$H^{\gamma\alpha} \equiv \frac{2h}{\sqrt{a}} s^{\alpha\beta}(\tilde{\sigma})|_{\xi=0} \quad ,$$

$$f^{\gamma} = \frac{1}{\sqrt{a}} (2hb_R^k|_{\xi=0} + \overset{+}{p}{}_R^k + \overset{-}{p}{}_R^k) r_{k,\beta} a^{\beta\gamma} \quad ,$$

$$i^{\gamma} = \frac{1}{\sqrt{a}} 2h\rho_R|_{\xi=0} \ddot{r}^k r_{k,\beta} a^{\beta\gamma} \quad , \tag{B9.10}$$

$$f^3 \equiv \frac{1}{\sqrt{a}} (2hb_R^k|_{\xi=0} + \overset{+}{p}{}_R^k + \overset{-}{p}{}_R^k) N_k \quad ,$$

$$i^3 \equiv \frac{1}{\sqrt{a}} 2h\rho_R|_{\xi=0} \ddot{r}^k N_k \quad .$$

Applying the projection π to Eqs. (B8.13) and taking into account Eqs. (B6.6), (B9.5) we obtain

$$H^{\gamma\alpha} n_{\alpha} = e^{\gamma} \tag{B9.11}$$

where we have denoted

$$e^{\gamma} \equiv \frac{2h}{\sqrt{a}} p_R^k|_{\xi=0} r_{k,\beta} a^{\beta\gamma} \quad . \tag{B9.12}$$

In Eqs. (B9.10), (B9.12) we have used the same denotations for the projections of the fields $H^{\gamma\alpha},\ldots,i^3$, e^{γ} as for the fields $H^{\gamma\alpha},\ldots,i^3$, e^{γ}, of the generalized Love-Kircchoff theory. Eqs. (B9.9), (B9.11) with the new denotations (B9.10), (B9.12) represent the field equations of the membrane shell theory. The obtained equations have been derived by using the concept of the effective constraints; if $A(x) = y$ stands for Eqs. $(B8.6)_{1-3}$, (B8.10) and Eqs. (B8.13) then $\bar{A}(x) = y$, $y \in \bar{\Delta}$, stands for (B9.9), (B9.11). Let us also observe that Eqs. $(B8.6)_3$ $(B8.13)_{2-4}$ are interrelated with the identities of the membrane theory, i.e. the projection π of these equations leads to the identities. The membrane shell theory can be used only for the special class of problems. The condition $\overset{o}{y} = A(x) - y \in \overset{o}{Y}_x$ has to define such solutions x of the "membrane" equation $\bar{A}(x) = y$, which, roughly speaking, are the "good approximations" of the solutions of the generalized Love-Kirchhoff shell theory, described by $A(x) = y$. Thus the multifunction (B9.2) has to determine the range of application of the membrane shell theory and the set Δ is the set of such external forces in the generalized Love-Kirchhoff shell theory for which this theory can be simplified to the "membrane" form.

CHAPTER C

NON-CLASSICAL PROBLEMS FOR PLATES, SHELLS AND RODS

In the treatement of the plate, shell or rod boundary-value problems we usually assume that the boundary conditions have the form of equalities and we take into account only such external forces which are the known loadings. However, we meet the phenomena for which the boundary conditions have to be given in the form of certain implications involving the inequalities. In some problems we also deal with the shell or rod "internal" conditions (for example, due to the unilateral contact of shell or rod with certain rigid bodies) or with the forces which "control" the motion of the shell or rod (for example, the effect of a friction). All these problems in which, roughly speaking, "certain tresholds are crossed or attained" (cf. [8], p.XIV), will be referred to as the "non-classical" problems. In this Chapter the general discussion of such problems will be carried on.

1. FORMULATION OF SPECIAL PROBLEMS

Throughout this section we shall assume that a certain plate or shell theory is known. Within this theory we are to formulate and to detail different, mainly non-classical problems for plates and shells. The general formulation of such problems also includes, as the special cases, the known boundary value problems.

1.1. General case

Every plate or shell theory (in the sense explained in the Prerequisites) is represented by the field equations (i.e., by the equations of motion and the kinetic boundary conditions) and by the constitutive relations. However, the shell theory itself provides no informations how the shell is loaded, how it is supported or how it can deform in the special situation under consideration. If the analytical form of such informations is known, then we shall say that a certain special problem of the plate or shell theory has been stated. In what follows we shall confine ourselves to the special problems analytical structure of which is determined by:

1. The field and constitutive relations of a certain plate or shell theory.
2. The analytical description of deformations which can be realized by a shell.
3. The analytical description of the loadings acting at the shell.
4. The interrelation between deformations which can be realized by the shell and the constraint reactions which are able to maintain these deformations.
5. The analytical description of other external forces acting at the shell (field reactions).

Thus the informations needed to formulate the special problems of the shell theory involve the new concepts of the loadings and the reactions. The physical sense of these concepts is rather clear and their analytical description will be introduced below. We shall understand the term "shell special problem" only in the sense outlined above and we shall deal with the "non-classical" as well as with the "classical" problems. We assume that the shell theory under consideration is determined by the field equations (A5.1), (A5.3) and the constitutive relations (A5.5), (A5.6). The form of functions i^{a}_{R}, $i^{a\alpha}_{R}$, $\tilde{H}^{a\alpha\beta}_{R}$, $\tilde{H}^{a\alpha}_{R}$, $\tilde{h}^{a}_{R}$, g_A, κ, ψ in Eqs. (A5.1), (A5.5), (A5.6) is assumed to be known.

Remark 1. In order to obtain the solution to the special problem we have also to take into account certain initial conditions for $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$. In what follows we may assume that either the initial conditions are stated independently on the formulation of problem or they are included into this formulation. In the latter case the initial conditions for $q_{(n)}$, due to the existence of the inertia terms, can be included into the description of deformations which can be realized by a shell. The initial conditions for $(e_{(r)}, \tau^{(N)}, \omega^{(p)})$, which depend on the form of the constitutive relations, can be also treated as certain constitutive assumptions. In this case the subscript "A" in Eqs. $(A5.6)_1$ has to run over $1,\dots,N+P+R$, where $g_A(\underset{\sim}{\theta}, e_{(r)}, \tau^{(N)}, \omega^{(p)}) = 0$, $A = N+P+1, \dots, N+P+R$, are the initial conditions for $(e_{(r)}, \tau^{(N)}, \omega^{(p)})$.

In the most cases not every shell deformation function $q_{(n)} = \{q_a(\underset{\sim}{\theta}, t),\ a = 1,\dots,n\}$ belonging to Q (cf. Secs. 1.1., 2.2. and 4.3. of the Chapter A) is admissible. Mostly we deal with the shells which are supported or for which some kinds of deformations are excluded by the action of certain external forces or by the properties of the material (material incompressibility, for example). This fact gives rise to the following assumption.

Assumption 1. In every shell problem there is known the non-empty subset $\overset{o}{Q}$ of Q, which is interpreted as the set of all shell deformation functions $q_{(n)}$ admissible in this problem.

The inclusion $\overset{o}{Q} \subset Q$ will be referred to as the shell *constraint inclusion*. The special case $\overset{o}{Q} = Q$ can be also taken into account.

In every shell problem we have to know the system of the shell external forces which characterizes the loading of the shell (in some problems the loadings can be equal to zero). Thus the second assumption about the shell problems will be stated as follows.

Assumption 2. In every shell problem there is known the system of shell external forces $\hat{y}_R = \{\hat{p}^a_R(q_{(n)}), \hat{p}^{a\alpha}_R(q_{(n)}), \hat{f}^a_R(q_{(n)}), \hat{f}^{a\alpha}_R(q_{(n)});$ $a = 1,\ldots,n\ ;\ \alpha = 1,2\}$, where $\hat{p}^a_R,\ldots,\hat{f}^{a\alpha}_R$ are, for every $\underset{\sim}{\theta} \in \Pi$, $t \in I$, the known functionals defined on $\overset{o}{Q}$ [1]. The system $\hat{y}_R$ will be interpreted as the shell *loading* system.

Let $y_R=\{p^a_R, p^{a\alpha}_R, f^a_R, f^{a\alpha}_R;\ a = 1,\ldots,m;\ \alpha = 1,2\}$ be the system of the total shell external forces (cf. the field equations (A5.1),(A5.3), (A5.4)).

Definition 1. The system of shell external forces $\overset{\vee}{y}_R = \{\overset{\vee}{p}{}^a_R, \overset{\vee}{p}{}^{a\alpha}_R,$ $\overset{\vee}{f}{}^a_R, \overset{\vee}{f}{}^{a\alpha}_R;\ a = 1,\ldots,n;\ \alpha = 1,2\}$, defined by

$$\overset{\vee}{y}_R \equiv y_R - \hat{y}_R \qquad \text{(C1.1)}$$

will be called the system of *shell reaction forces*.

Thus the shell external forces, in every problem under consideration, are the sum of the loadings and reactions. In some special problems the reactions can be due exclusively to the constraint inclusion $\overset{o}{Q} \subset Q$ but in the general case we may also deal with the reactions which are

---

(1) The form of these functionals is determined by Eqs. $(A2.24)_{4,5}$ and (A2.25), by the equations $p^k_R = \hat{p}^k_R(\underset{\sim}{p})$, $b^k_R = \hat{b}^k_R(\underset{\sim}{p})$ which define the loadings acting at the shell like body ($\hat{p}^k_R$, $\hat{b}^k_R$ are the known functionals) and by $\underset{\sim}{p} = \tilde{\underset{\sim}{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)})$, cf. Eq. $(A4.23)_1$.

not implied by the restriction of the deformations (for example, in the case of the reactions which are due to the friction). That is why we shall analyse two kinds of the reactions: the one maintaining the constraint and the other which is not related to the constraint inclusion.

For the time being let us assume that the set $\overset{o}{Q}$ of the shell deformation functions, which are admissible in the problem under consideration, is uniquely defined by the sets $\overset{o}{Q}_t$, $t \in I$, of the instant values $q_{(n)}(.,t) \in \overset{o}{Q}_t$ of these functions. The constraint inclusion $\overset{o}{Q} \subset Q$ given by $\overset{o}{Q}_t \subset Q_t$, $t \in I$,(1) will be called configurational. We shall denote $q_{(n)t} \equiv q_{(n)}(.,t)$ for every $t \in I$; we shall refer elements $q_{(n)t}$ of $\overset{o}{Q}_t$ as the admissible shell configurations at the time instant t. Let V be the linear space of the functions $v_{(n)} = \{v_a ; a = 1,\dots,n\}$ defined and continuous in $\bar{\Pi}$ and smooth in Π. Let $q_{(n)t} \in \overset{o}{Q}_t$; if there exists ε_o, $\varepsilon_o > 0$, such that $q_{(n)t} + \varepsilon v_{(n)} \in \overset{o}{Q}_t$ for every ε, $0 < \varepsilon < \varepsilon_o$, then the field $v_{(n)}$ will be called the virtual displacement for the admissible configuration $q_{(n)t}$ of the shell. The set of all virtual displacements will be denoted by $V_{q_{(n)t}}$.

Let us denote by $\langle y_R, v_{(n)} \rangle_t$ the rate of work at the time instant t of the external force system $y_R = \{p^a_R, p^{a\alpha}_R, f^a_R, f^{a\alpha}_R ; a = 1,\dots,n; \alpha = 1,2\}$ on an arbitrary field $v_{(n)}$, $v_{(n)} \in V$:

$$\langle y_R, v_{(n)} \rangle_t \equiv \oint_{\partial\Pi} (p^a_R v_a + p^{a\alpha}_R v_{a,\alpha}) dl_R + \int (f^a_R v_a + f^{a\alpha}_R v_{a,\alpha}) da_R .$$

The restriction of the set Q of all shell deformation functions to the subset $\overset{o}{Q}$ of the deformation functions which are admissible in the problem under consideration is always connected with the existence of certain systems $\overset{o}{y}_R = \{\overset{o}{p}{}^a_R, \overset{o}{p}{}^{a\alpha}_R, \overset{o}{f}{}^a_R, \overset{o}{f}{}^{a\alpha}_R ; a = 1,\dots,n ; \alpha = 1,2\}$ of the external forces. In the case of the configurational constraint inclusion the set $\overset{o}{Y}_{q_{(n)}}$ of all systems $\overset{o}{y}_R$ will be postulated in the form

$$\overset{o}{Y}_{q_{(n)}} := \{ \overset{o}{y}_R \mid \langle \overset{o}{y}_R, v_{(n)} \rangle_t \geq 0 \text{ for every } v_{(n)} \in V_{q_{(n)t}}, t \in I\} \quad \text{(C.1.2)}$$

where $q_{(n)t} \in \overset{o}{Q}_t$. An arbitrary element $\overset{o}{y}_R$ of $\overset{o}{Y}_{q_{(n)}}$ is called the constrant reaction on the shell deformation $q_{(n)}$, $q_{(n)} \in \overset{o}{Q}$. The definition

(1) Q_t is the set of all $q_{(n)}(.,t)$ such that $q_{(n)} \in Q$; $Q_{t_1} = Q_{t_2}$ for every t_1, $t_2 \in I$.

(C1.2) states that for the configurational constraints the total rate of work of the constraint reactions on an arbitrary virtual displacement is always non-negative.

If the constraint inclusion is not configurational then the sets $\overset{\circ}{Q}_t$, $t \in I$, do not determine the set $\overset{\circ}{Q}$ of all shell deformation functions $q_{(n)}$ admissible in the problem under consideration. Nevertheless we shall assume that for every $q_{(n)t} \in \overset{\circ}{Q}_t$ there exists the set $V_{q_{(n)t}}$ of the "test functions", such that Eq. (C1.2) holds (we take into account also anholomic case, cf. [45]). Now we shall formulate the next assumption describing the special shell problems under consideration.

<u>Assumption 3</u>. In every shell problem there are known the sets $\overset{\circ}{Y}_{q_{(n)}}$, $q_{(n)} \in \overset{\circ}{Q}$, of shell external forces $\overset{\circ}{y}_R = \{\overset{\circ a}{p}_R, \overset{\circ a\alpha}{p}_R, \overset{\circ a}{f}_R, \overset{\circ a\alpha}{f}_R ;\ a = 1,\dots,n;\ \alpha = 1,2\}$, $\overset{\circ}{y}_R \in \overset{\circ}{Y}_{q_{(n)}}$, which are defined by Eq. (C1.2), where $V_{q_{(n)t}}$ are the known sets of the test functions (defined for every $q_{(n)t} \in \overset{\circ}{Q}_t$.

The system of forces $\overset{\circ}{y}_R$, $\overset{\circ}{y}_R \in \overset{\circ}{Y}_{q_{(n)}}$, will be called the *shell constraint reactions.*

The foregoing assumption characterizes the interrelation between the constraints inclusion $\overset{\circ}{Q} \subset Q$ and the sets $\overset{\circ}{Y}_{q_{(n)}}$ of the shell constraint reactions, which, roughly speaking, can maintain the constraints or are due to them.

We have stated before that the constraint reactions may not be equal to the shell reactions $\overset{\vee}{y}_R$. It follows that we shall also deal with the other kind or reactions which are not due to the constraint inclusion $\overset{\circ}{Q} \subset Q$.

<u>Definition 2</u>. The system of external forces $\overset{*}{y}_R = \{\overset{*a}{p}_R, \overset{*a\alpha}{p}_R, \overset{*a}{f}_R, \overset{*a\alpha}{f}_R;\ a = 1,\dots,n\ ;\ \alpha = 1,2\}$, defined by

$$\overset{*}{y}_R \equiv \overset{\vee}{y}_R - \overset{\circ}{y}_R \tag{C1.3}$$

will be called the system of *shell field reactions.*

Thus the reaction forces are the sum of the constraint reactions and the field reactions. The field reactions $\overset{*}{y}_R$ will be interpreted as due to the existence of certain external systems of forces $y_R^{(\mu)}$,

$\mu=1,\ldots,m$, which, roughly speaking, "control" the deformations of the shell. The systems $y_R^{(\mu)}$, $\mu = 1,\ldots,m$, of such control forces are assumed to be determined by the motion of the shell, i.e., there are known the relations $y_R^{(\mu)} = \underset{o}{y}_R^{(\mu)}(q_{(n)})$, where $\underset{o}{y}_R^{(\mu)}(.)$, $\mu = 1,\ldots,m$ are the known functionals defined on the set Q of shell admissible deformation functions. We shall confine ourselves to the cases in which the field reactions can be determined by

$$\langle \overset{*}{y}_R, v_{(n)} \rangle_t \geq \langle y_R^{(\mu)}, v_{(n)} \rangle_t \text{ for every } v_{(n)} \in V_t^{(\mu)},\ \mu = 1,\ldots,m \tag{C1.4}$$

where $V_t^{(\mu)}$, $V_t^{(\mu)} \subset V$, are the known, for every $t \in I$, non-empty sets of the suitable chosen test functions, satisfying the conditions $V_t^{(\mu)} \cap V_t^{(\nu)} = \{\theta\}$ for every $\mu \neq \theta$ and such that $y_R^{(\mu)} = \theta$, $\mu = 1,\ldots,m$, implies that $\overset{*}{y}_R = \theta$. The sense of the assumption (C1.4) will be detailed in Sec. 3. We shall see later that for example, the field reactions due to the friction between the shell and some other bodies can be described in that way.

Let us denote $U_t \equiv \bigcup V_t^{(\mu)}$ and introduce the functions $\delta_t^{(\mu)} : U_t \to V_t^{(\mu)}$ putting $\delta_t^{(\mu)} v_{(n)} = v_{(n)}$ if $v_{(n)} \in V_t^{(\mu)}$ and $\delta_t^{(\mu)} v_{(n)} = \theta$ if $v_{(n)} \in U_t \setminus V_t^{(\mu)}$. Then

$$\langle \overset{*}{y}_R, v_{(n)} \rangle_t \geq \sum_{\mu=1}^{m} \langle y_R^{(\mu)}, \delta_t^{(\mu)} v_{(n)} \rangle_t .$$

Denoting by

$$J_{q_{(n)}t}(v_{(n)}) \equiv \sum_{\mu=1}^{m} \langle \underset{o}{y}_R^{(\mu)}(q_{(n)}), \delta_t^{(\mu)} v_{(n)} \rangle_t,\quad v_{(n)} \in U_t \tag{C1.5}$$

the rate of work of all control forces $y_R^{(\mu)}$, $\mu = 1,\ldots,m$, on the fields $\delta_t^{(\mu)} v_{(n)}$, we obtain

$$\langle \overset{*}{y}_R, v_{(n)} \rangle_t \geq J_{q_{(n)}t}(v_{(n)}) \text{ for every } v_{(n)} \in U_t,\ t \in I.$$

Now we can formulate the last assumption concerning the formulation of the special problems in the shell theories.

Assumption 4. In every shell problem there are known the sets $\overset{*}{Y}_{q_{(n)}}$, $q_{(n)} \in \overset{o}{Q}$, of the shell external forces (field reactions) $\overset{*}{y}_R = \{\overset{*a}{p}_R, \overset{*a\alpha}{p}_R, \overset{*a}{f}_R, \overset{*a\alpha}{f}_R;\ a = 1,\dots,n;\ \alpha = 1,2\}$, defined by

$$\overset{*}{Y}_{q_{(n)}} := \{\overset{*}{y}_R \,|\, \langle y_R, v_{(n)}\rangle_t \geq J_{q_{(n)t}}(v_{(n)}) \text{ for every } v_{(n)} \in U_t \quad \text{and every } t \in I\}\ , \tag{C1.6}$$

whre $J_{q_{(n)t}}(.)$ are the known functionals defined by Eq. (C1.5) and U_t, $U_t \subset V$, is the set of the test functions known for every $t \in I$.

Remark. Comparing Eq (C1.2) and Eq. (C1.6) we can suppose, that in some special cases the constraint reactions may be obtained as a "limit case" of the field reactions in which $J_{q_{(n)}} \to 0$, $U_t \to V_{q_{(n)t}}$ for every fixed $q_{(n)t} \in \overset{o}{Q}_t$ and $t \in I$. In these cases only the field reactions have to be taken into account and the approach corresponds to that known as the "penalty function method" in the optimization theory (cf. Sec. 3.3. of this Chapter and the examples detailed in [8]).

From the foregoing analysis it follows that, in every problem under consideration, the system $y_R = \{p_R^a, p_R^{a\alpha}, f_R^a, f_R^{a\alpha},\ a = 1,\dots,n;\ \alpha = 1,1,2\}$ of the shell external forces in Eqs. (A5.1) (A5.3) (A5.4) is the sum

$$y_R = \hat{y}_R + \check{y}_R = \hat{y}_R + \overset{o}{y}_R + \overset{*}{y}_R \tag{C1.7}$$

of the loadings, constraints reactions and field reactions. The loadings are determined by the shell deformation (cf. Assumption 2)

$$\hat{y}_R = \hat{y}_R(q_{(n)}),\ q_{(n)} \in \overset{o}{Q}\ , \tag{C1.8}$$

where $\hat{y}_R(.)$ are the known functionals defined on $\overset{o}{Q}$, and the reactions are determined by the inequalities (C1.2), (C1.6). Summing up we conclude that the general form of the informations needed to formulate the special problems within certain shell theory is given by the relations (*)

(*) We have tacitly assumed here that we deal with the second-order shell force system. For the simple shell force system (cf. Sec. 5.1. of the Chapter A) the terms $p_R^{a\alpha}, \hat{p}_R^{a\alpha}, \dots, f_R^{a\alpha}, \hat{f}_R^{a\alpha}, \dots$ drop out from the relations.

$$q_{(n)} \in \overset{o}{Q} \; ;$$

$$p_R^a \;=\; \hat{p}{}_R^a(q_{(n)}) + \overset{*}{p}{}_R^a + \overset{o}{p}{}_R^a \;,$$

$$p_R^{a\alpha} \;=\; \hat{p}{}_R^{a\alpha}(q_{(n)}) + \overset{*}{p}{}_R^{a\alpha} + p_R^{a\alpha} \;,$$

$$f_R^a \;=\; \hat{f}{}_R^a(q_{(n)}) + \overset{*}{f}{}_R^a + \overset{o}{f}{}_R^a \;,$$

$$f_R^a \;=\; \hat{f}{}_R^a\,(q_{(n)}) + \overset{*}{f}{}_R^a + \overset{o}{f}{}_R^a \;, \qquad \text{(C1.9)}$$

$$\oint_{\partial\Pi} (\overset{*}{p}{}_R^a u_a + \overset{*}{p}{}_R^{a\alpha} u_{a,\alpha})dl_R + \int_{\Pi} (\overset{*}{f}{}_R^a u_a + \overset{*}{f}{}_R^{a\alpha} u_{a,\alpha})da_R \geq J_{q_{(n)t}}(u_{(n)})$$

$$\text{for every } u_{(n)} \in U_t, \quad t \in I,$$

$$\oint_{\partial\Pi} (\overset{o}{p}{}_R^a v_a + \overset{o}{p}{}_R^{a\alpha} v_{a,\alpha})dl_R + \int_{\Pi} (\overset{o}{f}{}_R^a v_a + \overset{o}{f}{}_R^{a\alpha} v_{a,\alpha})da_R \geq 0$$

$$\text{for every } v_{(n)} \in V_{q_{(n)t}}, \quad t \in I \;,$$

where the sets $\overset{o}{Q}$, U_t, $V_{q_{(n)t}}$ and the functionals $\hat{p}{}_R^a$, $\hat{p}{}_R^{a\alpha}$, $\hat{f}{}_R^a$, $\hat{f}{}_R^{a\alpha}$, $J_{q_{(n)t}}$ are known.

Thus the problem within certain shell theory is described by the field equations (A5.1), A5.3) (cf. the last footnote), by the constitutive relations (A5.5), (A5.6) and by the relations (C1.9). If $V_{q_{(n)t}} \subset$ $\subset U_t$ for every $q_{(n)} \in \overset{o}{Q}$ and $t \in I$, then by the simple calculations from Eqs. (A5.1), (A5.3), $(C.1.9)_{2-7}$ we obtain the inequality

$$\int_{\Pi} (-\tilde{H}_R^{a\alpha\beta} v_{a,\alpha\beta} + \tilde{H}_R^{a} v_{a,\alpha} - \tilde{h}_R^a v_a)da_R \geq$$

$$\geq \oint_{\partial\Pi} (\hat{p}{}_{oR}^a v_a + \hat{p}{}_R^{a\underset{\sim}{N}} v_{a,\underset{\sim}{N}})dl_R + \int_{\Pi} [(\hat{f}{}_R^{a\alpha} - i_R^{a\alpha})v_{a,\alpha} + \qquad \text{(C1.10)}$$

$$+ (\hat{f}{}_R^a - i_R^a)v_a]da_R + J_{q_{(n)t}}(v_{(n)})$$

which has to be satisfied for every $v_{(n)} \in V_{q_{(n)t}}$, $q_{(n)} \in \overset{o}{Q}$, $t \in I$.

We have used here the earlier denotations

$$\hat{p}_R^{aN} \equiv \hat{p}_R^{a\alpha} n_{R\alpha}\,,\quad \hat{p}_{oR}^{a} \equiv \hat{p}_R^{a} - d(\hat{p}_R^{a\alpha} t_{R\alpha})/dl_R\,,\quad v_{a,\underset{\sim}{N}} \equiv v_{a,\alpha} n_R^{\alpha}\,,$$

and $\tilde{H}_R^{a\alpha\beta}$, $\tilde{H}_R^{a\alpha}$, $\tilde{h}_R^{a}$, as well as $\hat{p}_{oR}^{a}$, $\hat{p}_{R}^{a\underset{\sim}{N}}$, $\hat{f}_R^{a\alpha}$, $\hat{f}_R^{a}$ are the known functionals defined by Eqs. (A2.29) and introduced by Eq. (C1.8), respectively. Eq. (C1.10) will be called the fundamental inequality of the shell problem; it has to be considered together with Eqs. (A5.6) and the condition $q_{(n)} \in \overset{o}{Q}$. The basic unknowns in Eqs. (C1.10), (A5.6), (C1.9) are the fields $q_{(n)}$, $\tau^{(N)}$, $\omega^{(p)}$. If the material of the shell is simple then Eqs. (A5.6) are identities and the only basic unknown is $q_{(n)}$. In this case the shell problem is described by the inequality (C1.10)and by the condition $q_{(n)} \in \overset{o}{Q}$ (1).

The example of the application of the general approach to the special shell problems outlined above will be detailed in Sec. 3. Also in Sec. 3 we shall discuss some of the consequences of the inequality (C1.10).

Remark 2. Using the terminology of Sec. 4.0 of the Chapter A we can say that the formulation of problems within certain plate or shell theory coincides with imposing the special kind of semiconstraints on the relations of this theory. The semiconstraints under consideration are defined by the constraint inclusion $\overset{o}{Q} \subset Q$ and by the multifunction $\overset{o}{Q} \ni q_{(n)} \to Y_{q_{(n)}}$, where $Y_{q_{(n)}} = \{\hat{y}_R(q_{(n)}\} \oplus \overset{o}{Y}_{q_{(n)}} \oplus \overset{*}{Y}_{q_{(n)}}$.

Remark 3. Every formulation of the shell problem, described by the relations of the shell theory and by Eqs. (C1.7), (C1.8) as well as by the conditions (C1.2), (C1.6) should lead to the physically resonable solutions of this problem(it must exist, at least, one such solution). However, for the non-linear shell problems the sufficient conditions which ensure the existence of such solutions, are not known (cf. also Sec. 1.3 of this Chapter).

Remark 4. As the basic unknown in the problems described above we have assumed the triple $x=(q_{(n)},\tau^{(N)},\omega^{(p)})$. However, Eqs. (C1.10), (A5.6) are also starting point in the problems of stability (then $\hat{y}_R = \hat{y}_R(\varepsilon,q_{(n)})$ and we look for the values of the scalar parameter ε for

---

(1) We may assume here that the initial conditions have to be stated independently of the problem cf. Remark 1.

which there exist many physically reasonable solutions x),or in the optimization problems.

1.2. Application of the constraint functions

The approach to the special plate and shell problems given in Sec. 1.1. is very general. Now we shall analyse the more special class of plate and shell problems.

Let $\psi_{(m)} = (\psi_\mu,\ \mu = 1,\dots,m)$, $m < n$, be an arbitrary ordered set of the sufficiently regular functions defined on $\bar{\Pi} \times I$. We shall assume that the set $\overset{\circ}{Q}$ of shell deformation functions admissible in the problem under consideration is given by

$$\overset{\circ}{Q} := \{q_{(n)} \,|\, h_\nu(\underset{\sim}{\theta}, t, q_{(n)}, \nabla q_{(n)}, \psi_{(m)}, \nabla\psi_{(m)}) = 0,\ \nu = 1,\dots,N;$$

$$\underset{\sim}{\theta} \in \Pi,\ t \in I;\ b_\rho(\underset{\sim}{\theta}, t, q_{(n)}, \bar{\nabla}\bar{q}_{(n)}, \bar{\psi}_{(m)}, \bar{\nabla}\bar{\psi}_{(m)}) = 0\ ,$$

$$\rho = 1,\dots,R\ ;\ \underset{\sim}{\theta} \in \partial\Pi \quad \text{a.e.}\ ,\ t \in I\}\ , \qquad \text{(C1.11)}$$

where h_ν, b_ρ are the known independent ([1]) differentiable functions, $\bar{\nabla} \equiv d/dl_R$, $\bar{q}_{(n)} \equiv \tau q_{(n)}$, $\bar{\psi}_{(m)} \equiv \tau\psi_{(m)}$ and where τ is the linear transformation which assigns to the functions $q_{(n)}$, $\psi_{(m)}$, their boundary values on $\partial\Pi$. The relation $q_{(n)} \in \overset{\circ}{Q}$ can have many analytical representations of the form (C1.11). The unknown function $\psi_{(m)}$ plays here the role of a certain auxiliary function; introducing such function we can, for example, reduce the system of the k-th order differential equations with respect to $q_{(n)}$ to the system of the first order differential equations with respect to $q_{(n)}, \psi_{(m)}$. The functions $\psi_{(m)}$ can also be treated as the "slack variables" (used in the optimization and control theories) which are applied to convert the inequalities of the form $g(\underset{\sim}{\theta}, t, q_{(n)}, \nabla q_{(n)}) \geq 0$ into the equalities $h(\underset{\sim}{\theta}, t, q_{(n)}, \nabla q_{(n)}, \psi) = 0$, where $h(.) \equiv g(.) - (\psi)^2$.

We shall interpret the relations $h_\nu(.) = 0$, $\nu = 1,\dots,N$ and $b_\rho(.) = 0$, $\rho = 1,\dots,R$, as the known restrictions imposed on the internal and boundary values of shell deformation function $q_{(n)}$ and its material

---

([1])Denoting by z_A, $A = 1,\dots,M$, the arguments of $h_\nu(\underset{\sim}{\theta}, t, .)$, $b_\rho(\underset{\sim}{\theta}, t, .)$ we assume rank $(\partial h_\nu/\partial z_A) = N$, $N \leq M$ and rank $(\partial b_\rho/\partial z_A) = R$, $R \leq M$.

derivatives. The form of these relations depends on the special problem under consideration [1]. Equations $b_\rho = 0$ are often called the geometric boundary conditions.

Let us assume that we deal now the problems in which the control forces are equal to zero. Then the field reactions are equal to zero, $\overset{*}{y}_R = \theta$, and $y_R = \overset{o}{y}_R$, i.e., all reactions are only due to the constraints (C1.11). Let us also assume that for every $q_{(n)} \in \overset{o}{Q}$ the constraints are bilateral (cf. Sec. 4.0. of the Chapter A); then for every $v_{(n)} \in \in V_{q_{(n)t}}$ we also have $-v_{(n)} \in V_{q_{(n)t}}$ and the inequality in Eqa. (C1.2) can be related by the equality $\langle \overset{o}{y}_R, v_{(n)} \rangle_t = 0$. It means that

$$\oint_{\partial\Pi} (\overset{oa}{p}_R v_a + \overset{oa\alpha}{p}_R v_{a,\alpha}) dl_R + \int_{\Pi} (\overset{oa}{f}_R v_a + f_R^{a\alpha} v_{a,\alpha}) da_R = 0 \qquad (C1.12)$$

holds for an arbitrary test function $v_{(n)}$, $v_{(n)} \in V_{q_{(n)t}}$. The sets $V_{q_{(n)t}}$, $q_{(n)} \in \overset{o}{Q}$, of the tests functions will be assumed as the sets of the solutions of the following equations in Π

$$\frac{\partial h_\nu}{\partial q_a} v_a + \frac{\partial h_\nu}{\partial q_{a,\alpha}} v_{a,\alpha} + \frac{\partial h_\nu}{\partial \psi_d} u_d + \frac{\partial h_\nu}{\partial \psi_{d,\alpha}} u_{d,\alpha} = 0 \ , \ \nu = 1,\dots,N \qquad (C1.13)$$

and the conditions which have to hold almost everyhwere on $\partial\Pi$

$$\frac{\partial b_\rho}{\partial q_a} v_a + \frac{\partial b_\rho}{\partial \bar{\nabla} q_a} \bar{\nabla} v_a + \frac{\partial b_\rho}{\partial \psi_d} u_d + \frac{\partial b_\rho}{\partial \bar{\nabla} \psi_d} \bar{\nabla} u_d = 0 \ , \ \rho = 1,\dots,R, \quad (C1.14)$$

where a = 1,...,n and d = 1,...,m (summation convention holds).

Let us assume that we deal with the shell theory in which the simple shell force system is involved. In this case the terms with $\overset{oa\alpha}{p}_R$, $\overset{oa\alpha}{f}_R$ in Eqs. (C1.12) have to be neglected. Let $\lambda_R^\nu(\underset{\sim}{\theta}, t)$ be arbitrary differentiable functions for every $\underset{\sim}{\theta} \in \Pi$, defined and continuous in $\bar{\Pi}$ for every $t \in I$. Let us also denote by $\bar{\mu}_R^\nu(\underset{\sim}{\theta}, t)$, $\nu = 1,\dots,N$, $\mu_R^\rho(\underset{\sim}{\theta}, t)$, $\rho = 1,\dots,R$, arbitrary differentiable functions defined, for every $t \in I$, almost everywhere on $\partial\Pi$. Then the reaction forces $\overset{oa}{p}_R$, $\overset{oa}{f}_R$ satisfying the condition

---

(1) Instead of the restriction $b_\rho = 0$, in some problems we can also take into account the restrictions $b_\rho(\tau q_{(n)}, \tau\psi_{(m)}) = \theta$, where B_ρ, $\rho = 1,\dots,R$, are the differentiable (in the Gâteaux sense) operators from $L^2(\partial\Pi)$ to $L^2(\partial\Pi)$ and τf is the trace of the function f on the boundary .

$$\oint_{\partial\Pi} \overset{o}{p}{}^{a}_{R} v_a \, dl_R + \int_{\Pi} \overset{o}{f}{}^{a}_{R} v_a \, da_R = 0 \tag{C1.15}$$

for every v_a such that $(v_{(n)}, u_{(m)})$ is the solution of Eqs. (C1.13) (C1.14), can be assumed in the form

$$\begin{aligned} \overset{o}{f}{}^{a}_{R} &= -\lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial q_a} + \left(\lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial q_{a,\alpha}}\right)_{,\alpha}, \quad a = 1,\dots,n \; ; \\ \lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial \psi_d} &- \left(\lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial \psi_{d,\alpha}}\right)_{,\alpha} = 0 , \quad d = 1,\dots,m , \end{aligned} \tag{C1.16}$$

and

$$\begin{aligned} \overset{o}{p}{}^{a}_{R} = &-\lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial q_{a,\alpha}} n_{R\alpha} + \bar{\mu}^{\nu}_{R} \frac{\partial \bar{h}_\nu}{\partial q_a} - \bar{\nabla}\left(\bar{\mu}^{\nu}_{R} \frac{\partial \bar{h}_\nu}{\partial \bar{\nabla} q_a}\right) - \\ &-\mu^{\rho}_{R} \frac{\partial b_\rho}{\partial q_a} + \bar{\nabla}\left(\mu^{\rho}_{R} \frac{\partial b_\rho}{\partial \bar{\nabla} q_a}\right), \quad a = 1,\dots,n \; ; \\ &-\lambda^{\nu}_{R} \frac{\partial h_\nu}{\partial \psi_{d,\alpha}} n_{R\alpha} + \left(\bar{\mu}^{\nu}_{R} \frac{\partial \bar{h}_\nu}{\partial \psi_d} - \bar{\nabla}\left(\bar{\mu}^{\nu}_{R} \frac{\partial \bar{h}_\nu}{\partial \bar{\nabla} \psi_d}\right) - \right. \\ &-\mu^{\rho}_{R} \frac{\partial b_\rho}{\partial \psi_d} + \bar{\nabla}\left(\mu^{\rho}_{R} \frac{\partial b_\rho}{\partial \bar{\nabla} \psi_d}\right), \quad d = 1,\dots,m . \end{aligned} \tag{C1.17}$$

The definition of the functions $\bar{h}_\nu$ is given by Eq. (B7.11). The unknown functions λ^{ν}_{R}, $\bar{\mu}^{\rho}_{R}$, μ^{ρ}_{R} are called the constraint functions. Eqs. (C1.16), (C1.17) lead to the condition (C1.16) if the following jump conditions

$$\begin{aligned} \left[\!\left[\bar{\mu}^{\nu} \frac{\partial \bar{h}_\nu}{\partial \bar{\nabla} q_a} + \mu^{\rho} \frac{\partial b_\rho}{\partial \bar{\nabla} q_a} \right]\!\right] &= 0 , \quad a = 1,\dots,n \; ; \\ \left[\!\left[\bar{\mu}^{\nu} \frac{\partial \bar{h}_\nu}{\partial \bar{\nabla} \psi_d} + \mu^{\rho} \frac{\partial b_\rho}{\partial \bar{\nabla} \psi_d} \right]\!\right] &= 0 , \quad d = 1,\dots,m \end{aligned} \tag{C1.18}$$

hold in all points of $\partial\Pi$.

Summing up, the class of problems under consideration is described by the shell field equations and constitutive relations (with the simple force system), by the geometric boundary conditions $b_\rho(\underset{\sim}{\theta}, t, q_{(n)}, \bar{\nabla} q_{(n)}, \psi_{(m)}, \bar{\nabla}\psi_{(m)}) = 0$, $\rho = 1,\dots,R$, $\underset{\sim}{\theta} \in \partial\Pi$, by the geometric internal conditions $h_\nu(\underset{\sim}{\theta}, t, q_{(n)}, \nabla q_{(n)}, \psi_{(m)}, \nabla\psi_{(m)}) = 0$, $\nu = 1,\dots,N$, $\underset{\sim}{\theta} \in \Pi$, and by the decomposition rules of the shell external forces

$$p^a_R = \hat{p}^a_R(q_{(n)}) + \overset{o}{p}{}^a_R \;,$$
$$f^a_R = \hat{f}^a_R(q_{(n)}) + \overset{o}{f}{}^a_R \;, \qquad \text{(C1.19)}$$

where $\hat{f}^a_R(.)$, $p^a_R(.)$ are the known functionals which define the loadings and $\overset{o}{p}{}^a_R$, $\overset{o}{f}{}^a_R$ are given by Eqs. $(C1.17)_1$, $(C1.16)_1$, respectively. At the same time Eqs. $(C1.16)_2$, $(C1.17)_2$, (C1.18) have to be taken into account [1].

In many special shell problems we deal with the restrictions imposed on $q_{(n)}$ which are given exclusively by the geometric boundary conditions of the form $b_\rho(\underset{\sim}{\theta}, q_{(n)}, \bar{\nabla} q_{(n)}) = 0$. In this case we see that $\overset{o}{f}{}^a_R = 0$, $\overset{o}{p}{}^a_R$ is determined by the last two terms of Eq. $(C1.17)_1$ and that Eqs. $(C1.16)_2$, $(C1.17)_2$ are indentities.

Remark 1. The foregoing analysis remains valid if the functions h_ν, b_ρ depend also on the time derivatives $\dot{q}_{(n)}$, $\dot{\psi}_{(m)}$, $\nabla\dot{q}_{(n)}$, $\nabla\dot{\psi}_{(m)}$, i.e., if

$$h_\nu(\underset{\sim}{\theta}, \cdot\,, \dot{q}_{(n)}, \nabla\dot{q}_{(n)}, \dot{\psi}_{(m)}, \nabla\dot{\psi}_{(m)}) = 0, \quad \underset{\sim}{\theta} \in \Pi \;,$$
$$b_\rho(\underset{\sim}{\theta}, \cdot\,, \dot{q}_{(n)}, \bar{\nabla}\dot{q}_{(n)}, \dot{\psi}_{(m)}, \bar{\nabla}\dot{\psi}_{(m)}) = 0, \quad \underset{\sim}{\theta} \in \partial\Pi \quad \text{a.e.} \qquad \text{(C1.20)}$$

where dots stand for arguments mentioned in Eqs. (C1.11). Now we assume that $h_\nu(\theta, q_{(n)}, \nabla q_{(n)}, \psi_{(m)}, \nabla\psi_{(m)}, \cdot)$, $\nu = 1,\dots,N$, and $b_\rho(\underset{\sim}{\theta}, q_{(n)}, \bar{\nabla} q_{(n)}, \psi_{(m)}, \bar{\nabla}\psi_{(m)}, \bar{\nabla}\psi_{(m)}, \cdot)$, $\rho = 1,\dots,R$, are the independent (cf. the last footnote) differentiable functions. The derivatives $\partial h_\nu/\partial q_a, \dots$, $\partial b_\rho/\partial q_a, \dots$ in Eqs. (C1.13), (C1.14), (C1.16) - (C1.18) have to be replaced now by the derivatives $\partial h_\nu/\partial \dot{q}_a, \dots, \partial b_\rho/\partial \dot{q}_a, \dots$, respectively; all other relations remain valid. This is the case of the anholonomic constraint inclusion.

Remark 2. The foregoing results can be also modified by introducing other sets of the test functions. These sets may be obtained by replacing the derivatives $\partial h_\nu/\partial q_a, \dots, \partial b_\rho/\partial q_a, \dots \partial b_\rho/\partial \bar{\nabla}\psi_d$, in Eqs. (C1.13), (C1.14) by the known functions $h^a_\nu, \dots, b^a_\rho, \dots, \bar{b}^d_\rho$, respetively. These new functions $h^a_\nu, \dots, \bar{b}^d_\rho$ have also to replace the derivatives $\partial h_\nu/\partial q_a$, $\dots, \partial b_\rho/\partial \bar{\nabla}\psi_d$, respectively, in Eqs. (C1.16)- (C1.18). Such treatment

(1) We have tacitly assumed here that the initial conditions are not included into the description of the problem.

includes, as the special cases, the geometric and anholonomic constraints and can be also used if k-th time derivatives $\overset{(k)}{q}_{(n)}, \overset{(k)}{\psi}_{(m)}, \nabla\overset{(k)}{q}_{(n)}, \nabla\overset{(k)}{\psi}_{(m)}$ are the arguments of all functions h_ν, b_ρ defining the set $\overset{o}{Q}$. Then $h^a_\nu \equiv \partial b_\rho/\partial\overset{(k)}{q}, \ldots, \bar{b}^d_\rho \equiv \partial b_\rho/\partial\bar{\nabla}\overset{(k)}{\psi}_d$.

Remark 3. Comparing the results obtained above with those given is Sec. 7.4. we conclude that the formulation of some special problems can also include certain modification of the shell field equations.

1.3. Reliability of solutions

Let us assume that we have solved the special shell problem under consideration [1], i.e., that we have found the triple of functions $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$. In the non-linear problems it may happen that there exist many physically reasonable solutions $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$. Let us take into account an arbitrary but fixed solution. Now the question arises whether this solution is reliable from the point of view of the classical "three dimensional" mechanics of shell-like bodies. The estimation of the "errors" involved by using the shell theory instead of the classical solid mechanics was stated in [26] as one from the two main problems of the general theory of plates and shells (cf. [26],p. 444; the other main problem was stated as the development of the shell theories as approximate theories relative to shell-like bodies). In the engineering applications of the shell theories rather intuitive reasoning is usually used to evaluate the reliability of solutions. It follows from the fact that the general analytical methods of estimating the "errors" introduced by the shell theory are unknown. However, certain necessary conditions imposed on the solutions of the special shell problems can be formulated. To do this we shall assume that the shell theory under consideration was obtained by means of the formal approximation or constraint approaches. It means that if the fields $q_{(n)}, \tau^{(N)}, \omega^{(p)}$, have been found, then from Eqs. (A5.9) we can calculate the "three dimensional" fields $\underset{\sim}{p} = \underset{\sim}{p}(\underset{\sim}{X},t)$, $\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{X},t)$, $\lambda^{(s)} = \lambda^{(s)}(\underset{\sim}{X},t)$. After that we can calculate the field $a_{(m)}$ from Eqs. (A5.8)$_1$. If $j \leq 0, \varphi \leq 0$ for every $\underset{\sim}{C}_o, \underset{\sim}{T}_o$ with $j(\underset{\sim}{C}_o, \underset{\sim}{T}_o, \lambda^{(s)}) \leq 0$ and the

---

[1] cf. Remark 1 to Sec. 1.1 of this Chapter.

values of the fields a_μ are sufficiently small (for example, of an order of numerical errors of the calculations) then the shell solution $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$ can be treated as reliable from the point of view of the constitutive relations. Mind, that the fields a, α in Eqs. $(A5.8)_{2,3}$ cannot be uniquely determined within the mechanics of the shell-like bodies.

Next let us assume that the fields $\underset{\sim}{b}_R$, $\underset{\sim}{p}_R$ in Eqs. (A5.7) can be evaluated with the sufficient accuracy (1). Then from the field equations of the solid mechanics (A5.7) we can calculate the forces r_R, s_R. The external forces $(\underset{\sim}{b}_R, \underset{\sim}{p}_R)$, $(\underset{\sim}{r}_R, \underset{\sim}{s}_R)$ are elements of a certain linear space. Let us introduce in this space the finite system of pseudo-norms $\| (\cdot,\cdot) \|_\kappa$, $\kappa = 1,\ldots,K$, which characterize, roughly speaking, the magnitude of the external forces from the point of view of the quantities we are interested in. These pseudo-norms can be taken, for example, as the norms of the external forces acting on a certain parts of the shell like body at certain time instants or in certain time intervals, cf. [45]. If the conditions

$$\| (\underset{\sim}{r}_R, \underset{\sim}{s}_R) \|_\kappa \leq \varepsilon \, \| (\underset{\sim}{b}_R, \underset{\sim}{p}_R) \|_\kappa, \quad \kappa = 1,\ldots, K \tag{C1.21}$$

hold, ε being the known positive number sufficiently small with respect to unity, then we shall say that the shell solution $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$ is reliable form the point of view of the field equation. It must be stressed, however, that the formulation of the physically reasonable conditions (C1.21) can be difficult because rather intuitive reasoning has to be used in evaluating the fields $(\check{\underset{\sim}{b}}_P, \check{\underset{\sim}{p}}_R)$ (cf. the last footnote) and in introducing the suitable pseudo-norms $\| (\cdot,\cdot) \|_\kappa$. On the other hand, in the constraint approach the physical sense of the Eqs. (C1.21) is rather clear. It is the condition which postulates that the reaction forces maintaining the constraints leading to the shell theory have to be sufficiently small with respect to the external forces acting at the shell like body. The reaction forces $(\underset{\sim}{r}_R, \underset{\sim}{s}_R)$ are treated here as certain "imaginary" forces which have been introduced, together with the suitable constraint inclusion, only "to render the theory more treatable",[2]. Such constraints have been called in [2] the simplifying constraints. Some applications of the inequalities (C1.21) can

(1) We have $\underset{\sim}{b}_R = \hat{\underset{\sim}{b}}_R + \check{\underset{\sim}{b}}_R$, $\check{\underset{\sim}{p}}_R = \hat{\underset{\sim}{p}}_R + \check{\underset{\sim}{p}}_R$, where $\hat{\underset{\sim}{b}}_R$, $\hat{\underset{\sim}{p}}_R$ are the loadings which are known (if the function $\underset{\sim}{p}$ is known), and $\check{\underset{\sim}{b}}_R, \check{\underset{\sim}{p}}_R$ are the unknown reactions (usually $\check{\underset{\sim}{b}}_R = 0$) which can be evaluated only with a certain approximation.

be found in [45].

The solution $(q_{(n)}, \tau^{(N)}, \omega^{(p)})$ of the shell problem will be called reliable if it is reliable from the point of view of the constitutive equations and that of the field equations [1].

Remark. If we interpret the shell theory as a certain formal approximation of the solid mechanics relations for a shell-like body (cf. Sec. 2 of the Chapter A), then the fields $a_{(m)}$, $\underset{\sim}{r}_R$, $\underset{\sim}{s}_R$ are interpreted as certain "error" fields. The introduced above conditions (C1.21) (as well as the conditions imposed on the values of the fields a_μ) postulate that the "error" fields have to be sufficiently small. Thus the problem of reliability of the solutions has the same character in the constraint approach to the shell theory as in the case in which the shell theory is obtained from the formal approximation of the solid mechanics relations.

1.4. Some open questions

The problems in the general mechanics of plates and shells are connected mainly with the approximation and the constraint approaches (which have many common features, cf. Chapter A) because the direct approach seems to have rather theoretical meaning. Among these problems we shall mention the following:

1. To determine the form of functions $\underset{\sim}{p}$, $\underset{\sim}{T}_R$, $\underset{\sim}{\phi}^a$, $\underset{\sim}{\psi}^{a\alpha}$. etc. in the approximate and constraint approaches, which lead to the reasonable form of solutions to the different classes of problems.

2. To estimate the applicability of the different shell theories to the different classes of special problems by comparing the results obtained from the approximate and constraint approaches with those of the classical solid mechanics

3. Introducing the functional sequences of the form $\tilde{\underset{\sim}{p}} = \tilde{\underset{\sim}{p}}(\underset{\sim}{\theta}, \xi, q_{(n)})$, where "n" runs over certain infinite subsequence of the sequence 1,2,3,..., to evaluate the influence of the integer "n" on the solutions of special problems and to analyse the convergence problem when $n \to \infty$.

[1] We have tacitly assumed here that the special problem under consideration has been correctly stated and its solution is physically reasonable.

4. To give the correct formulations of the different special problems within the shell theories.

5. To find the conditions that ensure the existence, stability and, if needed, the uniqueness of the solutions to the special shell problems.

The problems mentioned above have been treated successfully only in some special situations mainly in the theory of small deformations or in the theory of linear elastic shells. The particulars can be found in [4,8,16,17] and for the general reviews of these problems the reader is referred to [18,26,29].

2. ALTERNATIVE APPROACH TO SPECIAL PROBLEMS

In the previous section we have assumed that a certain plate or shell theory is known and we have described the formulations of special problems within this theory. Now we are to assume that a certain special problem has been stated within the solid mechanics. We shall show how to pass from this "three-dimensional" statement of the problem directly to its formulation within certain shell theory. We shall also adopt the obtained results to derive the formulation of the "non-classical" problems for rods.

2.1. Statement of problems in solid mechanics

The governing relations of solid mechanics will be assumed in the form of the following field equations

$$\text{Div}\,(\nabla \underset{\sim}{p}\,\underset{\sim}{T}) + \underset{\sim}{b}_R = \rho_R \ddot{\underset{\sim}{p}} \;, \quad \underset{\sim}{T} = \underset{\sim}{T}^T \;,$$
$$(\nabla \underset{\sim}{p}\,\underset{\sim}{T})\underset{\sim}{n}_R = \underset{\sim}{p}_R \;, \qquad \text{(C2.1)}$$

and the constitutive relations

$$f_\mu(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) = 0, \quad \mu = 1,\dots m = 6 + S \;,$$
$$j\,(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)}) \le 0 \;,$$
$$\varphi(\underset{\sim}{X},\underset{\sim}{C},\underset{\sim}{T},\lambda^{(s)},\underset{\sim}{C}_o,\underset{\sim}{T}_o) \le 0 \;\text{ for every } \underset{\sim}{C}_o,\underset{\sim}{T}_o \;\text{ with} \qquad \text{(C2.2)}$$
$$j(\underset{\sim}{X},\underset{\sim}{C}_o,\underset{\sim}{T}_o,\lambda^{(s)}) \le 0 \;.$$

The foregoing relations have been discussed in Sec. 2.1. of the Chapter A and have to be interpreted here as the interrelation between the triple of the basic unknowns $(\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$, the deformation function $\underset{\sim}{p}$, the second Piola-Kirchhoff stress tensor T and the ordered set of "internal and kinematical parameters" $\lambda^{(s)}$ [1] and the external forces

[1] The fields $\lambda^1,\dots,\lambda^S$ are treated as the internal parameters (which are described by the constitutive relations $f_\mu = 0$, $\mu = 6+1, 6+S$) and the fields $\lambda^{S+1},\dots,\lambda^s$ are interpreted as the kinematical parameters, independent of the deformation function $\underset{\sim}{p}$. The integer S is known for each material, $0 \le S \le s$, cf. Sec. 2.1. of the Chapter A.

$(\underset{\sim}{p}_R, \underset{\sim}{b}_R)$ (the surface tractions $\underset{\sim}{p}_R$ and the body forces $\underset{\sim}{b}_R$). The set of all deformation functions which are admissible by Eqs. (C2.1), (C2.2) will be denoted by C (we can assume that $C := \{\underset{\sim}{p} | \det|\nabla \underset{\sim}{p} > 0\}$). The linear space of all external forces will be denoted by F.

We shall say that the special problem of the solid mechanics has been stated if the form of Eqs. (C2.2) is known and if the following conditions are satisfied:

1. There is known the non-empty subset $\overset{\circ}{C}$ of C which characterizes all motions of the solid admissible in the problem under consideration. The relation $\overset{\circ}{C} \subset C$ will be called the constraint inclusion.

2. There is know the system of external forces given by $(\hat{\underset{\sim}{p}}_R(\underset{\sim}{p}), \hat{\underset{\sim}{b}}_R(\underset{\sim}{p}))$, where $\hat{\underset{\sim}{p}}_R(.), \hat{\underset{\sim}{b}}_R(.)$ are the known functionals defined (for every $\underset{\sim}{X} \in \overline{\kappa_R(B)}$ and $t \in I$) on the set $\overset{\circ}{C}$. The external forces $(\hat{\underset{\sim}{p}}_R, \hat{\underset{\sim}{b}}_R)$ will be interpreted as the known loadings acting at the body.

3. For every $\underset{\sim}{p}, \underset{\sim}{p} \in \overset{\circ}{C}$, there is defined the non-empty subset $\overset{\circ}{F}_{\underset{\sim}{p}}$ of F which characterizes all reactions $(\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R)$ which can maintain the deformation $\underset{\sim}{p}$ admissible by the constraint inclusion (or which are due to the constraint inclusion, cf. Sec. 4 of the Chapter A). Thus the character of the sets $\overset{\circ}{F}_{\underset{\sim}{p}}$ depends on the set $\overset{\circ}{C}$.

4. For every $\underset{\sim}{p}, \underset{\sim}{p} \in \overset{\circ}{C}$, there is defined the non-empty subset $\overset{*}{F}_{\underset{\sim}{p}}$ of F characterizing all reactions $(\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R)$ acting at the body (in the motion described by $\underset{\sim}{p}$) which are not due to the constraint inclusion (for example, the forces of friction) and cannot be treated as the loadings ([1]).

At the same time we shall postulate that

$$\underset{\sim}{p}_R = \hat{\underset{\sim}{p}}_R(\underset{\sim}{p}) + \overset{\circ}{\underset{\sim}{p}}_R + \overset{*}{\underset{\sim}{p}}_R,$$
$$\underset{\sim}{b}_R = \hat{\underset{\sim}{b}}_R(\underset{\sim}{p}) + \overset{\circ}{\underset{\sim}{b}}_R + \overset{*}{\underset{\sim}{b}}_R. \qquad (C2.3)$$

([1]) The forces, which are defined by the relations of the form $\hat{\underset{\sim}{p}}_R = \hat{\underset{\sim}{p}}_R(\underset{\sim}{p})$ will be always interpreted here as the loadings.

Thus every special problem within the solid mechanics will be described by Eqs. (C2.1), (C2.2), by Eqs. (C2.3) and by the conditions

$$(\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R) \in \overset{\circ}{F}_{\underset{\sim}{p}} \quad , \quad (\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R) \in \overset{*}{F}_{\underset{\sim}{p}}$$
$$\underset{\sim}{p} \in \overset{\circ}{C} \, . \tag{C2.4}$$

We can observe that the special problems of solid mechanics (in the sense described above) are determined by the form of Eqs. (C2.2), by the set $\overset{\circ}{C}$, by the functionals $\hat{\underset{\sim}{p}}_R(.)$, $\hat{\underset{\sim}{b}}_R(.)$ and by the sets of external forces $\overset{\circ}{F}_{\underset{\sim}{p}}$, $\overset{*}{F}_{\underset{\sim}{p}}$, defined for every $\underset{\sim}{p} \in \overset{\circ}{C}$ (1). The pairs $(\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R)$, $(\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R)$ will be called the constraint reactions and the field reactions, respectively. Thus the constraint and field reactions are determined, in every problem under consideration, by the multifunctions

$$\overset{\circ}{C} \ni \underset{\sim}{p} \mapsto \overset{\circ}{F}_{\underset{\sim}{p}} \subset F \quad , \quad \overset{\circ}{C} \ni \underset{\sim}{p} \mapsto \overset{*}{F}_{\underset{\sim}{p}} \subset F . \tag{C2.5}$$

If $\overset{\circ}{C} = C$ then there are no constraint reactions and we shall assume that $\overset{\circ}{F}_{\underset{\sim}{p}} = \{(\underset{\sim}{\theta}, \underset{\sim}{\theta})\}$ for every $\underset{\sim}{p} \in \overset{\circ}{C}$. The constraint inclusion $\overset{\circ}{C} \subset C$ together with the multifunction

$$\overset{\circ}{C} \ni \underset{\sim}{p} \mapsto \overset{\vee}{F}_{\underset{\sim}{p}} \subset F \tag{C2.6}$$

where $\overset{\vee}{F}_{\underset{\sim}{p}} \equiv \overset{\circ}{F}_{\underset{\sim}{p}} \oplus \overset{*}{F}_{\underset{\sim}{p}}$, (i.e., $(\overset{\vee}{\underset{\sim}{p}}_R, \overset{\vee}{\underset{\sim}{b}}_R) = (\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R) + (\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R)$), define the constraints imposed on the solid body. Examples of constraints in solid mechanics can be found in [45] and the related papers (cf. also Sec. 4.1. of the Chapter A and examples below). In the special problems of solid mechanics analysed here we shall take into account exclusively the constraints describing the situations which can be interpreted as "physical". These constraints, however, can also include all idealizations of the problems which are usually met is solid mechanics. For example, in determination of $\overset{\circ}{C}$ we can assume that the supports of the body are absolutely rigid or the the material of the body is incompressible. On the other hand, we shall not introduce here the constraints solely in order to simplify the description of the problem

(1) The initial conditions for $(\underset{\sim}{p}, \underset{\sim}{T}, \lambda^{(s)})$ can be either stated independently of the formulation of a special problem or can be included into this formulations, cf. Remark 1 of the Sec. 1.1. of this Chapter

(for example the constraints leading to the theories of plates, shells or rods, cf. Sec. 4 of the Chapter A) (1).

Example 1. Let $\partial\kappa_R(\mathcal{B}) = \bar{S}_1 \cup \bar{S}_2$, $S_1 \cap S_2 = \phi$. The constraints given by

$$\overset{o}{C} := \{\underset{\sim}{p} | \underset{\sim}{p}|_{S_1} \text{ is known}\},$$

$$\overset{v}{\underset{\sim}{F}}_p := \{(\overset{v}{\underset{\sim}{p}}_R, \overset{v}{\underset{\sim}{b}}_R) \mid \overset{v}{\underset{\sim}{b}}_R = \underset{\sim}{\theta}, \quad \overset{v}{\underset{\sim}{p}}_R|_{S_2} = \underset{\sim}{\theta}\},$$

lead to the well known classical boundary value problems. We have here $\overset{v}{\underset{\sim}{F}}_p = \overset{o}{\underset{\sim}{F}}_p$, $\overset{*}{\underset{\sim}{F}}_p = \{\theta\}$ (i.e., there are only the constraint reactions) and the multifunction (C2.6) is constant.

Example 2. Let the solid body can move only in the region Ω of the reference space bounded by the smooth surface $\partial\Omega$ with interior normal $\underset{\sim}{n}$ and let the contact between the body and the surface $\partial\Omega$ be frictionless. Then

$$\overset{o}{C} := \{\underset{\sim}{p} | \underset{\sim}{p}(\underset{\sim}{X},t) \in \Omega \text{ for } \underset{\sim}{X} \in \kappa_R(\mathcal{B}) \text{ and } \underset{\sim}{p}(\underset{\sim}{X},t) \in \bar{\Omega}$$

$$\text{for } X \in \partial\kappa_R(\mathcal{B}) \quad , t \in I\} ,$$

$$\overset{v}{\underset{\sim}{F}}_p := \{(\overset{v}{\underset{\sim}{p}}_R, \overset{v}{\underset{\sim}{b}}_R) \mid \overset{v}{\underset{\sim}{b}}_R = \underset{\sim}{\theta} \ , \ \underset{\sim}{p}(\underset{\sim}{X},t) \in \Omega \Rightarrow \overset{v}{\underset{\sim}{p}}_R(\underset{\sim}{X},t) = \underset{\sim}{0} \text{ and}$$

$$\underset{\sim}{p}(\underset{\sim}{X},t) \in \partial\Omega \Rightarrow \overset{v}{\underset{\sim}{p}}_R(\underset{\sim}{X},t) = \lambda_R(\underset{\sim}{X},t)\underset{\sim}{n} \text{ for almost every}$$

$$\underset{\sim}{X} \in \partial\kappa_R(\mathcal{B}) \ , \ t \in I \text{ and for some } \lambda_P, \lambda_R \geq 0\} .$$

Here also $\overset{v}{\underset{\sim}{F}}_p = \overset{o}{\underset{\sim}{F}}_p$ but the multifunction (C2.6) is not constant.

2.2. General form of governing relations

The formulation of special problems introduced above is too general for the further analysis because it does not give any informations about the form of the multifunctions (C2.5). In what follows we are to deal with a certain class of these multifunctions. To describe this

---

(1) Such constraints in [1] have been called "simplifying" constraints. It must be stressed, that the terms "physical" and "simplifying" are related to the interpretation of certain phenomena within the mechanics and often may here rather pure conventional character, cf. Sec. 4.1. of the Chapter A.

class let us introduce the space D of the vector functions defined and continuous on $\bar{\kappa}_R(\mathcal{B})$ and differentiable in $\overset{\circ}{\kappa}_R(\mathcal{B})$. Let us also denote $\underset{\sim}{p}_t \equiv \underset{\sim}{p}(.,t)$ for every $t \in I$ and every $\underset{\sim}{p} \in \overset{\circ}{C}$. To interrelate the constraint inclusion $\overset{\circ}{C} \subset C$ with the sets of the constraint reactions $\overset{\circ}{F}_{\underset{\sim}{p}}$, $\underset{\sim}{p} \in \overset{\circ}{C}$, we shall firstly assign to every $\underset{\sim}{p}_t$ the suitably chosen non-empty subset $H_{\underset{\sim}{p}_t}$ of D; elements $\underset{\sim}{h}$ of $H_{\underset{\sim}{p}_t}$ will be called the test functions. The well known example of the test functions in mechanics are the virtual displacements. We shall characterize the multifunction $(C2.5)_1$ by means of the following assumption.

<u>Assumption 1</u>. The sets $\overset{\circ}{F}_{\underset{\sim}{p}}$, $\underset{\sim}{p} \in \overset{\circ}{C}$, of the constraint reactions $(\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R)$ are given by

$$\overset{\circ}{F}_{\underset{\sim}{p}} := \{ (\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R) \mid \oint_{\partial\kappa_R(\mathcal{B})} \overset{\circ}{\underset{\sim}{p}}_R \cdot \underset{\sim}{h}\, da_R + \int_{\kappa_R(\mathcal{B})} \overset{\circ}{\underset{\sim}{b}}_R \cdot \underset{\sim}{h}\, dv_R \geq 0 \text{ for} \tag{C2.7}$$

$$\text{every } \underset{\sim}{h} \in H_{\underset{\sim}{p}_t}, \ \underset{\sim}{p} \in \overset{\circ}{C} \text{ and } t \in I\} .$$

The foregoing assumption states that the total rate of work of the constraint reactions on an arbitrary "virtual displacement" $\underset{\sim}{h}$ is always non-negative.

<u>Example</u>. If the sets $\overset{\circ}{C}_t$ of all $\underset{\sim}{p}_t$ (where $\underset{\sim}{p} \in \overset{\circ}{C}$) are convex in D then we may assume that

$$H_{\underset{\sim}{p}_t} := \{h \mid h = \bar{\underset{\sim}{p}}_t - \underset{\sim}{p}_t \text{ for some } \bar{\underset{\sim}{p}}_t \in \overset{\circ}{C}_t\}$$

for every $\underset{\sim}{p}_t$, $t \in I$.

If the constraint are bilateral at $\underset{\sim}{p}$ (cf. Sec. 4 of the Chapter A), i.e., if $(\overset{\circ}{\underset{\sim}{p}}_R, \overset{\circ}{\underset{\sim}{b}}_R) \in \overset{\circ}{F}_p \Rightarrow (-\overset{\circ}{\underset{\sim}{p}}_R, -\overset{\circ}{\underset{\sim}{b}}_R) \in \overset{\circ}{F}_p$, then $\underset{\sim}{h} \in H_{p_t} \Rightarrow -\underset{\sim}{h} \in H_{p_t}$, $t \in I$. The inequality in Eq. (C2.7) can be then replaced by the equality and the total rate of work of constraint reactions can an arbitrary "virtual displacement" is equal to zero.

In order to give the description of the multifunction $(C2.5)_2$ we shall assume that the field reactions $(\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R)$ are due to certain systems $(\underset{\sim}{p}_R^{(\mu)}, \underset{\sim}{b}_R^{(\mu)})$, $\mu = 1,\ldots,m$, of the external forces which may act at the body and which, roughly speaking, "control" the motion of the body. That is why the systems $(\underset{\sim}{p}_R^{(\mu)}, \underset{\sim}{b}_R^{(\mu)})$ will be called the control forces. We shall assume that in every problem under consideration the instant values of the control forces are determined by certain known functionals defined on the sets $\overset{o}{C}_t$, i.e., that $\underset{\sim}{p}_R^{(\mu)} = \underset{\sim}{p}_R^{(\mu)}(\underset{\sim}{p}_t)$, $\underset{\sim}{b}_R^{(\mu)} = \underset{\sim}{b}_R^{(\mu)}(\underset{\sim}{p}_t)$, where $\underset{\sim}{p}_R^{(\mu)}(.)$, $\underset{\sim}{b}_R^{(\mu)}(.)$ are, for every $t \in I$, the known functionals. To every system $(\underset{\sim}{p}_R^{(\mu)}, \underset{\sim}{b}_R^{(\mu)})$, $\mu = 1,\ldots,m$, and every t, $t \in I$, we shall assign the subset $D_t^{(\mu)}$ of D of the virtual displacements on which the control forces do the work. Denoting

$$\delta_t^{(\mu)}\underset{\sim}{h} = \begin{cases} \underset{\sim}{h} & \text{if } \underset{\sim}{h} \in D_t^{(\mu)} \\ \underset{\sim}{\theta} & \text{if } \underset{\sim}{h} \quad D\backslash D_t^{(\mu)} \end{cases}$$

we introduce the functionals

$$I_{\underset{\sim}{p}_t}(\underset{\sim}{h}) \equiv \sum_{\mu=1}^{m} \Big[\oint_{\partial\kappa_R(\mathcal{B})} \underset{\sim}{p}_R^{(\mu)} \cdot \delta_t^{(\mu)}\underset{\sim}{h}\, da_R + \int_{\kappa_R(\mathcal{B})} \underset{\sim}{b}_R^{(\mu)} \cdot \delta_t^{(\mu)}\underset{\sim}{h}\, dv_R \Big] \qquad \text{(C2.8)}$$

defined on D for every $\underset{\sim}{p}_t$, $t \in I$ (such that $\underset{\sim}{p} \in \overset{o}{C}$) and every $t \in I$. The functionals (C2.8) represent the rate of work of all control forces. In order to interrelate the unknown field reactions with the control forces we shall introduce the following assumption.

<u>Assumption 2</u>. The sets $\overset{*}{F}_{\underset{\sim}{p}}$, $\underset{\sim}{p} \in \overset{o}{C}$, of the field reactions $(\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R)$ are given by

$$\overset{*}{F}_{\underset{\sim}{p}} := \Big\{ (\overset{*}{\underset{\sim}{p}}_R, \overset{*}{\underset{\sim}{b}}_R) \;\Big|\; \oint_{\partial\kappa_R(\mathcal{B})} \overset{*}{\underset{\sim}{p}}_R \cdot \underset{\sim}{h}\, da_R + \int_{\kappa_R(\mathcal{B})} \overset{*}{\underset{\sim}{b}}_R \cdot \underset{\sim}{h}\, dv_R \geq J_{\underset{\sim}{p}_t}(\underset{\sim}{h}) \qquad \text{(C2.9)}$$

$$\text{for every } \underset{\sim}{h} \in D, \quad \underset{\sim}{p} \in \overset{o}{C} \quad \text{and} \quad t \in I \Big\}$$

This assumption states that the total rate of work of the field reactions on an arbitrary displacement is never smaller then the corresponding total rate of work of the control forces. We see that if all control forces are equal to zero then the field reactions will be also equal to zero.

It must be stressed that both assumptions are based on certain physical premises and generalize many different situations connected with the descriptions of the various special problems of mechanics. They do not express the laws of mechanics but rather have the character of the constitutive hypothesis which define certain class of multifunctions (C2.5). To this class we shall confine ourselves in this Chapter.

Let us observe that in view of $H_{\underset{\sim}{p}_t} \subset D$ for every $\underset{\sim}{p}_t$, Eq. (C2.9) holds also for every $h \in H_{\underset{\sim}{p}_t}$. Then taking into account Eqs. (C2.9), (C2.7), (C2.3) and (C2.1) we obtain the inequality

$$\int_{\kappa_R(\mathcal{B})} \operatorname{tr}(\underset{\sim}{T}_R \nabla \underset{\sim}{h})\, dv_R \geq \oint_{\partial\kappa_R(\mathcal{B})} \hat{\underset{\sim}{p}}_R \cdot \underset{\sim}{h}\, da_R + \int_{\kappa_R(\mathcal{B})} (\hat{\underset{\sim}{b}}_R - \rho_R \ddot{\underset{\sim}{p}}) \cdot \underset{\sim}{h}\, dv_R + J_{\underset{\sim}{p}_t}(h) \tag{C2.10}$$

in which $\underset{\sim}{T}_R \equiv \nabla \underset{\sim}{p}\underset{\sim}{T}$ and which has to hold for every $\underset{\sim}{h} \in H_{\underset{\sim}{p}_t}$, $t \in I$ and $\underset{\sim}{p} \in \overset{o}{C}$. The inequality (C2.10) will be referred to as the basic inequality of the solid mechanics problem under consideration and has to be considered together with the constitutive relations (C2.2) and with the condition $\underset{\sim}{p} \in \overset{o}{C}$. If the material of the body is simple, i.e., if Eqs. (C2.2) reduce to the form

$$\underset{\sim}{T}_R = \tilde{\underset{\sim}{T}}_R(\underset{\sim}{X}, \nabla\underset{\sim}{p}(\underset{\sim}{X}, t-s)), \quad s \geq 0 \quad , \tag{C2.11}$$

where $\tilde{\underset{\sim}{T}}_R(\underset{\sim}{X}, \cdot)$ is the known reponse functional for every $\underset{\sim}{X} \in \kappa_R(\mathcal{B})$ then the special problems of solid mechanics (in the sense described in Sec. 2.1) will lead ot the inequality obtained by substitution of the RHS of Eq. (C2.11) into Eq. (C2.10)

Summing up, the general form of the governing relations for the special problems under consideration is given by Eqs. (C2.10), (C2.2) and by the condition $\underset{\sim}{p} \in \overset{o}{C}$. For the simple materials it reduces to Eq. (C2.10) (where the first Piola-Kirchhoff stress tensor $\underset{\sim}{T}_R$ is given by Eq. (C2.11)) and to the condition $\underset{\sim}{p} \in \overset{o}{C}$. For linear elastic bodies the analysis of the "unilateral" special problems can be found in [10] and for linear visco-elastic and elastic-plastic bodies some basic "non-classical" problems have been investigated in [8].

2.3. Passage to plate and shell problems

Now let us assume that $\mathcal{B}$ is the shell-like body, i.e., that $(\mathcal{B},\kappa_R)$ represent certain parametrized shell (cf. the Prerequisites). Then the question arises how to describe (at least, from the formal point of view) the special problems of solid mechanics, governed by Eqs. (C2.10), (C2.2), in term of functions which are independent of the material coordinate ξ, $\xi \in (h_-,h_+)$. We shall answer this question using the formal approximation procedure developed in Sec. 2 of the Chapter A. To this aid let us introduce the set P of deformation functions of the shell-like body $\mathcal{B}$, putting

$$P := \{\underset{\sim}{p} \,|\, \underset{\sim}{p} = \tilde{\underset{\sim}{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)}) \text{ for some } q_{(n)} \in Q\} \tag{C2.12}$$

where $\tilde{\underset{\sim}{p}}(\cdot)$ is the known sufficiently regular function and Q is the set of all $q_{(n)} = \{q_a(\underset{\sim}{\theta},t),\ \underset{\sim}{\theta} \in \bar{\Pi},\ t \in I,\ a = 1,\dots n\}$ such that $\det \nabla\tilde{\underset{\sim}{p}} > 0$ for every $q_{(n)} \in Q$. We shall assume that $P \cap \overset{o}{C} \neq \phi$ and denoting $\tilde{C} \equiv$ $\equiv P \cap \overset{o}{C}$ we shall also assume that the set $\tilde{C}$ can be given by

$$\tilde{C} := \{\underset{\sim}{p} \,|\, \underset{\sim}{p} = \tilde{\underset{\sim}{p}}(\underset{\sim}{X}, q_{(n)}, \nabla q_{(n)}) \quad \text{for some } q_{(n)} \in \overset{o}{Q}\} \tag{C2.13}$$

where $\overset{o}{Q}$ is the known subset of Q.

Let S be a set of the triples $s \equiv (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$ which are the solutions of Eqs. (C2.2). We shall "approximate" ([1]) the set $\tilde{S}$ by the set S of the triples $\tilde{s} \equiv (\tilde{\underset{\sim}{p}},\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)})$, such that $\tilde{\underset{\sim}{p}} \in \tilde{C}$ and $\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)}$ are determined by the RHS of Eqs. (A2.6), (A2.7), respectively. Then using the formal approximation procedure determined by Eqs. $(A2.8)_{4-6}$, (A2.9), $(A2.10)_{3-5}$, (A2.13), we shall arrive at the constitutive relations (A2.14) which can be interpreted as certain "shell approximation" of the constitutive relations (C2.2) of the solid mechanics.

In the general case the pair $(\tilde{\underset{\sim}{p}},\tilde{\underset{\sim}{T}})$ satisfying (C2.10) for every $\underset{\sim}{h} \in H_{\underset{\sim}{p}_t}$ and such that $(\tilde{\underset{\sim}{p}},\tilde{\underset{\sim}{T}},\tilde{\lambda}^{(s)}) \in S$ may not exist. Thus we have to "approximate" Eq. (C2.10). To this aid we shall define the subset H of the space D putting

([1]) Here and in what follows the "approximation" is understand in the sense of the formal approximation procedure, described in Sec. 2 of the Chapter A. This procedure can be also interpreted in terms of the constraint approach, cf. Sec. 4 of the Chapter A.

$$H := \{\underset{\sim}{h} | h_k = \phi_k^a v_a + \psi_k^{a\alpha} v_{a,\alpha} \quad \text{for some } v_{(n)} \in V\} \tag{C2.14}$$

where V is the linear space of the functions $v_{(n)} = \{v_a, a = 1,...,n\}$ defined and continuous in $\bar{\Pi}$ and smooth in Π, and where $\underset{\sim}{\phi}^a$, $\underset{\sim}{\psi}^{a\alpha}$ are the known functions introduced in Sec. 2.3. of the Chapter A. Apart from the conditions given in Chapter A we shall assume that $H_{\underset{\sim}{p}_t} \cap H \neq \phi$ for every $\underset{\sim}{p}_t$. Denoting $\tilde{H}_{\underset{\sim}{p}_t} \equiv H_{\underset{\sim}{p}_t} \cap H$ we shall also assume that the sets $\tilde{H}_{\underset{\sim}{p}_t}$ are given by

$$\tilde{H}_{\underset{\sim}{p}_t} := \{\underset{\sim}{h} | h_k = \phi_k^a v_a + \psi_k^{a\alpha} v_{a,\alpha} \quad \text{for some } v_{(n)} \in V_{q_{(n)t}}\} \tag{C2.15}$$

where $V_{q_{(n)t}}$ are the known subsets of V, defined for every $q_{(n)t} \equiv q_{(n)}(\cdot,t)$, $q_{(n)} \in \overset{o}{Q}$.

Let us observe that the conditions $P \cap \overset{o}{C} = \tilde{C}$, $H \cap H_{\underset{\sim}{p}_t} = \tilde{H}_{\underset{\sim}{p}_t}$ (where $\tilde{C}$, $\tilde{H}_{\underset{\sim}{p}_t}$ are given by Eqs. (C2.13), (C2.15), respectively), determining the formal approximation, modify the character of the special problem of solid mechanics under consideration. It follows that not all problems of the solid mechanics for the shell-like bodies, which are described by Eqs. (C2.10), (C2.2), can successfully treated by the relations independent of the material coordinate ξ.

Now restricting the sets $H_{\underset{\sim}{p}_t}$ of Eqs. (C2.10) to the sets $\tilde{H}_{\underset{\sim}{p}_t}$ given by Eqs. (C2.15), after simple calcualtions we arrive at the relation (C1.10) which has to hold for every $v_{(n)} \in V_{q_{(n)t}}$, $q_{(n)} \in \overset{o}{Q}$, $t \in I$, and where:

1. $\tilde{H}_R^{a\alpha\beta}, \tilde{H}_R^{a\alpha}, \tilde{h}_R^a$ are defined by Eqs. (A2.29) $\underset{\sim}{\tilde{g}}$ being defined by Eqs. (A2.6).

2. $\hat{p}_{OR}^a$, $\hat{p}_R^{a\underset{\sim}{N}}$, $\hat{f}_R^a$, $\hat{f}_R^{a\alpha}$ are the shell loadings, defined by

$$\hat{p}_{OR}^a \equiv \hat{p}_R^a - \frac{d}{dl_R}(\hat{p}_R^{a\alpha} t_{R\alpha}), \quad \hat{p}_R^{a\underset{\sim}{N}} \equiv \hat{p}_R^{a\alpha} n_{R\alpha}$$

   and by the RHS of Eqs. $(A2.24)_{4,5}$, (A2.25) in which in the places of b_R^k, p_R^k we have to substitute $\hat{b}_R^k$, $\hat{p}_R^k$, respectively.

3. i_R^a, $i_R^{a\alpha}$ are the shell inertia forces defined by Eqs. $(A2.24)_{6,7}$ for $q_{(n)} \in \overset{o}{Q}$.

4. The functional $J_{q_{(n)t}}$ is defined by

$$J_{q_{(n)t}}(v_{(n)}) = \sum_{\mu=1}^{m} \oint_{\partial\Pi} [(p_R^{(\mu)a}\delta_t^{(\mu)}v_a + p_R^{(\mu)a\alpha}\delta_t^{(\mu)}v_{a,\alpha})dl_R + \tag{C2.16}$$

$$+ \int_{\Pi} (f_R^{(\mu)a}\delta_t^{(\mu)}v_a + f_R^{(\mu)a\alpha}\delta_t^{(\mu)}v_{a,\alpha})da_R]$$

where $p_R^{(\mu)a}$, $p_R^{(\mu)a\alpha}$, $f_R^{(\mu)a}$, $f_R^{(\mu)a\alpha}$ are determined by the RHS of Eqs. (A2.25), (A2.24)$_{4,5}$ in which in the places of p_R^k, b_R^k we have to substitute $p_R^{(\mu)k}$, $b_R^{(\mu)k}$, respectively. Moreover, we have introduced here the denotation

$$\delta_t^{(\mu)}v_{(n)} = \begin{cases} v_{(n)} & \text{if } v_{(n)} \in V_t^{(\mu)} \\ \underset{\sim}{0} & \text{if } v_{(n)} \in V \backslash V_t^{(\mu)} \end{cases} \tag{C2.17}$$

where $V_t^{(\mu)}$ are the non-empty subsets of V, such that the sets $\tilde{D}_t^{(\mu)} \equiv D_t^{(\mu)} \cap H$ can be given by

$$\tilde{D}_t^{(\mu)} := \{\underset{\sim}{h} | h_k = \Phi_k^a v_a + \Psi_k^{a\alpha} v_{a,\alpha} \text{ for some } v_{(n)} \in V_t^{(\mu)}\} . \tag{C2.18}$$

Summing up, the special plate or shell problems are governed by $q_{(n)} \in \overset{o}{Q}$, by the condition (C1.10) and by the shell constitutive relations (A2.14). All fields in these problems are independent of the material coordinate ξ.

Remark 1. Let us observe that the formal structure of the governing relations for the special plate or shell problems obtained above is the same as that describing the special problems of the plate or shell theory which has been obtained in Sec. 1.1. of this Chapter (cf. Eqs. (C1.10),(A5.6)). However, the shell loadings p_R^a, $p_R^{a\alpha}$, f_R^a, $f_R^{a\alpha}$ and the control forces $\hat{p}_R^{(\mu)a}$, $\hat{p}_R^{(\mu)a\alpha}$, $\hat{f}_R^{(\mu)a}$, $\hat{f}_R^{(\mu)a\alpha}$, which in Sec. 1.1. have been assumed *a priori* as certain functionals of $q_{(n)}$, now are related to the loadings $(\hat{p}_R, \hat{b}_R)$ and to the control forces $(p_R^{(\mu)}, b_R^{(\mu)})$ of the shell-like body.

Remark 2. The comments concerning the relability of solutions which have been given in Sec. 1.3. remain valid also for the solutions of problems described above.

2.4. Passage to rod problems

Now let us assume that $\mathcal{B}$ is the rod-like body (cf. the Prerequisites). We shall try to describe the solid mechanics problems, governed by Eqs. (C2.10), (C2.2), in term of functions which are independent of the material coordinates $\underset{\sim}{\theta} = (\theta^1,\theta^2)$, $\underset{\sim}{\theta} \in \Pi$. Making use of the analogy between the formation of the rod theories and the shell theories (cf. Sec. 5.3. of the Chapter A) we shall apply the procedure which is analogous to that used in the Sec. 2.3. of this Chapter. Firstly, we shall introduce the set P of the deformation functions of the rod-like body $\mathcal{B}$ by the formula of the form (C2.12) in which Q is the set of all rod deformation functions $q_{(n)} = \{q_a(\xi,t),\ \xi \in \langle h_-,h_+\rangle,\ t \in I,\ a = 1,\dots,n\}$. It means that $Q = \{q_{(n)}\,|\det \nabla\underset{\sim}{\tilde{p}}(\theta,\xi,q_{(n)},q_{(n),3}) \geq 0\}$, where $\underset{\sim}{\tilde{p}} = \underset{\sim}{\tilde{p}}(\underset{\sim}{\theta},\xi,q_{(n)},q_{(n),3})$ is the known sufficiently regular function. We shall define the set $\tilde{C}$ putting $\tilde{C} \equiv P \cap \overset{\circ}{C}$ and assume that $\tilde{C} \neq \phi$ and that

$$\tilde{C} := \{\underset{\sim}{p}|\underset{\sim}{p} = \underset{\sim}{\tilde{p}}(\underset{\sim}{X},q_{(n)},q_{(n),3}) \text{ for some } q_{(n)} \in \overset{\circ}{Q}\}$$

where $\overset{\circ}{Q}$ is the known subset of Q.

Secondly, we shall "approximate" the set S of all solutions $s = (\underset{\sim}{p},\underset{\sim}{T},\lambda^{(s)})$ of Eqs. (C2.2) by the set $\tilde{S}$ of the triples $\tilde{s} = (\underset{\sim}{\tilde{p}},\underset{\sim}{\tilde{T}},\tilde{\lambda}^{(s)})$, putting $\underset{\sim}{\tilde{p}} \in \tilde{C}$ and assuming that $\underset{\sim}{\tilde{T}}$, $\tilde{\lambda}^{(s)}$ are determined by the relations obtained from Eqs. (A2.6), (A2.7) by replacing the arguments $\tau^{(N)}(\underset{\sim}{\theta},t)$, $\omega^{(p)}(\underset{\sim}{\theta},t)$ by $\tau^{(N)}(\xi,t)$, $\omega^{(p)}(\xi,t)$, respectively. Using the same procedure as in the previous section we shall arrive at the constitutive relations (A5.16). They constitute certain "rod approximation" of the constitutive relations (C2.2) of the solid mechanics.

Thirdly, we shall introduce the set H in the space D by the formula

$$H := \{\underset{\sim}{h}|h_k = \Phi^a_k v_a + \Psi^a_k v_{a,3} \text{ for some } v_{(n)} \in V\} \tag{C2.19}$$

in which V is the linear normed space of the sufficiently regular functions $v_{(n)} = \{v_a,\ a = 1,\dots,n\}$ defined on $\langle h_-,h_+\rangle$; the symbols $\underset{\sim}{\Phi}^a,\underset{\sim}{\Psi}^a$

stand for the known fields introduced in Sec. 5.3. of the Chapter A. Putting $\tilde{H}_{\underset{\sim}{p}_t} \equiv H_{\underset{\sim}{p}_t} \cap H$ we shall assume that $\tilde{H}_{\underset{\sim}{p}_t}$ are the non-empty sets given by

$$\tilde{H}_{\underset{\sim}{p}_t} := \{\underset{\sim}{h} | h_k = \Phi^a_k v_a + \Psi^a_k v_{a,3} \text{ for some } v_{(n)} \in V_{q_{(n)t}}\} \qquad \text{(C2.20)}$$

where $V_{q_{(n)t}}$ are the known subsets of V defined for every $q_{(n)t} = q_{(n)}(.,t)$, $q_{(n)} \in \overset{o}{Q}$. (In Eq. (C2.20) the function $\underset{\sim}{p}_t$ and $q_{(n)t}$ are inter-related by $\underset{\sim}{p}_t = \underset{\sim}{\tilde{p}}(\underset{\sim}{\theta},\xi,q_{(n)t},q_{(n)t,3})$.

At least, replacing the sets $H_{\underset{\sim}{p}_t}$ in Eqs. (C2.10) by the sets $\tilde{H}_{\underset{\sim}{p}_t}$ introduced above, after some calcualtions we obtain the relation

$$\int_{h_-}^{h_+} (-"\tilde{H}^a_R v_{a,33} + '\tilde{H}^a_R v_{a,3} - \tilde{h}^a_R v_a)\, d\xi \geq$$

$$\geq [\hat{p}^a_R v_a + '\hat{p}^a_R v_{a,3}]_{h_-} + [\hat{p}^a_R v_a + '\hat{p}^a_R v_{a,3}]_{h_+} + \qquad \text{(C2.21)}$$

$$+ \int_{h_-}^{h_+} [('\hat{f}^a_R - 'i^a_R) v_{a,3} + (\hat{f}^a_R - i^a_R) v_a] d\xi + J_{q_{(n)t}}(v_{(n)}) ,$$

which has to be satisfied for every $v_{(n)} \in V_{q_{(n)t}}$, $q_{(n)} \in \overset{o}{Q}$, $t \in I$, where

$$J_{q_{(n)t}}(v_{(n)}) \equiv \sum_{\mu=1}^{m} \{[p_R^{(\mu)a} \delta_t^{(\mu)} v_a + 'p_R^{(\mu)a} \delta_t^{(\mu)} v_{a,3}]_{h-} +$$

$$+ [p_R^{(\mu)a} \delta_t^{(\mu)} v_a + 'p_R^{(\mu)a} \delta_t^{(\mu)} v_{a,3}]_{h_+} + \qquad \text{(C2.22)}$$

$$+ \int_{h_-}^{h_+} (f_R^{(\mu)a} \delta_t^{(\mu)} v_a + 'f_R^{(\mu)a} \delta_t^{(\mu)} v_{a,3})\, d\xi \}$$

and where:

1. $"\tilde{H}^a_R$, $'\tilde{H}^a_R$, $\tilde{h}^a_R$ coincide with the RHS of Eqs. (A5.15),

2. $\hat{p}^a_R$, $'\hat{p}^a_R$, $\hat{f}^a_R$, $'\hat{f}^a_R$ are the rod loadings defined by the RHS of Eqs. (A5.14), $(A5.12)_{4,5}$ in which $\underset{\sim}{b}_R$, $\underset{\sim}{p}_R$ have to be replaced by $\hat{\underset{\sim}{b}}_R$, $\hat{\underset{\sim}{p}}_R$.

3. i^a_R, $'i^a_R$ are the rod inertia forces defined by Eqs. $(A5.12)_{7,8}$ [1].

Moreover, in Eq. (C2.22) we have used the denotations (C2.17), where $V^{(\mu)}_t$ are the non-empty subsets of V such that the sets $\tilde{D}^{(\mu)}_t \equiv D^{(\mu)}_r \cap H$ are defined by

$$\tilde{D}^{(\mu)}_t := \{\underset{\sim}{h} | h_k = \Phi^a_k v_a + \Psi^a_k v_{a,3} \quad \text{for some} \quad v_{(n)} \in V^{(\mu)}_t\} . \qquad \text{(C2.23)}$$

We have to remember that the symbols V, $V_{q_{(n)}t}$, $V^{(\mu)}_t$ denote now the sets of functions defined on $\langle h_-, h_+\rangle \times I$ and $q_{(n)}$ is now the rod deformation function.

The relation (C2.21) is independent of the material coordinates $\underset{\sim}{\theta} = (\theta^1, \theta^2)$. Thus the special rod problem is governed by the condition $q_{(n)} \in \overset{\circ}{Q}$, by Eq. (C2.21) and by the rod constitutive relations (A5.16). For the simple materials Eqs. (A5.16) are identities (cf. Sec. 5.3. of the Chapter A) and the problems under considerations are described exclusively by the inequality (C2.21) with the condition $q_{(n)} \in \overset{\circ}{Q}$.

Remark 1. Using the general analogy between the formation of the shell theories and the rod theories (cf. Sec. 5.3. of the Chapter A) we can easily apply the results of Sec. 1.2. of this Chapter (concerning the constraint functions) to the special rod problems.

Remark 2. The question of the relability of solutions of Eqs. (C2.21), (A5.16)(with the condition $q_{(n)} \in \overset{\circ}{\underset{\sim}{Q}}$) has to be treated analogously as in the case of the special plate and shell problems, cf. Sec. 1.3. of this Chapter.

Remark 3. The role of the initial conditions is analogous to that described in Sec. 1.1. of this Chapter, i.e., they may be either included into description of the special rod problems or treated separately.

---

[1] In the formulae of Sec. 5.3. of the Chapter A which are quoted here we have to assume that $\overline{d\Phi^a} = \Phi^a da_R$ or $= \Phi^a dl_R$, $\overline{d\Psi^a} = \Psi^a da_R$ or $= \Psi^a dl_R$.

3. PROBLEMS IN THE SCALAR PLATE THEORY

In this Section we shall illustrate the general consideration of Sec. 1.1. of this Chapter taking as an example the simplest plate theory, namely the scalar plate theory (cf. Sec. 1 of the Chapter B).

3.1. General formulation

We shall assume the field equations of the scalar plate theory in the form

$$H^{\alpha\beta}_{R,\alpha\beta} + H^{\alpha}_{R,\alpha} + h_R + f_R = i_R \tag{C3.1}$$

and

$$\begin{aligned} &H^{\alpha}_{R} n_{R\alpha} + \frac{d}{dl_R}(H^{\alpha\beta}_{R} t_{R\alpha} n_{R\beta}) + H^{\beta\alpha}_{R,\beta} n_{R\alpha} = p_{oR} \, , \\ &H^{\beta\alpha}_{R} n_{R\beta} n_{R\alpha} = -p^{N}_{R} \, , \end{aligned} \tag{C3.2}$$

where we denoted

$$p_{oR} \equiv p_R - \frac{d}{dl_R}(p^{\alpha}_{R} t_{R\alpha}) \, , \quad p^{N}_{R} \equiv p^{\alpha}_{R} n_{R\alpha} \, . \tag{C3.3}$$

In Eqs. (C3.1), (C3.2) we have neglected the terms with f^{α}_{R} and i^{α}_{R} in order to simplify the notations. The constitutive relations will be assumed in their general form described by

$$\begin{aligned} H^{\alpha\beta}_{R} &= \tilde{H}^{\alpha\beta}_{R}(\underset{\sim}{\theta}, q, \nabla q, \nabla^2 q, \tau^{(N)}), \\ H^{\alpha}_{R} &= \tilde{H}^{\alpha}_{R}(\underset{\sim}{\theta}, q, \nabla q, \nabla^2 q, \tau^{(N)}), \\ h_R &= \tilde{h}_R(\underset{\sim}{\theta}, q, \nabla q, \nabla^2 q, \tau^{(N)}), \end{aligned} \tag{C3.4}$$

and by Eqs. (A5.6) for $q_{(n)} \equiv q$; symbol $q = q(\underset{\sim}{\theta}, t)$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$, stands for the deflection of the undeformed plate midsurface (this midsurface coincides with the closed region $\bar{\Pi}$ on the parameter plane $x^3 = 0$). The set of all deflections will be denoted by Q. The inertia term i_R in Eqs. (C3.1) can be given by

$$i_R = \frac{d}{dt}\frac{\partial \kappa_R}{\partial \dot{q}} - \frac{\partial \kappa_R}{\partial q} \, , \tag{C3.5}$$

where $\kappa_R = \kappa_R(\underset{\sim}{\theta}, q, \dot{q})$ is the plate kinetic energy function. The function $\kappa_R(\cdot)$ as well as the functionals on the RHS of Eqs. (C3.4) and the form of Eqs. (A5.6) (with $q_{(n)} \equiv q$) are assumed to be known. By the basic unkowns we mean the plate deflection q and the functions $\tau^{(N)}$, $\omega^{(p)}$. The systems of external forces are represented by $y_R = (p_{oR}, p_{\underset{\sim}{R}}^{N}, f_R)$ and the linear space of all y_R will be denoted by Y. The system y_R will be related to the basic unknowns $(q, \tau^{(N)}, \omega^{(p)})$ by the formulation of the special problems of the scalar plate theory. The general formulation of such problems, realized within the analysis outlined in Sec. 1.1. of this Chapter, includes:

1. Determination of the set Q_o, $Q_o \subset Q$, of all admissible (in the problem under consideration) plate deflections.

2. Decomposition of the external forces $y_R = (p_{oR}, p_{\underset{\sim}{R}}^{N}, f_R)$ into the loadings $\hat{y}_R = (\hat{p}_{oR}, \hat{p}_{\underset{\sim}{R}}^{N}, \hat{f}_R)$ and reactions $\check{y}_R = (\check{p}_{oR}, \check{p}_{\underset{\sim}{R}}^{N}, \check{f}_R)$, where $p_{oR} = \hat{p}_{oR}(q)$, $\hat{p}_{\underset{\sim}{R}}^{N} = \hat{p}_{\underset{\sim}{R}}^{N}(q)$, $\hat{f}_R = \hat{f}_R(q)$ are the known functionals defined, for every $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$, on the set $\overset{o}{Q}$ of the admissible plate deflections (cf. Eq. (C1.1)).

3. Decomposition $\check{y}_R = \overset{o}{y}_R + \overset{*}{y}_R$ of the reactions $\check{y}_R$ into the constraint reactions $\overset{o}{y}_R = (\overset{o}{p}_{oR}, \overset{oN}{p}_{\underset{\sim}{R}}, \overset{o}{f}_R)$ and the field reactions $\overset{*}{y}_R = (\overset{*}{p}_{oR}, \overset{*N}{p}_{\underset{\sim}{R}}, \overset{*}{f}_R)$, cf. Eq. (C1.3), and determination of the sets $\overset{o}{Y}_q$, $\overset{*}{Y}_q$, $q \in \overset{o}{Q}$, of all $\overset{o}{y}_R$, $\overset{*}{y}_R$ by means of Eqs. (C1.2), (C1.6), respectively.

After that we can eliminate the system $y_R = (p_{oR}, p_{\underset{\sim}{R}}^{N}, f_R)$ of the plate external forces from the field equations (C3.1), (C3.2) and to obtain the final system of relations for the basic unknowns $q, \tau^{(N)}, \omega^{(p)}$ (cf. Eqs. (C1.10), (A5.6). If the plate is elastic or even simple then the argument $\tau^{(N)}$ drops out form Eqs. (C3.4) and Eqs. (A5.8) are identities (cf. Sec. 2.2 of the Chapter A).

In what follows we shall use the denotations $\overset{o}{Y}_q = \overset{o}{P}_q \times \overset{oN}{P}_{\underset{\sim}{q}} \times \overset{o}{F}_q$ and $\overset{*}{Y}_q = \overset{*}{P}_q \times \overset{*N}{P}_{\underset{\sim}{q}} \times \overset{*}{F}_q$, where $\overset{o}{p}_{oR} \in \overset{o}{P}_q$, $\overset{oN}{p}_{\underset{\sim}{R}} \in \overset{oN}{P}_{\underset{\sim}{q}}$, $\overset{o}{f}_R \in \overset{o}{F}_q$ and $\overset{*}{p}_{oR} \in \overset{*}{P}_q$, $\overset{*N}{p}_{\underset{\sim}{R}} \in \overset{*N}{P}_{\underset{\sim}{q}}$, $\overset{*N}{f}_{\underset{\sim}{R}} \in \overset{*}{F}_q$ for every $q \in \overset{o}{Q}$. We shall also denote $q_{,\underset{\sim}{N}} \equiv q_{,\alpha} n_R^{\alpha}$.

3.2. Examples of problems

We shall give now some examples of problems of the scalar plate theory. By Δ, Γ we shall denote the known smooth and open parts of $\partial\Pi$ (which may be also empty) and we shall use the denotation $\Delta_o \equiv \partial\Pi\setminus\bar{\Delta}$, $\Gamma_o \equiv \partial\Pi\setminus\bar{\Delta}$. Moreover, by K we denote the subset of Q such that for the stationary problems $K := \{q \mid \dot{q} \equiv 0\}$ and for the dynamical problems $K := \{q \mid q(\cdot,t_o) = \mathring{q}\ ,\ \dot{q}(\cdot,t_o) = \mathring{v}\}$ where $\mathring{q}, \mathring{v}$ are the known regular functions defined on $\bar{\Pi}$. We also assume that the plate is elastic and that the loadings $\hat{y}_R = (\hat{p}_{oR}, \hat{\underset{\sim}{p}}{}^N_R, \hat{f}_R)$ are known.

Example 1. The plate with the classical boundary conditions. In this case

$$\mathring{Q} := \{q \mid q \in K \text{ and } q|_\Delta = 0\ ,\ q_{,\underset{\sim}{N}}|_\Gamma = 0\}\ ;$$

$$\mathring{P}_q := \{\mathring{p}_{oR} \mid \mathring{p}_{oR}|_{\Delta_o} = 0\}\ ,\ \mathring{\underset{\sim}{P}}{}^N_q := \{\mathring{\underset{\sim}{p}}{}^N_R \mid \mathring{\underset{\sim}{p}}{}^N_R|_{\Gamma_o} = 0\}\ ,$$

$$\mathring{F}_q := \{\mathring{f}_R \mid \mathring{f}_R = 0\}\ ;$$

$$\overset{*}{Y}_q = \{\theta\}\ .$$

The sets $\mathring{Y}_q = \mathring{P}_q \times \mathring{\underset{\sim}{P}}{}^N_q \times \mathring{F}_q$ are here independent of q , $q \in \mathring{Q}$. Such problem will be referred to as the classical boundary-value problem and we shall denote $Q_{class} \equiv \mathring{Q}$, where $\mathring{Q}$ is defined above.

Example 2. The plate with the boundary conditions as in Example 1 and with partial interaction with the rigid surface.
Let $x^3 = \zeta(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $\zeta > 0$, be the rigid smooth surface situated "under" the undeformed plate. Neglecting the thickness of the plate we shall assume, that the plate is in a contact with this surface if $q(\underset{\sim}{\theta},t) = \zeta(\underset{\sim}{\theta})$. Then

$$\mathring{Q} := \{q \mid q \in Q_{class} \text{ and } q(\underset{\sim}{\theta},t) \leq \zeta(\underset{\sim}{\theta}),\ \underset{\sim}{\theta} \in \Pi,\ t \in I\}\ ,$$

$$\mathring{F}_q := \{\mathring{f}_R \mid q(\underset{\sim}{\theta},t) < \zeta(\underset{\sim}{\theta}) \Rightarrow \mathring{f}_R(\underset{\sim}{\theta},t) = 0 \text{ and } q(\underset{\sim}{\theta},t) = \zeta(\underset{\sim}{\theta}) \Rightarrow r_R(\underset{\sim}{\theta},t) \geq 0,\ \underset{\sim}{\theta} \in \Pi,\ t \in I\},$$

and $\mathring{P}_q$, $\mathring{\underset{\sim}{P}}{}^N_q$, $\overset{*}{Y}_q$ are given analogously as in Example 1.

Example 3. Unilateral kinematic boundary conditions on Δ and Γ. In this case

$$\overset{\circ}{Q} := \{q \mid q \in K \text{ and } q|_{\Delta} \leq 0 \text{ and } q_{,\underset{\sim}{N}}|_{\Gamma} \geq 0\}$$

$$\overset{\circ}{F}_q = \{0\}\ ,\ \overset{\circ}{P}_q := \{\overset{\circ}{p}_{OR} \mid q(\underset{\sim}{\theta},t) = 0 \Rightarrow \overset{\circ}{p}_{OR}(\underset{\sim}{\theta},t) \geq 0, \text{ and}$$

$$q(\underset{\sim}{\theta},t) < 0 \Rightarrow p_{OR}(\underset{\sim}{\theta},t) = 0\ ,\ \underset{\sim}{\theta} \in \Delta,\ t \in I, p_{OR}|_{\Delta_o} = 0\}$$

$$P^{\underset{\sim}{N}}_q := \{p^{\underset{\sim}{N}}_R \mid q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t) \Rightarrow \overset{\circ}{p}{}^{\underset{\sim}{N}}_R(\underset{\sim}{\theta},t) \geq 0 \text{ and } q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t) > 0 \Rightarrow \overset{\circ}{p}{}^{\underset{\sim}{N}}_R(\underset{\sim}{\theta},t) = 0$$

$$\theta \in \Gamma,\ t \in I, \overset{\circ}{p}{}^{\underset{\sim}{N}}_R|_{\Gamma_o} = 0\}\ ,$$

$$\overset{*}{Y}_q := \{\theta\}\ .$$

The foregoing unilateral conditions can be also combined with those given in Examples 1,2.

Example 4. Deflections and rotations with the friction on the boundary (static case). Assuming that there are no "rigid" supports on the boundary (i.e., of the type described in Example 1), we obtain $\overset{\circ}{Y}_q = \{\theta\}$ and

$$\overset{\circ}{Q} := \{q \mid \dot{q} = 0\}\ ,\quad \overset{\circ}{F}_q = \{\theta\}\ ,$$

$$\overset{*}{P}_q := \{\overset{*}{p}_{OR} \mid q(\underset{\sim}{\theta},t) = 0 \Rightarrow |\overset{*}{p}_{OR}(\underset{\sim}{\theta},t)| < k \text{ and } q(\underset{\sim}{\theta},t) \neq 0 \Rightarrow \overset{*}{p}_{OR}(\underset{\sim}{\theta},t) = -k,$$

$$\underset{\sim}{\theta} \in \Delta\ ,\ t \in I, \text{ and } \overset{*}{p}_{OR}|_{\Gamma_o} \equiv 0\}\ ,$$

$$\overset{*}{P}{}^{\underset{\sim}{N}}_q := \{\overset{*}{p}{}^{\underset{\sim}{N}}_R \mid q_{,N}(\underset{\sim}{\theta},t) = 0 \Rightarrow |\overset{*}{p}{}^{\underset{\sim}{N}}_R(\underset{\sim}{\theta},t)| > 1 \text{ and } q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t) \neq 0 \Rightarrow \overset{*}{p}{}^{\underset{\sim}{N}}_R(\underset{\sim}{\theta},t) =$$

$$= -\beta q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t)\ ,\ \theta \in \Gamma\ ,\ t \in I \text{ and } \overset{*}{p}{}^{\underset{\sim}{N}}_R|_{\Gamma_o} \equiv 0\},$$

where k and 1, β are certain known positive constants or functions defined on $\Delta \times I$ and $\Gamma \times I$, respectively.

Example 5. Deflection of the plate controlled by the system of forces. Suppose that the deflection of the plate (which is supported as in Example 1) in the $-x^3$ direction is controlled by the vertical force (1) with the extremal value $r_{(+)}(\underset{\sim}{\theta},t)$ and in the $+x^3$ direction by the vertical force with the extremal value $r_{(-)}(\underset{\sim}{\theta},t)$, where $r_{(+)} \geq 0$, $r_{(-)} \leq 0$. Then $\overset{\circ}{F}_q =$ $= \{\theta\}$, $\overset{*}{P}_q = \{\theta\}$, $\overset{*N}{P\!\!\sim}_q = \{\theta\}$ and

$$\overset{\circ}{Q} = Q_{class}\ ,$$

$$\overset{*}{F}_q := \{\overset{*}{f}_R | q(\underset{\sim}{\theta},t) > 0 \Rightarrow \overset{*}{f}_R(\underset{\sim}{\theta},t) = r_{(-)}(\underset{\sim}{\theta},t) \text{ and } q(\underset{\sim}{\theta},t) < 0 \Rightarrow$$

$$\Rightarrow \overset{*}{f}_R(\underset{\sim}{\theta},t) = r_{(+)}(\underset{\sim}{\theta},t) \text{ and } q(\underset{\sim}{\theta},t) = 0 \Rightarrow \overset{*}{f}_R(\underset{\sim}{\theta},t) \in \langle r_{(-)}(\underset{\sim}{\theta},t)\ ,$$

$$r_{(+)}(\underset{\sim}{\theta},t)\rangle\ ,\ \underset{\sim}{\theta} \in \Pi,\ t \in I\}\ ,$$

and $\overset{\circ}{P}_q$, $\overset{\circ N}{P\!\!\sim}_q$ are defined as in Example 1.

In examples 2 - 5 the multifunction $\overset{\circ}{Q} \ni q \to \overset{v}{Y}_q$, $\overset{v}{Y}_q = \overset{\circ}{Y}_{q_V} \oplus \overset{*}{Y}_q$, is not constant, i. e., for the different q, $q \in \overset{\circ}{Q}$, the sets $\overset{\circ}{Y}_q$ of the reaction forces $\overset{v}{y}_R$ are different.

3.3. Basic inequalities

Now we shall give the examples of inequalities in Eqs. (C1.2), (C1.6) and derive the basic inequality (C1.10) for the scalar plate theory. Let V be the linear space of the scalar functions defined and continuous in $\bar{\Pi}$ and smooth in Π (cf. Sec. 1.1.). Let us assume that the constraint inclusion $\overset{\circ}{Q} \subset Q$ is configurational, i.e., the set $\overset{\circ}{Q}$ is determined by the system of sets $\overset{\circ}{Q}_t$, $t \in I$, which define the admissible configurations $q_t \equiv q(.,t)$ of the plate (2) at the time instant t. We shall assign to every q_t, $q_t \in \overset{\circ}{Q}_t$, $t \in I$, the subset V_{q_t} of V, putting $v \in V_{q_t}$ iff there exsits $\varepsilon_o, \varepsilon_o > 0$, such that for every ε, $0 \leq \varepsilon \leq \varepsilon_o$, there is $q_t + \varepsilon v \in \overset{\circ}{Q}_t$. Elements of V_{q_t} will be called the plate virtual displacement at the configuration q_t. Thus Eqs. (C1.2) yields

---

(1) By the vertical force we meanhere the force acting in the direction of the x^3-axis.

(2) We have tacitly assumed here that the scalar plate theory had been obtained by the constraint approach, i.e., that the function $q(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$, determines the deformations $\underset{\sim}{p}$ of the plate. In the formal approximation the function q determines only the approximation $\underset{\sim}{\tilde{p}}$ of the deformations p of the plate, i.e., $\underset{\sim}{\tilde{p}} \in p_{\sim}$, cf. Sec. 2.0. of the Chapter A.

$$\overset{o}{Y}_q := \{\overset{o}{y}_R \equiv (\overset{o}{p}_{oR}, \overset{oN}{\underset{\sim}{p}}_R, \overset{o}{f}_R) \mid \oint_{\partial\Pi} (\overset{o}{p}_{oR} v + \overset{oN}{\underset{\sim}{p}}_R v_{,\underset{\sim}{N}}) dl_R + \tag{C3.6}$$

$$+ \int_{\Pi} \overset{o}{f}_R v \, da_R \geq 0 \text{ for every } v \in V_{q_t}, \; q_t \in \overset{o}{Q}_t, \; t \in I\} \; .$$

The external forces $\overset{o}{y}_R$, $\overset{o}{y}_R \in \overset{o}{Y}_q$, are said to be the constraint reactions; they are the forces which maintain (or which are due to) the constraint inclusion $\overset{o}{Q} \subset Q$.

Now let us determine the field reactions $\overset{*}{y}_R \equiv (\overset{*}{p}_{oR}, \overset{*N}{\underset{\sim}{p}}_R, \overset{*}{f}_R)$, $\overset{*}{y}_R \in \overset{*}{Y}_q$, $q \in \overset{o}{Q}$, which have been interpreted before as due to a certain system of forces which "control" the deflection of the plate. Putting $\overset{*}{Y}_q = \overset{*}{P}_q \times \overset{*N}{\underset{\sim}{P}}_q \times \overset{*}{F}_q$ we shall assume that

$$\begin{aligned}
\overset{*}{P}_q &:= \{\overset{*}{p}_{oR} \mid s_{(-)} \leq \overset{*}{p}_{oR} \leq s_{(+)}\} \; , \\
\overset{*N}{\underset{\sim}{P}}_q &:= \{\underset{\sim}{p}^N_R \mid \underset{\sim}{s}^N_{(-)} \leq \underset{\sim}{p}^N_R \leq \underset{\sim}{s}^N_{(+)}\} \; , \\
\overset{*}{F}_q &:= \{f_R \mid r_{(-)} \leq \overset{*}{f}_R \leq r_{(+)}\} \; , \; q \in \overset{o}{Q} \; ,
\end{aligned} \tag{C3.7}$$

where $(s_{(-)}, \underset{\sim}{s}^N_{(-)}, r_{(-)})$, $(s_{(+)}, \underset{\sim}{s}^N_{(+)}, r_{(+)})$ are the known values of the vertical "control" forces. We assume that $s_{(-)}, s_{(+)}, r_{(-)}, r_{(+)}$ depend on $\underset{\sim}{\theta}, t, q(\underset{\sim}{\theta}, t)$ and that $\underset{\sim}{s}^N_{(-)}, \underset{\sim}{s}^N_{(+)}$ depend on $\underset{\sim}{\theta}, t, q_{,N}(\underset{\sim}{\theta}, t)$; more general cases can be also taken into account.
Denoting $u^+(\underset{\sim}{\theta}) \equiv \sup(u(\underset{\sim}{\theta}), 0)$, $u^-(\underset{\sim}{\theta}) \equiv -\sup(-u(\underset{\sim}{\theta}), 0)$, where $u \in U$ (1), we obtain

$$(\overset{*}{p}_{oR} - s_{(-)})u^+ \geq 0, \; (\overset{*}{p}_{oR} - s_{(+)})u^- \geq 0, (\overset{*N}{\underset{\sim}{p}}_R - \underset{\sim}{s}^N_{(-)})u^+ \geq 0, \ldots, (\overset{*}{f}_R - r_{(-)})u^- \geq 0$$

and

$$\oint_{\partial\Pi} [(\overset{*}{p}_{oR} - s_{(-)})u^+ + (\overset{*}{p}_{oR} - s_{(+)})u^- + (\overset{*N}{\underset{\sim}{p}}_R - \underset{\sim}{s}^N_{(-)})u^+_{,\underset{\sim}{N}} + (\overset{*N}{\underset{\sim}{p}}_R - s^N_{(+)})u^-_{,\underset{\sim}{N}} \, dl_R +$$

$$+ \int_{\Pi} [(\overset{*}{f}_R - r_{(-)})u^+ + (\overset{*}{f}_R - r_{(+)})u^-] da_R \geq 0 \; , \quad t \in I \; .$$

---

(1) The denotations here are slightly different then those used in 8 . Symbol U stands for the space of all scalar functions defined and continuous in $\bar{\Pi}$ and smooth in Π.

Introducing the rate of work of the control forces

$$J_{q_t}(u) \equiv \oint_{\partial\Pi}(s_{(-)}u^+ + s_{(+)}u^- + s^N_{\sim(-)}u^+_{,\underset{\sim}{N}} + s^N_{\sim(+)}u^-_{,\underset{\sim}{N}})dl_R + \int_{\Pi}(r_{(-)}u^+ + r_{(+)}u^-)da_R, t \in I, \tag{C3.8}$$

we obtain the relation

$$\int_{\partial\Pi}(\overset{*}{p}_{OR}u + \overset{*N}{\underset{\sim}{p}}_R u_{,\underset{\sim}{N}})dL_R + \int_{\Pi}\overset{*}{f}_R u da_R \geq J_{q_t}(u),\ t \in I, \tag{C3.9}$$

which has to hold for every $u \in U$.

In the view of Eq. $(C1.9)_{2-4}$ we also have

$$
\begin{aligned}
p_{OR} &= \hat{p}_{OR}(q) + \overset{o}{p}_{OR} + \overset{*}{p}_{OR}, \\
\underset{\sim}{p}^N_R &= \overset{\wedge N}{\underset{\sim}{p}}_R(q) + \overset{oN}{\underset{\sim}{p}}_R + \overset{*N}{\underset{\sim}{p}}_R, \\
f_R &= \hat{f}_R(q) + \overset{o}{f}_R + \overset{*}{f}_R,
\end{aligned} \tag{C3.10}
$$

where $\hat{p}_{OR}(.)$, $\overset{\wedge N}{p}_R(.)$, $\hat{f}_R(.)$ are the known functionals (defined on $\overset{o}{Q}$) which describe the loading of the plate. The remaining terms in Eqs. (C3.10) are described by the inequalities (C3.6) and (C3.9).

Thus the special problem of the scalar plate theory can be stated as follows: find $(q,\tau^{(N)},\omega^{(p)})$ such that $q \in \overset{o}{Q}$ and Eqs. (C3.1), (C3.2), (C3.4), (A5.6) (i.e., the field and constitutive relations of the scalar plate theory) as well as Eqs. (C3.10), (C3.6), (C3.9) hold.

Using (C1.7), we shall denote the total reactions by

$$
\begin{aligned}
\overset{\vee}{p}_{OR} &\equiv \overset{o}{p}_{OR} + \overset{*}{p}_{OR}, \\
\overset{\vee N}{\underset{\sim}{p}}_R &\equiv \overset{oN}{\underset{\sim}{p}}_R + \overset{*N}{\underset{\sim}{p}}_R, \\
\overset{\vee}{f}_R &= \overset{o}{f}_R + \overset{*}{f}_R,
\end{aligned} \tag{C3.11}
$$

as the sums of the constraint reactions $\overset{o}{y}_R \equiv (\overset{o}{p}_{OR}, \overset{oN}{\underset{\sim}{p}}_R, \overset{o}{f}_R)$ and the field reactions $\overset{*}{y}_R \equiv (\overset{*}{p}_{OR}, \overset{*N}{\underset{\sim}{p}}_R, \overset{*}{f}_R)$. Now we are to obtain, from the system of relations mentioned above, the basic inequality (C1.10). Because of

$V_{q_t} \subset V$, from Eqs. (C3.10), (C3.6) and (C3.9) we obtain the relation

$$\oint_{\partial\Pi} (\overset{\vee}{p}_{oR} v + \overset{\vee N}{p}_R v_{,\underset{\sim}{N}}) dl_R + \int \overset{\vee}{f}_R v \, da_R \geq J_{q_t}(v) \quad , \tag{C3.12}$$

which has to hold for every virtual displacement $v \in V_{q_t}$. Eliminating $\overset{\vee}{p}_{oR}$, $\overset{\vee N}{p}_R$, $\overset{\vee}{f}_R$ from (C3.11) by means of Eqs. (C3.10) and taking into account Eqs. (C3.1), (C3.2), (C3.4), after some calculations we arrive at the relation

$$\begin{aligned} &\int_{\Pi} (-H_R^{\alpha\beta} v_{,\alpha\beta} + H_R^{\alpha} v_{,\alpha} - h_R v) da_R \geq \\ &\geq \oint_{\partial\Pi} (\hat{p}_R v + \hat{p}_R^N v_{,\underset{\sim}{N}}) dl_R + \int_{\Pi} (\hat{f}_R - i_R) v \, da_R + J_{q_t}(v) \quad , \end{aligned} \tag{C3.13}$$

which has to hold for every $v \in V_{q_t}$ where $q_t \in \overset{o}{Q}_t$, $t \in I$. This relation will be referred to as the basic variational inequality for the scalar plate theory problems. It constitutes the special case of the inequality (C1.10) and states, that the total virtual rate of work of the loadings and the inertia and control forces does not exceed the total virtual rate of work of the internal forces. If there are no field reactions (the control forces are equal to zero) then the variational inequality (C3.12) leads to the following principle of virtual work:

$$\begin{aligned} &\int_{\Pi} (-\tilde{H}_R^{\alpha\beta} v_{,\alpha\beta} + \tilde{H}_R^{\alpha} v_{,\alpha} - \tilde{h}_R v) da_R \geq \\ &\qquad \geq \oint_{\partial\Pi} (\hat{p}_R v + \hat{p}_R^N v_{,\underset{\sim}{N}}) dl_R + \int_{\Pi} (\hat{f}_R - i_R) v \, da_R \end{aligned} \tag{C3.14}$$

for every $v \in V_{q_t}$, $q_t \in \overset{o}{Q}_t$, $t \in I$.

We have arrived to the conclusion that every solution of the plate problem has to satisfy the condition $q \in \overset{o}{Q}$, the basic variational inequality (C3.13) and Eqs. (A5.6). Now the question arises when the solution of this problem can be reduced to the solution of the basic inequality (C3.13). We shall detail this problem in the next Section.

At the end of this Seciton we are to show that the constraints reactions can be obtained as the special case of the field reactions. To this aid let us assume, for the time being, that there are no constraint reactions. Let $\overset{o}{Q}$ be the set of function $q(.)$, which can be given by the conditions [1]

$$q(\underset{\sim}{\theta},t) \in Q_{(\underset{\sim}{\theta},t)}\ ,\ \underset{\sim}{\theta} \in \bar{\Pi},\ q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t) \in Q^{\underset{\sim}{N}}_{(\underset{\sim}{\theta},t)}\ ,\ \underset{\sim}{\theta} \in \partial\bar{\Pi},\ t \in I,$$

where $Q_{(\underset{\sim}{\theta},t)}$, $Q^{\underset{\sim}{N}}_{(\underset{\sim}{\theta},t)}$ are the known subsets of R. Let us also denote by $V_{q(\underset{\sim}{\theta},t)}$, $\underset{\sim}{\theta} \in \Pi$, the sets of values $u(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Pi$, satisfying the condition

$$r_{(-)}(\underset{\sim}{\theta},q(\underset{\sim}{\theta},t))u^{+} - r_{(+)}(\underset{\sim}{\theta},q(\underset{\sim}{\theta},t))u^{-} = 0, \quad \underset{\sim}{\theta} \in \Pi.$$

Analogously, by $V_q(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \partial\Pi$, we denote the sets of values $u(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \partial\Pi$, satisfying the condition

$$s_{(-)}(\underset{\sim}{\theta},q(\underset{\sim}{\theta},t))u^{+} + s_{(+)}(\underset{\sim}{\theta},q(\underset{\sim}{\theta},t))u^{-} = 0, \quad \underset{\sim}{\theta} \in \partial\Pi\ ,$$

and by $V_{q_{\underset{\sim}{N}}}(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \partial\Pi$, the sets of values $u_{\underset{\sim}{N}}(\underset{\sim}{\theta})$, satisfying the condition

$$s^{N}_{(-)}(\underset{\sim}{\theta},q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t)u^{+}_{,\underset{\sim}{N}} + s^{N}_{(+)}(\underset{\sim}{\theta},q_{,\underset{\sim}{N}}(\underset{\sim}{\theta},t))u^{-}_{,\underset{\sim}{N}} = 0,\ \underset{\sim}{\theta} \in \partial\Pi.$$

We shall consider three special cases:

1. $\bar{V}_q(\underset{\sim}{\theta},t) = R \Rightarrow r_{(-)} = r_{(+)} = 0,\ s_{(-)} = s_{(+)} = 0,\ \underset{\sim}{\theta} \in \bar{\Pi}$,

   $\bar{V}_{q_{\underset{\sim}{N}}}(\underset{\sim}{\theta},t) = R \Rightarrow s^{N}_{(-)} = s^{N}_{(+)} = 0 \qquad ,\ \underset{\sim}{\theta} \in \partial\Pi.$

2. $\bar{V}_q(\underset{\sim}{\theta},t) = R^{+}\cup\{0\} \Rightarrow r_{(-)} = 0,\ s_{(-)} = 0,\ \underset{\sim}{\theta} \in \bar{\Pi}$;

   $\bar{V}_{q_{\underset{\sim}{N}}}(\underset{\sim}{\theta},t) = R^{+}\cup\{0\} \Rightarrow s^{N}_{(-)} = 0 \qquad ,\ \underset{\sim}{\theta} \in \partial\Pi,$

3. $\bar{V}_q(\underset{\sim}{\theta},t) = R^{-}\cup\{0\} \Rightarrow r_{(+)} = 0,\ s_{(+)} = 0,\ \underset{\sim}{\theta} \in \bar{\Pi}$

   $\bar{V}_{q_{\underset{\sim}{N}}}(\underset{\sim}{\theta},t) = R^{-}\cup\{0\} \Rightarrow s^{N}_{(+)} = 0 \qquad ,\ \underset{\sim}{\theta} \in \partial\Pi.$

([1]) It means that no restrictions have been imposed on the gradients ∇q, $\nabla\nabla q$,... of the function q in the region Π.

Let us observe, that in the second case $r_{(+)}$, $s_{(+)}$, $\tilde{s}^{N}_{(+)}$ are arbitrary positive numbers and in the third case $s_{(-)}$, $\tilde{s}^{N}_{(-)}$ are arbitrary negative numbers. If $V_q(\underset{\sim}{\theta},t) = \{\theta\}$, $V_{q,\underset{\sim}{N}}(\underset{\sim}{\theta},t) = \{\theta\}$ then $r_{(-)}$, $r_{(+)}$, $s_{(-)}$, $s_{(+)}$, $\tilde{s}^{N}_{(-)}$, $\tilde{s}^{N}_{(+)}$ are arbitrary positiveor negative numbers.

Le us denote by V_{q_t} the set of functions $u(\underset{\sim}{\theta})$, $\theta \in \bar{\Pi}$, values of which belong to $V_q(\underset{\sim}{\theta},t)$, $\underset{\sim}{\theta} \in \bar{\Pi}$, and values of its normal derivatives on $\partial\Pi$ to $V_{q_N}(\underset{\sim}{\theta},t)$. We assume that $u(.)$ are sufficiently regular. Then from (C3.7) and (C3.8) we obtain the condition

$$\oint\limits_{\partial\Pi} (\overset{*}{p}_{oR}u + \overset{*N}{\tilde{p}_R}u_{,\underset{\sim}{N}})dl_R + \int \overset{*}{f}_R u\, da_R \geq 0 \qquad \text{(C3.15)}$$

which has to hold for every $u \in V_{q_t}$, $q_t \in \overset{o}{Q}_t$, $t \in I$. But this condition coincides with the condition (C3.6). We conclude that $f_R(\underset{\sim}{\theta},t)$, $p_{oR}(\underset{\sim}{\theta},t)$, $\tilde{p}^{N}_R(\underset{\sim}{\theta},t)$ satisfy the conditions of the form:

$$\overset{*}{\delta}(\underset{\sim}{\theta},t) \in R \quad \text{or} \quad \overset{*}{\delta}(\underset{\sim}{\theta},t) \in R^{+} \cup \{0\} \text{ or } \overset{*}{\delta}(\underset{\sim}{\theta},t) \in R^{-} \cup \{\theta\} ,$$

where $\overset{*}{\delta}$ stands for $\overset{*}{f}_R$, $\overset{*}{p}_{oR}$, $\overset{*N}{p}_R$, then the field reactions coincide with the constraint reactions. Thus we have proved that if the sets Q_t are uniquely defined by the sets $Q_{(\underset{\sim}{\theta},t)}$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $Q^{N}_{(\underset{\sim}{\theta},t)}$, $\underset{\sim}{\theta} \in \partial\Pi$, $t \in I$, then Eqs. (B1.18), (B1.19) can be replaced by the relations of the form (B1.17) in which $r_{(-)}$, $r_{(+)}$, $s_{(-)}$, $s_{(+)}$, $s^{N}_{(-)}$, $s^{N}_{(-)}$ are either the known functions as before or are certain constants which can attain arbitrary either non - positive or non-negative values.

3.4. Variational formulation

We shall denote by $X(\Pi)$, $X(\partial\Pi)$ the linear spaces of sufficiently regular functions defined on Π and almost everywhere on $\partial\Pi$, respectively. We shall confine ourselves to the special situations in which two following assumptions hold.

1. For every $q_t \in \overset{o}{Q}_t$, $t \in I$, there exists the function $\delta_R(q_t,.)$ defined and concave on $X(\Pi)$ and such that the condition

$$\begin{aligned} r_{(-)}u^{+} &\geq \delta_R(q_t,u^{+}) - \delta_R(q_t,0) \quad , \\ r_{(+)}u^{-} &\geq \delta_R(q_t,u^{-}) - \delta_R(q_t,0) \quad , \end{aligned} \qquad \text{(C3.16)}$$

   hold for every u, $u \in X(\Pi)$.

2. For every $q_t \in \overset{\circ}{Q}_t$, $t \in I$, there exists the function $\sigma_R(q_t,.,.)$, defined on $X(\partial\Pi) \times X(\partial\Pi)$ such that $\sigma_R(q_t,u,.)$, $\sigma_R(q_t,.,\underset{\sim}{u_N})$ are concave for every u , $\underset{\sim}{u_N} \in X(\partial\Pi)$ and such that the conditions

$$
\begin{aligned}
s_{(-)}u^+ &\geq \sigma_R(q_t,u^+,0) - \sigma_R(q_t,0,0) \quad , \\
s_{(+)}u^- &\geq \sigma_R(q_t,u^-,0) - \sigma_R(q_t,0,0) \quad , \\
s^N_{(-)}u^+_{,N} &\geq \sigma_R(q_t,0,u^+_N) - \sigma_R(q_t,0,0) \ , \\
s^N_{(+)}u^-_{,N} &\geq \sigma_R(q_t,0,u^-_N) - \sigma_R(q_t,0,0) \ ,
\end{aligned}
\qquad (C3.17)
$$

hold for every u , $u_N \in X(\partial\Pi)$.

Functions δ_R, σ_R can alos depend explicitly on the time coordinate t, $t \in I$. From Eqs. (C3.16), (C3.17) it follows that the values of the control forces $r_{(-)}$, $r_{(+)}$, $s_{(-)}$, $s_{(+)}$, $s^N_{(-)}$, $s^N_{(+)}$ are now determined as the derivatives of $\sigma_R(q_t,u(\underset{\sim}{\theta}),\ \underset{\sim}{u_N}(\underset{\sim}{\theta}))$, $\delta_R(q_t,u(\underset{\sim}{\theta}))$ when $u(\underset{\sim}{\theta}) \to \pm 0$, $\underset{\sim}{u_N}(\underset{\sim}{\theta}) \to \pm 0$.

Eqs. (C3.7) yield

$$
\begin{aligned}
\overset{*}{f}_R u &\geq r_{(-)}u^+ + r_{(+)}u^- \quad , \\
\overset{*}{p}_{OR} u &\geq s_{(-)}u^+ + s_{(+)}u^- \ , \\
\overset{*N}{\underset{\sim}{p}_R} u_{,N} &\geq \underset{\sim}{s}^N_{(-)}u^+_{,N} + \underset{\sim}{s}^N_{(+)}u^-_{,N} \quad ,
\end{aligned}
$$

and by virtue of Eqs. (C3.16), (C3.17) we obtain

$$
\begin{aligned}
\overset{*}{f}_R u &\geq \delta_R(q_t,u) - \delta_R(q_t,0) \ , \\
\overset{*}{p}_{OR} u &\geq \sigma_R(q_t,u,0) - \sigma_R(q_t,0,0), \\
\overset{*N}{\underset{\sim}{p}_R} u_{,\underset{\sim}{N}} &\geq \sigma_R(q_t,0,\underset{\sim}{u_N}) - \sigma_R(q_t,0,0)
\end{aligned}
$$

for every u , $\underset{\sim}{u_N}$. It means that the field reactions $\overset{*}{f}_R$, $\overset{*}{p}_{OR}$, $\overset{*N}{\underset{\sim}{p}_R}$ are subdifferentials of the functions $\delta_R(q_t,.)$, $\sigma_R(q_t,.,0)$, $\sigma_R(q_t,0,.)$, respectively. They will be referred to as the densities (related to Π or $\partial\Pi$) of the subpotential of the field reactions.

Combining Eqs. (C3.8), (C3.16), (C3.17) and denoting

$$J_{q_t}(u) \equiv \int_{\partial\Pi} \sigma_R(q_t, u, u_{,\underset{\sim}{N}}) dl_R + \int_{\Pi} \sigma_R(q_t, u) da_R \qquad (C3.18)$$

we obtain

$$J_{q_t}(u) \geq J_{q_t}(u) - J_{q_t}(0) \; . \qquad (C.3.19)$$

Every functional $J_{q_t}(u)$, $q_t \in \overset{o}{Q}_t$, $t \in I$, defined on X, will be called the subpotential of the field reactions. In view of Eqs. (C3.17), (C3.16), from Eq. (C3.18) it follows that

$$\oint_{\partial\Pi} (\overset{*}{p}_{oR} u + \overset{*N}{\underset{\sim}{p}}_{R} u_{,\underset{\sim}{N}}) dl_R + \int_{\Pi} \overset{*}{f}_R u \, da_R \geq J_{q_t}(u) - J_{q_t}(0) \qquad (C3.20)$$

for every $u(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \bar{\Pi}$; Eqs. (C3.20) describes the interrelation between the rate of work of the field reactions and the suitable increment of their subpotential.

<u>Remark</u>. If $r_{(-)} = r_{(+)}$, $s_{(-)} = s_{(+)}$, $s^N_{(-)} = s^N_{(+)}$, then

$$\overset{*}{f}_R = \frac{\partial \rho_R}{\partial q} \; , \; \overset{*}{p}_{oR} = \frac{\partial \sigma_R}{\partial q} \; , \; \overset{*N}{\underset{\sim}{p}}_R = \frac{\partial \sigma_R}{\partial q_{,N}} \; , \qquad (C3.21)$$

and the subpotential reduces to the potential of the field reactions.

The basic variational inequality (C3.13) of the scalar plate theory can be now rewritten in the form

$$\int_{\Pi} (-\tilde{H}^{\alpha\beta}_R v_{,\alpha\beta} + \tilde{H}^{\alpha}_R v_{,\alpha} - \tilde{h}^{\alpha}_R v) da_R \geq$$
$$\geq \oint_{\partial\Pi} (\hat{p}_R v + \overset{\wedge N}{\underset{\sim}{p}}_R v_{,\underset{\sim}{N}}) dl_R + \int (\hat{f}_R - i_R) v \, da_R + J_{q_t}(v) - J_{q_t}(0) \qquad (C3.22)$$

which has to hold for every virtual displacement $v \in V_{q_t}$, $q_t \in \overset{o}{Q}_t$, and for every $t \in I$. It states that the sum of the total virtual rate of work of effective and inertia forces and virtual increment of the subpotential of the field reactions does not exceed the total work of the internal forces.

By the direct calculations we can now verify that Eqs. (C3.22) leads to:

1. the inequalitites (C3.9), (C3.6),

2. the field equations (C3.1), (C3.2),

3. the constitutive equations (C3.4).

Thus we conclude that the problem of the scalar plate theory, in which there exists the subpotential of the field reactions, can be stated as follows: find $q \in \overset{\circ}{Q}$ and $\tau^{(N)}, \omega^{(p)}$ such that Eqs. (C3.22), (A5.6) hold for every $v \in V_{q_t}$, $q_t \in \overset{\circ}{Q}_t$, $t \in I$.

In the special cases the densities of the subpotential of field reactions can have the form

$$\delta_R = \overset{*}{\delta}_R(q_t) \ , \quad \sigma_R = \overset{*}{\sigma}_R(q_t, q_{t,\underset{\sim}{N}}) = \overset{*1}{\sigma}_R(q_t) + \overset{*2}{\sigma}_R(q_{t,\underset{\sim}{N}}) \ ,$$

and Eqs. (C3.16), (C3.17) interrelate the control forces $r_{(-)}, \ldots, \overset{N}{s}_{(+)}$ and the fields $\overset{*}{\sigma}_R$, $\overset{*}{\delta}_R$ by means of

$$r_{(-)} u^+ \geq \overset{*}{\delta}_R(q_t + u^+) - \delta_R(q_t), \ldots, \overset{N}{s}_{(+)} u^-_{,\underset{\sim}{N}} \geq \overset{*}{\sigma}_R(q_t, q_{t,\underset{\sim}{N}} + u^-_{\underset{\sim}{N}}) - \overset{*}{\sigma}_R(q_t, q_{t,\underset{\sim}{N}}) .$$

Then the subpotential of the field reactions (B11.18) has to be defined by

$$J_{q_t}(0) = J(q_t) \equiv \oint_{\partial\Pi} \overset{*}{\sigma}_R(q_t, q_{t,\underset{\sim}{N}}) dl_R + \int_{\Pi} \overset{*}{\rho}_R(q_t) da_R$$

and the increment of this subpotential in Eqs. (C 3.20), (C 3.21) have to be replaced by $J(q_t + u) - J(q_t)$. Thus basic variational inequality (C3.22) now yields

$$\int_{\Pi} (-\hat{H}^{\alpha\beta}_R v_{,\alpha\beta} + \hat{H}^{\alpha}_R v_{,\alpha} - \hat{h}_R v) da_R \geq$$
$$\geq \oint_{\partial\Pi} (\hat{p}_R v + \hat{p}^N_R v_{,\underset{\sim}{N}}) dl_R + \int_{\Pi} (\hat{f}_R - i_R) v \, da_R + J(q_t + v) - J(q_t) . \qquad \text{(C3.23)}$$

This special case of the obtained result was detailed in [8] under assumption that the plate is elastic and for every $t \in I$ the set $\overset{o}{Q}_t$ is convex in X. Then the virtual displacements can be defined by $V_{q_t} := \{v | v = w - q_t \text{ for every } w \in \overset{o}{Q}_t\}$, $t \in I$, i.e., Eq. (C3.23) holds for every $v \equiv w - q_t$ where w is arbitrary in $\overset{o}{Q}_t$. This case corresponds to the linear theory of elastic plates.

3.5. Examples

Let us return again to the Examples 2 and 5 of Sec. 3.2. of this Chapter. In the Example 2 the sets $\overset{o}{Q}_t$, $t \in I$, are convex. Moreover, $J(q_t) \equiv 0$, i.e., there are no field reactions. Thus the problem leads to the solution of inequality (C3.23) (with $J(.) \equiv 0$) for every $v = v(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \bar{\Pi}$, where $v|_\Delta = 0$, $v_{,\underset{\sim}{N}}|_\Gamma = 0$ and $v|_\Pi \leq \zeta$. The solution $q(\underset{\sim}{\theta}, t)$, $\underset{\sim}{\theta} \in \bar{\Pi}$, $t \in I$, of the problem has to satisfy, for every $t \in I$, also the conditions $q_t|_\Delta = 0$, $q_{t,\underset{\sim}{N}}|_\Gamma = 0$ and $q_t|_\Pi \leq \zeta$ as well as the condition $q \in K$ (cf. Sec. 3.2. of this Chapter).

In the Example 5 of Sec. 3.2. the sets $\overset{o}{Q}_t$ are also convex. The subpotential $J(.)$ of the field reactions has the form

$$J(q_t) = \int_\Pi (r_{(+)} q_t^- + r_{(-)} q_t^+) da_R$$

which can be derived directly from Eq. (C3.8). Thus the problem leads to the inequality

$$\int_\Pi (-\tilde{H}_R^{\alpha\beta} v_{,\alpha\beta} + \tilde{H}_R^{\alpha} v_{,\alpha} - \tilde{h}_R v) da_R \geq$$

$$\geq \oint_{\partial\Pi} (\hat{p}_R v + \hat{p}_R^N v_{,\underset{\sim}{N}}) dl_R + \int_\Pi (f_R - i_R) v da_R + \int_\Pi (r_{(+)} v^- - r_{(-)} v^+) da_R,$$

$$v \equiv w - q_t \; .$$

which has to hold for every $w = w(\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \bar{\Pi}$, such that $w|_\Delta = 0$, $w_{,\underset{\sim}{N}}|_\Gamma = 0$. The conditions $q_t|_\Delta = 0$, $q_{t,\underset{\sim}{N}}|_\Gamma = 0$, $q \in K$ (the function q is either time independent or satisfy the known initial conditions) have to be satisfied (for every $t \in I$) by the solution q of the foregoing inequality.

Both examples outlined here represent the non-linear problems even if $\tilde{H}_R^{\alpha\beta}$, $\tilde{H}_R^{\alpha}$, $\tilde{h}_R$, $\tilde{p}_R$, $\tilde{p}_R^N$, i_R are linear functions of q and its first and second derivatives, [8]. For the elastic plates for which $H_R^{\alpha\beta} = C_R^{\alpha\beta\gamma\delta} q_{,\gamma\delta}$, $H_R^{\alpha} = 0$, $h_R = 0$, there are known the conditions for the existence and uniqueness of solution, [8]. The analysis of problems of the linear plate theory is outside of the scope of this work because it may be found in [8].

REFERENCES [1]

[1] Antmann, S.S: The theory of rods, Handbuch der Physik-Encyclopedia of Physics VIa/2, 641-703, Berlin-Heidelberg-New York: Springer 1972.

[2] Antmann, S.S., Warner , W.H.: Dynamical theory of hyperelastic rods. Arch. Rational Mech. Anal. 23, 135 - 162 (1966).

[3] Boblewski, H., Bojda, K.H., Woźniak, Cz.: Plane equilibrium of certain spatial structures. Mech. Res. Comm. 3, 4 (1976).

[4] Budiansky, B., Sanders, H.L.: On the "best" first order linear shell theory. Progress in applied mechanics (The Prager Anniv. Vol.), pp. 129 - 140. 1963.

[5] Chien, W.Z.: The intrinsic theory of thin shells and plates (Parts I,II,III). Quart. Appl. Math. 1, 297 - 327; 2, 43 - 59; 3, 120-135 (1944).

[6] Cohen, H., De Silva, C.N.: Nonlinear theory of elastic surfaces. J. Math. Phys. 7, 246 - 253 (1966).

[7] Duszek, M: Equations of the large bending theory for plastic shells [in Polish]. Rozpr. Inż. 20, 389 - 407 (1972).

[8] Duvaut, G., Lions, J.L.: Inequalities in mechanics and physics [transl. from the 1972 French ed.]. Berlin - Heidelberg - New York: Springer 1976.

[9] Eringen, A.C.: Mechanics of continua. New York: Wiley 1967.

[10] Fichera, G.: Boundary value problems of elasticity with unilateral constraints. Handbuch der Physik - Encyclopedia of Physics VIa/2, 391 - 424. Berlin-Heidelberg-New York: Springer 1972.

[11] Galimov, K.Z.: Foundations of the nonlinear theory of thin shells [in Russian]. Kazan' 1975

[12] Gol'denveizer, A.L., Kolos, A.V.: On the derivation of two-dimensional equations in the theory of thin elastic plates. J. Appl. Math. Mech.(transl. of PMM) 29, 151 - 166 (1965).

[13] Green, A.E., Adkins, J.E.: Large elastic deformations. Oxford: Clarendon Press 1960.

[14] Green, A.E., Naghdi, P.M.: On superposed small deformations on a large deformation of an elastic Cosserat surface. J. Elasticity 1, 1-17 (1971).

([1]) The list below does not reflect the present state of the research on the subject being restricted only to works cited in the text.

[15] Hill, R.: Mathematical theory of plasticity. Oxford: University Press 1950.

[16] John, F.: Refined interior shell equations. Proc. Sec. IUTAM Symp. on thin shells (Copenhagen, 1967), 1-14 (1969).

[17] John F.: Estimates for the derivatives of the stresses in a thin shell and interior shell equations. Comm. Pure and Appl. Math. 18, 235-267 (1965).

[18] Koiter, W.T.: On the nonlinear theory of thin elastic shells. Proc. Konikl. Ned. Akad. Wetenschap. Ser. B 69, 1 - 54 (1966).

[19] Koiter, W.T.: A consistent first approximation in the general theory of thin elastic shells. Proc. IUTAM Symp. on thin shells (Delft 1959). Amsterdam: North Holl. Publ. Comp., 12 - 33 (1960).

[20] Koiter, W.T., Simmonds, J.G.: Foundation of shell theory. Proc. 13-th IUTAM Congress (Moscow 1972). Berlin - Heidelberg-New York: Springer, 150 - 175 (1972).

[21] Kilchevskii, M.O.: The analytical theory of shells [in Russian] Kiev 1970.

[22] Librescu, L.: Elastostatics and kinetics of anisotropic and heterogeneous shell-type structures. Leyden: Nordhoff Int. 1975.

[23] Mandel, G.: Cours de mêchanique des milieux continus (Vol. 1 and 2). Paris: Gauthier-Villars 1966.

[24] Mushtari, K.M., Galimov, K.Z.: Nonlinear theory of elastic shells [in Russian]. Kazan: Tatknigizdat 1957.

[25] Naghdi, P.M.: Foundation of elastic shell theory. Progress in Solid Mechanics 4, 1 - 90. Amsterdam: North Holl. Publ. Comp. 1963.

[26] Naghdi, P.M.: The theory of plates and shells. Handbuch der Physik - Encyclopedia of Physics VIa/2, 425 - 640. Berlin-Heidelberg-New York: Springer 1972.

[27] Novotny, B.: On the asymptotic integration of the three dimensional nonlinear equations of thin elastic plates and shells. Int. J. Solid Structure 6, 433 - 451 (1970).

[28] Pietraszkiewicz, W.: Introduction ot the non-linear theory of shells. Ruhr-Universität Bochum, Mitteilungen aud dem Institut für Mechanik, 10, 1977.

[29] Pietraszkiewicz, W.: Finite rotations and Lagrangean description in the non-linear theory of shells. Warszawa-Poznań: PWN 1979.

[30] Prager,W.: Problèmes de plasticité théorique. Paris: Dunod 1958.

[31] Prager, W.: On ideal locking materials. Transaction of the Society of Rheology 1, 169 - 175 (1957).

[32] Sanders, J.L.: Nonlinear theories for thin shells. Quart. Appl. Math. 21, 21 - 36 (1963).

[33] Truesdell, C., Noll, W.: The non-linear field theories of mechanics. Handbuch der Physik-Encyclopedia of Physics III/3. Berlin-Heidelberg-New-York: Springer 1965.

[34] Truesdell, C., Toupin, R.: The classical field theories. Handbuch der Physik-Encyclopedia of Physics III/1, 226-793. Berlin-Götting-Heidelberg: Springer 1960.

[35] Volmir, A.S.: Non-linear dynamics of plates and shells [in Russian], Moscow: Nauka 1972.

[36] Volterra, E.: Equations of motion for curved elastic bars by the use of the "method of internal constraints". Ing. Arch. 23, 402 - 409 (1955).

[37] Volterra, E.: Equations of motion for curved and twisted elastic bars deduced by the use of "method of internal constraints". Ing.Arch. 24, 392 - 400 (1956).

[38] Volterra, E.: Second approximation of the method of internal constraints and its applications. Int. J. Mech. Sci. 3, 47 - 67 (1961).

[39] Wempner, G.A.: Mechanics of solids. New York: McGraw Hill 1973.

[40] Woźniak, Cz.: Foundations of dynamics of deformable bodies [in Polish]. Warszawa: PWN 1969.

[41] Woźniak, Cz.: Constrained continuous media (Parts I,II,III). Bull. Acad. Polon. Sci., Sér. Sci. Techn. 21; 1, 109-116; 2, 167 - 173; 3, 175 - 182 (1973).

[42] Woźniak, Cz.: Bodies with the binary structure. Bull. Acad. Polon. Sci., Sér. Sci. Techn. 25, 383 - 389 (1977).

[43] Woźniak, Cz.: On the formation of simplified theories of solid mechanics. Arch. Mech. 31, 6 (1979).

[44] Woźniak, Cz.: On the concept of abstract constraints, Bull. Acad. Polon. Sci., Sér. Sci. Techn.(in press).

[45] Woźniak, Cz., Kleiber, M.: Non-linear structural mechanics [in Polish] (in press).

[46] Zerna, W.: Über eine nichtlineare allgemeine Theorie der Schalen. Proc. IUTAM Symp. on the theory of shells (Delft 1959), 34 - 42 (1960)!

Mitteilungen aus dem Institut für Mechanik
RUHR-UNIVERSITÄT BOCHUM
Nr. 20