RUHR-UNIVERSITÄT BOCHUM

Maria K. Duszek

Problems of Geometrically Non-Linear Theory of Plasticity

Heft Nr. 21

Mitteilungen aus dem Institut für Mechanik

INSTITUT FÜR MECHANIK

R U H R - U N I V E R S I T Ä T B O C H U M

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JUNI 1 9 8 0

Editor:
Institut für Mechanik der Ruhr-Universität Bochum

ACKNOWLEDGMENTS

The present work is an extension of the lecture notes prepared during author's stay at the Institut für Mechanik II of the Ruhr-Universität Bochum as a Visiting Professor in the period April-July 1979.

The stimulating discussions with the research staff of the Institute are greatfully acknowledged. The author is particularly indebted to Prof. Dr.-Ing. H. Stumpf for his kind invitation and friendly support.

Thanks are also due to Mrs. E. Mönikes for her excellent job of typing the manuscript.

SUMMARY

The paper is divided into two parts.

Part I contains the foundations of non-linear deformation and stress analysis in the Eulerian and in the Lagrangian descriptions. The various sets of objective and conjugate constitutive variables are constructed and discussed.

Part II is devoted to the presentation of the consistent, geometrically nonlinear description of elastic-plastic and rigid-plastic material behaviour. The presentation is limited to the purely mechanical theory and to quasi-static deformation processes. The attention is focused on the problems concerning geometrical non-linearities. In particular, the material stability concept is widely discussed.

The elaboration contains a number of new results in various chapters.

ZUSAMMENFASSUNG

Die Arbeit besteht aus zwei Teilen.

Teil I beinhaltet die Grundlagen nichtlinearer Deformations- und Spannungsanalysis in Euler'scher und in Lagrange'scher Beschreibung. Die verschiedenen Gruppen objektiver und konjugierter konstitutiver Variablen werden hergeleitet und diskutiert.

Teil II ist der Darstellung der konsistenten geometrisch nichtlinearen Beschreibung elastisch-plastischen und starrplastischen Materialverhaltens gewidmet. Die Darstellung ist dabei auf die rein mechanische Theorie und auf quasi-statische Deformationsprozesse beschränkt. Hauptaugenmerk ist auf Probleme gerichtet, die geometrische Nichtlinearitäten betreffen. Insbesondere wird das Konzept der Materialstabilität ausführlich diskutiert.

Die Arbeit enthält in verschiedenen Kapiteln eine Reihe neuer Ergebnisse.

CONTENTS

PART I

NON-LINEAR STRAIN AND STRESS ANALYSIS

1. INTRODUCTORY REMARKS

Elastic-plastic deformation is a very complicated process. In analysing this problem one might wish to include such phenomena as non-linear and anisotropic memory, thermal effects, strain rate sensitivity and so on. It is impossible, however, to construct a theory which is at the same time general, detailed enough and applicable in engineering practice. Therefore, for practical purposes some simplified models have been proposed. These models describe only the basic properties of elastic-plastic deformation processes, neglecting more complex ones which are supposed to be of minor importance in particular applications.

In general, the elastic-plastic theory is non-linear both physically and geometrically, and therefore non-linear functional relationships between stress, strain and deformation history as well as between strain and displacement fields are involved. The problem may be physically linearized by using linear material models.

Geometrical simplifications result from assumptions concerning the magnitude of strain and lead to the specific elastic-plastic theories. Let us consider such situations.

1. Elastic and plastic components of strain are both small and of the same order of magnitude. The maximum elastic strain component is given by the yield stress divided by Young's modulus, $\varepsilon^e = \sigma_o/E$, and is usually of the order 10^{-3}. Therefore, the plastic strain components of the order of magnitude 10^{-3} are also allowed. In this case the linear strain analysis can usually be applied and the total strain can be considered as a sum of the elastic and plastic components, $\varepsilon = \varepsilon^e + \varepsilon^p$. Such theory will be called *infinitesimal elastic-plastic.*

2. Elastic components of strain are still infinitesimal ($\varepsilon^e \lesssim 10^{-3}$), but plastic components are supposed to be finite. The additive

decomposition of the total strain measure into elastic and plastic parts is still allowed. This situation often occurs when dealing with metal plasticity, unless the hydrostatic pressure is very high. Therefore the problems of finite deformations of engineering structures usually belong to this category. The theory describing such cases will be called *elastic-finite plastic.*

3. Elastic and plastic components of strain are each large (finite). Such condition can be realized in an explosive forming of metals, since then a high hydrostatic pressure occurs (the elastic deformation caused by the hydrostatic pressure is not limited by the onset of yielding). Since the finite strain components are non-linear expressions in terms of displacements, the assumption of additive decomposition of strain into elastic and plastic parts cannot be applied directly. Such theory will be called *finite elastic-plastic.*

In this paper attention will be focused on some particular aspects of the elastic-plastic theory, namely on the problems concerning geometrical non-linearities. However, the assumption of small elastic strains will be retained and, therefore, we shall deal with the elastic-finite plastic theory as well as with some problems of inifinitesimal elastic-plastic theory. In particular, in the analysis of structural stability the rigorous, geometrically non-linear approach is required. Also when the representative stress level attains a magnitude comparable to the magnitude of the slope of stress-strain curve (in the plastic range the hardening modulus appears to be often of the same order of magnitude as the stress itself), some problems may be poorly described as a result of the linearization. In other words, the necessity of geometrically non-linear formulation results sometimes more from the shape of the member under consideration (structural stability problems) and from the intrinsic material properties (small hardening parameter) than from the smallness or largeness of the strains themselves.

An important question which arises in the geometrically non-linear analysis is: which description, Lagrangian or Eulerian, should be used when formulating the constutitive equations. This question should be answered on the basis of experimental test results which would suggest the forms of constitutive relations and the appropriate variables in terms of which to express them.

In the paper the development of the constitutive relations will be given both in the initial (Lagrangian) and in the current (Eulerian) description, and next the results will be compared and discussed. The presentation will be limited to the purely mechanical (isothermal) theory and quasi-static deformation processes.

Although the theory of elasto-plasticity is not new, it is only in the last decade that more attention has been paid to geometrically non-linear description of deformation process of elastic-plastic materials. The interest in the non-linear analysis has been stimulated by increasing requirements for better predicition of structural behaviour on the one hand, and by the fact that our capability of solving the non-linear problems has grown with the advent of large high-speed computers on the other.

2. KINEMATICS

2.1. Motion

Let us consider a material body which in its initial (natural stress-free) configuration at time t_o occupies a region B_o in the three-dimensional physical space. To identify material points we can use their coordinates X_K at time t_o with respect to some fixed Cartesian frame of reference. Points in the space will be identified by their Cartesian coordinates x_k (Fig. 2.1).

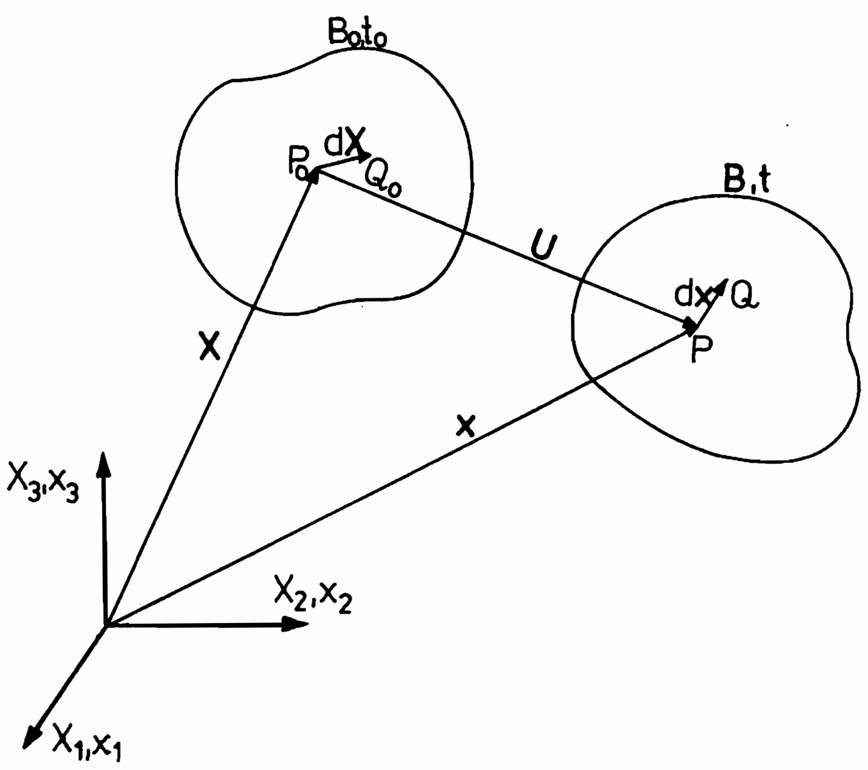

Fig. 2.1

During the motion the material points $\underset{\sim}{X}$ are displaced to various positions $\underset{\sim}{x}$ in the space. At the time t they occupy a region B.

In general, the quantities associated with the initial (undeformed) configuration of the body will be denoted by upper case Latin letters or by the subscript "o" throughout the paper. Similarly, quantities associated with the actual (deformed) position of the body will be denoted by lower case Latin letters.

The one-parameter family of transformations

$$x_k = x_k(X_K, t) \quad \text{or} \quad \underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) \tag{2.1}$$

is known as a motion. We assume that the functional relations (2.1) have continuous partial derivatives up to whatever order desired, expect possibly at some singular points, curves and surfaces. Moreover, they are single-valued and have a unique inverse

$$X_K = X_K(x_k, t) \tag{2.2}$$

in the neighbourhood of the material point P.
The unique inverse (2.2) exists if and only if the jacobian

$$j \equiv \left| \frac{\partial x_k}{\partial X_K} \right| \neq 0 \tag{2.3}$$

is not identically zero. The assumption (2.3) is known as the *axiom of continuity*. It expresses the *indestructibility* and the *inpenetrability* of matter. (No region of positive, finite volume is deformed into zero or infinite volume; and one portion of matter never penetrates into another).

2.2. The Eulerian and Lagrangian description

The deformation necessarily involves both the initial and final configurations, but there is a choice of coordinates to be used as independent variables.

Depending on whether we want to describe such quantities as stresses, strains and velocities for some particles of the body $\underset{\sim}{X}$ or for some points of physical space $\underset{\sim}{x}$, we have to apply Lagrangian (material) or Eulerian (spatial) description.

The movement of cars along a one-way street may serve as the one-dimensional example, given by W. Prager [1], to illustrate the difference between these two ways of describing the same motion. The Eulerian description corresponds to the observations of traffic policemen, who report on the velocities with which cars pass their fixed observation stations. The Lagrangian descripton, however, corresponds to the observations of drivers, who report on their velocities and progress along the street.

In the Lagrangian analysis the initial position of the particle $\underset{\sim}{X}$ and the time t are taken as independent variables. They are called the Lagrangian or material variables. The functions $\Phi = \Phi(\underset{\sim}{X},t)$ expressed in terms of Lagrangian variables, describe the variation of physical parameters for a given particle during its wandering through the space. The Lagrangian analysis is used primarily when considering geometrically non-linear behaviour of elastic and plastic structures since then the boundary conditions are usually referred to the initial configuration.

In the Eulerian analysis the actual position of the particle $\underset{\sim}{x}$ and the time t are used as independent variables. They are called Eulerian or spatial variables. The functions $\varphi = \varphi(\underset{\sim}{x},t)$ describe variation of physical parameters of the body at a given point of the physical space. The Eulerian analysis appears to be convenient for the description of a flow process, in which the initial configuration is immaterial (for example a metal forming process).

It is worth emphasizing that, whenever we know the motion of each particle of the body, we can easily pass from one description to the other.

2.3. Strain analysis

2.3.1. Deformation and displacement gradients

The motion, which carries a fixed material point through various spatial positions, may be expressed by

$$x_i = x_i(X_K,t) \qquad \text{in the Lagrangian description} \qquad (2.4)$$

or

$$X_K = X_K(x_i,t) \qquad \text{in the Eulerian description} \qquad (2.5)$$

Differentiating (2.4) with respect to X_K, we obtain a tensor

$$F_{iK} \equiv \frac{\partial x_i}{\partial X_K} \equiv x_{i,K} \tag{2.6}$$

which is called the *material deformation gradient.*

Similarly, differentiating (2.5) with respect to x_i we obtain a tensor

$$L_{Ki} \equiv \frac{\partial X_K}{\partial x_i} \equiv X_{K,i} \tag{2.7}$$

which is called the *spatial deformation gradient.*

Indices after a comma indicate differentiation with respect to X_K when they are upper case letters, and with respect to x_i when they are lower case letters.

The material and spatial deformation gradients are interrelated through the chain rule for partial differentiation,

$$\frac{\partial x_i}{\partial X_K}\frac{\partial X_K}{\partial x_j} = \delta_{ij} \quad , \quad \frac{\partial x_i}{\partial X_K}\frac{\partial X_L}{\partial x_i} = \delta_{KL} \quad . \tag{2.8}$$

Making use of (2.4) and (2.5) we have:

$$dx_i = x_{i,K}dX_K \quad , \quad dX_K = X_{K,i}dx_i \quad . \tag{2.9}$$

The vector $\underset{\sim}{U}$, joining the points P_o and P in Fig. 2.1 (the initial and final position of the particle) is known as the *displacement vector*. This vector may be expressed as the difference of coordinates in the initial and final position:

$$\underset{\sim}{U}(\underset{\sim}{X},t) = \underset{\sim}{x}(\underset{\sim}{X},t) - \underset{\sim}{X} \text{ in the Lagrangian description,} \tag{2.10}$$

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \underset{\sim}{x} - \underset{\sim}{X}(\underset{\sim}{x},t) \text{ in the Eulerian description.} \tag{2.11}$$

Differentiating the displacement vector with respect to the coordinates we obtain either the *material displacement gradient*

$$U_{K,L} = x_{i,L}\delta_{iK} - \delta_{KL} \tag{2.12}$$

or the *spatial displacement gradient*

$$u_{i,j} = \delta_{ij} - X_{K,j}\delta_{Ki} \; . \tag{2.13}$$

2.3.2. Deformation and finite strain tensors

The neighbouring material points P_o and Q_o in the initial (undeformed) configuration (Fig. 2.1) move to the point P and Q, respectively, in the final (deformed) configuration.

The square of length of the line element P_oQ_o is

$$(dS)^2 = d\underset{\sim}{X} \cdot d\underset{\sim}{X} = dX_K dX_K \tag{2.14}$$

whereas, in the deformed configuration, the square of the line element PQ is

$$(ds)^2 = d\underset{\sim}{x} \cdot d\underset{\sim}{x} = dx_i dx_i \; . \tag{2.15}$$

Substitution of $(2.9)_2$ into (2.14) and $(2.9)_1$ into (2.15) yields

$$(dS)^2 = X_{K,i}X_{K,j}dx_i dx_j \tag{2.16}$$

and

$$(ds)^2 = x_{i,K}x_{i,L}dX_K dX_L \, . \tag{2.17}$$

A body is said to undergo a rigid-body motion whenever

$$(ds)^2 = (dS)^2 \tag{2.18}$$

for all material points, (the deformation has not changed the distance of any pair of neighbouring material points). Therefore, the difference $(ds)^2 - (dS)^2$ is used as a measure of the strains produced during the motion.

Making use of (2.14) and (2.17) this difference may be expressed in the Lagrangian description in the form

$$(ds)^2 - (dS)^2 = (x_{i,K}x_{i,L} - \delta_{KL})dX_K dX_L = 2E_{KL}dX_K dX_L \qquad (2.19)$$

where the second-order tensor

$$E_{KL}(X_K,t) \equiv \frac{1}{2}(x_{i,K}x_{i,L} - \delta_{KL}) \qquad (2.20)$$

is called *the Lagrangian (or Green's) finite strain tensor.*

Using (2.15) and (2.16), the difference $(ds)^2 - (dS)^2$ may be expressed in the Eulerian description in the form

$$(ds)^2 - (dS)^2 = (\delta_{ij} - X_{K,i}X_{K,j})dx_i dx_j = 2e_{ij}dx_i dx_j \qquad (2.21)$$

where the second-order tensor

$$e_{ij}(x_i,t) \equiv \frac{1}{2}(\delta_{ij} - X_{K,i}X_{K,j}) \qquad (2.22)$$

is called *the Eulerian (or Almansi's) finite strain tensor.*

It follows from the definitions (2.20) and (2.22) that the Lagrangian and the Eulerian strain tensors are symmetric

$$E_{KL} = E_{LK} \quad , \quad e_{ij} = e_{ji} \quad . \qquad (2.23)$$

The finite strain tensors may be expressed as functions of the displacement gradients. Thus, the substitution of $x_{i,K}$ from (2.12) into (2.20) leads to the Lagrangian strain tensor given by

$$E_{KL} = \frac{1}{2}(U_{K,L} + U_{L,K} + U_{M,K}U_{M,L}). \qquad (2.24)$$

Similarly, the substitution of $X_{K,i}$ from (2.13) into (2.22) gives the Eulerian strain tensor in the form

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}). \qquad (2.25)$$

In view of the relations (2.19) and (2.21) the necessary and sufficient condition for a rigid body motion is that the Lagrangian and/or

Eulerian strain tensors be identically equal to zero for all material points of a continuum.

2.3.3. Infinitesimal strains and rotations

Let us introduce the infinitesimal strain tensors $\tilde{E}_{KL}$, $\tilde{e}_{kl}$ and the infinitesimal rotation tensors $\tilde{R}_{KL}$, $\tilde{r}_{kl}$ as symmetric and skew-symmetric parts of the displacement gradients, respectively, in the Lagrangian and the Eulerian descriptions:

$$\tilde{E}_{KL} \equiv \frac{1}{2}\,(U_{K,L} + U_{L,K}) \quad , \quad \tilde{e}_{kl} \equiv \frac{1}{2}\,(u_{k,l} + u_{l,k}) \ , \tag{2.26}$$

$$\tilde{R}_{KL} \equiv \frac{1}{2}\,(U_{K,L} - U_{L,K}) \quad , \quad \tilde{r}_{kl} \equiv \frac{1}{2}\,(u_{k,l} - u_{l,k}) \ . \tag{2.27}$$

The above quantities are the strains and rotations of the classical linear continuum theory.

Equations (2.26) und (2.27) result in

$$U_{K,L} = \tilde{E}_{KL} + \tilde{R}_{KL} \quad , \tag{2.28}$$

$$u_{k,l} = \tilde{e}_{kl} + \tilde{r}_{kl} \quad . \tag{2.29}$$

Substituting (2.28) into (2.24) and (2.29) into (2.25) we obtain

$$E_{KL} = \tilde{E}_{KL} + \frac{1}{2}\,(\tilde{E}_{MK} + \tilde{R}_{MK})\,(\tilde{E}_{ML} + \tilde{R}_{ML}) \ , \tag{2.30}$$

$$e_{kl} = \tilde{e}_{kl} - \frac{1}{2}\,(\tilde{e}_{mk} + \tilde{r}_{mk})\,(\tilde{e}_{ml} + \tilde{r}_{ml}) \ . \tag{2.31}$$

It follows from (2.30) und (2.31) that, in general, $\tilde{E}_{KL} = 0$ and/or $\tilde{e}_{kl} = 0$ does not imply $E_{KL} = 0$ and/or $e_{kl} = 0$; that is, the vanishing of infinitesimal strains is not sufficient for a rigid-body motion. Therefore, $\tilde{E}_{KL}$ and $\tilde{e}_{kl}$ are not strain measures for a general case of finite deformations.

2.3.4. Rotation tensor and other strain tensors

For the description of the local rotation of a given particle we now define a rotation tensor $\underset{\sim}{R}$.

Let $\underset{\sim}{N}^{\alpha}$ be an orthogonal triad along the principal directions of strain at a point P_o. After deformation the original triad $\underset{\sim}{N}^{\alpha}$ is rotated into the orthogonal triad $\underset{\sim}{n}^{\alpha}$ lying along the principal directions of strain at P (Fig. 2.2).

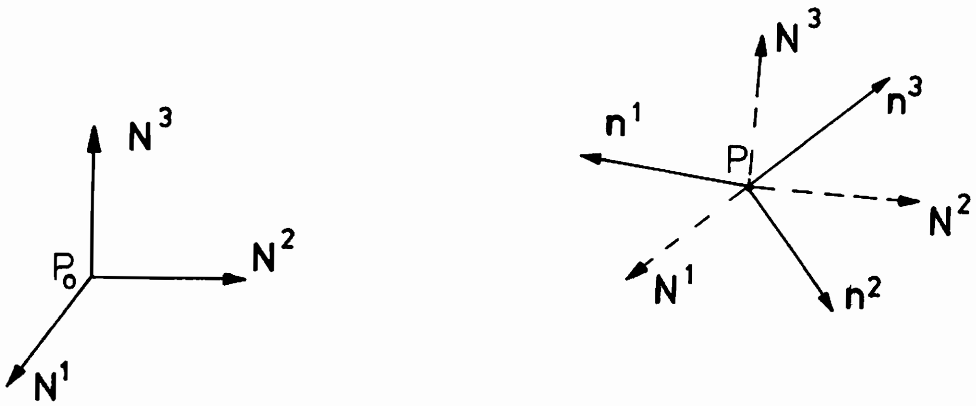

Fig. 2.2

We define a so-called *rotation tensor* $\underset{\sim}{R}$ as the unit, orthogonal tensor which transforms (shifts and rotates) $\underset{\sim}{N}^{\alpha}$ into $\underset{\sim}{n}^{\alpha}$:

$$n_k^{\alpha} = R_{kK} N_K^{\alpha} \quad , \tag{2.32}$$

$$N_K^{\alpha} = R_{kK}^{-1} n_k^{\alpha} \quad . \tag{2.33}$$

It is now clear that the necessary and sufficient condition for pure strain is

$$R_{kK} = \delta_{kK} \tag{2.34}$$

Let us apply the so-called polar decomposition theorem [2] to the deformation gradient $x_{i,K}$. Then the result may be written as

$$x_{i,K} = R_{iL} U_{LK} = V_{ij} R_{jK} \tag{2.35}$$

where R_{iL} is the rotation tensor, and U_{KL} and V_{ij} are positive definite symmetric tensors known as the *right (or Lagrangian) stretch tensor* and the *left (or Eulerian) stretch tensor*, respectively.

Substitution of $(2.35)_1$ into (2.12) gives

$$u_{K,L} = R_{KP}U_{PL} - \delta_{KL} \quad . \tag{2.36}$$

According to the definitions (2.26) and (2.27) the infinitesimal strain and rotation tensors in the Lagrangian description may now be written as

$$\tilde{E}_{KL} = R_{(KP}U_{PL)} - \delta_{KL} \,, \tag{2.37}$$

$$\tilde{R}_{KL} = R_{[KP}U_{PL]} \tag{2.38}$$

where parentheses enclosing indices indicate the symmetric part of the quantity and brackets the antisymmetric part.

Solving (2.36) for R_{KP} and U_{PL} and substituting (2.28), we obtain

$$R_{KP} = (\delta_{KL} + \tilde{E}_{KL} + \tilde{R}_{KL})U^{-1}_{PL} \,, \tag{2.39}$$

$$U_{PL} = (\delta_{KL} + \tilde{E}_{KL} + \tilde{R}_{KL})R^{-1}_{KP} \tag{2.40}$$

Formulae dual to these involving the Eulerian representation are not difficult to find.

The vector dX_K at a point P_o (Fig. 2.1) is carried by the deformation into

$$dx_i = x_{i,K}dX_K \,. \tag{2.41}$$

Substituting (2.35) into (2.41), we have

$$dx_i = R_{iL}U_{LK}dX_K = V_{ij}R_{jK}dX_K . \tag{2.42}$$

The deformation of any line material element may be considered, therefore, as resulting from a translation, a rigid rotation of the principal axes of strain and stretches along these axes. The deformation (2.41) may be decomposed into translation, rotation and stretch in two different ways:

The vector $d\underset{\sim}{X}$ is first rigidly translated and rotated into $d\overset{(R)}{\underset{\sim}{x}}$ (Fig. 2.3).

$$d\overset{(R)}{x}_j = R_{jK} dX_K \; . \tag{2.43}$$

Next, the vector $d\overset{(R)}{\underset{\sim}{x}}$ is stretched into $d\underset{\sim}{x}$,

$$dx_i = V_{ij} d\overset{(R)}{x}_j \; . \tag{2.44}$$

Substituting of (2.43) into (2.44) yields $(2.42)_2$.

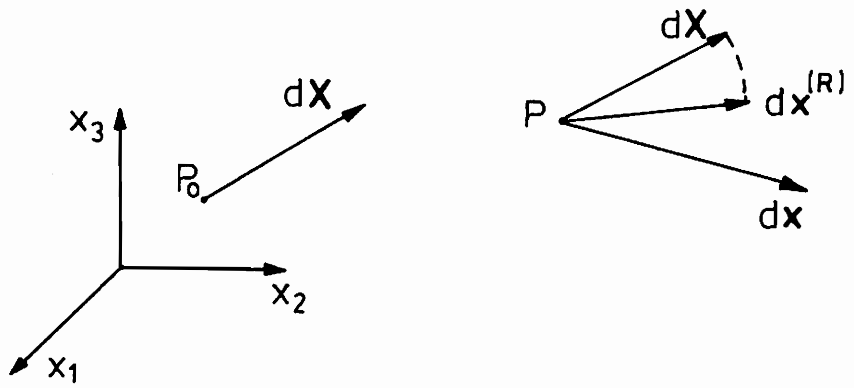

Fig. 2.3

Note that the stretching (2.44) involves, in general, further rotation of the vector $d\overset{(R)}{\underset{\sim}{x}}$, except when $d\underset{\sim}{x}$ is taken along the principal axes of V_{ij}.

The relation (2.41) may be decomposed in another way as follows (Fig. 2.4). The vector $d\underset{\sim}{X}$ is first stretched into $d\overset{(s)}{\underset{\sim}{X}}$

$$d\overset{(s)}{X}_L = U_{LK} dX_K \; . \tag{2.45}$$

The vector $d\overset{(s)}{\underset{\sim}{X}}$ is then rigidly rotated into the vector $d\underset{\sim}{x}$ and shifted to the point P,

$$dx_i = R_{iL} d \overset{(s)}{X}_L \; . \tag{2.46}$$

Substitution of (2.45) into (2.46) gives $(2.42)_1$.

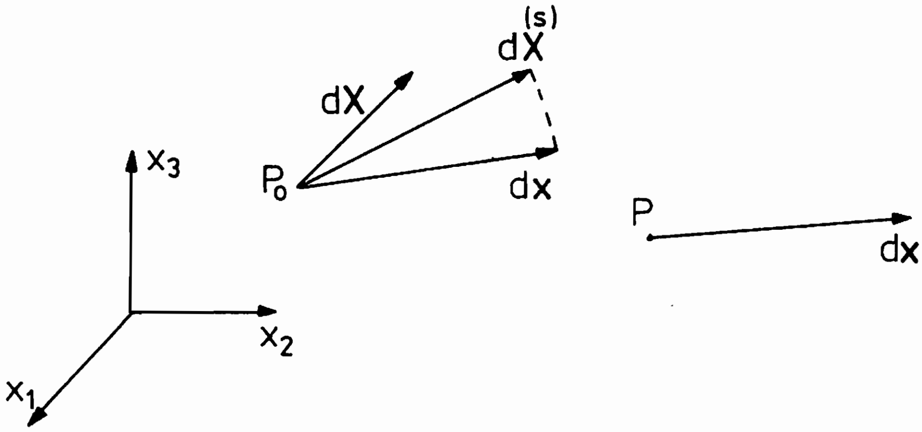

Fig. 2.4

As we can see, the tensorial measures of the foregoing transformations are dependent on the order in which they are applied.

Let us define the *stretch* λ_n in the direction $\underset{\sim}{n}$ as the ratio of final ds to initial dS length of an infinitesimal line element $\underset{\sim}{dx}$ of direction $\underset{\sim}{n}$,

$$\lambda_n = \frac{ds}{dS} \; . \tag{2.47}$$

Let $\lambda_i (i = 1,2,3)$ be *principal stretches* (that is stretches in principal directions).

An acceptable tensor measure of finite strain $\underset{\sim}{\varepsilon}$ may be obtained [3] by defining its principal values ε_i as

$$\varepsilon_i = f(\lambda_i) \quad \text{where } f(1) = 0 \;,\; f'(1) = 1 \tag{2.48}$$

and $f(\lambda_i)$ is any sufficiently smooth monotone function.

The most commonly used strain measures are included in the one parameter family:

$$\varepsilon_i = \begin{cases} \frac{1}{2m}(\lambda_i^{2m} - 1) & \text{for } m \neq 0 , \\ \ln\lambda_i & \text{for } m = 0 . \end{cases} \tag{2.49}$$

For m = 1 we obtain the principal values of the Lagrangian (or Green's) strain tensor $\underset{\sim}{E}$:

$$\varepsilon_i = \frac{1}{2}(\lambda_i^2 - 1). \tag{2.50}$$

For m = -1 we have the principal values of the Eulerian (or Almansi) strain tensor $\underset{\sim}{e}$:

$$\varepsilon_i = \frac{1}{2}(1 - \lambda_i^{-2}) . \tag{2.51}$$

For $m = \frac{1}{2}$ we get the principal values of the Cauchy (or engineering) strain measure:

$$\varepsilon_i = \lambda_i - 1 . \tag{2.52}$$

For m = 0 we get the principal values of the Hencky (or logarithmic) strain measure:

$$\varepsilon_i = \ln\lambda_i . \tag{2.53}$$

Similarly, any of the above considered strain measures may be expressed in terms of the displacement gradient by putting m = 1 or m = -1 or $m = \frac{1}{2}$ or m = 0, respectively, in the relation

$$\varepsilon_{KL} = \tilde{E}_{KL} + \frac{1}{2} U_{M,K}U_{M,L} + (m - 1)\tilde{E}_{MK}\tilde{E}_{ML} + \ldots \tag{2.54}$$

where polynomial up to the second order in the displacement gradient is indicated.

2.3.5. Approximations

Various approximate theories of the nonlinear mechanics of continua may be obtained by neglecting some of the nonlinear terms in the geometrical relations (the terms of second order importance). The following approximations are presented to provide only an intuitive ground in this theory.

1. *Pure finite strains*

Since the necessary and sufficient condition for pure strain is that

$$R_{KL} = \delta_{KL} \tag{2.55}$$

the equation (2.40) takes now the form

$$U_{KL} = \delta_{KL} + \tilde{E}_{KL} + \tilde{R}_{KL} \quad . \tag{2.56}$$

From the symmetry of the stretch tensor it follows that in this case

$$\tilde{R}_{KL} = 0 \tag{2.57}$$

and

$$U_{KL} = \delta_{KL} + \tilde{E}_{KL} \tag{2.58}$$

Substitution of (2.57) into (2.30) yields

$$E_{KL} = \tilde{E}_{KL} + \frac{1}{2}\tilde{E}_{MK}\tilde{E}_{ML} \quad . \tag{2.59}$$

2. *Small rotations and large strains*

When the rotation of the principal axes is small, the rotation tensor $\underset{\sim}{R}$ is close to unity and the infinitesimal rotation tensor $\underset{\sim}{\tilde{R}}$ is small.

The Green strain tensor (2.30) may then be approximated by

$$E_{KL} \cong \tilde{E}_{KL} + \frac{1}{2}(\tilde{E}_{MK}\tilde{E}_{ML} + \tilde{E}_{MK}\tilde{R}_{ML} + \tilde{E}_{ML}\tilde{R}_{MK}) \, . \tag{2.60}$$

3. *Small principal extensions and large rotations*

In this case the extensions in principal directions are supposed to be small. Hence, the stretch tensors are close to unity

$$U_{KL} \cong \delta_{KL} \quad , \quad V_{kl} \cong \delta_{kl} \ . \tag{2.61}$$

The rotation tensors (2.39) can now be approximated by

$$R_{KL} \cong (\delta_{KL} + \tilde{E}_{KL} + \tilde{R}_{KL}) \ . \tag{2.62}$$

On comparison of the symmetric and skew-symmetric parts of both sides of equation (2.62) we get

$$R_{(KL)} \cong \delta_{KL} + \tilde{E}_{KL} \ , \tag{2.63}$$

$$R_{[KL]} \cong \tilde{R}_{KL} \ . \tag{2.64}$$

As simple example a thin bar may serve which is bent into a ring without large extension.

In some situations certain components of the rotation tensor may be considered to be small as compared with others; then the geometrical relations may be further simplified by dropping the terms containing these components. This is used to obtain the second-order plate theory known as the von Kârmân - Timoshenko theory.

4. *Small strains and small rotations (small deformation)*

This is equivalent to an assumption of small displacement gradients:

$$U_{K,L} << 1 \quad , \quad u_{k,l} << 1 \ . \tag{2.65}$$

Then all nonlinear terms in (3.30) and (3.31) may be dropped and the finite strain tensors reduce to the infinitesimal strain tensors:

$$E_{KL} = \tilde{E}_{KL} \quad , \quad e_{kl} = \tilde{e}_{kl} \ . \tag{2.66}$$

The resulting equations represent the so-called *small deformation theory* of continuous media.

5. *Small strains, rotations and displacements*

If both the displacement gradients and the displacements themselves are small, then disappears the difference between material and spatial coordinates and hence between material and spatial descriptions. Thus

$$E_{KL} = \tilde{E}_{KL} = e_{kl} = \tilde{e}_{kl} \tag{2.67}$$

and we have the classical linear theory of infinitesimal deformations.

The above classification of approximate theories is based on the assumptions imposed on the strain and rotation tensors. Direct application of these theories presents some difficulties since the compatibility conditions have to be satisfied.

The strain and rotation tensors are expressible in terms of the displacement gradients. Therefore, another classification of approximate theories may be obtained by neglecting some of the non-linear terms in the strain-displacement equations (2.24) or (2.25). Then, the three displacement components and their gradients (or the wave lengths of deformation patterns) are employed as a basis of approximation. For example, moderately large deflection and small tangential displacement theory is known as the Donnell-Vlasov theory, when applied to shallow shells. This approach is particularly convenient to formulate the approximate plastic shell theories.

2.4. Rate of deformation

The definition of the velocity vector is given by (2.4) as $\underset{\sim}{v} \equiv \dot{\underset{\sim}{x}}$. An alternative form of the same vector may be obtained by substituting $\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{U}$ into (2.4). This leads to

$$\underset{\sim}{v} \equiv \dot{\underset{\sim}{x}} = (\underset{\sim}{X} + \underset{\sim}{U})^{\cdot} = \dot{\underset{\sim}{U}} \tag{2.68}$$

since $\underset{\sim}{X}$ does not depend upon time.

According to (2.68), the velocity field in the Lagrangian and in the Eulerian descriptions is given, respectively, by

$$v_K(\underset{\sim}{X},t) = \dot{U}_K(\underset{\sim}{X},t) = \frac{\partial U_K}{\partial t} , \tag{2.69}$$

$$v_i(\underset{\sim}{x},t) = \dot{u}_i(\underset{\sim}{x},t) = \frac{\partial u_i}{\partial t} + v_k \frac{\partial u_i}{\partial x_k} . \tag{2.70}$$

In (2.70) the velocity is not given explicitly since it appears also as a factor in the second term on the right-hand side.

The material derivative of the velocity is the *acceleration*. In the Lagrangian description it is given by

$$a_K(\underset{\sim}{X},t) \equiv \dot{v}_K(\underset{\sim}{X},t) = \frac{\partial v_K}{\partial t} , \tag{2.71}$$

and in the Eulerian description by

$$a_i(\underset{\sim}{x},t) \equiv \dot{v}_i(\underset{\sim}{x},t) = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} . \tag{2.72}$$

The spatial gradient of the velocity field, called *velocity gradient*, may be decomposed into its symmetric and skew-symmetric parts,

$$v_{i,j} = \frac{1}{2} (v_{i,j} + v_{j,i}) + \frac{1}{2} (v_{i,j} - v_{j,i}). \tag{2.73}$$

The symmetric part

$$d_{ij} \equiv \frac{1}{2} (v_{i,j} + v_{j,i}) \tag{2.74}$$

is called the *deformation rate tensor*, whereas the skew-symmetric part

$$w_{ij} \equiv \frac{1}{2} (v_{i,j} - v_{j,i}) \tag{2.75}$$

is called the *spin or vorticity tensor*.

It is easy to show that, if the displacement gradient is small as compared to unity, $u_{i,k} << 1$, the material derivative of the Eulerian infinitesimal strain tensor $\tilde{e}_{ij}$ is approximately equal to the deformation rate tensor d_{ij},

$$\dot{\tilde{e}}_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})^{\cdot} = \frac{1}{2} (v_{i,j} - u_{i,k}v_{k,j}$$

$$+ v_{j,i} - u_{j,k}v_{k,i}) \simeq \frac{1}{2}(v_{i,j} + v_{j,i}) = d_{ij} \qquad (2.76)$$

It can similarly be shown that, if $u_{i,k} \ll 1$, the spin tensor w_{ij} is approximately equal to the material derivative of the Eulerian infinitesimal rotation tensor $\dot{\tilde{r}}_{ij}$,

$$\dot{\tilde{r}}_{ij} \simeq \frac{1}{2} (v_{i,j} - v_{j,i}) = w_{ij}. \qquad (2.77)$$

According to the equations (2.19) and (2.21) the difference between the squares of line elements before and after deformation may be expressed in the Lagrangian and the Eulerian description, respectively, as follows:

$$ds^2 - dS^2 = 2E_{KL}dX_K dX_L \,, \qquad (2.78)$$

$$ds^2 - dS^2 = 2e_{kl}dx_k dx_l \;. \qquad (2.79)$$

The material derivative of the equation (2.78) gives

$$(ds^2)^{\cdot} = 2\dot{E}_{KL}dX_K dX_L \quad , \qquad (2.80)$$

whereas the material derivative of (2.79) leads to

$$(ds^2)^{\cdot} = 2\dot{e}_{kl}dx_k dx_l + 2e_{kl}dv_k dx_l + 2e_{kl}dx_k dv_l =$$

$$= 2(\dot{e}_{kl} + e_{il}v_{i,k} + e_{ki}v_{i,l})dx_k dx_l \;. \qquad (2.81)$$

On the other hand, the material derivative of the square of the differential line element $d\chi_i$ may be calculated as follows:

$$(ds^2)^{\cdot} = (dx_i dx_i)^{\cdot} = 2dx_i dv_i = 2dx_i v_{i,j} dx_j = 2d_{ij}dx_i dx_j \;, \qquad (2.82)$$

since

$$(v_{i,j} - v_{j,i})dx_i dx_j = 0 \quad .$$

Comparison of (2.80) with (2.82) furnishes

$$d_{ij} = \dot{E}_{KL} X_{K,i} X_{L,j} \tag{2.83}$$

whereas comparison of (2.81) with (2.82) leads to the relation

$$d_{ij} = \dot{e}_{ij} + e_{ik} v_{k,j} + e_{jk} v_{k,i} \equiv \overset{\nabla C-R}{e_{ij}} \tag{2.84}$$

The right-hand side of the above equation is called the Cotter-Rivlin time derivative and will be denoted by $\overset{\nabla C-R}{e_{ij}}$ (1).

According to the definition introduced in the section 2.3.2, a motion is regarded to be rigid-body motion if and only if the deformation does not change the distance of any pair of neighbouring material points,that is if

$$(ds^2)^{\cdot} = 0. \tag{2.85}$$

By comparison of (2.85) with (2.80), (2.82) and (2.84) we arrive at the conclusion that the necessary and sufficient condition for the motion of a body to be rigid is $d_{kl} = 0$, or $\overset{\nabla C-R}{e_{kl}} = 0$, or $\dot{E}_{KL} = 0$, but not $\dot{e}_{kl} = 0$.

The deformation rate rensor d_{kl} is a measure of the instantaneous rate of change of lengths of material elements and angles between them. To see this, let us substitute $dx_i = n_i ds$ into the equation (2.82) to obtain

$$2ds(ds)^{\cdot} = 2d_{ij} n_i n_j (ds)^2 \tag{2.86}$$

or

$$\frac{(ds)^{\cdot}}{ds} = d_{ij} n_i n_j = d_{(n)} \tag{2.87}$$

---

(1) Objective time derivatives will be considered in chapter 3.3. The material derivative of Almansi strain tensor $\dot{e}_{ij}$ is not objective.

where $d_{(n)}$ is the rate of stretching in the direction $\underset{\sim}{n}$. If, for example, $\underset{\sim}{n}$ is taken along the $\underset{\sim}{x}_1$ axis, then

$$d_{(n)} = d_{11}. \tag{2.88}$$

Thus the diagonal components of the deformation rate tensor appear to be the rates of stretching in the coordinate directions. It can be shown in a similar way that off-diagonal components d_{ij} are halves of the shear rates in the orthogonal coordinate directions,

$$d_{12} = -\frac{1}{2}\dot{\theta}_{(1,2)} \; .$$

2.5. Change of volume and surface elements

During the motion from some initial configuration at time t_o to the current configuration at time t, the volume element dV_o is deformed into dV. If the initial volume element is taken as the parallelepiped specified by three vectors $d\underset{\sim}{X}^{(1)}$, $d\underset{\sim}{X}^{(2)}$ and $d\underset{\sim}{X}^{(3)}$ (Fig. 2.5), then

$$dV_o = d\underset{\sim}{X}^{(1)}\times d\underset{\sim}{X}^{(2)}\cdot d\underset{\sim}{X}^{(3)} = \epsilon_{KLN}dX_K^{(1)}dX_L^{(2)}dX_N^{(3)} \tag{2.89}$$

where ϵ_{KLN} is the permutation tensor. Due to the motion the line element $d\underset{\sim}{X}$ becomes $d\underset{\sim}{x}$ and the parallelepiped deformes into a skewed parallelepiped having edges $d\underset{\sim}{x}^{(1)}$, $d\underset{\sim}{x}^{(2)}$, $d\underset{\sim}{x}^{(3)}$ and a volume given by the box product

$$dV = d\underset{\sim}{x}^{(1)}\times d\underset{\sim}{x}^{(2)}\cdot d\underset{\sim}{x}^{(3)} = \epsilon_{ijk}dx_i^{(1)}dx_j^{(2)}dx_k^{(3)} \tag{2.90}$$

Because of the relationship $dX_K = X_{K,i}dx_i$ between the initial and the deformed line elements the initial volume element dV_o, defined by (2.89), may be transformed to take the form

$$dV_o = \epsilon_{KLN}X_{K,i}X_{L,j}X_{N,k}dx_i^{(1)}dx_j^{(2)}dx_k^{(3)} \quad . \tag{2.91}$$

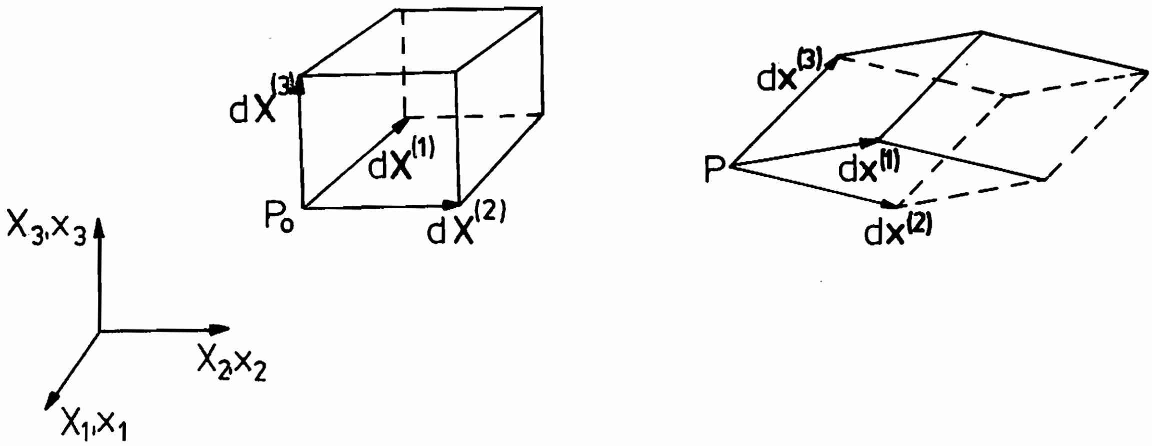

Fig. 2.5

The jacobian of the transformation $dX_K = X_{K,i}dx_i$ is defined as

$$J \equiv |X_{K,i}| = \frac{1}{6}\,\epsilon_{KLN}\epsilon_{ijk}X_{K,i}X_{L,j}X_{N,k}\ . \tag{2.92}$$

Multiplication of equ. (2.92) by ϵ_{ijk} and use of the relation $\epsilon_{ijk}\epsilon_{ijk} = 6$ leads to

$$J\epsilon_{ijk} = \epsilon_{KLN}X_{K,i}X_{L,j}X_{N,k}\ . \tag{2.93}$$

Substituting (2.93) into (2.91), we obtain the inital volume element dV_o in the form

$$dV_o = J\epsilon_{ijk}dx_i^{(1)}dx_j^{(2)}dx_k^{(3)}\ . \tag{2.94}$$

Finally, the comparison of (2.94) with (2.90) leads to the relation between initial and final volume elements

$$\frac{dV_o}{dV} = J. \tag{2.95}$$

By using the principle of mass conservation

$$dV_o \rho_o = dV \rho \tag{2.96}$$

in the equ. (2.95), the jacobian J may also be expressed as the ratio of the instantaneous and the initial mass densities

$$J = \frac{\rho}{\rho_o} \; . \tag{2.97}$$

In order to describe the deformation of a surface element, let us consider again the material line elements represented by the vectors $d\underset{\sim}{X}^{(1)}$ and $d\underset{\sim}{X}^{(2)}$ in the initial state and by the vectors $d\underset{\sim}{x}^{(1)}$ and $d\underset{\sim}{x}^{(2)}$ in the instantaneous state, (Fig. 2.6).

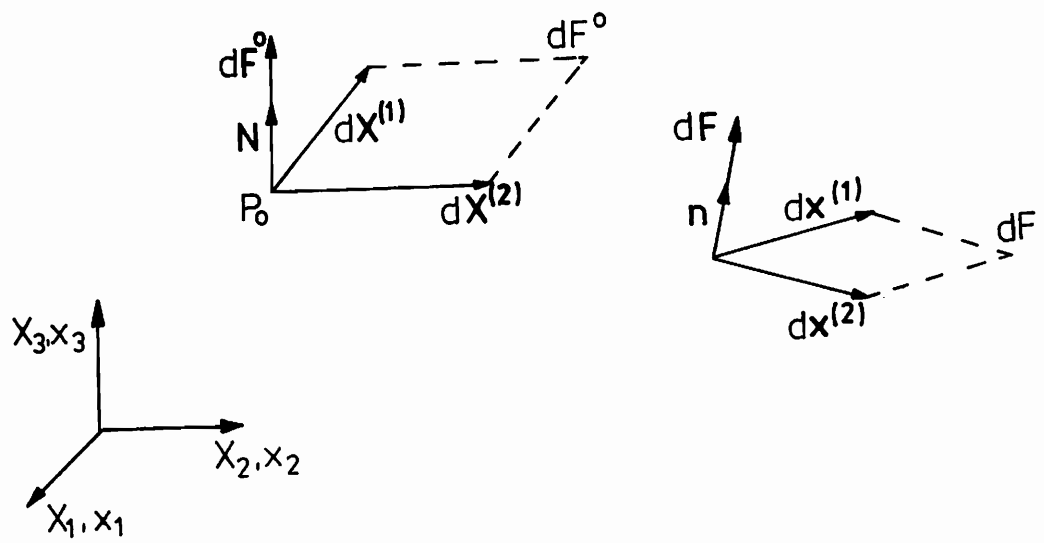

Fig. 2.6

If dF^o denotes the area and $\underset{\sim}{N}$ normal to the plane of the parallelogram specified by the vectors $dX^{(1)}$, $dX^{(2)}$, then the surface element vector $d\underset{\sim}{F}^o$ is given by the vector product

$$d\underset{\sim}{F}^o = \underset{\sim}{N}\, dF^o = d\underset{\sim}{X}^{(1)} \times d\underset{\sim}{X}^{(2)}$$

or

$$dF_K^o = N_K dF^o = \epsilon_{KLN} dX_L^{(1)} dX_N^{(2)} \; . \tag{2.98}$$

In the similar manner, the deformed surface element vector $d\underset{\sim}{F}$ of the area dF and normal $\underset{\sim}{n}$ can be interpreted as the vector product

$$d\underset{\sim}{F} = \underset{\sim}{n}\,dF = d\underset{\sim}{x}^{(1)} \times d\underset{\sim}{x}^{(2)}$$

or

$$dF_k = n_k dF = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} \quad . \tag{2.99}$$

Substitution of the relation $dX_L = X_{L,j} dx_j$ into (2.98) yields

$$dF_K^o = \epsilon_{KLN} X_{L,j} X_{N,k} dx_j^{(1)} dx_k^{(2)} \quad . \tag{2.100}$$

Next, mulitplication of the relation (2.100) by $X_{K,1}$ and use of (2.93) furnishes

$$dF_K^o X_{K,i} = \epsilon_{KLN} X_{L,j} X_{N,k} X_{K,i} dx_j^{(1)} dx_k^{(2)} = J\epsilon_{ijk} dx_j^{(1)} dx_k^{(2)} \quad . \tag{2.101}$$

Finally, substituting (2.99) into (2.101), we obtain:

$$dF_K^o = J dF_i x_{i,K} \tag{2.102}$$

or by using (2.97) we may also write the relation between the initial and the deformed surface elements in the form

$$dF_K^o = \frac{\rho}{\rho_o} dF_i x_{i,K} \tag{2.103}$$

2.6. Strain analysis in curvilinear coordinates

It is sometimes convenient (e.q. in the shell theory) to analize the deformation process in curvilinear coordinates system $\{x^k\}$.

At each point of the physical space the base vectors $\underset{\sim}{g}_k$ are now defined as partial derivatives of radius-vector $\underset{\sim}{p}$ with respect to the coordinates (Fig. 2.7),

$$\underset{\sim}{g}_k \equiv \frac{\partial \underset{\sim}{p}}{\partial x^k} \quad . \tag{2.104}$$

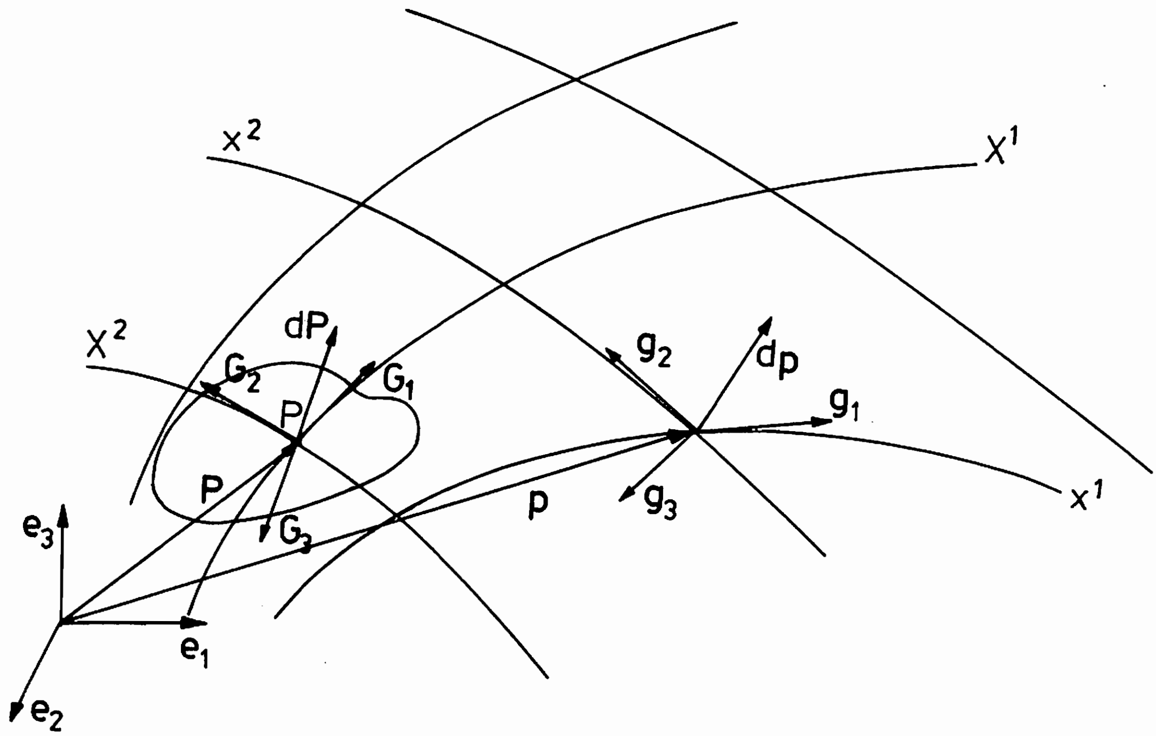

Fig. 2.7

To identify material points we use their curvilinear coordinates at time t_o, denoted by X^K. The base vectors for the points of physical space, which are occupied by the body in its initial state, are defined as the partial derivatives of radius-vector $\underset{\sim}{P}$ with respect to the coordinates X^K,

$$\underset{\sim}{G}_K \equiv \frac{\partial \underset{\sim}{P}}{\partial X^K} \quad . \tag{2.105}$$

According to the definitions, the base vectors are tangential to the coordinate curves.

The infinitesimal vectors $d\underset{\sim}{P}$ and $d\underset{\sim}{p}$ in the initial and the actual configurations, respectively, may be expressed as [1]

$$d\underset{\sim}{P} = \frac{\partial \underset{\sim}{P}}{\partial X^K} dX^K = \underset{\sim}{G}_K dX^K \quad , \tag{2.106}$$

$$d\underset{\sim}{p} = \frac{\partial \underset{\sim}{p}}{\partial x^k} dx^k = \underset{\sim}{g}_k dx^k \quad . \tag{2.107}$$

([1]) The summation convention is here implied by every diagonally repeated index.

Therefore, the length of an infinitesimal line element before and after deformation can be written as

$$dS^2 = d\underset{\sim}{P} \cdot d\underset{\sim}{P} = \underset{\sim}{G}_K \cdot \underset{\sim}{G}_L dX^K dX^L = G_{KL} dX^K dX^L \ , \tag{2.108}$$

$$ds^2 = d\underset{\sim}{p} \cdot d\underset{\sim}{p} = \underset{\sim}{g}_k \cdot \underset{\sim}{g}_l dx^k dx^l = g_{kl} dx^k dx^l \tag{2.109}$$

where G_{KL} and g_{kl} are covariant components of the metric tensor in the initial and the actual configurations, respectively, defined as scalar products of the base vectors

$$G_{KL} \equiv \underset{\sim}{G}_K \cdot \underset{\sim}{G}_L \ , \tag{2.110}$$

$$g_{kl} \equiv \underset{\sim}{g}_k \cdot \underset{\sim}{g}_l \ . \tag{2.111}$$

The relations between the contravariant components of the infinitesimal line elements before and after deformation have now the similar form as for the Cartesian components

$$dX^K = X^K_{\ ,i} dx^i \ , \tag{2.112}$$

$$dx^i = x^i_{\ ,K} dX^K \ . \tag{2.113}$$

By using (2.112) und (2.113) in (2.108) and (2.109), the difference between the squares of the line element before and after deformation may be written as

$$ds^2 - dS^2 = g_{ij} x^i_{\ ,K} x^j_{\ ,L} dX^K dX^L - G_{KL} dX^K dX^L = (g_{ij} x^i_{\ ,K} x^j_{\ ,L} - G_{KL}) dX^K dX^L =$$

$$= 2E_{KL} dX^K dX^L \tag{2.114}$$

or

$$ds^2 - dS^2 = g_{ij} dx^i dx^j - G_{KL} X^K_{\ ,i} X^L_{\ ,j} dx^i dx^j = (g_{ij} - G_{KL} X^K_{\ ,i} X^L_{\ ,j}) dx^i dx^j =$$

$$= 2e_{kl} dx^k dx^l \tag{2.115}$$

where the Lagrangian and the Eulerian strain tensors are defined as:

$$E_{KL}(\underset{\sim}{X},t) \equiv \frac{1}{2}\,(g_{ij}x^i_{,K}x^j_{,L} - G_{KL}) \ , \tag{2.116}$$

$$e_{ij}(\underset{\sim}{x},t) \equiv \frac{1}{2}\,(g_{ij} - G_{KL}X^K_{,i}X^L_{,j}) \ . \tag{2.117}$$

In the Cartesian coordinates we have

$$G_{KL} = g_{kl} = \delta_{kl} \tag{2.118}$$

and equations (2.116), (2.117) coincide with (2.20), (2.22).

For later use, let us introduce two additional base vectors:

$$\underset{\sim}{c}_k(\underset{\sim}{x},t) \equiv \frac{\partial \underset{\sim}{P}}{\partial x^k} = \frac{\partial \underset{\sim}{P}}{\partial X^K}\ \frac{\partial X^K}{\partial x^k} = \underset{\sim}{G}_K X^K_{,k} \ , \tag{2.119}$$

$$\underset{\sim}{C}_K(\underset{\sim}{X},t) \equiv \frac{\partial \underset{\sim}{p}}{\partial X^K} = \frac{\partial \underset{\sim}{p}}{\partial x^k}\ \frac{\partial x^k}{\partial X^K} = \underset{\sim}{g}_k x^k_{,K} \ . \tag{2.120}$$

The relation (2.119) between $\underset{\sim}{c}_k$ and $\underset{\sim}{G}_K$ indicates that $\underset{\sim}{c}_k$ are vectors into which the base vectors $\underset{\sim}{G}_K$ transform after deformation. Similarly, it follows from (2.120) that the vectors $\underset{\sim}{C}_K$ after deformation become $\underset{\sim}{g}_k$. Therefore, $\underset{\sim}{c}_k$ may be treated as the base vectors in *convected* (embedded in the material and deforming with it) or *intrinsic* coordinate system.

3. STATE OF STRESS

3.1. Stress tensors

A stress tensor referred to the current state of a body is a natural physical concept. The use of Eulerian variables enables us, therefore, to treat all questions of statics in a particularly simple manner. On the other hand, from the standpoint of kinematics, the Lagrangian variables seem to be particularly suitable for the description of motion of a body, especially if the boundary conditions are referred to the initial state.

Since the constitutive equations relate stresses to strains (or strain rates), therefore, in order to construct a consistent theory, it is necessary to express both stresses and strains in the same description. Hence, if strains are referred to the initial state of a continuum (as in the Lagrangian description) it is required to use stress measures defined also with respect to the initial configuration. Such stress measures are physically artificial, though mathematically consistent.

If $d\underset{\sim}{F}^{o}$ denotes the initial surface element vector of area dF^{o} with the corresponding unit outward normal vector $\underset{\sim}{N}$, and $d\underset{\sim}{F}$ is the surface element vector in the current state of area dF with the unit outward normal vector $\underset{\sim}{n}$ (Fig. 3.1), then

$$dF^{o}_{K} = dF^{o} N_{K} \ , \tag{3.1}$$

$$dF_{k} = dF \, n_{k} \ . \tag{3.2}$$

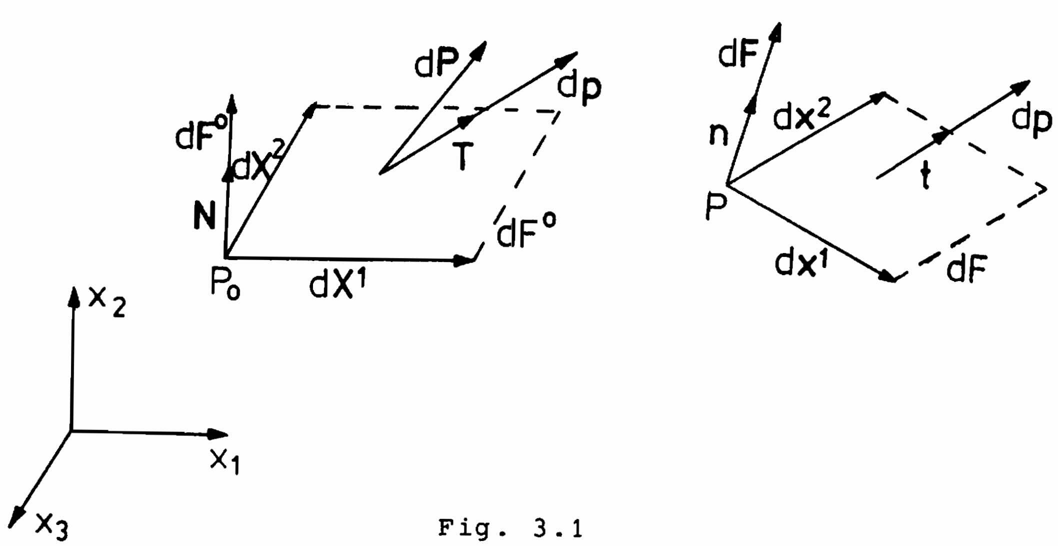

Fig. 3.1

Let us denote by $d\underset{\sim}{p}$ a force vector acting on a surface element $d\underset{\sim}{F}$. Then the force vector $\underset{\sim}{t}$ acting on the unit surface element is given by the relation

$$t_i = \frac{dp_i}{dF} \tag{3.3}$$

and is called a *true stress vector*. The *true or Cauchy stress tensor* σ_{ki} is defined by the equation

$$dp_i = \sigma_{ki} dF_k \tag{3.4}$$

or

$$t_i = \sigma_{ki} n_k \ . \tag{3.5}$$

Since σ_{kl} is referred to the current state, it constitutes a stress measure in the Euler description.

Equation (3.4) suggests the following way of defining a stress tensor T_{Ki} referred to the initial state:

$$dp_i = T_{Ki} dF_K^o \tag{3.6}$$

or

$$T_i = T_{Ki} N_K \tag{3.7}$$

where

$$T_i = \frac{dp_i}{dF^o} \tag{3.8}$$

denotes the force vector acting on the unit surface element in the initial configuration. T_i is called the *nominal stress vector*, and T_{Ki} is called the *first Piola-Kirchhoff or nominal stress tensor*.

Substitution of dF_j given by (2.103) into (3.4) and comparison with (3.6) furnishes

$$T_{Ki} = \frac{\rho_o}{\rho} \sigma_{ij} X_{K,j} \quad \text{or} \quad \sigma_{ij} = \frac{\rho}{\rho_o} T_{Ki} x_{j,K} . \tag{3.9}$$

The relation (3.9) indicates that the first Piola-Kirchhoff stress tensor T_{Ki} is not symmetric, as a rule. It is therefore inconvenient to use this tensor in the constitutive equations. To avoid this difficulty, let us modify the tensor T_{Ki} so as to obtain a symmetric tensor, which is also a suitable stress measure in the Lagrangian description. To this end, we subject the force vector dp_i to the same transformation that changes the vector dx_i (in the current state) into the vector dX_K (in the initial state). Therefore, in analogy to relation $dX_K = X_{K,i}dx_i$ the transformed force dP_K is given by

$$dP_K = X_{K,i}dp_i \ . \qquad (3.10)$$

A modified stress tensor $\underset{\sim}{S}$ is now defined by the relation

$$dP_K = S_{KL}dF_L^o \ . \qquad (3.11)$$

S_{KL} - is called *second Piola-Kirchhoff (or Kirchhoff) stress tensor.*

Substitution of dp_i given by (3.6) inot (3.10) and comparison with (3.11) furnishes the relation between first and second Piola-Kirchhoff stress tensors:

$$S_{KL} = X_{K,i}T_{Li} \ . \qquad (3.12)$$

Similarly, substitution of T_{Li} form $(3.9)_1$ into (3.12) yields the relation between the second Piola-Kirchhoff and the Cauchy stress tensors

$$S_{KL} = \frac{\rho_o}{\rho} X_{K,i}X_{L,j}\sigma_{ij} \ . \qquad (3.13)$$

The relation (3.13) clearly exhibits the symmetry of the tensor S_{KL}. By using the relation $X_{K,i}x_{j,K} = \delta_{ij}$ we can solve the equation (3.13) with respect to the Cauchy stress tensor to obtain

$$\sigma_{ij} = \frac{\rho}{\rho_o} x_{i,K}x_{j,L}S_{KL} \ . \qquad (3.14)$$

In order to elucidate the differences between the introduced stress measures, let us present these definitions in a more compact way.

The *Cauchy or true stress tensor* $\underset{\sim}{\sigma}$ expresses the relation between the surface element vector $d\underset{\sim}{F}$ in the deformed configuration and the force vector $d\underset{\sim}{p}$ acting on this element, that is

$$dp_i = \sigma_{ij} dF_j \tag{3.15}$$

or

$$t_i = \sigma_{ij} n_j \quad \text{where} \quad t_i = \frac{dp_i}{dF} \tag{3.16}$$

The first Piola-Kirchhoff or nominal stress tensor $\underset{\sim}{T}$ expresses the relation between the surface element vector $d\underset{\sim}{F}^o$ in the initial configuration and the force vector $d\underset{\sim}{p}$ in the deformed configuration,

$$dp_i = T_{Ki} dF^o_K \tag{3.17}$$

or

$$T_i = t_{Ki} N_K \quad \text{where} \quad T_i = \frac{dp_i}{dF^o} \tag{3.18}$$

The second Piola-Kirchhoff stress tensor $\underset{\sim}{S}$ expresses the relation between the surface element vector $d\underset{\sim}{F}^o$ in the initial configuration and the transformed force vector $d\underset{\sim}{P}$,

$$dP_K = S_{KL} dF^o_L \tag{3.19}$$

where

$$dP_K = X_{K,i} dp_i \ .$$

3.2. Stress representation in the convected reference frame

The second Piola-Kirchhoff stress tensor has been defined by its components referred to the initial configuration of the body. Therefore in the initial curvilinear coordinates with the base vectors $\underset{\sim}{G}_K$, this tensor may be expressed as

$$\underset{\sim}{S} = S^{KL} \underset{\sim}{G}_K \underset{\sim}{G}_L \ . \tag{3.20}$$

In order to find the representation of the second Piola-Kirchhoff stress tensor $\underset{\sim}{S}$ in the convected coordinate system, let us transform the base vectors G_K into the current configuration. According to (2.119) we have $\underset{\sim}{G}_K = \underset{\sim}{x}^k_{,K}\, \underset{\sim}{c}_k$. After substitution the relation (3.20) takes the form

$$\underset{\sim}{S} = S^{KL} x^k_{,K} x^l_{,L} \underset{\sim}{c}_k \underset{\sim}{c}_l \ . \tag{3.21}$$

Next using (3.14), written in the form $\sigma^{ij} = \frac{\rho}{\rho_o} x^i_{,K} x^j_{,L} S^{KL}$, in (3.2 1) we obtain

$$\underset{\sim}{S} = \frac{\rho_o}{\rho} \sigma^{kl} \underset{\sim}{c}_k \underset{\sim}{c}_l \, . \tag{3.22}$$

The above relation indicates that $\frac{\rho_o}{\rho}\sigma^{kl}$ may be treated as the representation of the second Piola-Kirchhoff stress tensor in the convected coordinate system. It is why both stress measures S^{KL} and $\frac{\rho_o}{\rho}\sigma^{kl}$ are sometimes called the Kirchhoff stress tensor. The difference between tensors and their representations in various coordinate systems have to be clearly distinguished in order to avoid misunderstanding.

3.3. Stress rates

When formulating physical laws and constitutive relations it is desirable to use such tensor fields which do not depend on the observer (or on the frame of reference). Such quantities are called *objective*.

It is not difficult to show that the material rate of change of the Cauchy stress components (calculated with respect to a fixed coordinate system)

$$\dot{\sigma}_{ij}(\underset{\sim}{x},t) = \frac{\partial \sigma_{ij}}{\partial t} + v_k \sigma_{ij,k} \tag{3.23}$$

is not objective. To this end, let us consider a bar in simple tension, along x axis (Fig. 3.2a), the stresses being $\sigma_{xx} = k$, $\sigma_{yy} = 0$. After a rigid rotation of 90^o about the z axis, the stresses become: $\sigma_{xx} = 0$, $\sigma_{yy} = k$ (Fig. 3.2.b). We therefore see that the stress components with respect to the fixed reference frame have changed and $\dot{\sigma}_{xx} \neq 0$, $\dot{\sigma}_{yy} \neq 0$. However, from the standpoint of the moving bar, the stress state has remained constant.

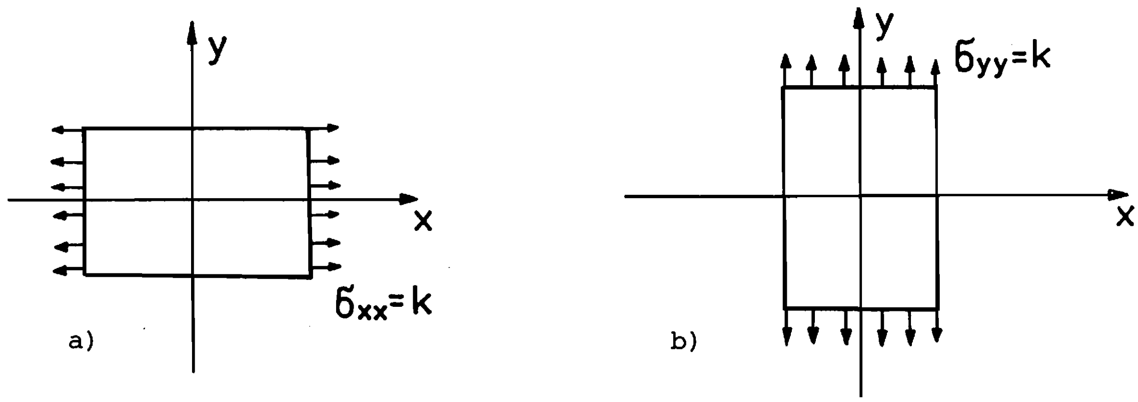

Fig. 3.2

We shall now construct objective stress rate tensors which can replace the material time derivative of the Cauchy stress tensor in the constitutive relations. Such objective tensors will be called stress fluxes.

Let us introduce the following definitions [4]:

Definition 1

Two motions $x_k(\underset{\sim}{X},t)$ and $x'_k(\underset{\sim}{X},t)$ *are called equivalent if*

$$x'_k(\underset{\sim}{X},t) = Q_{kl}(t)x_l(\underset{\sim}{X},t) + b_k(t) \ , \tag{3.24}$$

$$t' = t - a \tag{3.25}$$

where Q(t) is an arbitrary, nonsingular orthogonal transformation

$$Q_{kl}Q_{ml} = Q_{lk}Q_{lm} = \delta_{km} \ , \quad |Q_{kl}| = 1 \ , \tag{3.26}$$

$b_k(t)$ is an arbitrary vector, and a is a constant.

In the rectangular coordinates, Q_{kl} and b_k represent, respectively, the rotation and the translation of one frame with respect to the other. No such physical interpretation is possible in curvilinear coordinates.

Definition 2

A tensorial quantity is said to be objective if in any two equivalent motions it obeys the appropriate tensor transformation law for all times.

According to the definition, a vector $\underset{\sim}{v}$, a second-order tensor $\underset{\sim}{t}$ and, generally, a n-th order tensor $\underset{\sim}{c}$ are all objective if their componenents in objectively equivalent motions are given by the respective relations:

$$v'_k(\underset{\sim}{X},t) = Q_{kl}(t)v_l(\underset{\sim}{X},t) \quad , \tag{3.27}$$

$$t'_{kl}(\underset{\sim}{X},t) = Q_{km}(t)Q_{ln}(t)t_{mn}(\underset{\sim}{X},t) \quad , \tag{3.28}$$

$$C'_{ab}\dots(\underset{\sim}{X},t) = Q_{ar}(t)Q_{bs}(t)\dots C_{rs}\dots(\underset{\sim}{X},t) \quad . \tag{3.29}$$

For time-independent tensorial quantities, the transformation law, under the coordinate change (3.24), is as given by (3.27) - (3.29). Thus, for such quantities the objectivity readily applies. For time-dependent quantities, however, this is not always the case. Consider, for example, the velocity vector $\underset{\sim}{v} = \dot{\underset{\sim}{x}}$. Differentiation of (3.24) with respect to time yields

$$v'_k = Q_{kl}v_l + \dot{Q}_{kl}x_l + \dot{b}_k \quad . \tag{3.30}$$

Comparison of (3.30) with (3.27) indicates that the velocity vector is not objective. It is easy to show that neither the acceleration vector nor the spin tensor are objective [4].

According to the definition 2 an objective Cauchy stress rate measure $\overset{\nabla}{\sigma}_{ij}$ in equivalent motions must transform as

$$(\overset{\nabla}{\sigma}_{kl})' = Q_{km}Q_{ln}\overset{\nabla}{\sigma}_{mn} \quad . \tag{3.31}$$

Since the definition of an objective stress rate is not unique, various stress fluxes have been frequently used in the recent literature. The so-called Jaumann derivative was the first one, derived by Zaremba [6] and Jaumann [7] independently. We shall denote it by $\underset{\sim}{\sigma}^{\nabla J}$.

To see the meaning of the Jaumann derivative let us consider a typical particle at the point P and choose P as the common origin of the coordinate systems x_i and x'_i. At an instant of time t these systems coincide; the first is fixed the second participates in the rotation of the neighbourhood of P with instantaneous velocity w_{ij}. Thus, the coordinates x_i and x'_i are related by the transformation

$$x'_i = x_i + w_{ij}x_j dt = (\delta_{ij} + w_{ij}dt)x_j \quad . \tag{3.32}$$

At the instant of time t let the stress at the particle P be denoted by $\sigma'_{ij}(t)$ and at a later instant of time t + dt by $\sigma'_{ij}(t + dt)$ as referred to the rotating axes x'_i. Jaumann defined the stress rate as

$$(\sigma_{ij})^{\nabla J} \equiv \lim_{dt \to o} \frac{1}{dt} [\sigma'_{ij}(t + dt) - \sigma'_{ij}(t)] \tag{3.33}$$

whereas the material time derivative is defined as

$$\dot{\sigma}_{ij} \equiv \lim_{dt \to o} \frac{1}{dt} [\sigma_{ij}(t + dt) - \sigma_{ij}(t)] \tag{3.34}$$

where $\sigma_{ij}(t)$ and $\sigma_{ij}(t + dt)$ denote the stresses referred to the fixed coordinate system.

According to these definitions, the Jaumann derivative of the tensor σ_{ij} is the rate of change of this tensor from the point of view of an observer taking part in a rigid rotation of a particle, whereas the material derivative of the tensor σ_{ij} is a rate of change of this tensor from the point of view of an observer fixed at the point P.

The stress tensor at the paricle P at the instant of time t + dt referred to the fixed coordinates x_j is

$$\sigma_{ij}(t + dt) = \sigma_{ij}(t) + \dot{\sigma}_{ij}(t)dt \tag{3.35}$$

whereas, when referred to the coordinate system x'_j it becomes

$$\sigma'_{ij}(t + dt) = \sigma_{ij}(t) + (\sigma_{ij})^{\nabla J}(t)dt. \tag{3.36}$$

Transforming the stress tensor $\sigma_{ij}(t + dt)$ under the coordinate transformation (3.32) to the x'_j axes, we obtain:

$$\sigma'_{ij}(t + dt) = (\delta_{ik} + w_{ki}dt)(\delta_{jl} + w_{lj}dt)\sigma_{kl}(t + dt) \qquad (3.37)$$

Next substituting, $\sigma_{ij}(t + dt)$ from (3.35) into (3.37), we find:

$$\sigma'_{ij}(t + dt) = (\delta_{ik} + w_{ki}dt)(\delta_{jl} + w_{lj}dt)[\sigma_{kl}(t) + \dot{\sigma}_{kl}dt] =$$

$$= \sigma_{ij}(t) + (\dot{\sigma}_{ij} + w_{ki}\sigma_{kj} + w_{lj}\sigma_{il})dt + O(dt^2) + \dots \qquad (3.38)$$

Comparison of (3.38) and (3.36) furnishes the following result

$$\boxed{(\sigma_{ij})^{\nabla J} = \dot{\sigma}_{ij} - w_{ik}\sigma_{kj} - w_{jk}\sigma_{ki}} \qquad (3.39)$$

if the terms of higher order with respect to dt are neglected.

In the recent literature various definitions of the stress rate are frequently used. Since the deformation rate tensor d_{ij} vanishes if the neighbourhood of the considered particle moves as a rigid body, others possible expression for the stress rates that are objective may be obtained by adding terms $\pm(\sigma_{ik}d_{jk} + \sigma_{kj}d_{ik})$ to $(\sigma_{ij})^{\nabla J}$.

In view of further applications it appears convenient to construct the objective stress rate tensors following Noll's derivation [5].

The "convected" or the Oldroyd derivative of the Cauchy stress tensor $\underset{\sim}{\sigma}$ was derived by Oldroyd [8]. We shall denote it by $\underset{\sim}{\sigma}^{\nabla O}$ and obtain by mapping the contravariant components σ^{kl} onto the curvilinear reference frame with the base vectors $\underset{\sim}{C}_K$ (which are transforming into the spatial base vectors $\underset{\sim}{g}_l$ by the deformation gradient: $\underset{\sim}{g}_l = \underset{\sim}{C}_K X^K_{,l}$), taking material time derivative, and then mapping the result again onto the actual configuration:

$$(\sigma^{kl})^{\nabla O} \equiv (\sigma^{ij}X^K_{,i}X^L_{,j})^{\cdot}x^k_{,K}x^l_{,L} = [\dot{\sigma}^{ij}X^K_{,i}X^L_{,j} + \sigma^{ij}(X^K_{,i})^{\cdot}X^L_{,j} + \sigma^{ij}X^K_{,i}(X^L_{,j})^{\cdot}]x^k_{,K}x^l_{,L} =$$

$$= \dot{\sigma}^{kl} - \sigma^{kp}v^l_{,p} - \sigma^{lp}v^k_{,p} \qquad (3.40)$$

Use has here been made of the relations:

$$X^L_{,j} x^l_{,L} = \delta^l_j \quad , \quad (X^L_{,l})^{\cdot} = -X^L_{,p} v^p_{,l} \quad . \tag{3.41}$$

The equ. $(3.41)_2$ may be proved by differentiating $(3.41)_1$ with respect to time and mulitplying the result by $X^L_{,k}$.

The Cotter-Rivlin derivative [9] of the Cauchy stress tensor, denoted by $\underset{\sim}{\sigma}^{\nabla C-R}$, may be obtained in the similar way as the Oldroyd derivative, but by using the covariant components of the stress tensor σ_{kl} instead of the contravaiant ones:

$$(\sigma_{kl})^{\nabla C-R} \equiv (\sigma_{ij} x^i_{,K} x^j_{,L})^{\cdot} X^K_{,k} X^L_{,l} = \dot{\sigma}_{kl} + \sigma_{kp} v^p_{,l} + \sigma_{lp} v^p_{,k} \quad . \tag{3.42}$$

The Jaumann derivative of the Cauchy stress tensor $\underset{\sim}{\sigma}$ may be obtained again similarly as the Oldroyd derivative. To this end let us assume that the deformation results from translation and rigid rotation of the principal direction only, then

$$X^K_{,k} = R^K_k \quad , \quad x^i_{,K} = R^i_K \quad . \tag{3.43}$$

Substitution of (3.43) into $(3.41)_2$ yields

$$(R^K_k)^{\cdot} = -R^K_p v^p_{,k} = -R^K_p w^p_k \quad . \tag{3.44}$$

Now, we are taking material time derivative of the contravariant Cauchy stress tensor components in the reference frame which after rigid body rotation becomes an actual spatial reference frame with the base vectors $\underset{\sim}{g}_k$, and then mapping the result again onto the actual configuration:

$$(\sigma^{kl})^{\nabla J} \equiv (\sigma^{ij} R^K_i R^L_j)^{\cdot} R^k_K R^l_L = [\dot{\sigma}^{ij} R^K_i R^L_j + \sigma^{ij} (R^K_i)^{\cdot} R^L_j + \sigma^{ij} R^K_i (R^L_j)^{\cdot}] R^k_K R^l_L = \dot{\sigma}^{kl} - \sigma^{kp} w^l_p - \sigma^{lp} w^k_p \tag{3.45}$$

Relations (3.44) have been applied here.

It follows from the derivation that the Oldroyd and Cotter-Rivlin (known as "convected") time derivatives of a tensor are time rates of the spatial (contravariant and covariant, respectively,) components of this tensor from the point of view of an observer taking part in the deformation of the body; whereas the Jaumann or "co-rotational" derivative of a tensor is (as it was already mentioned) a time rate of the spatial components of this tensor from the point of view of an observer taking part in the rigid motion of the body.

For the use in the constitutive equations of plasticity the Jaumann definition is preferable to the other stress fluxes since, if it vanishes, the stress invariants have to be stationary:

$$\underset{\sim}{\sigma}^{\nabla J} = 0 \Rightarrow \dot{I}_{\underset{\sim}{\sigma}} = \dot{II}_{\underset{\sim}{\sigma}} = \dot{III}_{\underset{\sim}{\sigma}} = 0 . \tag{3.46}$$

This condition is not fulfilled by other definitions of stress rates.

3.4. Equilibrium conditions

We consider the static equilibrium of a body subjected to a body force f_i per unit mass and surface tractions. In the Eulerian description the surface traction t_i is referred to the unit area of a deformed surface with the unit outer normal n_i. In the Lagrangian description the surface traction T_i is referred to the unit area in the initial state with the unit outer normal N_K.

The resultant body force acting on the region V is

$$\int_V f_i \rho dV = \int_{V_o} f_i \rho_o dV_o . \tag{3.47}$$

The resultant of the surface tractions acting on the surface F is

$$\int_F t_i dF = \int_{F^o} T_i dF^o . \tag{3.48}$$

At the static equilibrium, the sum of the resultant body force and the resultant of the surface tractions vanishes. Hence, on account of (3.47) and (3.48), we have

$$\int_V f_i \rho dV + \int_F t_i dF = 0 \tag{3.49}$$

in the Eulerian description, or

$$\int_{V^o} f_i \rho_o dV^o + \int_{F^o} T_i dF^o = 0 \tag{3.50}$$

in the Lagrangian description.

Making use of the relations (3.5) and (3.7), and the Gauss theorem, the equations of equilibrium (3.49) and (3.50) can be rewritten to take the forms

$$\int_V (f_i \rho + \sigma_{ij,j}) dV = 0 \tag{3.51}$$

$$\int_{V^o} (f_i \rho_o + T_{Ki,K}) dV^o = 0 \quad . \tag{3.52}$$

Since these equations must be valid for an arbitrary region V and V^o, the equations of equilibrium may be written as:

$$\sigma_{ij,j} + f_i \rho = 0 \qquad \text{in } V \; , \tag{3.53}$$

$$\sigma_{ij} n_j = t_i \qquad \text{on } F \tag{3.54}$$

in the Eulerian description, and

$$T_{Ki,K} + f_i \rho_o = 0 \qquad \text{in } V^o \; , \tag{3.55}$$

$$T_{Ki} N_K = T_i \qquad \text{on } F^o \tag{3.56}$$

in the Lagrangian description.

Substituting $T_{Ki} = S_{KL}x_{i,L}$, which follows from (3.12), into (3.55) and (3.56), we obtain the equations of equilibrium in the Lagrangian description expressed in terms of the second Piola-Kirchhoff stress tensor

$$(S_{KL}x_{i,L})_{,K} + f_i\rho_o = 0 \quad , \tag{3.56}$$

$$S_{KL}x_{i,L}N_K = T_i \ . \tag{3.57}$$

Making use of the relation (2.12) between the displacement and deformation gradients, the equations (3.56), (3.57) may be written as:

$$(S_{KL}\delta_{iL} + S_{KL}u_{i,L})_{,K} + f_i\rho_o = 0 \tag{3.58}$$

$$(S_{KL}\delta_{iL} + S_{KL}u_{i,L})N_K = T_i \tag{3.59}$$

Differentiation of the equilibrium equations with respect to time leads to the equilibrium rate equations. Equations (3.53), (3.54) furnish

$$\dot{\sigma}_{ij,j} - \sigma_{ij,l}v_{l,j} + \dot{f}_i\rho - f_i\rho v_{k,k} = 0 \ , \tag{3.60}$$

$$(\dot{\sigma}_{ij} + \sigma_{ik}v_{k,j} - \sigma_{ij}d_{(n)})n_j = \dot{t}_i \tag{3.61}$$

where the following relations are used:

$$(\sigma_{ij,j})^{\cdot} = (\sigma_{ij,K}X_{K,j})^{\cdot} = \dot{\sigma}_{ij,K}X_{K,j} + \sigma_{ij,K}(X_{K,j})^{\cdot} =$$

$$= \dot{\sigma}_{ij,j} - \sigma_{ij,K}X_{K,l}v_{l,j} = \dot{\sigma}_{ij,j} - \sigma_{ij,l}v_{l,j}, \tag{3.62}$$

$$\dot{\rho} = -\rho v_{k,k}, \tag{3.63}$$

$$(n_j)^{\cdot} = (\frac{dx_j}{ds})^{\cdot} = \frac{1}{ds}(dx_j)^{\cdot} - \frac{1}{ds^2}(ds)^{\cdot}dx_j = v_{j,i}n_i - d_{kl}n_kn_ln_j =$$

$$= v_{j,i}n_i - d_{(n)}n_j \ . \tag{3.64}$$

The relation (3.63) follows from the mass conservation requirement. Differentiating (3.55) - (3.59) with respect to time, we obtain the equilibrium rate quations in the Lagrangian description:

$$\dot{T}_{Ki,K} + \dot{f}_i \rho_o = 0 \quad , \tag{3.65}$$

$$\dot{T}_{Ki} N_K = \dot{T}_i \quad , \tag{3.66}$$

$$(\dot{S}_{KL} \delta_{iL} + \dot{S}_{KL} u_{i,L} + S_{KL} v_{i,L})_{,K} + \dot{f}_i \rho_o = 0 \; , \tag{3.67}$$

$$(\dot{S}_{KL} \delta_{iL} + \dot{S}_{KL} u_{i,L} + S_{KL} v_{i,L}) N_K = \dot{T}_i \tag{3.68}$$

As we can see from the above relations, only the equilibrium rate equations (3.65), (3.66) expressed in terms of the first Piola-Kirchhoff stress tensor have the form usually applied in the classical linear mechanics of continua.

4. CONJUGATE VARIABLES

A process of isothermal deformation of solids may be described using different stress, strain , stress rate and strain rate measures. This choice, however, is not arbitrary, but constrained by some invariant requirements.

Starting with the definition of deformation energy, Hill introduced the concept of a stress measure conjugate to a given strain rate measure [3]. This concept can be generalized for stress rates and strain accelerations. Such generalization is particularly useful when the Eulerian description is used since then the objective stress rate and the strain acceleration measures are not uniquely defined, even near the reference configuration. Application of the definition of conjugate variables helps to choose the proper set of objective measures.

According to Hill's definition [3], *a stress measure* $\underset{\sim}{\tau}$ *is conjugate to a given strain measure* $\underset{\sim}{\varepsilon}$, *when any infinitesimal increment of deformation energy* dW *per unit reference volume is expressible as the scalar product of* $\underset{\sim}{\tau}$ *and* $d\underset{\sim}{\varepsilon}$

$$dW = \underset{\sim}{\tau} \cdot d\underset{\sim}{\varepsilon} \tag{4.1}$$

or in rate form:

$$\dot{W} = \underset{\sim}{\tau} \cdot \dot{\underset{\sim}{\varepsilon}} \quad . \tag{4.2}$$

Since the scalar product of the Cauchy stress tensor $\underset{\sim}{\sigma}$ and the deformation rate tensor $\underset{\sim}{d}$ gives the rate of deformation energy per unit *current* volume, the rate of deformation energy per unit *reference* volume may be expressed as

$$\dot{W} = \sigma_{ij} d_{ij} \frac{dV}{dV_o} \tag{4.3}$$

or

$$\dot{W} = \frac{\overset{o}{\rho}}{\rho} \sigma_{ij} d_{ij} \quad , \tag{4.4}$$

where the mass conservation law is used.

Equation (4.4) shows that the variables σ_{ij}, d_{ij} commonly used in the Euler description, are not the conjugate variables in the sense of definition (4.2). Introducing, however, the stress measure

$$t_{ij} \equiv \frac{\rho_o}{\rho} \sigma_{ij} \quad , \tag{4.5}$$

called the Kirchhoff or Trefftz stress tensor, the equation (4.4) can be written in the form

$$\dot{W} = t_{ij} d_{ij} \tag{4.6}$$

which is consistent with the definition (4.2). Therefore, the variables t_{ij}, d_{ij} are conjugate variables of the Euler description. For a rigid-plastic material, which is assumed incompressible, we have $\rho = \rho^o$ and, therefore, $t_{ij} = \sigma_{ij}$.

Substituting (2.83) and (3.14) into (4.4), the rate of deformation energy per unit inital volume can be written as:

$$\dot{W} = S_{KL} x_{i,L} x_{j,K} \dot{E}_{MN} X_{M,i} X_{N,j} = S_{KL} \dot{E}_{KL} \tag{4.7}$$

whereas, substituting $(3.9)_2$ into (4.4), we obtain

$$\dot{W} = T_{Ki} x_{j,K} d_{ij} = T_{Ki} v_{i,K} . \tag{4.8}$$

Relations (4.7) and (4.8) indicate that the variables in the Lagrangian description S_{KL}, $\dot{E}_{KL}$ and T_{Ki}, $v_{i,K}$ are conjugate.

Differentiation of equ. (4.2) with respect to time furnishes the power rate equation which defines conjugate stress rate and strain acceleration measures,

$$\ddot{W} = (\underset{\sim}{\tau} \cdot \dot{\underset{\sim}{\varepsilon}})^{\cdot} = \dot{\underset{\sim}{\tau}} \dot{\underset{\sim}{\varepsilon}} + \underset{\sim}{\tau} \ddot{\underset{\sim}{\varepsilon}} . \tag{4.9}$$

This condition, written for the conjugate variables in the Eulerian description t_{ij}, d_{ij}, takes the form

$$\ddot{W} = (t_{ij} d_{ij})^{\cdot} = \dot{t}_{ij} d_{ij} + t_{ij} \dot{d}_{ij} . \tag{4.10}$$

The conditions (4.2) and (4.9) are not, however, the only requirements which have to be satisfied by the constitutive variables in order to obtain a consistent theory. The other one is the invariant requirement under superimposed rigid body motion. This condition is not satisfied when the material time derivatives of the components of the Kirchhoff stress tensor $\dot{t}_{ij}$ and of deformation rate tensor $\dot{d}_{ij}$ in the current configuration are applied as stress rate and strain acceleration measures.

Despite widespread interest in recent years in various definitions of the objective stress rates [4] - [12], the choice of suitable definitions when formulating the constitutive relations still remains a matter of taste or convenience. Therefore, we shall now attempt to construct such stress rate and strain acceleration measures that will be conjugate and objective.

Since the rotation tensor $\underset{\sim}{R}$ is the proper orthogonal tensor, the following relations take place:

$$R_{iK}R_{Li} = \delta_{KL} \quad , \quad R_{iL}R_{Lj} = \delta_{ij} \ . \tag{4.11}$$

In view of (4.11) the equation (4.6) can be rewritten to become

$$\dot{W} = t_{kl}R_{Kk}R_{Ll}R_{pK}R_{rL}d_{pr} \tag{4.12}$$

Differentiating (4.12) with respect ot time, we obtain

$$\ddot{W} = (t_{kl}R_{Kk}R_{Ll})^{\cdot}R_{pK}R_{rL}d_{pr} + $$

$$+ t_{kl}R_{Kk}R_{Ll}(R_{pK}R_{rL}d_{pr})^{\cdot} \ . \tag{4.13}$$

The Jaumann (or co-rotational) derivative of the Cauchy stress tensor was defined in the section 3.3 by the equation (3.45). Analogously, we can define the Jaumann derivative of the Kirchhoff stress tensor and the Jaumann derivative of the deformation rate tensor as:

$$(t_{pr})^{\nabla J} \equiv (t_{kl}R_{Kk}R_{Ll})^{\cdot}R_{pK}R_{rL} = \dot{t}_{pr} - t_{pm}w_{rm} - t_{rm}w_{pm} \ , \tag{4.14}$$

$$(d_{kl})^{\nabla J} \equiv R_{Kk}R_{Ll}(R_{pK}R_{rL}d_{pr})^{\cdot} = \dot{d}_{kl} - d_{km}w_{lm} - d_{lm}w_{km} \,. \tag{4.15}$$

Now, by using the definitions $(4.14)_1$ and $(4.15)_1$, the equation (4.13) can be rewritten to become

$$\ddot{W} = (t_{kl})^{\nabla J}d_{kl} + t_{kl}(d_{kl})^{\nabla J} \tag{4.16}$$

The relations (4.6) and (4.16) provide the first set of conjugate variables in the Euler descritpion which are invariant under superimposed rigid body motion:

I set: t_{kl}, d_{kl}, $(t_{kl})^{\nabla J}$, $(d_{kl})^{\nabla J}$.

Now, let us attempt to construct other sets of conjugate and objective constitutive variables. Following Green's and Naghdi's [13] (or perhaps ealier Noll's [5]) idea, let us introduce co-rotational or rigid-body components $\bar{t}_{KL}$ of the Kirchhoff stress tensor (components in the reference frame which takes part in the rigid rotation of the body)

$$\bar{t}_{KL} \equiv t_{kl}R_{Kk}R_{Ll} \tag{4.17}$$

and co-rotational components $\bar{d}_{KL}$ of the deformation rate tensor

$$\bar{d}_{KL} \equiv R_{pK}R_{rL}d_{pr} \,. \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.12) and into (4.13), we obtain the rate of energy and the rate of power equations in the forms:

$$\dot{W} = \bar{t}_{KL}\bar{d}_{KL} \,, \tag{4.19}$$

$$\ddot{W} = \dot{\bar{t}}_{KL}\bar{d}_{KL} + \bar{t}_{KL}\dot{\bar{d}}_{KL} \tag{4.20}$$

The relations (4.19) and (4.20) furnish the second set of objective conjugate variables:

II set: $\bar{t}_{KL}$, $\bar{d}_{KL}$, $\dot{\bar{t}}_{KL}$, $\dot{\bar{d}}_{KL}$

(all these quantities are unlatered by superimposed rigid body motion).

Comparison of (4.14) and (4.15) with (4.17) and (4.18) furnishes the following results:

$$(\bar{t}_{KL})^{\cdot}\bar{d}_{KL} = (t_{kl})^{\nabla J} d_{kl} \quad , \tag{4.21}$$

$$\bar{t}_{KL}(\bar{d}_{KL})^{\cdot} = t_{kl}(d_{kl})^{\nabla J} \quad . \tag{4.22}$$

Now, let us consider the other way of constructing the conjugate variables, the way concerned with convected measures. To this end, we make use of the identity:

$$x_{k,L}X_{L,l} \equiv \delta_{kl} \quad . \tag{4.23}$$

The energy rate equation (4.6) can be, therefore, written as

$$\dot{W} = t_{kl}X_{K,k}X_{L,l}x_{p,K}x_{r,L}d_{pr} \quad . \tag{4.24}$$

Differentiating (4.24) with respect to time, we obtain

$$\ddot{W} = (t_{kl}X_{K,k}X_{L,l}x_{p,K}x_{r,L}d_{pr})^{\cdot} =$$

$$= (t_{kl}X_{K,k}X_{L,l})^{\cdot}x_{p,K}x_{r,L}d_{pr} +$$

$$+ t_{kl}X_{K,k}X_{L,l}(x_{p,K}x_{r,L}d_{pr})^{\cdot} \quad . \tag{4.25}$$

According to the definition of "convected" time derivative introduced in the section 3.3 by equations (3.40) and (3.42), the Oldroyd derivative of the Kirchhoff stress tensor and the Cotter-Rivlin derivative of the deformation rate tensor can be written as: [1]

$$(t_{pr})^{\nabla o} \equiv (t_{kl}X_{K,k}X_{L,l})^{\cdot}x_{p,K}x_{r,L} = \dot{t}_{pr} - t_{pm}v_{r,m} - t_{rm}v_{p,m} \quad , \tag{4.26}$$

---

[1] The Cotter-Rivlin derivative of the deformation rate tensor, defined by equ. (4.27) was first introduced in joint paper by Rivlin and Ericksen and, therefore, some time is called Rivlin-Ericksen derivative.

$$(d_{pr})^{\nabla C-R} \equiv (d_{kl}x_{k,K}x_{l,L})^{\cdot}X_{K,p}X_{L,r} = \dot{d}_{pr} + d_{pl}v_{l,r} + d_{rl}v_{l,p} \qquad (4.27)$$

where use was made of the relations:

$$(X_{L,l})^{\cdot} = -X_{L,p}v_{p,l} \ , \quad (x_{p,K})^{\cdot} = x_{l,K}v_{p,l} . \qquad (4.28)$$

Substitution of $(4.26)_1$ and $(4.27)_1$ into (4.25) yields the relation

$$\ddot{W} = (t_{kl})^{\nabla o}d_{kl} + t_{kl}(d_{kl})^{\nabla C-R} \qquad (4.29)$$

which furnishes the next, third, set of conjugate, objective variables:

III set: t_{kl}, d_{kl}, $(t_{kl})^{\nabla o}$, $(d_{kl})^{\nabla C-R}$.

In view of equ. (2.84) the deformation rate tensor d_{ij} is equal to the Cotter-Rivlin time derivative of the Almansi strain tensor $(e_{ij})^{\nabla C-R}$. Therefore, the III set of conjugate, objective variables can be written as

(III)' set: t_{kl}, $(e_{kl})^{\nabla C-R}$, $(t_{kl})^{\nabla o}$, $(e_{kl})^{\nabla\nabla C-R}$.

The "convected" components of the tensors $\underset{\sim}{t}$ and $\underset{\sim}{d}$ are defined as (see e.g. [13])

$$\hat{t}_{KL} \equiv t_{kl}X_{K,k}X_{L,l} \ , \qquad (4.30)$$

$$\hat{d}_{KL} \equiv d_{kl}x_{k,K}x_{l,L} \ . \qquad (4.31)$$

They are obtained by mapping the spatial components t_{kl} of the Kirchhoff stress tensor, and d_{kl} of the deformation rate tensor onto the reference state according to the transformation rule such as $dX_K = X_{K,i}dx_i$ and $dF_K^o = \frac{\rho}{\rho_o} x_{i,K}dF_i$, respectively.

Substitution of (4.30) and (4.31) into (4.24) and (4.25) provides the rate of energy and the rate of power equations in the forms

$$\dot{W} = \hat{t}_{KL}\hat{d}_{KL} \ , \qquad (4.32)$$

$$\ddot{W} = (\hat{t}_{KL})^{\cdot}\hat{d}_{KL} + \hat{t}_{KL}(\hat{d}_{KL})^{\cdot} \quad . \tag{4.33}$$

The relations (4.32) and (4.33) indicate the fourth set of objective, conjugate variables:

IV set: $\hat{t}_{KL}$, $\hat{d}_{KL}$, $(\hat{t}_{KL})^{\cdot}$, $(\hat{d}_{KL})^{\cdot}$.

Comparison of (4.26) and (4.27) with (4.30) and (4.31) proves the following relations:

$$(\hat{t}_{KL})^{\cdot}\hat{d}_{KL} = (t_{kl})^{\nabla o}d_{kl} \ , \tag{4.34}$$

$$\hat{t}_{KL}(\hat{d}_{KL})^{\cdot} = t_{kl}(d_{kl})^{\nabla C\text{-}R} \ . \tag{4.35}$$

In view of the relations (3.13) and (4.5) we have

$$S_{KL} = t_{kl}X_{K,k}X_{L,l} \quad . \tag{4.36}$$

Comparing (4.36) with (4.30) and (2.83) with (4.31), we see that

$$\hat{t}_{KL} = S_{KL} \quad , \quad \hat{d}_{KL} = \dot{E}_{KL} \quad , \tag{4.37}$$

that is convected components of the Kirchhoff stress tensor $\hat{t}_{KL}$ and the deformation rate tensor $\hat{d}_{KL}$ coincide respectively with the components of the second Piola-Kirchhoff stress tensor S_{KL} and the Green strain rate tensor $\dot{E}_{KL}$ in the Lagrangian (initial) frame of reference.

The (IV)' set of the conjugate, objective variables can be, therefore, written as:

(IV)' set: S_{KL}, $\dot{E}_{KL}$, $\dot{S}_{KL}$, $\ddot{E}_{KL}$

and the relations (4.34), (4.35) take now the form: ([1])

$$\dot{S}_{KL}\dot{E}_{KL} = (t_{kl})^{\nabla o}d_{kl} \ , \tag{4.38}$$

([1]) The relations (4.38) and (4.39) were derived in another manner by H. Stolarski [14]

$$S_{KL}\ddot{E}_{KL} = t_{kl}(d_{kl})^{\nabla C-R} \ . \qquad (4.39)$$

For a general case of curvilinear coordinates, the considerations analogous to the above lead to the following sets of the constitutive variables which are objective and conjugate in the sense of the definitions (4.2) and (4.9).

In the Eulerian description:

1. <u>Co-rotational (rigid-body) measures</u>

a) $\bar{t}^{KL}$, $\bar{d}_{KL}$, $(\bar{t}^{KL})^{\cdot}$, $(\bar{d}_{KL})^{\cdot}$

- contravariant components of the Kirchhoff stress tensor $\underset{\sim}{t}$ and covariant components of the deformation rate $\underset{\sim}{d}$ in the co-rotational reference frame and their material time derivatives.

b) t^{kl}, d_{kl}, $(t^{kl})^{\nabla J}$, $(d_{kl})^{\nabla J}$

- respective components of $\underset{\sim}{t}$ and $\underset{\sim}{d}$ tensors in the Eulerian (spatial) reference frame and their Jaumann(co-rotational) derivatives.

2. <u>Convected measures</u>

a) $\hat{t}^{KL}$, $\hat{d}_{KL}$, $(\hat{t}^{KL})^{\cdot}$, $(\hat{d}_{KL})^{\cdot}$

- contravariant components of the Kirchhoff stress tensor $\underset{\sim}{t}$ and covariant components of the deformation rate tensor $\underset{\sim}{d}$ in the convected reference frame (embedded in the material and deforming with it) and their material time derivatives.

b) t^{kl}, d_{kl}, $(t^{kl})^{\nabla O}$, $(d_{kl})^{\nabla C-R}$

- respective componenets of $\underset{\sim}{t}$ and $\underset{\sim}{d}$ tensors in the Eulerian (spatial) reference frame and their convected (Oldroyd and Cotter-Rivlin, respectively) derivatives.

In the Lagrangian description:

S^{KL}, $\dot{E}_{KL}$, $\dot{S}^{KL}$, $\ddot{E}_{KL}$

- contravariant components of the second Piola-Kirchhoff stress tensor $\underset{\sim}{S}$ and covariant components of the Green strain rate tensor $\underset{\sim}{E}$ in the Lagrangian (initial) coordinate frame of reference and their material time derivatives.

According to (4.37) the Lagrangian measures concide with the convected measures (2a).

The second set of Lagrangian conjugate measures, that is the first Piola-Kirchhoff stress tensor T_{Ki}, the velocity gradient $v_{i,K}$ and their material time derivatives are inconvenient to be used as constitutive variables since they are neither ojective (not invariant with respect to the rigid body motion) nor symmetric.

Hill considered the relations between constitutive equations when different stress rate and strain rate measures are used. He has shown [10] that the existence of a homogeneous quadratic rate potential (which leads to a symmetric stiffness matrix) for a material when the constitutive law is expressed in terms of a set of conjugate stress rate and strain rate measures, implies the existence of similar quadratic potentials when the constitutive laws are expressed in terms of any other set of conjugate stress and strain rate measures. Therefore, the transformation of the constitutive law established, for example, in the Lagrangian description under the assumption of existence of a quadratic rate potential into the Eulerian description may lead (for compressible materials) to non-symmetric stiffness matrix when as constitutive variables the Cauchy stress tensor and deformation rate tensor are used.

As it will be shown in the part II, co-rotational and convected (or Lagrangian) measures do not lead to the dual description of the same material even for infinitesimal strains.

PART II

CONSTITUTIVE RELATIONS FOR ELASTIC-PLASTIC AND RIGID-PLASTIC MATERIALS

1. INTRODUCTORY REMARKS AND MODELS OF MATERIALS

An internally consistent axiomatic theory of elastic-plastic media may be constructed with the use of any conjugate variables. The Cauchy stress tensor, however, is usually chosen as a stress measure since it is a "true" stress (force per unit area) in a current configuration.

As the theory of elastic-plastic material has been developed mainly for the description of the behaviour of structural metals, it is the experiments that should decide which variables are most suitable for the description of material models reflecting known properties of metals. Therefore, before we discuss the constitutive relations in the plastic regime, a very brief outline of some experimental facts will be given.

When a bar of ductile metal is stressed in simple tension, its mechanical behaviour is described by the load-elongation curve. Numerous experiments show typical load-elongation relationships, as indicated in the diagrams of Fig. 1.1.

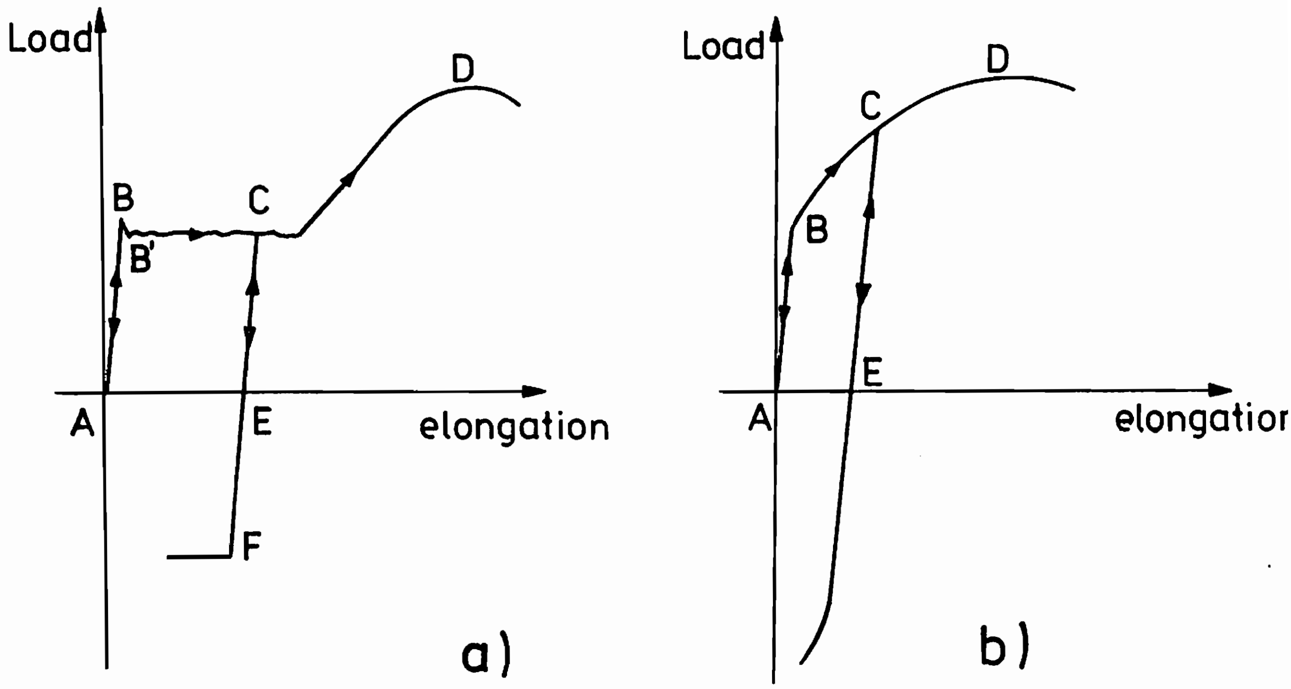

Fig. 1.1

For sufficiently small values of the load (before point B is reached), the relationship is linear and reversible and the law of linear elasticity is expected to hold. For higher load values - BCD curves - the relationship becomes nonlinear and irreversible. The behaviour in this region is termed plastic. Mild steel shows an upper yield point B and a flat yield plateau BC, see Fig. 1.1(a). Most other metals do not have such a flat yield platform and their behaviour is illustrated in Fig. 1.1(b).

The curves shown in Fig. 1.1 are too complicated to be used as mathematical models of materials, so it is clearly desirable to approximate them by simpler relations. An obvious suggestion is to apply the linear approximations as presented in Figs. 1.2 and 1.3. As before, point B represents the yield point load. If BC is horizontal, as in Fig. 1.2(a) and 1.3(a), we obtain perfectly plastic material models. Finally, if the total strains are large as compared to the elastic strains, the latter may be neglected. Then, we obtain rigid plastic models of actual materials - Fig. 1.3.

When we consider unloading, the elastic material will retrace its load-elongation curve to the origin, whereas the plastic process is irreversible. Thus, if the material is loaded into the plastic range along ABC and then unloaded, the slope of CE is assumed to be the same as the initial slope of AB.

If the unloading is continued, the material will eventually flow plastically in compression. For the perfectly-plastic material models, Fig. 1.2(a) and Fig. 1.3(a), the compressive yield load will be the same as the tensile yield load CE = EF. For hardening material models, Fig. 1.2(b) and 1.3(b), the compressive yield load depends upon the amount of tensile hardening. This property will be discussed later on.

Another important difference between elasticity and plasticity is the question of uniqueness. For the plastic range, the stress is no longer uniquely determined by the strain, and vice versa.

The load and the elongation represent the axial stress and axial strain of the specimen under uniaxial stress. Therefore, the material

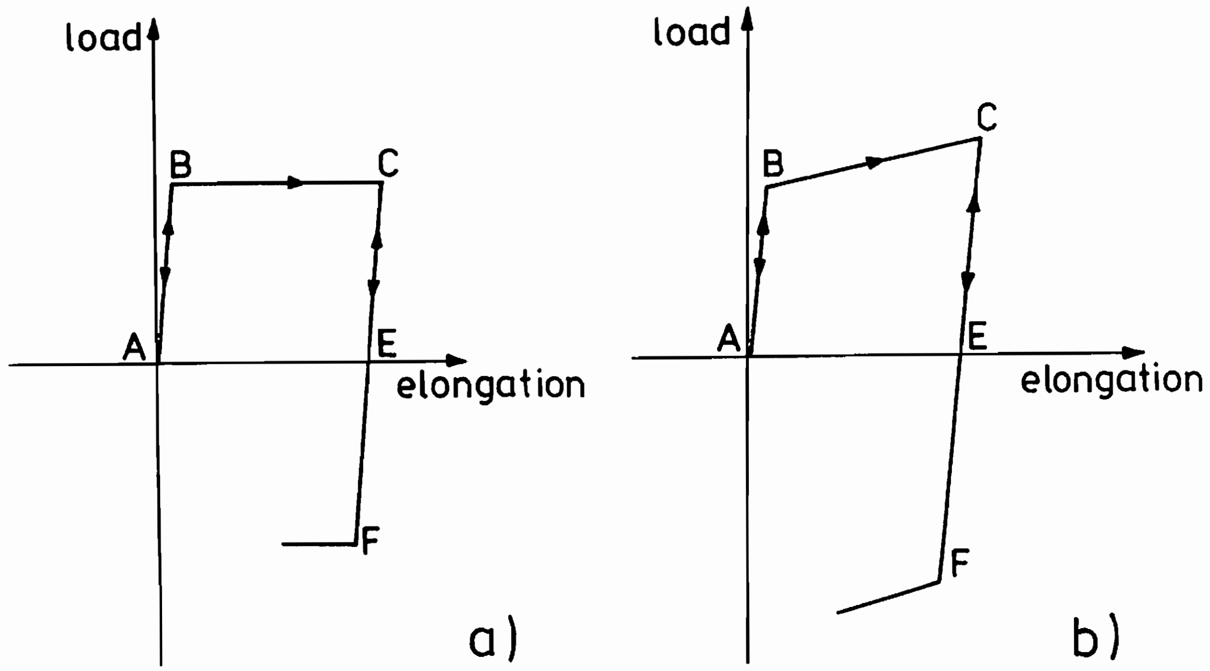
load
B
C
A
E
elongation
F
a)
load
C
B
A
E
elongation
F
b)

Fig. 1.2

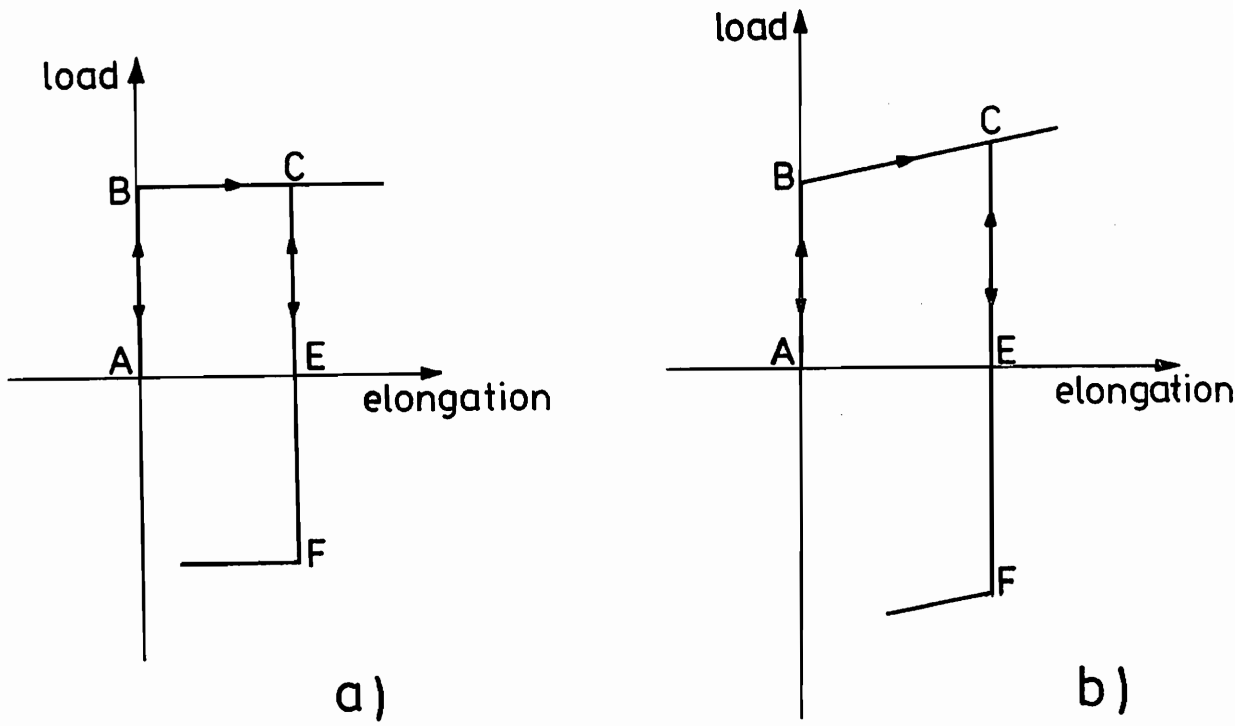
load
C
B
A
E
elongation
F
a)
load
C
B
A
E
elongation
F
b)

Fig. 1.3

models are often defined by the stress-strain relations analogous to those presented in Figs. 1.2 - 1.3. Now, the theory of elastic-plastic solids in terms of conjugate variables $\underset{\sim}{\tau}$ - $\underset{\sim}{\varepsilon}$ will be constructed and next, by the identification of conjugate variables with those of the Eulerian or Lagrangian description, the characteristic properties of defined materials will be analysed and compared.

2. YIELD CONDITION

2.1. Yield function

Existence of a yield(or loading)function in the theory of plasticity is usually introduced as an assumption ([1]). We postulate that there exists a scalar function $f(\underset{\sim}{\tau}, \underset{\sim}{\varepsilon}^p, \kappa)$ which depends on the state of stress, plastic strain and a parameter κ. The parameter κ is called a hardening parameter and depends on the plastic deformation history.

If we regard the stress state $\underset{\sim}{\tau}$ as a point with components τ_{ij} in the nine-dimensional stress space, then, for a given value $\underset{\sim}{\varepsilon}^p$ and κ, the equation

$$f(\underset{\sim}{\tau}, \underset{\sim}{\varepsilon}^p, \kappa) = 0 \tag{2.1}$$

represents a surface in the above space and is called a yield condition or a yield surface. If we assume that plastic deformation is independent of hydrostatic pressure, what is in agreement with experiamental data, then the yield surface takes in the principal stress space σ_1, σ_2, σ_3 the form of an infinitely long cylinder or prism with the axis inclined at the same angle to each of the three axes of principal stresses (Fig. 2.1).

Changes in plastic deformations occur only when $f = 0$. No changes in plastic deformations occur when $f < 0$. No meaning is associated with $f > 0$.

A history of loading may be regarded as a path in the stress space and the corresponding deformation history as a path in the strain space with axes ε_{ij}.

The yield criteria that are in excellent agreement with experiments for most ductile metals are the Huber-Mises and the Tresca conditions. For most metals the Huber-Mises yield condition fits the data more closely, but because of linearity the Tresca law is frequently simpler to use in analytical considerations.

([1])It may be shown [15], however, that existance of a yield function is a consequence of the requirement that the stress is an isotropic, homegeneous of the order zero, tensorial function of the strain rate.

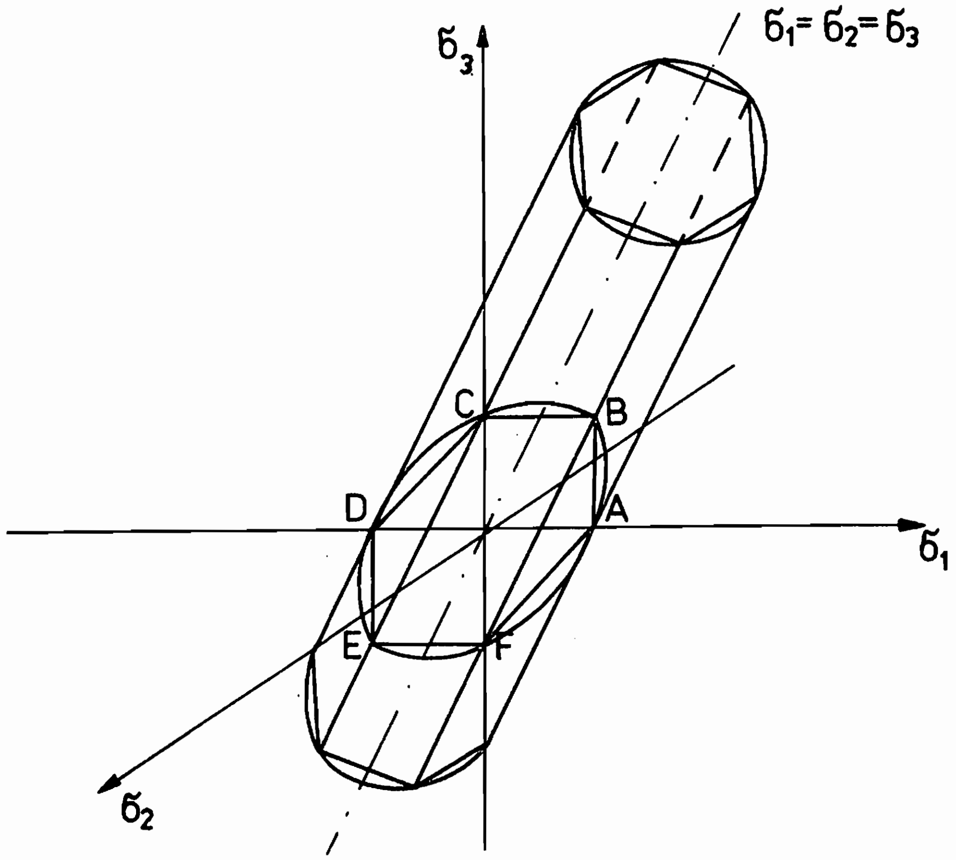

Fig. 2.1

2.2. The Tresca yield condition

For mild steel and for some others metals, it has been observed that plastic deformations basically consist of slips in crystals. Hence, it was supposed (by Tresca in 1864) [16] that yielding occurs when the maximum shear stress reaches a certain critical value.

In a multiaxial stress state with the principal stresses σ_1, σ_2, σ_3 the Tresca yield condition can be written as

$$\max \left[\frac{|\sigma_1-\sigma_2|}{2} , \frac{|\sigma_2-\sigma_3|}{2} , \frac{|\sigma_3-\sigma_1|}{2} \right] = k \qquad (2.2)$$

where k is the yield locus in the pure shear test. Considering the uniaxial tension test with the tensile stress σ, we find the following relation between the values of yield point in pure shear and uniaxial tension tests:

$$k = \frac{\sigma_o}{2} \tag{2.3}$$

where σ_o is the yield limit for tension. Consequently, the condition (2.2) is represented in the Cartesian space with axes σ_1, σ_2, σ_3 by the hexagonal prism whose axis is equally inclined to the coordinate axes (Fig. 2.1).

For plane stress state one of the principal stresses vanishes, say $\sigma_3 = 0$, and the hexagonal prism reduces to the hexagon ABCDEF obtained by intersection of the prism with the plane $\sigma_3 = 0$. The yield condition then becomes

$$\max\,[\,|\sigma_1| \;,\; |\sigma_2| \;,\; |\sigma_1 - \sigma_2|\,] = 0 \;. \tag{2.4}$$

2.3. The Huber-Mises yield condition

In 1904 M.T. Huber assumed that it is a certain critical amount of the shear energy in an elastic body that should be responsible for the onset of yielding, irrespective of the type of stress state [17]. Huber's idea was independently expressed by R. von Mises in 1913 [18] in the different form. He suggested that yielding of the material begins when the shear stresses intensity $\sigma_i = \sqrt{J_2}$ (where J_2 is the second invariant of the deviatoric part of stress tensor) reaches a critical value k. Thus the Huber-Mises yield condition assumes a very simple form of the general relation (2.1), namely

$$J_2 = k^2 \tag{2.5}$$

or

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2 \tag{2.6}$$

when referred to the principal stresses. The value of k may be obtain in a simple way, by means of the uniaxial tension test. Let us assume that only the principal stress σ_1 having the value of the yield point σ_o is present, while the two remaining principal stresses $\sigma_2 = \sigma_2 = 0$. From the relation (2.6) we obtain

$$k = \frac{\sigma_o}{\sqrt{3}} \tag{2.7}$$

Geometrical interpretation of Huber-Mises yield condition is a circular cylinder in the principal stress space σ_1, σ_2, σ_3 with the axis inclined at the same angle to each of the three axes of principal stresses (Fig. 2.1).

For a plane stress state ($\sigma_3 = 0$) the yield condition (2.6) reduces to the equation

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_o^2 \tag{2.8}$$

represented by an elipse in the σ_1, σ_2 plane (Fig. 2.2).

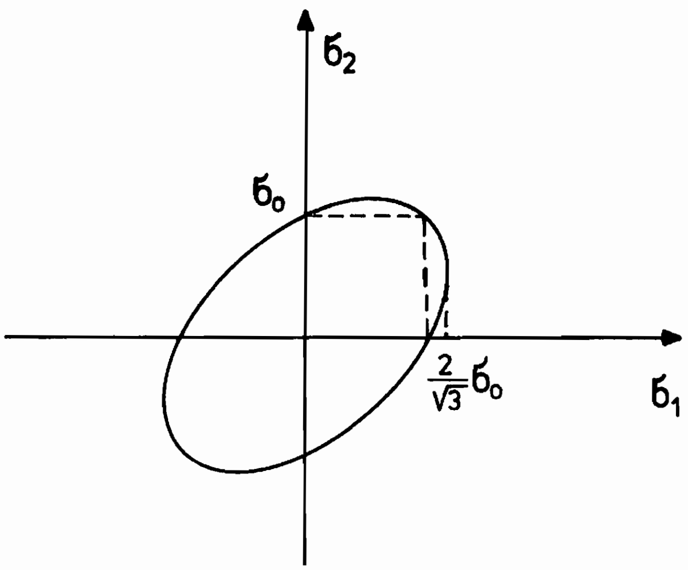

Fig. 2.2

3. LOADING AND UNLOADING CRITERIA

To explain the meaning of loading and unloading in a plastic state, consider a state in which (2.1) is satisfied. The time rate of the yield function can then be written as

$$\dot{f} = \frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} + \frac{\partial f}{\partial \underset{\sim}{\varepsilon}^{p}} \dot{\underset{\sim}{\varepsilon}}^{p} + \frac{\partial f}{\partial \kappa} \dot{\kappa} \quad . \tag{3.1}$$

A point in the stress space lies on the surface (2.1) and is about to move inward if $f = 0$ and $\dot{f} < 0$. Such a change leads to an elastic (or rigid) state and is natural to call it unloading. However, unloading is a purely elastic process and, therefore, no plastic strain occurs, $\dot{\underset{\sim}{\varepsilon}}^{p} = 0$, and the rate of change of the hardening parameter must also vanish, $\dot{\kappa} = 0$. Hence, in view of (3.1), *the criterion for unloading from a plastic state* is

$$f = 0 \text{ and } \dot{f} \equiv \frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} < 0 \quad . \tag{3.2}$$

Since $\dot{\underset{\sim}{\varepsilon}}^{p}$ and $\dot{\kappa}$ vanish also for hardening materials when a stress point lies on the yield surface and is about to move in the tangential direction *the criterion for neutral loading from a plastic state* is

$$f = 0 \quad \text{and} \quad \dot{f} \equiv \frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} = 0 \quad , \tag{3.3}$$

otherwise the loading process takes place. Thus the *criterion for loading from a plastic state* is

$$f = 0 \quad , \quad \dot{f} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} > 0 \quad . \tag{3.4}$$

A stress point lies on the yield surface and is about to move outwards, together with the yield surface (because the end of a stress vector cannot go outside).

A simple geometric interpretation of these criteria is shown in Fig. 3.1. For a stress state represented by a point on the yield surface, loading, unloading or neutral loading take place, according to whether the stress increment vector is directed outward, inward, or along the tangent to the yield surface, respectively.

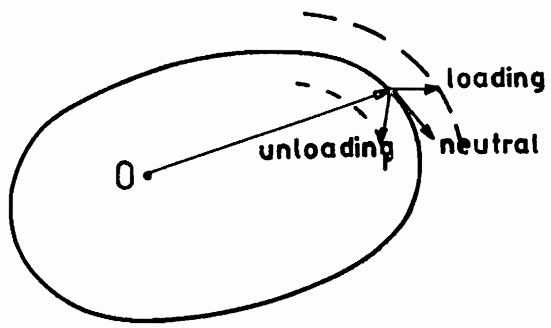

Fig. 3.1

For perfectly plastic solid, the yield function f depends neither on the plastic strain nor the strain history, therefore (2.1) may be written as

$$f(\underset{\sim}{\tau}) = \phi(\underset{\sim}{\tau}) - k = 0 \tag{3.5}$$

where k is a constant.

In this case the equation (3.5) represents a surface in the stress space which remains constant during the whole plastic flow process. The unloading criterion has the same form (3.2) as for the hardening plastic material, whereas the loading criterion now becomes

$$f = 0 \quad , \quad \dot{f} \equiv \frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} = 0 \quad . \tag{3.6}$$

Note that (3.6) coincides with the neutral loading criterion for hardening materials. In view of (3.5) the situation in which f = 0 and $\frac{\partial f}{\partial \underset{\sim}{\tau}} \dot{\underset{\sim}{\tau}} > 0$ is not possible for perfectly plastic materials.

4. THE DRUCKER POSTULATE

The Drucker postulate plays a fundamental role in the mathematical theory of plasticity. The normality, convexity and material stability conditions, all follow directly and simply from this postulate.

The postulate proposed by Drucker [10] may be formulated in the following way. Consider a body or a system at rest, made of time-independent material, loaded by a set of surface tractions $\underset{\sim}{T}^{o}$ and body forces $\underset{\sim}{f}^{o}$ in equilibrium. An external agency is supposed to generate an independent set of conservative forces $\Delta\underset{\sim}{T} = \underset{\sim}{T} - \underset{\sim}{T}^{o}$ and $\Delta\underset{\sim}{f} = \underset{\sim}{f} - \underset{\sim}{f}^{o}$ in the equilibrium which cause a change in the displacements $\Delta\underset{\sim}{u}$. *The system is said to be stable when the work done by the external agency on the displacements it produces is positive or zero whatever the set of added forces.* This statement may be written as

$$\Delta W = \int_{F^o}\int_{o}^{\Delta u_i} (T_i - T_i^o)\, du_i\, dF^o + \int_{V^o}\int_{o}^{\Delta u_i} \rho_o (f_i - f_i^o)\, du_i\, dV^o \geq 0 \qquad (4.1)$$

or

$$\Delta W = \int_{F^o}\int_{t_o}^{t} (T_i - T_i^o) v_i\, dt\, dF^o + \int_{V^o}\int_{t_o}^{t} \rho_o (f_i - f_i^o) v_i\, dt\, dV^o \geq 0 \qquad (4.2)$$

where F^o and V^o are the surface and the volume of the body at the time t_o, T_i is the surface traction referred to the unit surface F^o ($T_i = \frac{dp_i}{dF^o}$) and f_i is a body force referred to the unit mass. Denoting by N_k the unit vector normal to dF^o, we have the known relations:

$$T_i = T_{Ki} N_K \quad , \quad T_i^o = T_{Ki}^o N_K \qquad (4.3)$$

where T_{Ki} is the nominal or the first Piola-Kirchhoff stress tensor.

As F^o and V^o are time-independent we can interchange in (4.2) the time and the surface as well as the time and volume integrals. Next, substituting (4.3) into (4.2) and making use of the Gauss theorem, we obtain

$$\Delta W = \int_{t_o}^{t} \int_{F^o} (T_{Ki} - T^o_{Ki}) N_K v_i \, dF^o \, dt + \int_{t_o}^{t} \int_{V^o} \rho_o (f_i - f^o_i) v_i \, dV^o \, dt =$$

$$= \int_{t_o}^{t} \int_{V^o} [(T_{Ki} - T^o_{Ki}) v_{i,K} + (T_{Ki} - T^o_{Ki})_{,K} v_i + (f_i - f^o_i) \rho_o v_i] dV^o \, dt \geq 0. \quad (4.4)$$

With the use of equilibrium conditions the relation (4.4) becomes

$$\Delta W = \int_{t_o}^{t} \int_{V_o} (T_{Ki} - T^o_{Ki}) v_{i,K} \, dV^o \, dt \geq 0 \quad . \quad (4.5)$$

When the body is under homogeneous stress and strain, the volume integral in (4.5) may be omitted and the criterion defines a class of materials for which

$$\int_{t_o}^{t} (T_{Ki} - T^o_{Ki}) v_{i,K} \, dt \geq 0 \quad . \quad (4.6)$$

To eliminate from the criterion the elastic part of strains, the closed stress cycle is considered - ABCD or A'B'C'D' - as shown in Fig. 4.1 for simple tension.

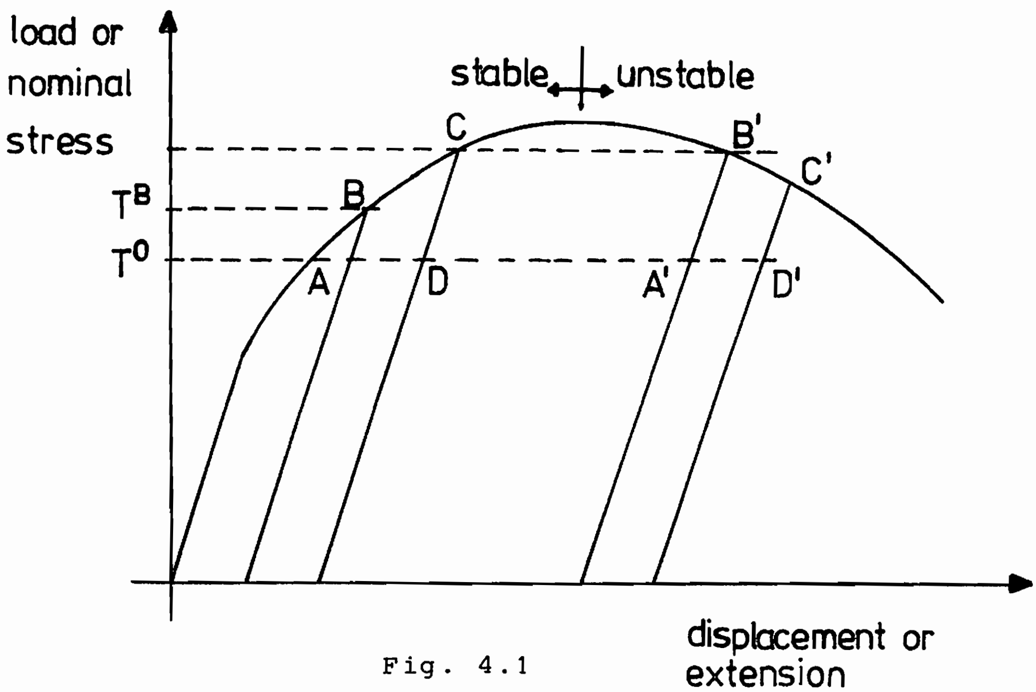

Fig. 4.1

If the elastic properties of the material are uneffected by plastic strain, then the elastic recovery at the end of the cycle is complete and only the plastic deformation $du^p_{i,K}$ remains. The relation (4.6) takes then the form

$$\int_{t_o}^{t} (T_{Ki} - T^o_{Ki}) v^p_{i,K} \, dt \geq 0 \tag{4.7}$$

where v^p is the plastic part of velocity.

With the assumption of small strains the expression on the left hand side of inequality (4.7) can be expanded into the Taylor series at the point B. To within an accuracy of the second order terms, the inequality (4.7) becomes

$$(T^B_{Ki} - T^o_{Ki}) v^p_{i,K} dt + \frac{1}{2} \dot{T}_{Ki} \Big|_{T_{Ki} = T^B_{Ki}} v^p_{i,K} (dt)^2 + \frac{1}{2} (T^B_{Ki} - T^o_{Ki}) \dot{v}^p_{i,K} (dt)^2 \geq 0 \tag{4.8}$$

It may be shown [19] that the inequality

$$(T^B_{Ki} - T^o_{Ki}) v^p_{i,K} \geq 0 \tag{4.9}$$

is the only condition which follows from (4.8) (that is from the Drucker postulate) when it is satisfied for the closed stress cycle. Indeed, even if the second term in (4.8) is negative,

$$\dot{T}_{Ki} v^p_{i,K} < 0 \quad , \tag{4.10}$$

(falling branch of $T_{Ki} - u_{i,K}$ curve in Fig. (4.1)), the inequality (4.8) cannot be violated by holding $v^p_{i,K}$ and $\dot{T}^B_{Ki}$ fixed and moving T^o_{Ki} closer to T^B_{Ki}. If $|T^B_{Ki} - T^O_{Ki}|$ is too small in comparison with $\dot{T}^B_{Ki}$, it is not possible to close the stress cycle.

5. NORMALITY AND CONVEXITY

As a consequence of the condition (4.9) (i.e. the requirement of positive work done over a stress cycle), the normality of strain rate vector to the yield surface and convexity of the yield surface can be proved at the assumption of small strains.

Since near to the reference state all the stress measures coincide and all the strain rate measures coincide too, the inequality (4.9) may be written in the similar form for any conjugate stress and strain rate measures,

$$(\underset{\sim}{\tau}^B - \underset{\sim}{\tau}^o)\dot{\underset{\sim}{\varepsilon}}^p \geq 0 \ . \qquad (5.1)$$

In particular, for the variables $t_{ij} - d_{ij}$ which are conjugate in the Euler description the inequality (5.1) takes the form

$$(t_{ij}^B - t_{ij}^o)d_{ij} \geq 0 \qquad (5.2)$$

whereas, for the variables $S_{KL} - \dot{E}_{KL}$ conjugate in the Lagrangian description we have

$$(S_{KL}^B - S_{KL}^o)\dot{E}_{KL} \geq 0 \ . \qquad (5.3)$$

The inequality (5.1) may be expressed in the form

$$|\underset{\sim}{\tau}^B - \underset{\sim}{\tau}^o||\dot{\underset{\sim}{\varepsilon}}^p| \cos \ \psi \geq 0 \qquad (5.4)$$

where ψ is the angle between $(\underset{\sim}{\tau}^B - \underset{\sim}{\tau}^o)$ and $\dot{\underset{\sim}{\varepsilon}}^p$ vectors. This requirement means that the vector $(\underset{\sim}{\tau}^B - \underset{\sim}{\tau}^o)$ for each interior point $\underset{\sim}{\tau}^o$ has to make an acute or right angle with the vector $\dot{\underset{\sim}{\varepsilon}}^p$, that is

$$|\psi| \leq 90^o \ . \qquad (5.5)$$

The vector $\dot{\underset{\sim}{\varepsilon}}^p$ depends upon $\underset{\sim}{\tau}^B$ but is independent of $\underset{\sim}{\tau}^o$. Therefore, if a plane is drawn through point B (Fig. 5.1) perpendicular to $\dot{\underset{\sim}{\varepsilon}}^p$, then all the admissible points O must lie on one side of this plane. This is clearly the definition of a convex surface. In this way the convexity of yield surface has been proved.

Next, let us take the convex yield surface and discuss the restrictions on the plastic strain rate vector $\dot{\underset{\sim}{\varepsilon}}$. Since $\dot{\underset{\sim}{\varepsilon}}$ must make a nonabtuse angle with every vector $(\underset{\sim}{\tau}^B - \underset{\sim}{\tau}^O)$ this can only be satisfied if $\dot{\underset{\sim}{\varepsilon}}$ is in the direction of the normal (assuming the yield surface to be smooth at the point B). Thus

$$\dot{\underset{\sim}{\varepsilon}}^p = \Lambda \frac{\partial f}{\partial \underset{\sim}{\tau}} \tag{5.5}$$

where Λ is a nonnegative scalar function which may depend on stress, stress rate, strain and strain history.

At a corner or edge of the yield surface (point B' in Fig. 5.1) the normal and thus the direction of $\dot{\underset{\sim}{\varepsilon}}^p$ is not unique. Such surface is composed of a number of individual smooth yield surfaced $f_\alpha = 0$ which intersect at B'. Then the strain rate vector $\dot{\underset{\sim}{\varepsilon}}^p$ lies within a cone bounded by normals to the surfaces $f_\alpha = 0$ at point B' and can be written as follows

$$\dot{\varepsilon}^p = \Lambda_\alpha \frac{\partial f_\alpha}{\partial \underset{\sim}{\tau}} \ . \tag{5.6}$$

The relation (5.5) is known as the flow rule and (5.6)as the generalized (or Koiter's) flow rule.

Hence, the convexity of a yield surface in any stress space and the normality of conjugate strain rate vector at the assumption of small strains have been proved.

Entirely equivalent results can be obtained in the load space. From the formulation of Drucker's postulate, it immediately follows that the inequality analogous ot (4.9) may be written for the load $\underset{\sim}{T}$ and velocity $\underset{\sim}{v}^p$ vectors:

$$(T_i^B - T_i^O)\underset{\sim}{v}_i^p \geq 0. \tag{5.7}$$

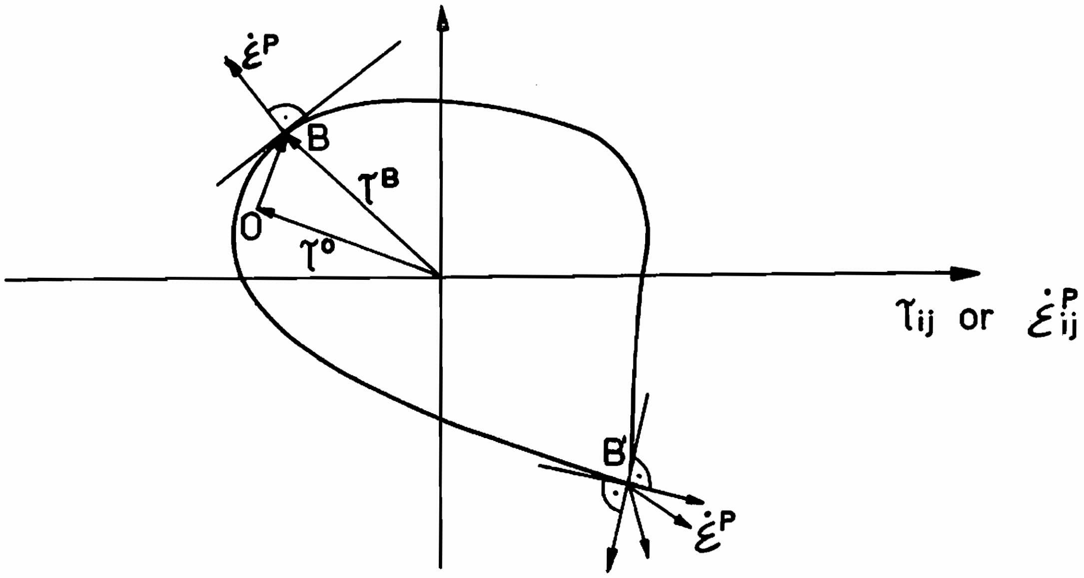

Fig. 5.1

Convexity of the load surface and normality of the plastic velocity vector may be proved in the same way as above (Fig. 5.2).

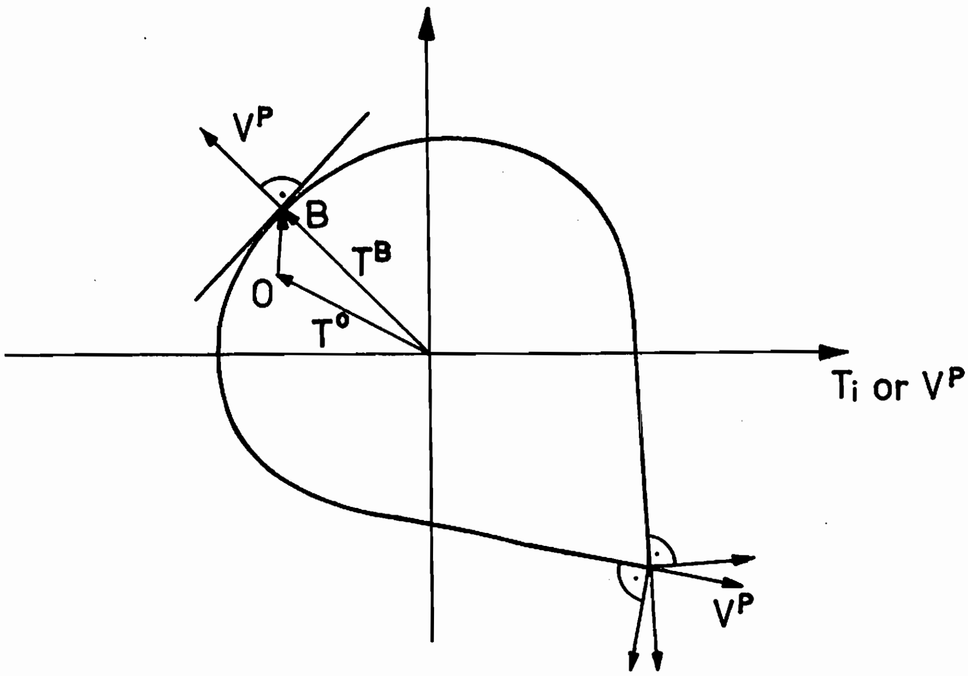

Fig. 5.2

Now, let us draw some conclusions from the Drucker postulate when finite elastic strains are allowed. Then the different stress measures and different strain rate measures do not, in general, coincide and the relations (5.2) - (5.3) do not follow from (4.9) any more.

From the definition of conjugate variables (the requirement of invariant energy rate) we can state that

$$T^B_{Ki} v^p_{i,K} = t^B_{ki} d^p_{ki} = S^B_{KL} \dot{E}^p_{KL} . \tag{5.8}$$

The second term at the left-hand side of inequality (4.9) is, however, not an invariant measure when finite deformations take place, thus

$$T^o_{Ki} v^p_{i,K} \neq t^o_{ki} d^p_{ki} \neq S^o_{KL} \dot{E}^p_{KL} . \tag{5.9}$$

Therefore no conclusion about normality and convexity in the space of Kirchhoff (or Cauchy) stress tensor and in the space of the second Piola-Kirchhoff stress tensor can be drawn directly from the Drucker postulate if finite elastic strains are allowed.

However, it can be shown that from the assumption of normality

$$d_{ij} = \Lambda \frac{\partial f(t_{ij})}{\partial t_{ij}} \tag{5.10}$$

when the Eulerian variables are use to define a material, follows the normality of the Green strain rate tensor $\dot{E}_{KL}$ to the yield surface which is obtained by the transformation of equation $f(t_{ij}) = 0$ into the second Piola-Kirchhoff stress space. Thus we have

$$\dot{E}_{KL} = \Lambda \frac{\partial f(S_{MN} x_{k,M} x_{l,N})}{\partial S_{KL}} \tag{5.11}$$

where, in view of the relationship $t_{ij} = S_{KL} x_{i,K} x_{j,L}$,

$$f(S_{KL} x_{i,K} x_{j,L}) = f(t_{ij}) . \tag{5.12}$$

Making use of (5.10), (2.83) from part I, and known rules of differential calculus, we may transform the right-hand side of (5.11) as follows:

$$\Lambda \frac{\partial f(S_{MN}x_{k,M}x_{l,N})}{\partial S_{KL}} = \Lambda \frac{\partial f(S_{MN}x_{k,M}x_{l,N})}{\partial (S_{MN}x_{k,M}x_{l,N})} \cdot \frac{\partial (S_{KL}x_{k,M}x_{l,N})}{\partial S_{KL}}$$

$$= \Lambda \frac{\partial f(t_{kl})}{\partial t_{kl}} \cdot x_{k,K}x_{l,L} = d_{kl}x_{k,K}x_{l,L} = \dot{E}_{KL} \; . \qquad (5.13)$$

This proves the statement (5.11).

Similarly, one would like to show that from the assumption of convexity of the yield surface in the Kirchhoff stress space, the convexity of the yield surface in the second Piola-Kirchhoff stress space does also follows. However, this implication at finite deformations is not so simple and requires additional restrictions.

6. MATERIAL STABILITY

Although the concept of material stability in the plastic range was introduced by Drucker almost 30 years ago, this definition appears to be not precise enough and therefore the question still arises of what exactly a "stable material" means.

In the recent literature the following two approaches to the definition of the material stability of time-independent materials are most commonly used.

The first one is concerned with Drucker's concept. The material stability condition is then derived from the energy criterion of stability of a body or a system when homogeneous stress and strain states are assumed.

The second approach consists in generalization of the definition of material stability under one-dimensional tension to cover an arbitrary stress state. According to this definition the material is said to be stable when the stress-strain curve is rising (the OAB curve, Fig. 6.1(a)) and it is said to be unstable when the curve is falling (the BC curve, Fig. 6.1(a)). For the 3-dimensional stress state the material is said to be stable when the yield surface is swelling locally at the plastic flow process, and is said to be unstable when the yield surface is shrinking locally (Fig. 6.1(b)).

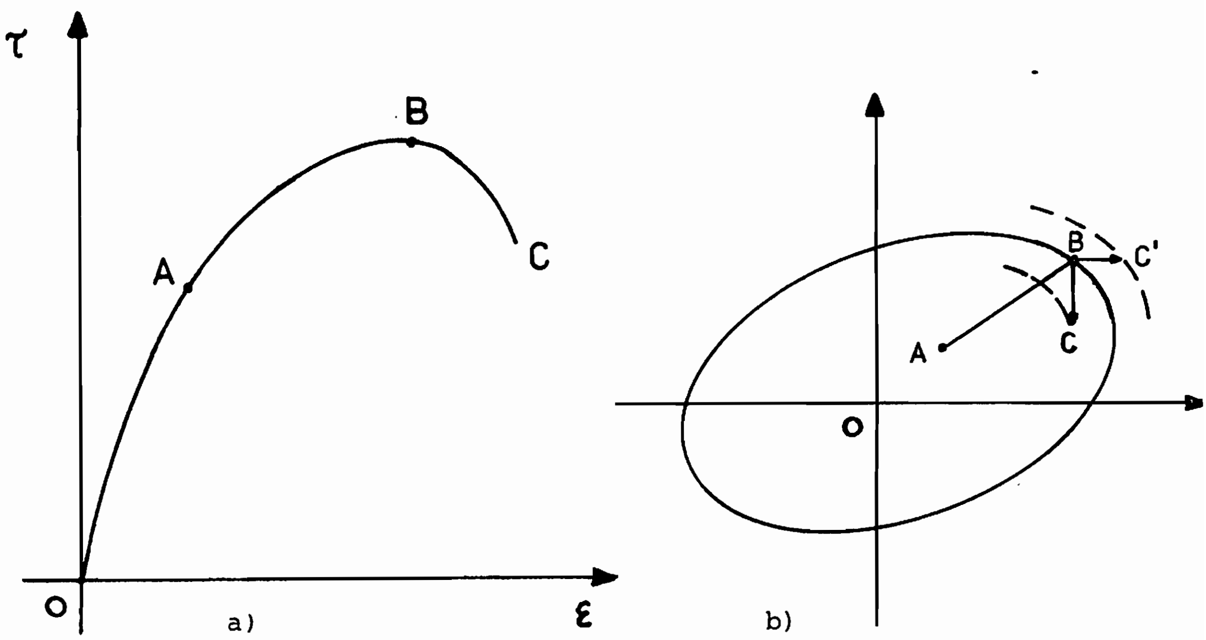

Fig. 6.1

This idea can be expressed analytically as a requirement that the scalar product of conjugate stress rate and strain rate tensors be non-negative for stable material

$$\overset{\nabla}{\tau}_{ij}\overset{\nabla}{\varepsilon}_{ij} \geq 0 \tag{6.1}$$

where ∇ denotes an objective time derivative.

This definition does not, however, describe the material properties in a unique way since the material stability depends here upon the choice of measures of the stresses, strains and the objective stress and strain rates. (The scalar product $\overset{\nabla}{\tau}_{ij}\overset{\nabla}{\varepsilon}_{ij}$ is not invariant under the change of stress, strain, stress rate and strain rate measures).

Usually in the literature both approaches are treated as leading to the same results when small strains are considered. It will be shown, however, that in general it is not the case. The differences in response of structural elements made of plastic materials which are stable in different senses may be substantial when the stress level has attained a magnitude comparable to that of the plastic hardening modulus.

Moreover, it will be shown that a rigorous non-linear analytical formulation of the Drucker postulate leads to the unique choice of the stress rate and the strain acceleration measures in the inequality (6.1).

As it was shown in chapter 4, part II, the Drucker postulate satisfied in a stress cycle guarantees the normality and convexity only, but not the material stability. To ensure the sufficient condition of the material stability in Drucker's sense, the requirement of positive work done has to be satisfied not only in a stress cycle but for *any stress path* from an initial state T^{O}_{Ki}. Then, instead of (4.7) the relation (4.6) has to be used in further considerations. After expanding into the Taylor series at the point B to within an accuracy of the second order terms, the inequality (4.6) becomes

$$(T^B_{Ki} - T^o_{Ki})v_{i,K}dt + \frac{1}{2}\dot{T}_{Ki}\Big|_{T_{Ki}=T^B_{Ki}} v_{i,K}(dt)^2 + \frac{1}{2}(T^B_{Ki} - T^o_{Ki})v_{i,K}(dt)^2 \geq 0 \ .$$

For $T^B_{Ki} = T^o_{Ki}$ this relation reduces to the requirement

$$\dot{T}_{Ki}v_{i,K} \geq 0 \qquad (6.2)$$

which expresses the material stability condition in Drucker's sense.

For an elastic-plastic material we are, therefore, not able to eliminate from the criterion the elastic part of strains.

For simplified, rigid-plastic material model, the Drucker postulate satisfied for *any stress path* in the nominal stress space leads to the condition

$$\dot{T}_{Ki}v_{i,K} = \dot{T}_{Ki}v^p_{i,K} \geq 0 \qquad (6.3)$$

which is both sufficient and necessary for the material stability in the Drucker sense.

However, in view of the lack of symmetry and objectivity, the tensors $\dot{T}_{Ki}$ and $v_{i,K}$ are rather inconvenient variables for the material description. Therefore, we shall express the material stability condition in the Drucker sense (6.3) in terms of the variables of the Eulerian and the Lagrangian description.

The material stability condition in the Drucker sense in the Lagrangian description:

Making use of the relation between the first and the second Piola-Kirchhoff stress tensors (equ. (3.12) part I),

$$T_{Ki} = S_{KL}x_{i,L}, \qquad (6.4)$$

the left-hand side of the inequality (6.3) can be rewritten in the form

$$\dot{T}_{Ki}v_{i,K} = (S_{KL}x_{i,L})^{\cdot}v_{i,K} = \dot{S}_{KL}x_{i,L}v_{i,K} + S_{KL}v_{i,L}v_{i,K} =$$

$$= \frac{1}{2}\dot{S}_{KL}(x_{i,L}v_{i,K} + x_{i,K}v_{i,L}) + S_{KL}v_{i,L}v_{i,K} \quad . \qquad (6.5)$$

In view of the definition (2.20), part I, of the Green strain tensor

$$E_{KL} \equiv \frac{1}{2}(x_{i,K}x_{i,L} - \delta_{KL}) \qquad (6.6)$$

the Green strain rate tensor may be expressed as

$$\dot{E}_{KL} = \frac{1}{2}(v_{i,K}x_{i,L} + x_{i,K}v_{i,L}) \quad . \qquad (6.7)$$

With the use of (6.7) the relation (6.5) can be shown to be

$$\dot{T}_{Ki}v_{i,K} = \dot{S}_{KL}\dot{E}_{KL} + S_{KL}V_{N,K}V_{N,L} \quad . \qquad (6.8)$$

Substitution of (6.8) into (6.3) yields

$$\boxed{\dot{S}_{KL}\dot{E}_{KL} + S_{KL}V_{N,K}V_{N,L} \geq 0} \qquad (6.9)$$

which is both sufficient and necessary condition for the material stability in the Drucker sense in the Lagrangian description.

The material stability condition in the Drucker sense in the Eulerian description:

In view of the relation (4.36), part I, between the second Piola-Krichhoff and the Kirchhoff stress tensors,

$$S_{KL} = t_{kl}X_{K,k}X_{L,l} \quad , \qquad (6.10)$$

the second term on the left-hand side of the inequality (6.9) can be rewritten in the form

$$S_{KL}v_{N,K}v_{N,L} = t_{kl}X_{K,k}X_{L,l}v_{N,K}v_{N,L} = t_{kl}v_{n,k}v_{n,l} \quad . \tag{6.11}$$

According to (4.38), part I, the first term on the left-hand side of (6.9) can be expressed as

$$\dot{S}_{KL}\dot{E}_{KL} = (t_{kl})^{\nabla o} d_{kl} \tag{6.12}$$

where $(t_{kl})^{\nabla o}$ is the Oldroyd derivative of the Kirchhoff stress tensors. Substitution of (6.11) and (6.12) into (6.9) yields

$$\boxed{(t_{kl})^{\nabla o} d_{kl} + t_{kl}v_{n,k}v_{n,l} \geq 0} \tag{6.13}$$

which furnishes the material stability condition in the Drucker sense expressed in the Eulerian description.

In the literature, however, the Jaumann stress derivative is usually chosen to be used in the constitutive relations. Therefore, making use of the relation between the Jaumann and the Oldroyd stress derivatives (which can be obtained from (3.39) and (3.40), part I)

$$(t_{kl})^{\nabla o} = (t_{kl})^{\nabla J} - t_{kp}d_{lp} - t_{lp}d_{kp} \; , \tag{6.14}$$

the material stability condition (6.13) can be transformed to become

$$\boxed{(t_{kl})^{\nabla J} d_{kl} - t_{kl}v_{n,k}v_{l,n} \geq 0 \; .} \tag{6.15}$$

In particular, when the solid is plastically isotropic, the principal axes of t_{kl} and d_{kl} coincide (then $t_{kp}d_{kl} = t_{kl}d_{kp}$ and $t_{kp}d_{kl}w_{lp} = t_{lp}d_{kl}w_{kp} = 0$ as a scalar product of symmetric and antisymmetric tensors [20]) and (6.15) can be rewritten as

$$\boxed{(\dot{t}_{kl}d_{kl} - t_{kl}v_{n,k}v_{l,n}) \geq 0 \; .} \tag{6.16}$$

Volume integral of (6.16) leads to the sufficient stability condition derived by Hill [21] for a rigid-plastic body.

The validity of the Drucker postulate is reasonable only when strains are small. The conventional small strain formulation of this postulate leads to the inequality (6.1). Therefore, the requirement (6.1) written in the form

$$\dot{t}_{kl} d_{kl} \geq 0 \qquad \text{or} \qquad \dot{S}_{KL} \dot{E}_{KL} \geq 0 \quad . \tag{6.17}$$

is commonly presented as the Drucker material stability condition.

It was shown, however, that the rigorous non-linear formulation of the Drucker postulate leads to the material stability condition (6.3) (called material stability in the Drucker sense) which coincides neither with $(6.17)_1$ nor $(6.17)_2$.

The constitutive inequality

$$(t_{kl})^{\nabla J} d_{kl} \geq 0 \tag{6.18}$$

will in further considerations be called *the material stability condition in the Eulerian description or in the Jaumann sense*, whereas the constitutive inequality

$$\dot{S}_{KL} \dot{E}_{KL} \geq 0$$

will be referred to as *the material stability condition in the Lagrangean description or in Oldroyd sense* (since $\dot{S}_{KL} \dot{E}_{KL} = (\sigma_{kl})^{\nabla O} d_{kl}$).

7. PERFECTLY PLASTIC MATERIAL

Since the perfectly plastic material has so far been introduced as a plastic solid for which the neutral stability condition is satisfied, in view of the above considerations, such definition does not describe the material properties in a unique way. Therefore, it seems reasonable to introduce the following definitions of the perfectly plastic materials in various senses.

We say that material *is perfectly plastic in the Drucker sense* if

$$\dot{T}_{Ki} v_{i,K} = 0 \tag{7.1}$$

or

$$\dot{S}_{KL} \dot{E}_{KL} + S_{KL} V_{N,K} V_{N,L} = 0 \tag{7.2}$$

or

$$(t_{kl})^{\nabla o} d_{kl} + t_{kl} v_{n,k} v_{n,l} = 0 \tag{7.3}$$

or

$$(t_{kl})^{\nabla J} d_{kl} - t_{kl} v_{n,k} v_{l,n} = 0 \tag{7.4}$$

Material is *perfectly plastic in the Eulerian description if*

$$(t_{ij})^{\nabla J} d_{ij} = 0 \tag{7.5}$$

or

$$\dot{t}_{ij} d_{ij} = 0 \tag{7.6}$$

provided the material remains isotropic.

Material *is perfectly plastic in the Lagrangian description if*

$$\dot{S}_{KL} \dot{E}_{KL} = 0. \tag{7.7}$$

The significance of the above differences in the definitions of perfectly plastic materials can be demonstrated in simple tension of a bar made of the isotropic perfectly-plastic material in the Drucker sense (Fig. 7.1).

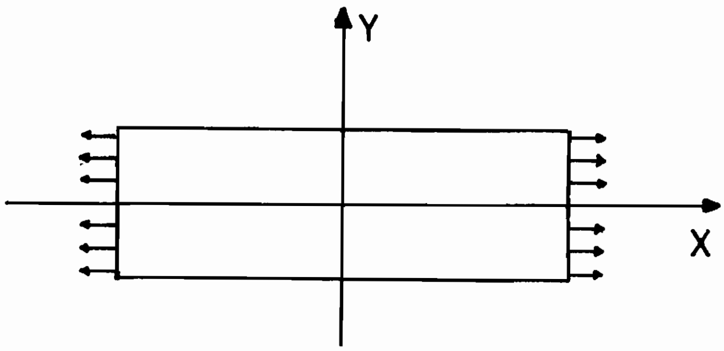

Fig. 7.1

The relations (7.1), (7.2), (7.4) can now be written as:

$$\dot{T}_{xx} v_{x,x} = 0 \ , \tag{7.8}$$

$$\dot{S}_{xx} \dot{E}_{xx} + S_{xx} (v_{x,x})^2 = 0 \ , \tag{7.9}$$

$$\dot{\sigma}_{xx} d_{xx} - \dot{\sigma}_{xx} (v_{x,x})^2 = 0 \ . \tag{7.10}$$

Since for small deformations all the strain rate measures coincide $v_{x,x} \simeq d_{xx} \simeq \dot{E}_{xx} \simeq \dot{\varepsilon}_{xx}$, the equations (7.8) - (7.10) can be rewritten as:

$$\dot{T}_{xx} \dot{\varepsilon}_{xx} = 0 \ , \tag{7.11}$$

$$\dot{S}_{xx} \dot{\varepsilon}_{xx} + S_{xx} (\dot{\varepsilon}_{xx})^2 = 0 \ , \tag{7.12}$$

$$\dot{\sigma}_{xx} \dot{\varepsilon}_{xx} - \sigma_{xx} (\dot{\varepsilon}_{xx})^2 = 0 \ . \tag{7.13}$$

For time-independent materials the extension ε_{xx} may be taken as the time measure, then the equations (7.11) - (7.13) become:

$$\frac{dT_{xx}}{d\varepsilon_{xx}} = 0 \ , \tag{7.14}$$

$$\frac{dS_{xx}}{d\varepsilon_{xx}} = - S_{xx} \ . \tag{7.15}$$

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}} = \sigma_{xx} \ . \tag{7.16}$$

Satisfying the initial conditions:

$$T_{xx} = T^{o}_{xx} \quad , \quad S_{xx} = S^{o}_{xx} \quad , \quad \sigma_{xx} = \sigma^{o}_{xx} \text{ for } \varepsilon_{xx} = 0 \ ,$$

the solution of equations (7.14) - (7.16) leads to the relations:

$$T_{xx} = T^{o}_{xx} \ , \tag{7.17}$$

$$\varepsilon_{xx} = - \ln S_{xx} + \ln S^{o}_{xx} = \ln \frac{S^{o}_{xx}}{S_{xx}} \ , \tag{7.18}$$

$$\varepsilon_{xx} = \ln \sigma_{xx} - \ln \sigma^{o}_{xx} = \ln \frac{\sigma_{xx}}{\sigma^{o}_{xx}} \tag{7.19}$$

as illustrated in Fig. 7.2.

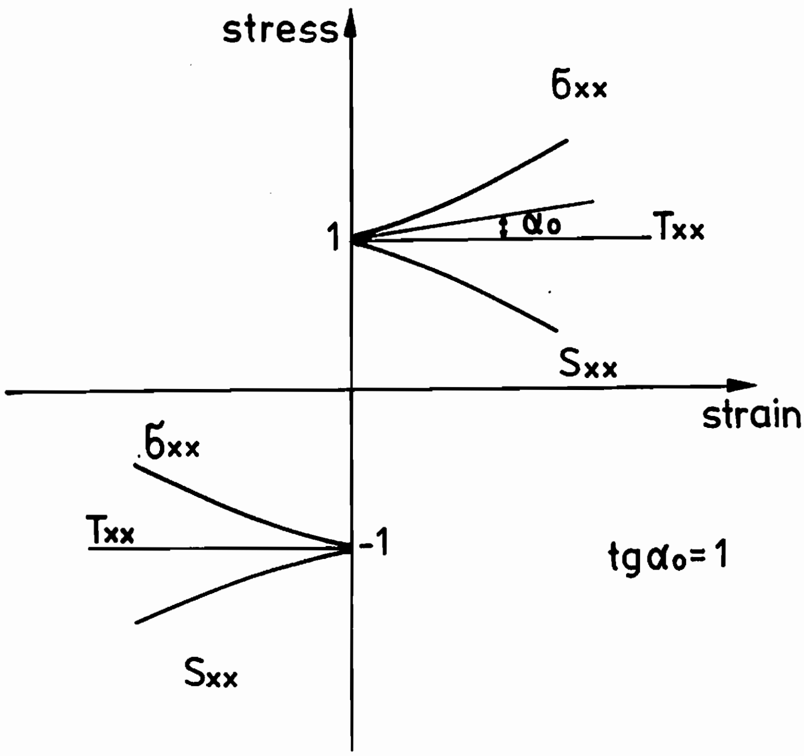

Fig. 7.2

As we can see in Fig. 7.2, the perfectly plastic material in the Drucker sense turns out to be strain-hardening for tension and strain-softening for compression in the Eulerian description whereas in the Lagrangian description it is strain-softening for tension and strain hardening for compression. The hardening (or softening) parameter h at the onset of yielding is equal to the yield-point load $\sigma_o (tg\alpha_o = 1)$.

8. HARDENING RULES

In the previous sections we have discussed the yield surface, the flow rule and material stability conditions. Now, let us consider the *hardening rules* which describe how the plastic deformation $\underset{\sim}{\varepsilon}^p$ enters the yield function

$$f(\underset{\sim}{\tau}, \underset{\sim}{\varepsilon}^p, \kappa) = 0 \tag{8.1}$$

or, how the yield surface changes in size and in shape during the plastic deformation process. We shall discuss several most common hardening rules.

8.1. Isotropic hardening

For *isotropic hardening* we assume that the yield surface maintains its shape, centre and orientation, but expands uniformly about the origin (Fig. 8.1). The subsequent yield surface may be written as

$$f(\underset{\sim}{\tau}, \underset{\sim}{\varepsilon}^p, \kappa) = f_o(\underset{\sim}{\tau}) - \kappa = 0 \tag{8.2}$$

where $f_o(\underset{\sim}{\tau}) = 0$ is the initial yield surface (for $\underset{\sim}{\varepsilon}^p = 0$), and κ is the hardening parameter which depends on the plastic strain history.

There are two widely discussed hypotheses proposed for computing the hardening parameter κ. The first one, known as the *work-hardening* hypothesis, states that the hardening is a monotonically increasing function of the total plastic work

$$\kappa = \kappa(W^p) \quad , \quad \frac{d\kappa}{dW^p} > 0 \tag{8.3}$$

where

$$W^p = \int_0^{\varepsilon^p} \underset{\sim}{\tau}\, d\,\underset{\sim}{\varepsilon}^p \ . \tag{8.4}$$

It is usually assumed that

$$\kappa = W^p \ . \tag{8.5}$$

The second isotropic hardening hypothesis, known as the *strain-hardening* hypothesis has it that κ depends on the amount of plastic strain

$$\kappa = \kappa(\underset{\sim}{\varepsilon}^p) \quad . \tag{8.6}$$

This dependence is usually assumed in the form

$$\kappa = \int_0^{\varepsilon^p} d\,\bar{\varepsilon}^p \tag{8.7}$$

where

$$d\,\bar{\varepsilon}^p = \left(\frac{2}{3}\, d\underset{\sim}{\varepsilon}^p \cdot d\underset{\sim}{\varepsilon}^p \right)^{1/2} . \tag{8.8}$$

For the Huber-Mises yield condition both hypothesis lead to the same results (Fig. 8.1)

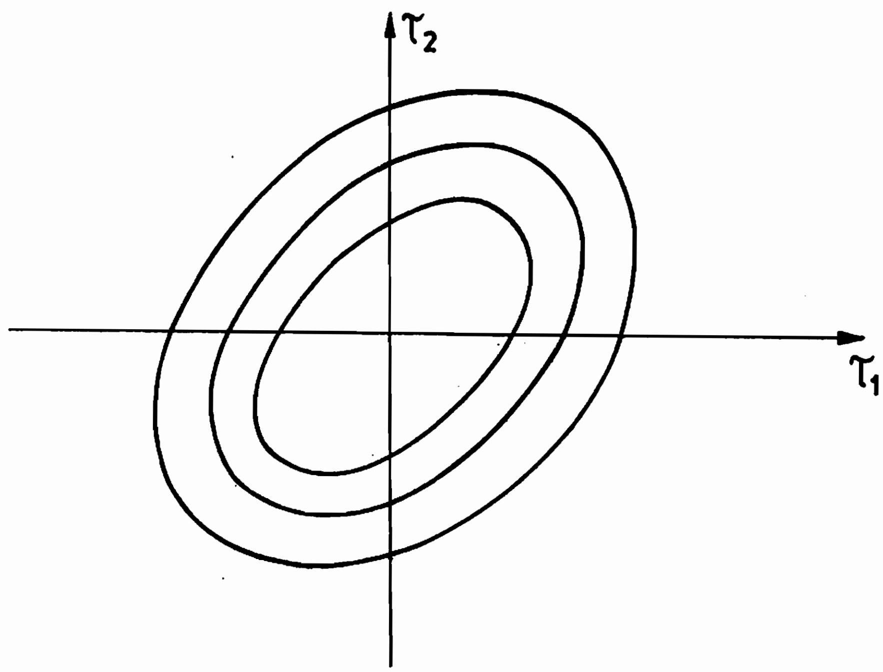

Fig. 8.1

8.2. Kinematic hardening

According to the kinematic hardening hypothesis proposed by W. Prager [22] the initial yield surface translates in the stress space preserving its size and orientation. Any motion must be normal to the edge in contact with the stress vector. When a corner of the yield surface is reached by the stress point, the yield surface moves in the direction of stress vector. Fig. 8.2 illustrates Prager's hypothesis for the Tresca yield condition shown for two segments of proportional loading.

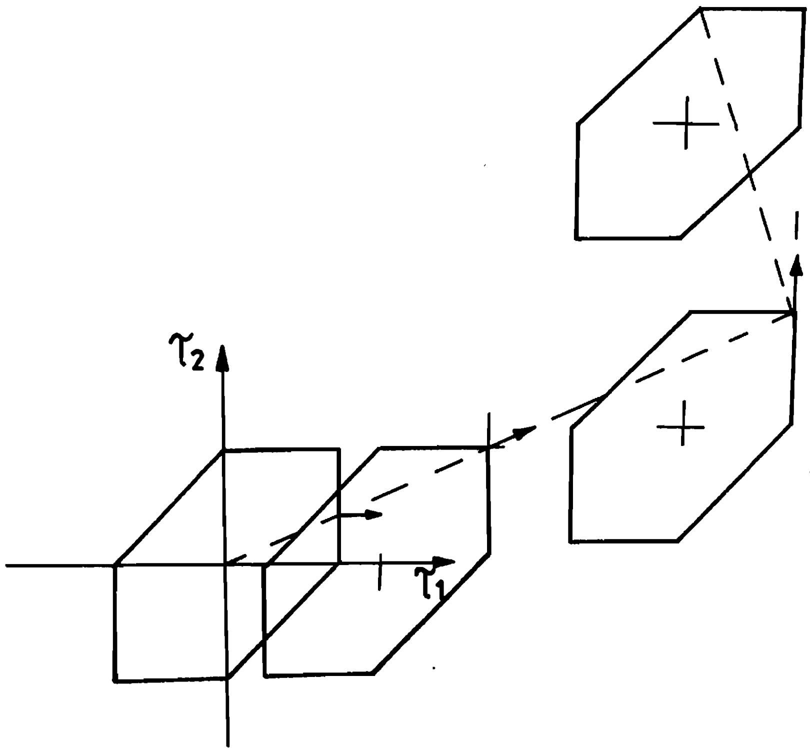

Fig. 8.2

Prager's concept of kinematical hardening may be formulated analytically as follows

$$f(\underset{\sim}{\tau},\underset{\sim}{\varepsilon}^{p},\kappa) = f(\underset{\sim}{\tau} - \underset{\sim}{\alpha}) = 0 \qquad (8.9)$$

where $\underset{\sim}{\alpha}$ represents the translation of the centre of the initial yield surface.

If linear hardening is assumed, then

$$\dot{\alpha}_{ij} = c\dot{\varepsilon}_{ij} \quad (8.10)$$

where c is a constant.

H. Ziegler [23] modified Prager's rule by suggesting that the direction of motion of the yield surface agrees with the radius vector OB that joins the centre of the yield surface with the yield point representing an acutal state of stress (Fig. 8.3)

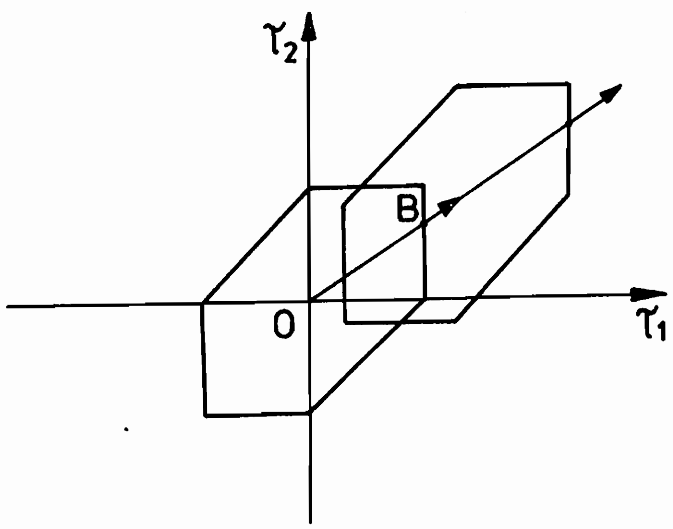

Fig. 8.3

Ziegler's hardening rule may be expressed analytically if (8.10) is replaced by the relation

$$\dot{\alpha}_{ij} = \dot{\mu}(\tau_{ij} - \alpha_{ij}) \quad (8.11)$$

where $\dot{\mu} > 0$.

More complex hardening rule was proposed by P. Hodge [24], who extended the kinematic hardening to include an expansion of the yield surface simultaneously with its translation.

9. OBJECTIVE AND CONSISTENT DESCRIPTION OF PLASTIC DEFORMATION PROCESS

9.1. Invariance requirements

As it was already said in chapter 4, part I, the choice of suitable stress, stress rate, strain rate and strain acceleration measures, when formulating constitutive laws, is not arbitrary but constrained by some invariance requirements.

The invariance requirement under superimposed rigid body motion leads to the formulation of objective stress rate and strain acceleration measures, whereas the invariance requirement of the rate of deformation energy was the base at the defining conjugate variables.

In chapter 4, part I, the conjugate an objective sets of stress, stress rate strain rate and strain acceleration measures were constructed. When the constitutive relationships for elastic-plastic and rigid-plastic solids are formulated, it turns out that these sets of measures do have to be employed to obtain consistent and objective material behaviour description.

The following invariant descriptions of the plastic deformation process can be proposed.

9.2. Co-rotational formulation

The yield condition is formulated in terms of co-rotational (rigid-body) components $\bar{t}_{KL}$ of the Kirchhoff stress tensor by the equation

$$f(\bar{t}_{KL},\kappa) = 0 \ . \qquad (9.1)$$

The process of plastic flow is then described by the relations:

$$\bar{d}^p_{KL} = 0 \qquad \text{if} \quad f < 0 \qquad \text{elastic state}$$

$$\text{or if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{KL}} \dot{\bar{t}}_{KL} < 0 \qquad \text{unloading}$$

$$\text{or if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{KL}} \dot{\bar{t}}_{KL} = 0 \qquad \text{neutral loading} \tag{9.2}$$

$$\bar{d}^p_{KL} = \Lambda \frac{\partial f}{\partial t_{KL}} , \quad \Lambda \geq 0 \quad \text{if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{KL}} \dot{\bar{t}}_{KL} > 0 \qquad \text{loading} \tag{9.3}$$

Making use of (9.2), (9.3) it may be proved that the following relation has to be satisfied

$$\bar{d}^p_{KL} \dot{\bar{t}}_{KL} \geq 0 \quad . \tag{9.4}$$

Indeed, if $\bar{d}^p_{KL} = 0$ then (9.4) is satisfied with the sign of equality. If $\bar{d}^p_{KL} \neq 0$ then, according to (9.3),

$$\bar{d}^p_{KL} \dot{\bar{t}}_{KL} = \Lambda \frac{\partial f}{\partial t_{KL}} \dot{\bar{t}}_{KL} \geq 0 \tag{9.5}$$

because $\quad \Lambda \geq 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{KL}} \dot{\bar{t}}_{KL} > 0$.

The co-rotational yield condition is one which is most commonly used in the theory of plastic structures. Indeed, constitutive relations for isotropic, plastic shells and plates are formulated in terms of stress and strain rate components tangent to the middle surface. This formulation is in agreement with (9.1) - (9.3), therefore it does coincide with the co-rotational yield condition.

If the actual state coincides with the reference configuration, then

$$\bar{t}_{KL} = t_{kl} \quad , \quad \bar{d}_{KL} = d_{kl} \tag{9.6}$$

and, according to (4.14) - (4.15), part I,

$$\dot{\bar{t}}_{KL} = (t_{kl})^{\nabla J} , \quad \dot{\bar{d}}_{KL} = (d_{kl})^{\nabla J} \quad . \tag{9.7}$$

The yield condition (9.1) and the plastic flow rule may now be written as

$$f(t_{kl},\kappa) = 0 \quad , \tag{9.8}$$

$$d^p_{kl} = 0 \qquad \text{if} \quad f < 0$$

$$\text{or if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{kl}} (t_{kl})^{\nabla J} \leq 0 \; , \tag{9.9}$$

$$d^p_{kl} = \Lambda \frac{\partial f}{\partial t_{kl}} \, , \Lambda \geq 0 \;\; \text{if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t_{kl}} (t_{kl})^{\nabla J} > 0 \quad . \tag{9.10}$$

It may be shown, in the same way as before, that, as a consequence of (9.9) - (9.10), the following relation takes place,

$$d^p_{kl}(t_{kl})^{\nabla J} \geq 0 \quad . \tag{9.11}$$

The inequality (9.11) is called in chapter 5 the material stability condition in the Euler description (or in the Jaumann sense). According to (4.21), part I, the condition (9.4) concides with (9.11).

If the considered material is assumed to be isotropic then [20]

$$d^p_{kl}(t_{kl})^{\nabla J} = d^p_{kl}\dot{t}_{kl} \tag{9.12}$$

and the Jaumann derivative of the Kirchhoff stress tensor $(t_{kl})^{\nabla J}$ in the relations (9.9) - (9.11), may be substituted by the material derivative $\dot{t}_{kl}$. Therefore the co-rotational formulation of constitutive relations for initially isotropic material coincides with the conventional one if small deformations are considered.

9.3. <u>Convected formulation</u>

Let us now assume that the yield condition can be formulated in terms of convected components $\hat{t}_{KL}$ of the Kirchhoff stress tensor in the form

$$f(\hat{t}_{KL},\kappa) = 0 \; . \tag{9.13}$$

Then, according to the flow rule, the process of plastic flow is described by the relations:

$$\hat{d}^{p}_{KL} = 0 \qquad \text{if } f < 0$$

$$\text{or if } f = 0 \text{ and } \frac{\partial f}{\partial \hat{t}_{KL}} \dot{\hat{t}}_{KL} \leq 0 \quad , \tag{9.14}$$

$$\hat{d}^{p}_{KL} = \Lambda \frac{\partial f}{\hat{t}_{KL}} \, , \Lambda \geq 0 \text{ if } f = 0 \text{ and } \frac{\partial f}{\partial \hat{t}_{KL}} \dot{\hat{t}}_{KL} > 0 \; . \tag{9.15}$$

If the actual state coincides with the reference configuration, then

$$\hat{t}_{KL} = t_{kl} \quad \text{and} \quad \hat{d}_{KL} = d_{kl} \tag{9.16}$$

and, according to (4.26) - (4.27), part I,

$$\dot{\hat{t}}_{KL} = (t_{kl})^{\nabla_{O}} \quad , \quad \dot{\hat{d}}_{KL} = (d_{kl})^{\nabla_{C-R}} \; . \tag{9.17}$$

On substituting (9.16) and (9.17) into (9.13) - (9.15) the yield condition and flow rule may be written as

$$f(t_{kl}\, , \kappa) = 0 \quad , \tag{9.18}$$

$$d^{p}_{kl} = 0 \qquad \text{if } f < 0$$

$$\text{or if } f = 0 \text{ and } \frac{\partial f}{\partial t_{kl}} (t_{kl})^{\nabla_{O}} \leq 0 \quad , \tag{9.19}$$

$$d^{p}_{kl} = \Lambda \frac{\partial f}{\partial t_{kl}} \, , \Lambda \geq 0 \text{ if } f = 0 \text{ and } \frac{\partial f}{\partial t_{kl}} (t_{kl})^{\nabla_{O}} > 0 \; . \tag{9.20}$$

Similarly as before it follows from (9.14) and (9.15) that

$$\dot{\hat{d}}^{p}_{KL} \, \dot{\hat{t}}_{KL} \geq 0 \tag{9.21}$$

and, according to (4.34), part I,) or directly from (9.19) and (9.20)

$$d^{p}_{kl}(t_{kl})^{\nabla o} \geq 0 \quad . \tag{9.22}$$

9.4. The Lagrangian formulation

As it was shown in chapter 4, part I, convected components of the Kirchhoff stress tensor $\hat{t}_{KL}$, convected components of the deformation rate tensor $\hat{d}_{KL}$ and their material time derivatives coincide, respectively, with the components of the second Piola-Kirchhoff stress tensor S_{KL}, the Green strain rate tensor $\dot{E}_{KL}$ and their material time derivatives in the Lagrangian (initial) frame of reference:

$$\hat{t}_{KL} = S_{KL} \quad , \quad \hat{d}_{KL} = \dot{E}_{KL} \quad , \quad \dot{\hat{t}}_{KL} = \dot{S}_{KL} \quad , \quad \dot{\hat{d}}_{KL} = \ddot{E}_{KL} \quad . \tag{9.23}$$

Therefore, the relations (9.13) - (9.15) can be written with the use of the Lagrangian description variables in the form:

$$f(S_{KL},\kappa) = 0 \quad , \tag{9.24}$$

$$\dot{E}^{p}_{KL} = 0 \qquad \text{if} \quad f < 0$$

$$\text{or if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial S_{KL}} \dot{S}_{KL} \leq 0 \quad , \tag{9.25}$$

$$\dot{E}^{p}_{KL} = \Lambda \frac{\partial f}{\partial S_{KL}} \, , \; \Lambda \geq 0 \quad \text{if} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial S_{KL}} \dot{S}_{KL} > 0 \quad . \tag{9.26}$$

The constitutive inequality (9.21) may now be written as

$$\dot{E}^{p}_{KL} \dot{S}_{KL} \geq 0 \quad . \tag{9.27}$$

The inequality (9.27) is called in chapter 6 the material stability condition in the Lagrangian description. According to (4.34) and (4.38), part I, the condition (9.21) coincides with (9.22) and (9.27).

9.5. Conclusions

Comparison of the above considerations with those presented in chapter 4, part I, shows that the stresses,strains and their rate measures used in the formulation of the constitutive relations of plastic flow process coincide with those which were derived in chapter 4, part I, as conjugate and objective constitutive variables. Thus the presented theory turns out to be a consistent and objective one. Furthermore, the constitutive inequalities (9.11) and (9.27), which were regarded in chapter 6 as the material stability conditions in the Eulerian and the Lagrangian descriptions respectively, are not introduced here as more assumptions but are derived as necessary conditions when the yield function and the flow law are formulated in terms of particular variables. As it was shown in chapter 6, none of the constitutive inequalities (9.4), (9.11), (9.21), (9.22) and (9.27) coincide with the material stability condition in Drucker's sense.

10. THE LINEAR FUNCTIONAL RELATION BETWEEN STRESS RATE AND STRAIN RATE

10.1. General formulation

Now, we shall see that the linear functional relation between a conjugate stress rate $\dot{\underset{\sim}{\tau}}$ and strain rate $\dot{\underset{\sim}{\varepsilon}}$ in the form

$$\dot{\tau}_{ij} = A_{ijkl}\dot{\varepsilon}_{kl} \qquad (10.1)$$

where A_{ijkl} is a known function of the current state and is called an elastic-plastic hipooperation, follows as a consequence of the following assumptions:

1. Elastic strains are small, so the total strain rate may be written as the sum of elastic and plastic components

   $$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{p}_{ij} \quad . \qquad (10.2)$$

   This is not generally true, but for static deformations of metals the assumption is reasonable.

2. The elastic response of the material can be described by linear functional relation between the stress rate and the strain rate,

   $$\dot{\tau}_{ij} = A^{e}_{ijkl}\dot{\varepsilon}^{e}_{kl} \qquad (10.3)$$

   where A^{e}_{ijkl} is called an elastic hipooperator. Hooke's material belongs to this category.

3. Plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^{p}$ is described by the flow rule

   $$\dot{\varepsilon}^{p}_{ij} = \Lambda \frac{\partial f}{\partial \tau_{ij}} \qquad (10.4)$$

   where

   $$f(\tau_{ij}, \varepsilon^{p}_{ij}, \kappa) = 0 \qquad (10.5)$$

   is the yield condition.

We shall now derive an appropriate form for A_{ijkl} in (10.1) for particular material properties.
For simplicity, let us consider an isotropic, work-hardening material ([1]), then

$$f(\tau_{ij},\kappa) = \phi(\tau_{ij}) - \kappa(W^p) \qquad (10.6)$$

where

$$W^p = \int \tau_{ij}\varepsilon^p_{ij}\,dt \qquad (10.7)$$

is the energy dissipated during the plastic deformation process.

With the above restriction, A_{ijkl} is formed exactly as in the small strain analysis. Substituting $\dot{\varepsilon}^e_{ij}$ from (10.2) into (10.3) and then (10.4) into (10.3), we obtain

$$\dot{\tau}_{ij} = A^e_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) = A^e_{ijkl}(\dot{\varepsilon}_{kl} - \Lambda \frac{\partial f}{\partial \tau_{kl}}) \ . \qquad (10.8)$$

The condition $\dot{f} = 0$ (necessary for plastic state to occur) can be now written as

$$\dot{f} = \frac{\partial f}{\partial \tau_{ij}} \dot{\tau}_{ij} + \frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial W^p} \dot{W}^p = 0. \qquad (10.9)$$

Next, substitution of (10.8) into (10.9) yields

$$\dot{f} = A^e_{ijkl} \frac{\partial f}{\partial \tau_{ij}} (\dot{\varepsilon}_{kl} - \Lambda \frac{\partial f}{\partial \tau_{kl}}) + \frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial W^p} \dot{W}^p = 0 \ . \qquad (10.10)$$

In view of (10.6) and (10.7) the equation (10.10) can be rewritten to become

$$\dot{f} = A^e_{ijkl} \frac{d\phi}{d\tau_{ij}} (\dot{\varepsilon}_{kl} - \Lambda \frac{d\phi}{d\tau_{kl}}) - \frac{d\kappa}{dW^p} \tau_{ij}\dot{\varepsilon}^p_{ij} = 0 \ . \qquad (10.11)$$

([1]) Non-isotropic hardening might be included in the formalism, however, this would lead to severe complications.

Substituting once again $\dot{\varepsilon}^p_{ij}$ from (10.4) into (10.11) and taking into account (10.6), we obtain

$$\dot{f} = A^e_{ijkl} \frac{d\phi}{d\tau_{ij}} (\dot{\varepsilon}_{kl} - \Lambda \frac{d\phi}{d\tau_{kl}}) - \frac{d\kappa}{dW^p} \tau_{ij} \Lambda \frac{d\phi}{d\tau_{ij}} = 0 \quad . \tag{10.12}$$

From this equation the factor Λ can be calculated as

$$\Lambda = \frac{A^e_{ijkl} \frac{d\phi}{d\tau_{ij}} \dot{\varepsilon}_{kl}}{(A^e_{ijkl} \frac{d\phi}{d\tau_{kl}} + \frac{d\kappa}{dW^p} \tau_{ij}) \frac{d\phi}{d\tau_{ij}}} \quad . \tag{10.13}$$

Finally, substituting Λ from (10.13) into (10.8) we arrive at the relation (10.1), where the elastic-plastic hipooperator A_{ijkl} is described by the relation:

$$A_{ijkl} = A^e_{ijkl} - \frac{A^e_{ijkl} A^e_{prmn} \frac{d\phi}{d\tau_{mn}} \frac{d\phi}{d\tau_{pr}}}{(A^e_{ijkl} \frac{d\phi}{d\tau_{kl}} + \frac{d\kappa}{dW^p} \tau_{ij}) \frac{d\phi}{d\tau_{ij}}} \quad . \tag{10.14}$$

Due to symmetry of the elastic hipooperator A^e_{ijkl} it follows from (10.14) that the elastic-plastic hipooperator A_{ijkl} is also symmetric, thus

$$A_{ijkl} = A_{klij} \quad . \tag{10.15}$$

Let us note that in the obtained constitutive relation the elastic-plastic hipooperator A_{ijkl} is the function of the current stress tensor $\underset{\sim}{\tau}$ only, but not the current strain tensor. This is only the case when isotropic work-hardening is assumed. In general case, however, A_{ijkl} will be a function of the current strain tensor as well.

To obtain a mathematically consistent theory of elastic-plastic material, the constitutive equation (10.1) should be expressed in terms of any set of conjugate, objective variables derived in chapter 4, part I. Application of different variables may lead, however, to the definitions of elastic-plastic materials whose behaviour in the plastic range will be quite different.

Now, the constitutive relations (10.1), (10.14) in terms of the most commonly used variables of the Eulerian and the Lagrangian descriptions will be presented and next the results for small strains and small rotations approximation will be compared and discussed.

10.2. The Eulerian description

Let us consider the following set of conjugate measures in the Eulerian description

$$t_{kl},\ (t_{kl})^{\nabla J},\quad d_{kl},\ (d_{kl})^{\nabla J} \quad . \tag{10.16}$$

Then the constitutive relations (10.1),(10.14) can be written in the known form

$$(t_{ij})^{\nabla J} = A_{ijkl} d_{kl} \tag{10.18}$$

where

$$A_{ijkl} = A^{e}_{ijkl} - \frac{A^{e}_{ijkl} A^{e}_{prmn} \dfrac{d\phi}{dt_{mn}} \dfrac{d\phi}{dt_{pr}}}{(A^{e}_{ijkl} \dfrac{d\phi}{dt_{kl}} + \dfrac{d\kappa}{dW^{p}} t_{ij}) \dfrac{d\phi}{dt_{ij}}} \tag{10.19}$$

and the yield condition (10.6) takes the form

$$f = \phi(t_{kl}) - \kappa(W^{p}) = 0 \quad . \tag{10.20}$$

In view of (4.4) and (4.5), part I, the following relations between the Jaumann derivatives of the Cauchy and the Kirchhoff stress tensors take place:

$$(t_{ij})^{\nabla J} = \frac{\rho_o}{\rho} (\sigma_{ij})^{\nabla J} + \frac{\rho_o}{\rho} v_{k,k} \sigma_{ij} \quad . \tag{10.21}$$

Substitution of (10.21) into (10.18) leads to

$$(\sigma_{ij})^{\nabla J} = \frac{\rho}{\rho_o} (A_{ijkl} - \sigma_{ij} \delta_{kl}) d_{kl} \tag{10.22}$$

or

$$(\sigma_{ij})^{\nabla J} = D_{ijkl} d_{kl} \qquad (10.23)$$

where

$$D_{ijkl} = \frac{\rho}{\rho_o} (A_{ijkl} - \sigma_{ij}\delta_{kl}) \qquad (10.24)$$

It should be noted that, whereas the elastic-plastic hipooperator A_{ijkl} is symmetric (as is indicated by equation (10.15), the hipooperator D_{ijkl} is non-symmetric,

$$D_{ijkl} \neq D_{klij} \qquad (10.25)$$

Hill [10] has shown that the existence of a homoegeneous quadratic rate potential $\Phi = \frac{1}{2} \dot{\underset{\sim}{t}} \cdot \dot{\underset{\sim}{\varepsilon}}$, such that

$$\dot{\tau} = \frac{\partial \phi}{\partial \dot{\underset{\sim}{\varepsilon}}} \quad , \qquad (10.26)$$

leads to the symmetric stiffness(elastic-plastic hipooperator) provided the constitutive law is expressed in terms of the conjugate stress and strain measures. The Cauchy stress tensor $\underset{\sim}{\sigma}$ and the deformation rate $\underset{\sim}{d}$ are not, however, conjugate measures (as it was shown in chapter 4, part I). Therefore, the lack of symmetry of the hipooperator D_{ijkl} in (10.23) is not surprising.

However, in view of the assumptions of small elastic strains and incompressibility in the plastic range, the term $\sigma_{ij}\delta_{kl}$ in (10.22) may be regarded small as compared with A_{ijkl} and then

$$D_{ijkl} \simeq A_{ijkl} \quad . \qquad (10.27)$$

10.3. The Lagrangian description

Let us take the set of conjugate variables in the Lagrangian description

$$S_{KL},\ \dot{S}_{KL},\ \dot{E}_{KL},\ \ddot{E}_{KL} \quad . \tag{10.28}$$

The constitutive relations (10.1), (10.14) can now be written as

$$\dot{S}_{KL} = C_{KLRS}\dot{E}_{RS} \tag{10.29}$$

where

$$C_{KLRS} = C^{e}_{KLRS} - \frac{C^{e}_{KLRS}C^{e}_{MNPZ}\dfrac{d\phi}{dS_{MN}}\dfrac{d\phi}{dS_{PZ}}}{\left(C^{e}_{KLRS}\dfrac{d\phi}{dS_{KL}} + \dfrac{d\kappa}{dW^{P}}S_{RS}\right)\dfrac{d\phi}{dS_{RS}}} \tag{10.30}$$

and the yield condition (10.6) becomes

$$f = \phi(S_{KL}) - \kappa(W^{P}) = 0 \quad . \tag{10.31}$$

Let us now convert the constitutive equations (10.18) - (10.19) expressed in the Euler description into the variables of the Lagrangian description. To this end, we make use of the relations (2.83), (3,14), (3.39) and (4.5), part I, to obtain:

$$d_{ij} = \dot{E}_{KL}X_{K,i}X_{L,j} \quad , \tag{10.32}$$

$$t_{ij} = S_{KL}x_{i,K}x_{j,L} \quad , \tag{10.33}$$

$$(t_{ij})^{\nabla J} = \dot{t}_{ij} - w_{ik}t_{kj} - w_{jk}t_{ki} \ . \tag{10.34}$$

Material time differentiation of (10.33) leads to the relation

$$\dot{t}_{ij} = \dot{S}_{KL}x_{i,K}x_{j,L} + t_{kj}v_{i,k} + t_{ik}v_{j,k} \quad . \tag{10.35}$$

Substituting $\dot{t}_{ij}$ from (10.35) into (10.34), we obtain

$$(t_{ij})^{\nabla J} = \dot{S}_{KL} x_{i,K} x_{j,L} + t_{kj} d_{ik} + t_{ki} d_{jk} \ . \tag{10.36}$$

Next, substitution of (10.36) into (10.18) yields

$$\dot{S}_{KL} x_{i,K} x_{j,L} = A_{ijkl} d_{kl} - t_{kj} d_{ik} - t_{ki} d_{jk} \tag{10.37}$$

Finally, substituting d_{kl} from (10.32) and t_{kj} from (10.33) into (10.37) we arrive at the following result:

$$\dot{S}_{KL} = (A_{ijkl} X_{K,i} X_{L,j} X_{R,k} X_{S,l} - S_{RL} X_{K,i} X_{S,i} - S_{RK} X_{L,i} X_{S,i}) \dot{E}_{RS} \tag{10.38}$$

or

$$\dot{S}_{KL} = B_{KLRS} \dot{E}_{RS} \tag{10.39}$$

where

$$B_{KLRS} = A_{ijkl} X_{K,i} X_{L,j} X_{R,k} X_{S,l} - S_{RL} X_{K,i} X_{S,i} - S_{RK} X_{L,i} X_{S,i} \tag{10.40}$$

The form of the right-hand side of the above equation indicates that

$$B_{KLRS} = B_{RSKL} \ . \tag{10.41}$$

Comparison of equations (10.29) with (10.38) clearly reveals the differences between elastic-plastic materials defined with the use of different sets of conjugate variables.

In order to better understand the order of magnitude of the errors introduced when one set measures is replaced by the other without suitable change of the constitutive function, let us consider the following approximations.

1. *Small strain, large rotation approximation*

Making use of polar decomposition of the deformation gradient, we can characterize the motion of any material line element as a rigid body motion $\underset{\sim}{R}$ and a pure deformation $\underset{\sim}{U}$,

$$X_{K,l} = R_{Km}U_{ml} \quad . \tag{10.42}$$

For small strain, large rotation approximation, we can assume that

$$U_{ml} \simeq \delta_{ml} \quad \text{and} \quad X_{K,l} \simeq R_{Kl} \quad . \tag{10.43}$$

Then according to (10.40), the tensor B_{KLRS} takes the form

$$B_{KLRS} = A_{ijkl}R_{Ki}R_{Lj}R_{Rk}R_{Sl} - S_{RL}R_{Ki}R_{Si} - S_{RK}R_{Li}R_{Si} \quad . \tag{10.44}$$

It is readily seen that the formulation for finite strains is not much more complicated than that for small strains, large rotations.

2. *Small strain, small rotation approximation*

If both strains and rotations are small as compared with unity, then the constitutive equation (10.44) may be further simplified. Substituting

$$X_{K,l} = \delta_{Kl} \tag{10.45}$$

into (10.40), we obtain

$$B_{KLRS} = A_{KLRS} - S_{RL}\delta_{KS} - S_{RK}\delta_{LS} \quad . \tag{10.46}$$

In the elastic range, A_{KLRS} is a tensor of elastic moduli and hence the additional stress terms in (10.46) are negligible as compared with it. Then $B_{KLRS} = A_{KLRS}$, and both the Lagrangian and the Eulerian formulation leads to the definition of the same material. However, in the plastic range the additional terms are no longer negligible since the slope of the stress-strain curve, described by tensor A_{KLRS}, is often of the same (or smaller) order of magnitude as the stress itself.

Therefore, no matter how small the strains and rotations are in the plastic range, the Lagrangian and the Eulerian formulations given by relations (10.18) - (10.20) and (10.29) - (10.31), respectively, lead to the definitions of *two different materials*. Though the yield point loads of two geometrically identical structures made of these two materials are in general the same, their post-yield behaviour may be entirely different. Whereas a structure made of the material defined by relations (10.18) - (10.20) exhibits structural stability, the behaviour of the same structure made of the material (10.29) - (10.31) may be unstable at the onset of yielding and vice versa.

In particular, the use of $(t_{ij})^{\nabla J}$ and d_{ij} measures and the constitutive relations (10.18) - (10.20) for defining a plastic material which exhibits plastic hardening rate of h would imply, in uniaxial tension or compression, a plastic hardening rate of h-2S when expressed in terms of the rates of the second Piola-Kirchhoff stress tensor and the Green strain tensors, Fig. 10.1. Since S can be for plastic material of the same order of magnitude as h, this is in conflict with the approximately symmetric tensile and compressive response of initially isotropic materials.

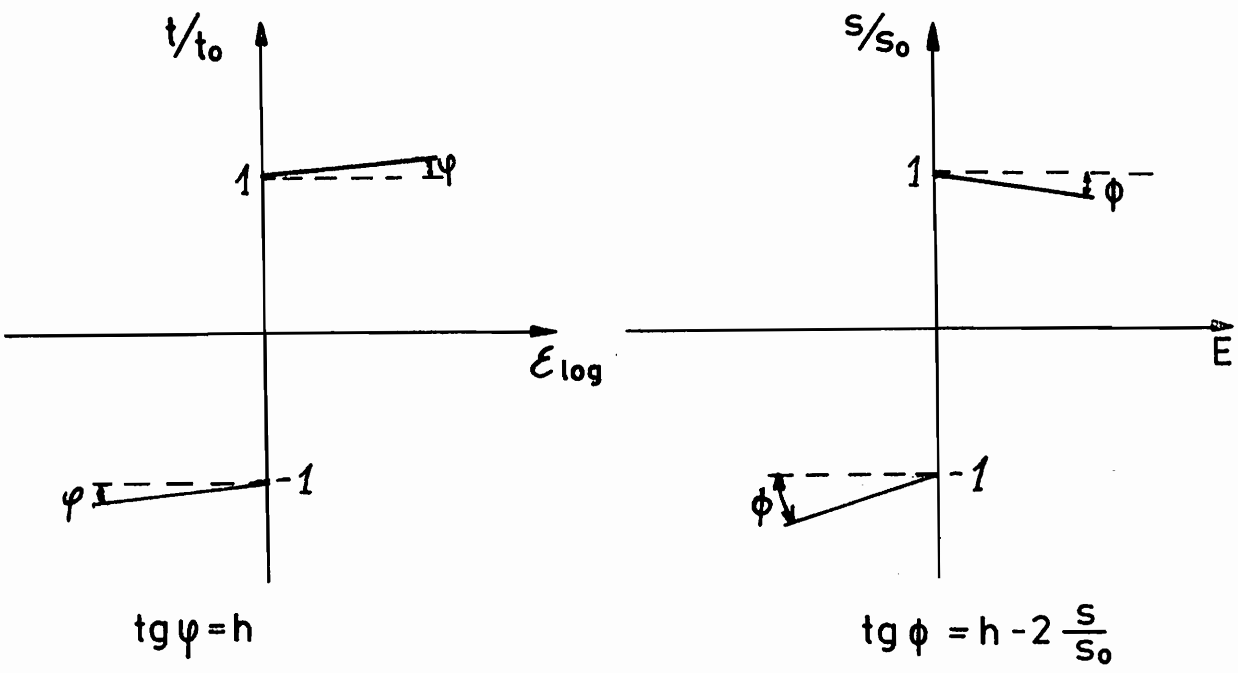

Fig. 10.1

10.4. The Prandtl-Reuss equations

If the constitutive relations (10.18) - (10.20) are coupled with the Huber-Mises yield condition, then we obtain the so-called Prandtl-Reuss equations. These equations are most commonly used for description of the elastic-plastic material behaviour.

For isotropic, work-hardening material the Huber-Mises yield condition can be written in the form

$$3J_2 \equiv \frac{3}{2} t'_{ij} t'_{ij} = \kappa(W^p) \qquad (10.47)$$

where

$$t'_{ij} = t_{ij} - \frac{1}{3} t_{kk} \delta_{ij} \quad . \qquad (10.48)$$

Comparing $(10.47)_2$ with (10.6) we see that

$$\phi(t_{ij}) = \frac{3}{2} t'_{ij} t'_{ij} \quad . \qquad (10.49)$$

According to the flow rule (10.4) the plastic part of the strain rate is now defined by

$$\dot{\varepsilon}^p_{ij} = d^p_{ij} = \Lambda \frac{\partial \phi}{\partial t_{ij}} = 3 \Lambda t'_{ij} \quad . \qquad (10.50)$$

After differentiation with respect to time the equation $(10.47)_2$ can be written as

$$3\, t'_{ij} (t'_{ij})^{\nabla J} = \frac{d\kappa}{dW^p} \dot{W}^p \quad . \qquad (10.51)$$

In view of (10.7) and (10.50) the equation (10.51) can be rewritten to become

$$t'_{ij} (t'_{ij})^{\nabla J} = \frac{d\kappa}{dW^p} \Lambda t'_{ij} t'_{ij} \quad . \qquad (10.52)$$

From this we get

$$\Lambda = \frac{t'_{ij}(t'_{ij})^{\nabla J}}{\frac{d\kappa}{dW^p} t'_{ij} t'_{ij}} \quad . \tag{10.53}$$

Substitution of (10.53) into (10.50) yields

$$d^p_{ij} = \frac{3\, t'_{kl}(t'_{kl})^{\nabla J}}{\frac{d\kappa}{dW^p} t'_{kl} t'_{kl}} t'_{ij} \tag{10.54}$$

or

$$d^p_{ij} = \frac{t'_{kl}(t'_{kl})^{\nabla J}}{h\, t'_{kl}\, t'_{kl}} \quad t'_{ij} \tag{10.55}$$

where

$$h = \frac{1}{3} \frac{d\kappa}{dW^p} \tag{10.56}$$

is the slope of the simple tension stress-strain curve in plastic range.

The elastic component of the deformation rate tensor is defined by Hooke's law as

$$d^e_{ij} = \frac{1+\nu}{E} (t_{ij})^{\nabla J} - \frac{\nu}{E} \delta_{ij} (t_{kk})^{\nabla J} \tag{10.57}$$

where ν is the Poisson's ratio and E is the Young's modulus.

Since for small elastic strains the deformation rate tensor may be decomposed additively into elastic and plastic parts,

$$d_{ij} = d^e_{ij} + d^p_{ij} \ , \tag{10.58}$$

substitution of (10.55) and (10.57) into (10.58) provides the Prandtl-Reuss equations in the form

$$d_{ij} = \frac{1+\nu}{E} (t_{ij})^{\nabla J} - \frac{\nu}{E} \delta_{ij} (t_{kk})^{\nabla J} + \frac{t'_{kl}(t'_{kl})^{\nabla J}}{h\, t'_{kl}\, t'_{kl}} \; t'_{ij} \tag{10.59}$$

for plastic loading and

$$d_{ij} = \frac{1+\nu}{E} (t_{ij})^{\nabla J} - \frac{\nu}{E} \delta_{ij} (t_{kk})^{\nabla J} \qquad (10.60)$$

for elastic loading or unloading.

The equations (10.59), (10.60) can be inverted, then the Prandtl-Reuss material is described be the constitutive relations in the form

$$(t_{ij})^{\nabla J} = \frac{E}{1+\nu} [\delta_{ik}\delta_{jl} + \frac{\nu}{1+2\nu} \delta_{ij}\delta_{kl} - \frac{t'_{ij}t'_{kl} \frac{E}{(1+\nu)}}{t'_{kl}t'_{kl}(\frac{3}{2}h + \frac{E}{(1+\nu)})}] d_{kl}$$

(10.61)

for plastic loading and

$$(t_{ij})^{\nabla J} = \frac{E}{1+\nu} (\delta_{ik}\delta_{jl} + \frac{\nu}{1+2\nu} \delta_{ij}\delta_{kl}) d_{kl} \qquad (10.62)$$

for elastic loading or unloading.

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Mitteilungen aus dem Institut für Mechanik
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