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Variational Formulation and Solution of Boundary-Value Problems in the Theory of Plasticity and Application to Plate Problems

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VARIATIONAL FORMULATION AND SOLUTION OF BOUNDARY - VALUE PROBLEMS IN THE THEORY OF PLASTICITY AND APPLICATION TO PLATE PROBLEMS

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SUMMARY

Starting from the formulation of the initial boundary value problem and the rate boundary value problem for elasto-plastic bodies at finite displacements and small deformations by means of the first Piola stress tensor a couple of minimum principles are derived. The necessary restrictive assumptions for constitutive relations, e.g. additivity of plastic and elastic part of deformation and description of plastic behaviour by means of the first Piola stress tensor are discussed in extension. In the application of the derived principles to plate problems the possibility of approximate description of arbitrary nonlinear stress distribution over the thickness of the plate was a matter of special interest. Several examples illustrate the numerical application of the derived methods. Range of technical application could be those elasto-plastic plate problems to which load history is known but no predictions about development of stress distribution (namely proportional development) can be made. Extension to shell problems seems to be possible with high evidence.

Efficiency and practicability of the developped methods with view on technical application depends in how far the existing computer programms, up to now only valid for simple special cases, will be developped.

ZUSAMMENFASSUNG

Ausgehend von der Formulierung des Anfangsrandwertproblems und des Zuwachsrandwertproblems für elasto-plastische Körper bei endlichen Verschiebungen und kleinen Deformationen unter Benutzung des ersten Piola'schen Spannungstensors werden mehrere Minimalprinzipe hergeleitet. Die notwendigen Annahmen für die konstitutiven Beziehungen wie zum Beispiel die Additivität der plastischen und elastischen Anteile der Deformation sowie die Beschreibung des plastischen Verhaltens mittels des ersten Piola'schen Spannungstensors werden ausführlich diskutiert. Bei der Anwendung der hergeleiteten Prinzipe auf Plattenprobleme war die Möglichkeit der approximativen Beschreibung beliebig nichtlinearer Spannungsverteilungen über den Querschnitt der Platte von besonderem Interesse. Mehrere Beispiele illustrieren die numerische Anwendung der hergeleiteten Methoden. Technischer Anwendungs-

bereich könnten jene Plattenprobleme sein, bei denen die Belastungsgeschichte bekannt ist aber keine Voraussagen über die Spannungsentwicklung (insbesondere proportionale Spannungsentwicklung) möglich sind. Die Ausweitung auf Schalenprobleme scheint mit hoher Wahrscheinlichkeit möglich zu sein.

Effizienz und Praktikabilität der hergeleiteten Methoden im Hinblick auf technische Anwendungen hängen davon ab, in wie weit die bestehenden Rechenprogramme, bis jetzt nur gültig für einfache Spezialfälle, weiterentwickelt werden.

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NOTATION

Throughout the entire work we refer to fixed cartesian coordinates using small and captial letters to distinguish between inital and deformed configuration resp.. Latin indices take the values 1,2,3, greek indices the values 1,2. We use index notation and, for brevity, symbolic notation parallely. Then the following expressions are equivalent:

$\underset{\sim}{(.)} \cdot \underset{\sim}{(.)} \mathrel{\hat{=}} (.)_{IJ}(.)_{JK}$; $\underset{\sim}{(.)} \cdot\cdot \underset{\sim}{(.)} \mathrel{\hat{=}} (.)_{iJ}(.)_{Ji}$,

$(.)_{iJ}$, $(.)_{IJ}$ denote the measures of considered tensors.

We have to distinguish throughout the entire work between locally defined vectors and tensors and vector- and tensorfields as well as between locally and globally defined potentials. As those quantities are explicitly defined in each chapter we use for locally and globally defined vectors and tensors the same symbol. Locally and globally defined potentials are denoted by small and capital letters respectively.

In the rate problem lower index $(.)_o$ denotes quantities which refer to reference state, denoted by $(.)_o$. Upper index $(.)^o$ denotes elastic part of the quantity under consideration. Lower index $(.)_s$ is applied if only symmetric part of a tensor of second degree is used: $(\underset{\sim}{\alpha})_s \mathrel{\hat{=}}$ $\mathrel{\hat{=}} \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$. Two-dimensional representations of three-dimensional quantities are denoted by capital letters; upper index $(.)^p$ denotes herein the order of defined representative according to (3.1). For rate quantities we use three symbols: $(\dot{.})$,$(\overset{\frown}{.})$, $\delta(.)$, which mean derivation with respect to time $\frac{\partial}{\partial\tau}$, finite difference of two neighboured configurations and infinitesimal increment respectively. In application to infinitesimal rate problem $(\dot{.})$ and $\delta(.)$ are equivalent.

We introduce first Piola stress tensor $\underset{\sim}{t} \mathrel{\hat{=}} t_{iJ}$ and second Piola-Kirchhoff stress tensors $\underset{\sim}{\sigma} \mathrel{\hat{=}} \sigma_{IJ}$ as stress measures. As strain measures serve deformationgradient $\underset{\sim}{F} \mathrel{\hat{=}} F_{iJ}$, displacement gradient Grad $\underset{\sim}{u} \mathrel{\hat{=}} u_{i,J}$ $= F_{iJ} - \delta_{iJ}$ and Green's strain tensor $\underset{\sim}{\varepsilon} \mathrel{\hat{=}} \varepsilon_{IJ}$. Derivations with respect to coordinates of the undeformed reference system are denoted by $(.)_{,K}$, where index K gives the direction of coordinates. We introduce symbols of differentiation Div$\underset{\sim}{(.)} \mathrel{\hat{=}} (.)_{iJ,J}$, Grad$(.) \mathrel{\hat{=}} \underset{\sim}{(.)}_{i,J}$.

If we consider only infinitesimal deformations, deformed and undeformed reference system coincide and we use the symbols $\underset{\sim}{\sigma} \mathrel{\hat{=}} \sigma_{ij}$ and $\underset{\sim}{\varepsilon} \mathrel{\hat{=}} \varepsilon_{ij}$ as stress and strain measure respectively.

1. INTRODUCTION

Aim of the work and general assumptions

In this work extremum- and stationarity principles for the solution of the initial boundary value problem and the rate boundary value problem of elastic-plastic bodies are investigated in general and in application to plate problems. We restrict our considerations in many aspects:

- Only quasistatic processes are regarded with external forces which can be derived from potentials ("dead loads").
- All deformations are assumed to be so small that plastic and elastic parts of deformation are additive. This allows to consider plates with moderate rotations. However, when we discuss the rate boundary value problem we assume only additivity of elastic and plastic part of strain-rate (2.6.30-53), (3.4).
- In chapter (2.1.6) we shall assume that small plastic deformations are superimposed on finite elastic deformations. Though usually, e.g. in metal forming, limit-load-analysis [6-8], theory of stability of plastic structures [9,10], elastic part of strain is assumed to be neglectible with respect to plastic part, our approach is useful for example if plates and shells perform large elastic deformations before plastification starts.
- We deal with adiabatic and isothermal processes [1], though plastic dissipation causes always change of temperature and theory needs for consistency a derivation of constitutive relations from thermodynamics [14-15,17,20]. The influence of external change of temperature [11,12] [20] is neglected in our approach also.
- We deal with "phenomenological plasticity". Though plastic behaviour is a microscopic phenomena, we assume media to be continuous.
- We assume in the entire work isotropical and time-independent material behaviour, though by introduction of internal parameters according to [21] the relations in chapters (2.8, 3.2) could be extended to viscous material behaviour. In chapters (2.1 - 4, 2.7 - 8, 3.3) we assume elasto-idealplastic material behaviour with convex region of admissible stress and validity of normality-rule. In (2.3 - 4) we discuss the differences in using the first Piola or second Piola-

Kirchhoff-stress tensor. In chapters (2.8,3.2), where we deal with the theory of infinitesimal deformations [39] linear hardening material behaviour is assumed, described by generalized stress and strain measures [21]. - In chapters (2.5 - 2.6) and (3.4) dealing with the rate problem according to [53 - 54] we use a material law given by [25] describing a broad class of hardening material behaviour by three parameters.

Some geometrical considerations

Always if finite displacements are considered where the history of deformation during the loading-process has to be taken into account, the question which stress and strain measure, connected with the question of chosen reference-configuration is essential. On one hand material properties have local nature, so that description by use of Cauchy-stress tensor in Eulerian coordinates [22 - 23] and Almansi-strain tensor is rather obvious. On the other hand equations of motion, here degenerated to statical equilibrium conditions can only be related to a fixed system of coordinates. From this follows that either equilibrium conditions or constitutive relations are submitted to coordinate transformations if we want a unified description. In this work attention is focussed on the solution of the boundary value problem where only the bounds of the initial configuration are exactly known. At best in case of the rate boundary value problem it may be assumed that the bounds of the reference-configuration are known. Here we refer all quantities to the initial configuration as the formulation of the problem then becomes relatively easy. If more than that we use first Piola stress tensor and displacementgradient as measures for stress and strain then the problem becomes accessible to functional-analytic considerations as there exists a linear differential relation between displacements and deformations [88]. However then we need very restrictive assumptions in formulating constitutive relations: We assume not only convexity/strict convexity of elastic strain energy density (2.1 - 4,2.8) but also convexity of the region of admissible stresses and validity of normality rule, expressed by the first Piola stress tensor. As we restrict our considerations to small strains these assumption are justified.

Splitting of deformation into elastic and plastic part is especially for finite strains subject of continuous discussion [24]. Whilst in [14 - 16] Green's strain tensor is additively splitted into a purely plastic and a not purely elastic part, in [13 , 18] a multiplicative decomposition of deformationgradient into purely elastic and purely plastic parts is proposed, where the second describes an unloaded intermediate configuration, determined only up to an unknown rigid rotation of the considered material element [18]. In [17,20] three strain measures are derived from the comparison of the metrics of the considered configurations. This way Green's strain tensor, Almansi's strain tensor and the " natural " (logarithmic) strain tensor can be derived.

Variational methods

Our comparably simple representation of material behaviour can only be appreciated in connection with formulation and solution of the boundary value problem. Previous propositions for the solution of the initial boundary value problem where restricted (besides incremental methods, which will be discussed in the following) to the special case of proportionally loaded bodies [2,3]. In [4] variational methods on the basis of deformation theory, comparable to nonlinear elasticity [1], for the initial boundary value problem are given and applied in [76] to plates. This method, however, lacks consistency for non-proportional developing stresses which may occur also for proportionally increasing external loads. Anyways this method is of practical importance.

A considerable progress in the development of solution techniques was achieved by application of convex analysis [32,89] to elastic-plastic problems [33 - 44]. When beforehand inequalities in yield-condition and derivation with respect to time in the flow-law could only be governed for arbitrary loading processes by incremental methods, now, briefly speaking by considering the four-dimensional space-time continuum, the replacement of Legendre-transformation by Fenchel-transformation and by a suitable transition from local to global functionalanalytic formulation we may consider the initial boundary value problem to be solved at least for a restricted class of problems assuming infinitesimal deformations [41]: In linear theory of elasticity duality of spaces of stresses and strains by the inner product induced by energy-norm is well known. Legendre transformation is in this case the equivalent representation of Hooke's law. Replacing Legendre-transformation by Fenchel-transformation plastic flow-law and yield-condition are described equivalently [35 - 42]. For the first time in [39] the general initial boundary value problem for infinitesimal deformations for elasto-plastic bodies had been solved. This concept was extended in [40] by introduction of internal parameters according to [21] to linear hardening, linear viscous material behaviour. In this paper we try for the first time to extend this concept to finite deformations in chapter (2.8). In (3.3) we deal with the rate-problem of the plate according to the von Kármán theory in a similar way. An application to the rate-problem at infinitesimal deformations has been given in [79]. Here accordingly we treat the initial-value problem of the plate for the first time in (3.2) and give numerical illustrations in (4.1,4.2).

The incremental approach

In spite of the genuine difficulties of incremental methods (see (2.7)) these are broadly and successfully used [56 - 59,61 - 63,66]. In [47 - 49] mixed functionals for the rate problem analogous to Reissner's variational functional in elasticity [50] assuming a stationary value for the solution are formulated, using material law according to [25]. Besides, duality of functionals expressed only by stresses and only by strains are mentioned however without discussion of the conditions for extremum properties (see 2.5 - 6, 3.4). In [55] variational functionals are derived systematically from a generalized functional and their applicability to finite element methods is discussed. In [53 - 54] starting from the theory of adjoint operators a very general and complete derivation of stationarity and extremum principles is given which we use in (2.6,3.4) to formulate variational functionals for the von Kármán plate applied in (4.3) to two illustrative numerical examples.

Plates

In opposition to elastic plates, for elasto-plastic plates no one-to-one correspondence between strains and stresses exists and for arbitrary load history even under assumption of the Kirchhoff-Love hypothesis the shape of stress-distribution over the crossection of the plate cannot

be predicted. This problem can be tackled in several ways:
In [49,80] the plate is devided into sheets and to each of them a plane stress-state is attributed. In [81] Prager's normality-rule is used to describe also hardening material behaviour in terms of moments and membrane forces. However this proceeding is bound to proportional increasing inner forces and moments; unloading states are not allowed for [81]. Similarly in [77] a linear approximation of yield-condition expressed in moments and membrane forces is introduced assuming that beyond the yield-limit the plate is fully plastified. Whereas in [81] variational methods with subsidary conditions are used for the construction of the solution, in [77] a method of finite differences is proposed. In [75] starting from the Kirchhoff-Love hypothesis constitutive relations for plates are formulated by introduction of rates of plastic curvature and membrane strains and additional plastic parameters depending on the coordinate orthogonal to the midspan of the plate. This description is shown to be compatible with the concept of "generalized standard material" [21]. In [78] a variational functional based on equilibrium finite elements and description of material behaviour only by moments is applied to the calculation of plate bending. In [29,30] the surfaces seperating elastic and plastic regions of shells are determined by a free parameter measuring their distance to the midsurface and the unknown shape of stress-distribution over the crossection of the shell is approximated by polynomical test-functions.

We discuss in (3.1) several possibilities of representation of stress- and strain distribution over the crossection of the plate in view of compability between two-dimensional representation with general three-dimensional theory. In (3.2,3.3) we chose polynomial, in (3.4) sheet-modell approach. Whereas in (3.2,3.3) the Kirchhoff-Love hypothesis is not used as constraint in the sense of [70,71] for the derivation of the important conditions of statical admissibility, it has been imposed in (3.4). However this choice is arbitrary and the derived relations can be modified immediately by change of adjoint compatible strain fields.

By this introduction of two-dimensional representatives for three-dimensional fields and functional analytic considerations (2.7), we derive in (3.2) a minimum principle for residual stresses in geometrically linear plates for arbitrary load-history, numerically applied in

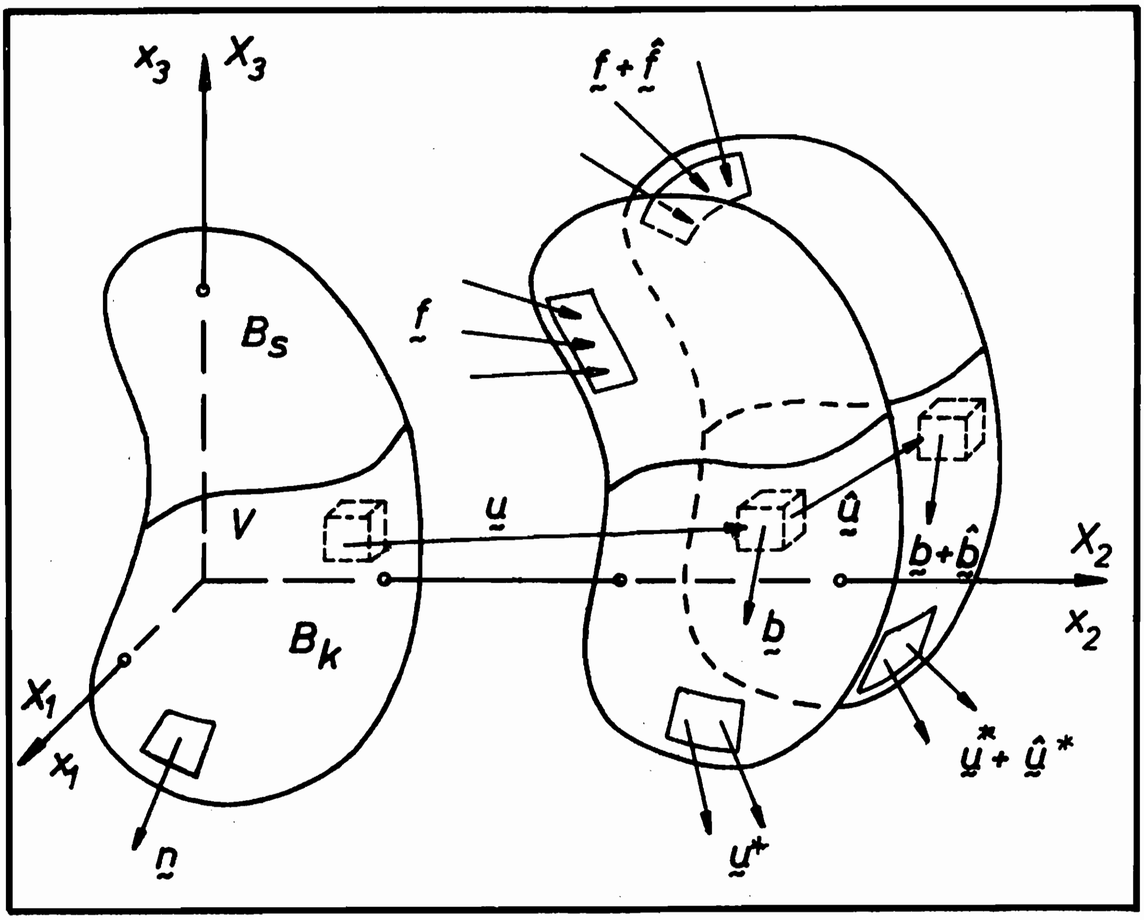
(4.1 - 2). - A first application of the methods developped in (2.1 - 2.5) for finite deformation is given in (3.3) by the formulation of the rate problem for the von Kármân plate theory.

Outlook

The solution of the initial boundary value problem for more general assumption seems to be of special interest for future works. Especially hindering are the assumptions of convexity of the chosen potentials in (2.1 - 2.4) and assumption of independance of plastic and elastic part of deformation from each other. In numerical application a large field is open for coming works: Choice of test-functions of higher order for stress-representatives defined in (3.2), consideration of linear hardening, extension to viscous and nonlinear hardening material behaviour are examples for future developments. An application of finite-element technique could render the herein derived methods attractive to technical application.

2. THE THREEDIMENSIONAL BOUNDARY VALUE PROBLEM

As we established in the introduction we use the Lagrangean description of the problem referring all quantities to the initial state of the considered body.



initial state reference state neighbour state

fig. 1

Herein V denotes volume, B_s and B_k the parts of surface where statical and kinematical conditions resp. are prescribed, $\underset{\sim}{n}$ denotes outer normal vector, $\underset{\sim}{b}$ and $\underset{\sim}{f}$ prescribed volume- and surface forces resp., $\underset{\sim}{u}^*$ prescribed surfacedisplacements. $\underset{\sim}{u}(\underset{\sim}{x})$ is the displacementvector of an arbitrary point in V from reference - to neighbour configuration, $(\hat{\ })$ denotes increments of quantities from reference - to neighbour state, not necessarily infinitesimal.

Even if changes of field quantities from reference state to neighbour state are finite, differential equations of equilibrium - and compatibility - condition are linear. We obtain the following descriptions of the problem:

i) The initial boundary value problem as transition from initial state to actual (neighbour) state.

$$Div\,(\underset{\sim}{t}+\hat{\underset{\sim}{t}})+(\underset{\sim}{b}+\hat{\underset{\sim}{b}}) = 0 \qquad \text{in } V$$

$$\underset{\sim}{n}\cdot(\underset{\sim}{t}+\hat{\underset{\sim}{t}})-(\underset{\sim}{f}+\hat{\underset{\sim}{f}}) = 0 \qquad \text{on } B_s \tag{2.0.1}$$

$$\underset{\sim}{d}+\hat{\underset{\sim}{d}}-Grad\,(\underset{\sim}{u}+\hat{\underset{\sim}{u}}) = 0 \qquad \text{in } V$$

$$\underset{\sim}{u}+\hat{\underset{\sim}{u}}-\underset{\sim}{u}^{*}-\hat{\underset{\sim}{u}}^{*} = 0 \qquad \text{on } B_k \tag{2.0.2}$$

Constitutive relations, describing material behaviour in the transition from initial to actual state where $\underset{\sim}{t}$ denotes the first Piola stress tensor and $\underset{\sim}{d}$ denotes displacementgradient. (2.0.3)

Only for hyperelastic behaviour, i. e. if there exists some potential $\psi(\underset{\sim}{t})$ allowing a one-to-one correspondence of stress and strain measure of actual state independant of history of load with

$$\underset{\sim}{d} = \frac{\partial\psi(\underset{\sim}{t})}{\partial\underset{\sim}{t}} \tag{2.0.4}$$

a direct solution of the initial boundary value problem is possible. For special load-histories (proportional loading) [1,4] we can find a potential $\tilde{\psi}$ even if a general potential $\psi(t)$ independant of load history does not exist. Deformation theory, almost similar to theory of physically nonlinear elasticity [1], is based on the assumption of such potentials $\tilde{\psi}$.

In case of general loading however this method fails, such that normally an incremental formulation of the problem is used in order to construct an approximate solution of the initial boundary value problem by a sequence of solution of the rate boundary value problem. For the formulation of the rate boundary value problem one assumes that all quantities determining the (mechanical) reference state are exactly known. This leads to the

ii) Rate boundary value problem as transition from reference state to actual state.

$$Div\,\hat{\underset{\sim}{t}}+\hat{\underset{\sim}{b}} = 0 \qquad \text{in } V$$

$$\underset{\sim}{n}\cdot\hat{\underset{\sim}{t}}-\hat{\underset{\sim}{f}} = 0 \qquad \text{on } B_s \tag{2.0.5}$$

$$\hat{\underset{\sim}{d}} - \text{Grad}\, \hat{\underset{\sim}{u}} = 0$$

$$\hat{\underset{\sim}{u}} - \hat{\underset{\sim}{u}}^{*} = 0 \qquad (2.0.6)$$

Constitutive relations describing material behaviour in the transition from reference to actual state. } (2.0.7)

Instead of $\underset{\sim}{t}$ and $\underset{\sim}{d}$ we may also use $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ to formulate relations (2.0.5-7). With

$$\underset{\sim}{F} = \text{Grad}\, \underset{\sim}{u} + \underset{\sim}{1} \;\hat{=}\; F_{iA} = u_{i,A} + \delta_{iA}$$

allowing for

$$+\infty > \text{Det}\, \underset{\sim}{F} > 0$$

to assure that only deformations are taken into account which lead to positive volume of every subregion of the considered body after deformation, characterized by $\underset{\sim}{u}$.

Using

$$\underset{\sim}{t} = \underset{\sim}{F}.\underset{\sim}{\sigma} \;;\; \hat{\underset{\sim}{t}} = \hat{\underset{\sim}{F}}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\hat{\underset{\sim}{\sigma}} + \hat{\underset{\sim}{F}}.\hat{\underset{\sim}{\sigma}} \qquad (2.0.8)$$

and

$$2\underset{\sim}{\varepsilon} = \underset{\sim}{F}^{T}.\underset{\sim}{F} - \underset{\sim}{1} \;;\quad 2\hat{\underset{\sim}{\varepsilon}} = \hat{\underset{\sim}{F}}^{T}.\underset{\sim}{F} + \underset{\sim}{F}^{T}.\hat{\underset{\sim}{F}} + \hat{\underset{\sim}{F}}^{T}.\hat{\underset{\sim}{F}} \qquad (2.0.9)$$

we obtain instead of (2.0.5.-7)

$$\text{Div}\, \hat{\underset{\sim}{t}} = \text{Div}\,(\hat{\underset{\sim}{F}}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\hat{\underset{\sim}{\sigma}} + \hat{\underset{\sim}{F}}.\hat{\underset{\sim}{\sigma}}) = -\hat{\underset{\sim}{b}} \quad \text{in } V$$

$$\underset{\sim}{n}.\hat{\underset{\sim}{t}} = \underset{\sim}{n}.(\hat{\underset{\sim}{F}}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\hat{\underset{\sim}{\sigma}} + \hat{\underset{\sim}{F}}.\hat{\underset{\sim}{\sigma}}) = \hat{\underset{\sim}{f}} \quad \text{on } B_s \qquad (2.0.10)$$

$$2\hat{\underset{\sim}{\varepsilon}} = \underset{\sim}{F}^{T}.\hat{\underset{\sim}{F}} + \hat{\underset{\sim}{F}}^{T}.\underset{\sim}{F} + \hat{\underset{\sim}{F}}^{T}.\hat{\underset{\sim}{F}} \quad \text{in } V$$

$$\hat{\underset{\sim}{u}} = \hat{\underset{\sim}{u}}^{*} \quad \text{on } B_k \qquad (2.0.11)$$

constitutive relations (2.0.12)

If we focus on infinitesimal neighbourhoods of reference state, we replace $(\hat{\cdot})$ as arbitrary difference-state by first variation $\delta(.)$, and we obtain a linearized form of differentialequations. This means that effects of higher order are neglected such that relations (2.0.13-14) are valid only in the infinitesimal neighbourhood of $(\ )_o$.

$$\begin{aligned} &Div\,(\delta \underset{\sim}{F}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\delta\underset{\sim}{\sigma}) + \delta\underset{\sim}{b} &&= 0 &&\text{in } V \\ &\underset{\sim}{n}.(\delta\underset{\sim}{F}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\delta\underset{\sim}{\sigma}) - \delta\underset{\sim}{f} &&= 0 &&\text{on } B_s \end{aligned} \qquad (2.0.13)$$

$$\begin{aligned} &\delta\underset{\sim}{\varepsilon} - (\underset{\sim}{F}^T.\delta\underset{\sim}{F})_s &&= 0 &&\text{in } V \\ &\delta\underset{\sim}{u} - \delta\underset{\sim}{u}^* &&= 0 &&\text{on } B_k \end{aligned} \qquad (2.0.14)$$

2.1. Elastic behaviour

Assuming the existence of a lower semicontinuous convex elastic strain energy density $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$, bounded from below, the second Piola-Kirchhoff stress tensor $\underset{\sim}{\sigma} \in S \subset R^6$ is an element of the subgradient of $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$

$$\underset{\sim}{\sigma} \in \partial\tilde{\psi}(\underset{\sim}{\varepsilon}^e) \qquad (2.1.1)$$

So polar energy density $\tilde{\psi}^*(\underset{\sim}{\sigma})$ is defined by

$$\tilde{\psi}^*(\underset{\sim}{\sigma}) = \sup_{\underset{\sim}{\varepsilon}^e} [\underset{\sim}{\sigma}..\underset{\sim}{\varepsilon}^e - \tilde{\psi}(\underset{\sim}{\varepsilon}^e)] \qquad (2.1.2)$$

if bilinear form $\underset{\sim}{\sigma}..\underset{\sim}{\varepsilon}^e$ is used which puts spaces $S \subset R^6$ and $E^e \subset R^6$ of stresses $\underset{\sim}{\sigma}$ and elastic strains $\underset{\sim}{\varepsilon}^e$ into duality. $\tilde{\psi}^*(\underset{\sim}{\sigma})$ is convex and bounded from below [33]. For differentiable $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$ the Fenchel-transformation (2.1.2) reduced to Legendre-transformation:

$$\tilde{\psi}(\underset{\sim}{\varepsilon}^e) + \tilde{\psi}^*(\underset{\sim}{\sigma}) = \underset{\sim}{\sigma}..\underset{\sim}{\varepsilon}^e \qquad (2.1.3)$$

Equivalently holds:

$$\underset{\sim}{\sigma} = \frac{\partial \tilde{\psi}(\underset{\sim}{\varepsilon}^e)}{\partial \underset{\sim}{\varepsilon}^e} \quad ; \qquad \underset{\sim}{\varepsilon}^e = \frac{\partial \tilde{\psi}^*(\underset{\sim}{\sigma})}{\partial \underset{\sim}{\sigma}} \tag{2.1.4}$$

Assuming invariance of $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$ with respect to change of coordinate system we define:

$$\psi(\underset{\sim}{d}^e) = \tilde{\psi}(\underset{\sim}{\varepsilon}^e) \tag{2.1.5}$$

The assumption of existence of elastic strain energy density $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$ independant from plastic deformations is very restrictive: If the Green strain $\underset{\sim}{\varepsilon}$ is described to be composed from elastic and plastic part of displacement-gradient $\underset{\sim}{d}$, we obtain:

$$\begin{aligned} 2\varepsilon_{MJ} = u_{M,J} + u_{J,M} + u_{k,J}\, u_{k,M} = d^e_{JM} + d^e_{MJ} + d^e_{Jk}\, d^e_{kM} + \\ d^p_{JM} + d^p_{MJ} + d^p_{Jk}\, d^p_{kM} + d^e_{Jk}\, d^p_{kM} + d^p_{Jk}\, d^e_{kM} \end{aligned} \tag{2.1.6}$$

Uncoupling of elastic and plastic part needs the additional assumption that either elastic or plastic deformations are comparatively small so that quadratic and coupled terms can be neglected. Whereas in metal forming for example the assumption of infinitesimal elastic and finite plastic displacements is usual, we assume that infinitesimal plastic deformations are superimposed on moderately finite elastic displacements. This assumption is specially reasonable if we consider thin-walled structures which may undergo finite elastic deformations before plastic deformation occurs.

Using bilinear form $\underset{\sim}{d}^e \cdot\cdot\, \underset{\sim}{t}$ we may now define polar elastic energy density $\psi^*(\underset{\sim}{t})$:

$$\psi^*(\underset{\sim}{t}) = \underset{\sim}{t} \cdot\cdot\, \underset{\sim}{d}^e - \psi(\underset{\sim}{d}^e) \tag{2.1.7}$$

According to (2.1.4) hold equivalently:

$$\underset{\sim}{t} = \frac{\partial \psi(\underset{\sim}{d}^e)}{\partial \underset{\sim}{d}^e} \quad ; \qquad \underset{\sim}{d}^e = \frac{\partial \psi^*(\underset{\sim}{t})}{\partial \underset{\sim}{t}} \tag{2.1.8}$$

Only for strictly convex functions polar function is identical with complementary function [33]. In our case from strict convexity of $\tilde{\psi}(\underset{\sim}{\varepsilon}^e)$ does not follow convexity of $\psi(\underset{\sim}{d}^e)$. According to [72] we shall assume strict convexity of $\psi(\underset{\sim}{d}^e)$ though all excluded cases cannot be determined à priori. Condition for strict convexity of $\psi(\underset{\sim}{d}^e)$ is that $\underset{\sim}{M}^{-1}$ is

$$\underset{\sim}{M}^{-1} = \frac{\partial^2 \psi(\underset{\sim}{d}^e)}{\partial \underset{\sim}{d}^e \, \partial \underset{\sim}{d}^e} \qquad\qquad M^{-1}_{ijkL} = \frac{\partial^2 \psi(\underset{\sim}{d}^e)}{\partial d^e_{ij} \, \partial d_{kL}} \tag{2.1.9}$$

Necessary and sufficient condition for hyperelastic behaviour is symmetry of $\underset{\sim}{M}^{-1}$ [68]: $M^{-1}_{ijkL} = M^{-1}_{kLij}$

2.2. Potentials of elastic rate quantities

Here we assume that elastic strain enery density $\psi(\underset{\sim}{d}^e)$ defined in (2.1) is strictly convex and differentiable up to order n, just like complementary energy-density $\psi^*(\underset{\sim}{t})$. We expand ψ and ψ^* into Taylor-series in the neighbourhood of reference state $(\;)_o$, for which all quantities determining the mechanical state of the considered body are assumed to be known. Here n denotes an arbitrary integer number.

$$\psi(\underset{\sim}{d}^e + \hat{\underset{\sim}{d}}^e) = \psi(\underset{\sim}{d}^e)/_o + \frac{\partial \psi}{\partial \underset{\sim}{d}^e}/_o \, \hat{\underset{\sim}{d}}^e + \frac{1}{2} \frac{\partial^2 \psi}{\partial \underset{\sim}{d}^e \partial \underset{\sim}{d}^e}/_o \, \hat{\underset{\sim}{d}}^e \hat{\underset{\sim}{d}}^e + \\ + \dots + \frac{1}{n!} \frac{\partial^n \psi}{(\partial \underset{\sim}{d}^e)^n}/_o \, (\hat{\underset{\sim}{d}}^e)^n + R_{n+1} \tag{2.2.1}$$

$$\psi^*(\underset{\sim}{t} + \hat{\underset{\sim}{t}}) = \psi^*(\underset{\sim}{t})/_o + \frac{\partial \psi^*}{\partial \underset{\sim}{t}}/_o \, \hat{\underset{\sim}{t}} + \frac{1}{2} \frac{\partial^2 \psi^*}{\partial \underset{\sim}{t} \, \partial \underset{\sim}{t}}/_o \, \hat{\underset{\sim}{t}} \, \hat{\underset{\sim}{t}} + \\ + \dots + \frac{1}{n!} \frac{\partial^n \psi^*}{(\partial \underset{\sim}{t})^n}/_o \, (\hat{\underset{\sim}{t}})^n + R^*_{n+1} \tag{2.2.2}$$

If we now introduce the quantities $\underset{\sim}{d}^e + \hat{\underset{\sim}{d}}^e$ and $\underset{\sim}{t} + \hat{\underset{\sim}{t}}$ from the neighbourhood to reference state $(\;)_o$ into the Legendre-transformation (2.1.7) we obtain:

$$\psi(\underset{\sim}{d}^e + \hat{\underset{\sim}{d}}^e) + \psi^*(\underset{\sim}{t} + \hat{\underset{\sim}{t}}) = \underset{\sim}{d}^e \cdot\cdot \underset{\sim}{t} + \underset{\sim}{d}^e \cdot\cdot \hat{\underset{\sim}{t}} + \hat{\underset{\sim}{d}}^e \cdot\cdot t + \hat{\underset{\sim}{d}}^e \cdot\cdot \hat{\underset{\sim}{t}} \tag{2.2.3}$$

Using Taylor-expansions (2.2.1-2) this becomes:

$$\psi(\underset{\sim}{d}^e)/_o + \psi^*(\underset{\sim}{t})/_o + \frac{\partial \psi}{\partial \underset{\sim}{d}^e}/_o \hat{\underset{\sim}{d}}^e + \frac{\partial \psi^*}{\partial \underset{\sim}{t}}/_o \hat{\underset{\sim}{t}} +$$

$$+ \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial \underset{\sim}{d}^e \partial \underset{\sim}{d}^e}/_o \hat{\underset{\sim}{d}}^e \hat{\underset{\sim}{d}}^e + \frac{\partial^2 \psi^*}{\partial \underset{\sim}{t}\, \partial \underset{\sim}{t}}/_o \hat{\underset{\sim}{t}}\, \hat{\underset{\sim}{t}} \right) + \cdots$$

$$+ \frac{1}{n!}\left(\frac{\partial^n \psi}{(\partial \underset{\sim}{d}^e)^n}/_o (\hat{\underset{\sim}{d}}^e)^n + \frac{\partial^n \psi^*}{(\partial \underset{\sim}{t})^n}/_o (\hat{\underset{\sim}{t}})^n \right) + \cdots$$

$$+ R_{n+1} + R^*_{n+1} \tag{2.2.4}$$

Then because of

$$\underset{\sim}{t}/_o = \frac{\partial \psi}{\partial \underset{\sim}{d}^e}/_o \; ; \qquad \underset{\sim}{d}^e/_o = \frac{\partial \psi^*}{\partial \underset{\sim}{t}}/_o \tag{2.2.5}$$

holds:

$$\frac{1}{2}\left(\frac{\partial^2 \psi}{\partial \underset{\sim}{d}^e \partial \underset{\sim}{d}^e}/_o \hat{\underset{\sim}{d}}^e \hat{\underset{\sim}{d}}^e + \frac{\partial^2 \psi^*}{\partial \underset{\sim}{t}\, \partial \underset{\sim}{t}}/_o \hat{\underset{\sim}{t}}\, \hat{\underset{\sim}{t}} \right) + \cdots$$

$$\frac{1}{n!}\left(\frac{\partial^n \psi}{(\partial \underset{\sim}{d}^e)^n}/_o (\hat{\underset{\sim}{d}}^e)^n + \frac{\partial^n \psi^*}{(\partial \underset{\sim}{t})^n}/_o (\hat{\underset{\sim}{t}})^n \right) +$$

$$R_{n+1} + R^*_{n+1} = \hat{\underset{\sim}{t}} \cdot\cdot \hat{\underset{\sim}{d}}^e \tag{2.2.6}$$

Neglecting in Taylor-expansion terms of order n > 2, we obtain the Legendre-transformation for the rate quantities $\hat{\underset{\sim}{t}}$ and $\hat{\underset{\sim}{d}}^e$:

$$\underbrace{\frac{1}{2} \frac{\partial^2 \psi}{\partial \underset{\sim}{d}^e \partial \underset{\sim}{d}^e}/_o \hat{\underset{\sim}{d}}^e \hat{\underset{\sim}{d}}^e}_{P_1(\hat{\underset{\sim}{d}}^e)} + \underbrace{\frac{1}{2} \frac{\partial^2 \psi^*}{\partial \underset{\sim}{t}\, \partial \underset{\sim}{t}}/_o \hat{\underset{\sim}{t}}\, \hat{\underset{\sim}{t}}}_{P_1^*(\hat{\underset{\sim}{t}})} = \hat{\underset{\sim}{t}} \cdot\cdot \hat{\underset{\sim}{d}}^e \tag{2.2.7}$$

$P_1(\hat{\underset{\sim}{d}}^e)$ is strictly convex because of assumption of strict convexity of $\psi(\underset{\sim}{d}^e)$. This implies that $P_1^*(\hat{\underset{\sim}{t}})$ is complementary to $P_1(\underset{\sim}{d}^e)$. We define

$$(\underset{\sim}{M}^o)^{-1} = \frac{\partial^2 \psi}{\partial \underset{\sim}{d}^e \partial \underset{\sim}{d}^e}\Big|_o \tag{2.2.8}$$

Then, equivalently to (2.1.8) holds:

$$\begin{aligned} \hat{\underset{\sim}{t}} &= \frac{\partial P_1(\hat{\underset{\sim}{d}}^e)}{\partial \hat{\underset{\sim}{d}}^e} = (\underset{\sim}{M}^o)^{-1} \cdot\cdot\, \hat{\underset{\sim}{d}}^e \\ \hat{\underset{\sim}{d}}^e &= \frac{\partial P_1^*(\hat{\underset{\sim}{t}})}{\partial \hat{\underset{\sim}{t}}} = \underset{\sim}{M}^o \cdot\cdot\, \hat{\underset{\sim}{t}} \end{aligned} \tag{2.2.9}$$

Inversion of $(\underset{\sim}{M}^o)^{-1}$ is possible if $(\underset{\sim}{M}^o)^{-1}$ is positive definite, i. e. $P_1(\hat{\underset{\sim}{d}}^e)$ is strictly convex. Spaces $D \in \hat{\underset{\sim}{d}}^e$ and $T \in \hat{\underset{\sim}{t}}$ are tangent spaces referring to reference state $(\ )_o$, dual to each other by bilinear form $\hat{\underset{\sim}{t}} \cdot\cdot\, \hat{\underset{\sim}{d}}^e$; $D \subset R^9$, $T \subset R^9$. We should emphasize that in $\underset{\sim}{M}^o$ not only material properties but also informations about the state of deformation of the reference state are included.

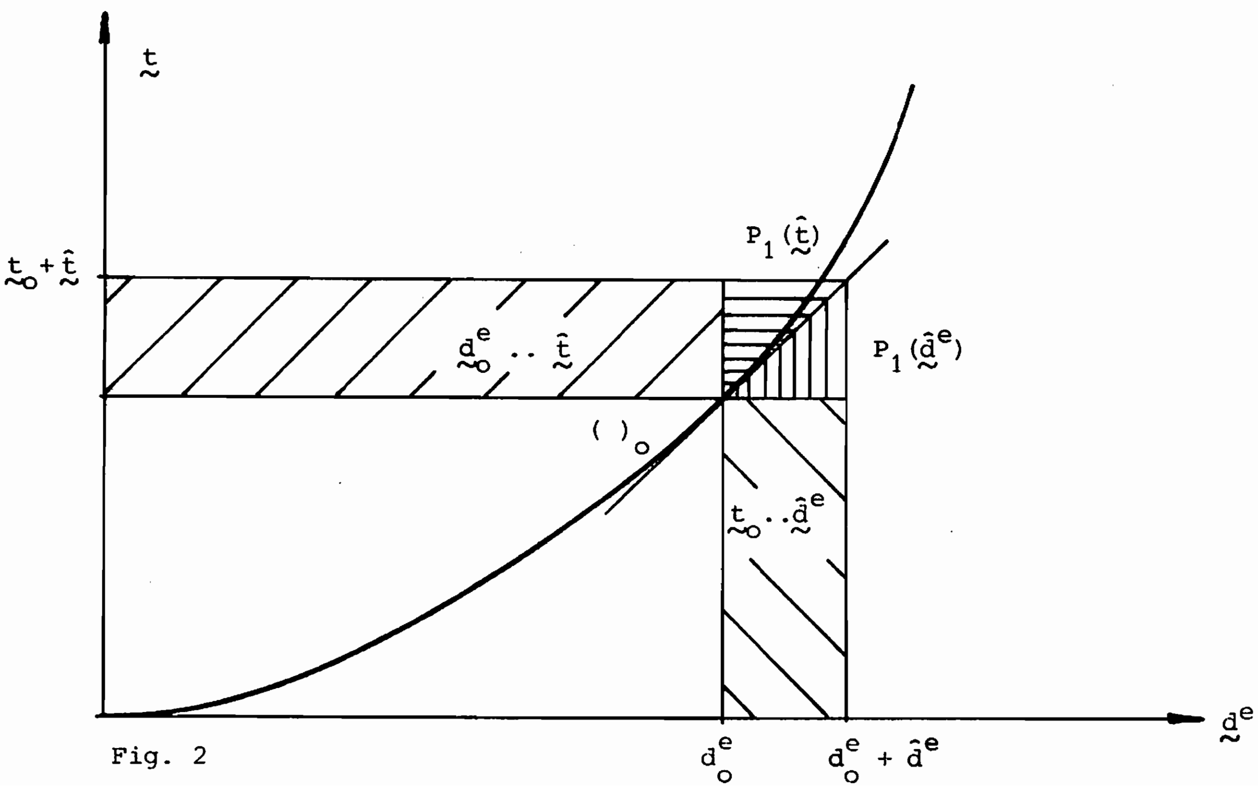


Fig. 2

As we are going to operate with nondifferentiable functions the introduction of notion of the subdifferential as an extension of notion of the differential is useful:
We consider $x \in X$ and $y \in Y$ as elements of dual spaces X and Y with the bilinear form $\langle x,y \rangle$ and $f(x)$ as a not necessarily differentiable convex function over X. Then holds equivalently: $f^*(y) = \sup_{x^* \in X}[\langle x^*,y \rangle - f(x^*)]$ and $y \in \partial f(x)$, $x \in \partial f^*(y)$ [33]. In our example subdifferential in (1) consists out of all elements $y \in A$, such that equivalently holds: $y_1 \in \partial f(x)|_1 \Leftrightarrow y_1 \in A$, whereas in (2) subdifferential consists of only one element identical with the differential at this point.

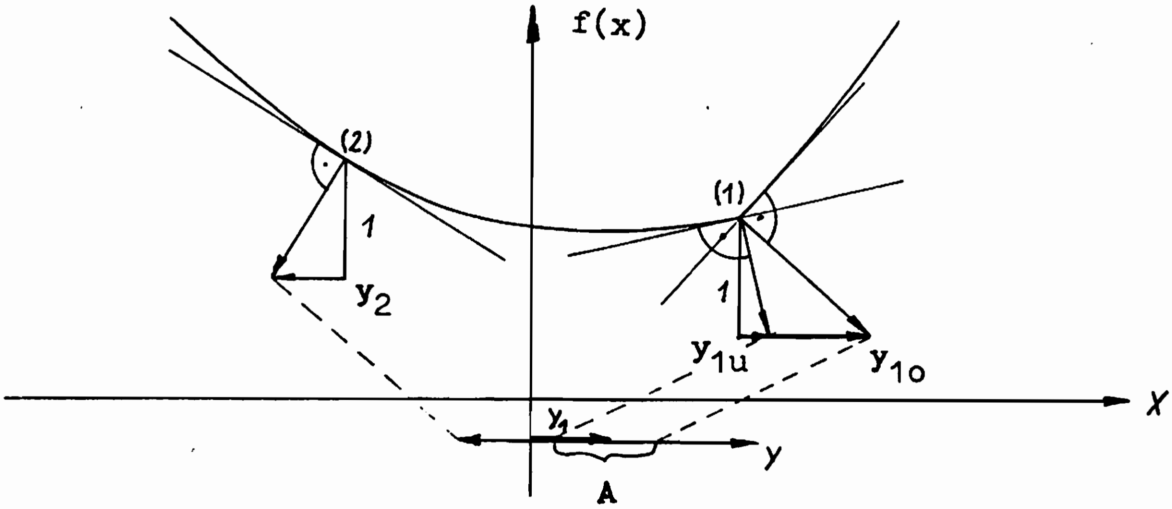


Fig. 3a

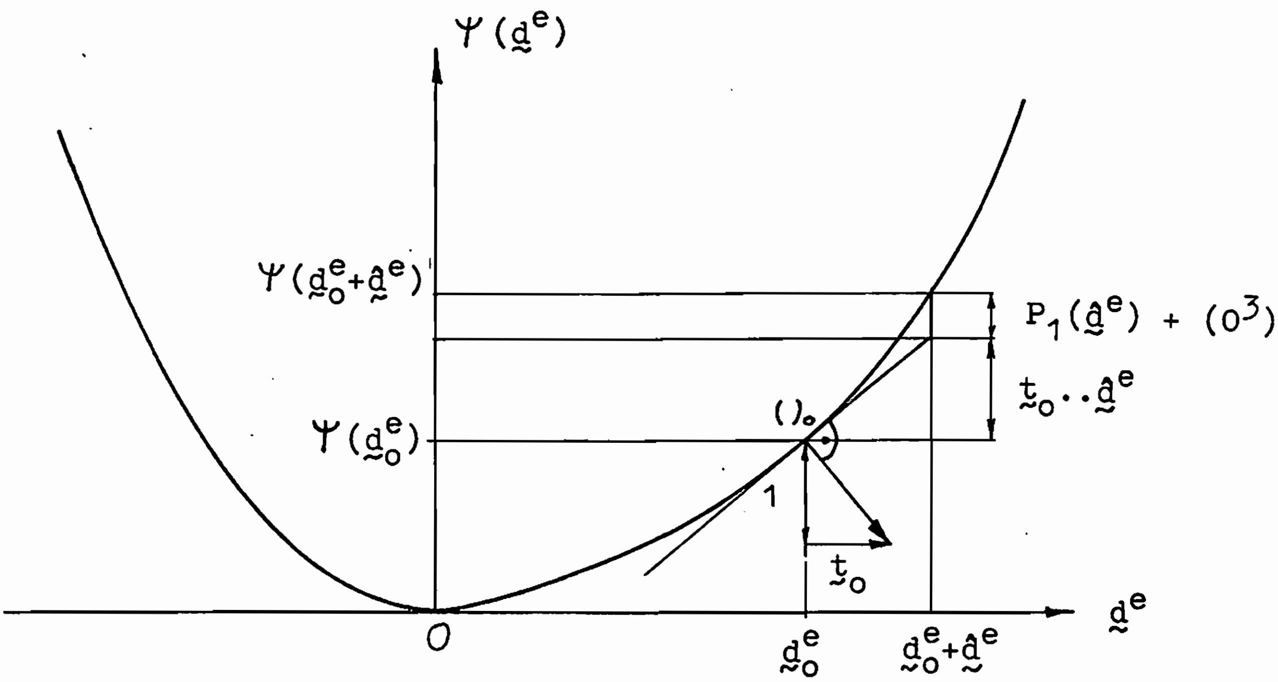


Fig.3b

Substitution of $\underset{\sim}{t}$ and $\underset{\sim}{d}^e$ by $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}^e$ in the infinitesimal neighbourhood of the reference state

For linear elastic material behaviour tensor of constant elastic coefficients $\underset{\sim}{L}$ defines for small strains the relation between $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}^e$ [14]. These coefficients can be determined by experience so that formulation of the rate problem by use of the second Piola-Kirchhoff stress tensor is interesting. In infinitesimal neighbourhood of reference state characterized by vanishing variations of order n > 1, we obtain the following expression instead of (2.2.7) using the relation $\delta(\underset{\sim}{F}\cdot\underset{\sim}{\sigma})\cdot\cdot\delta\underset{\sim}{d}^e =$ $= (\delta\underset{\sim}{F}\cdot\underset{\sim}{\sigma} + \underset{\sim}{F}\cdot\delta\underset{\sim}{\sigma})\cdot\cdot\delta\underset{\sim}{d}^e$:

$$(\delta\underset{\sim}{F}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\delta\underset{\sim}{\sigma})\cdot\cdot\delta\underset{\sim}{d}^e = \frac{1}{2}[\underset{\sim}{M}^o\cdot\cdot\delta(\underset{\sim}{F}.\underset{\sim}{\sigma})]\cdot\cdot\delta(\underset{\sim}{F}.\underset{\sim}{\sigma})$$

$$+ \frac{1}{2}((\underset{\sim}{M}^o)^{-1}\cdot\cdot\delta\underset{\sim}{d}^e)\cdot\cdot\delta\underset{\sim}{d}^e \tag{2.2.10}$$

Appendix A1 shows that

$$\frac{1}{2}[\underset{\sim}{M}^o\cdot\cdot\delta(\underset{\sim}{F}.\underset{\sim}{\sigma})]\cdot\cdot\delta(\underset{\sim}{F}.\underset{\sim}{\sigma}) = \frac{1}{2}[(\underset{\sim}{M}^o.\underset{\sim}{F}.\underset{\sim}{F})\cdot\cdot\delta\underset{\sim}{\sigma}]\cdot\cdot\delta\underset{\sim}{\sigma}$$

$$+\underset{\sim}{\sigma}.\delta\underset{\sim}{F}.\delta\underset{\sim}{d}^e + (\underset{\sim}{\alpha}\cdot\cdot\delta\underset{\sim}{\sigma})\cdot\cdot\delta\underset{\sim}{\sigma} \tag{2.2.11}$$

with $\underset{\sim}{\alpha} = 0$ for linear elastic material behaviour. With the definitions:

$$(\underset{\sim}{\tilde{L}}^o)^{-1} = [(\underset{\sim}{M}^o)^{-1}.\underset{\sim}{F}^{-1}].\underset{\sim}{F}^{-1}; \quad \underset{\sim}{L}^o = (\underset{\sim}{M}^o.\underset{\sim}{F}).\underset{\sim}{F}$$

$$\delta\underset{\sim}{\tilde{\varepsilon}}^e = \underset{\sim}{F}.\delta\underset{\sim}{d}^e \quad \Rightarrow \quad \delta\underset{\sim}{d}^e = \underset{\sim}{F}^{-1}.\delta\underset{\sim}{\tilde{\varepsilon}}^e \tag{2.2.12}$$

we obtain

$$\delta\underset{\sim}{\sigma}\cdot\cdot\delta\underset{\sim}{\varepsilon}^e + \underset{\sim}{\sigma}.\delta\underset{\sim}{F}.\delta\underset{\sim}{d}^e = \frac{1}{2}(\underset{\sim}{\tilde{L}}^o\cdot\cdot\delta\underset{\sim}{\sigma})\cdot\cdot\delta\underset{\sim}{\sigma} +$$

$$\frac{1}{2}((\underset{\sim}{\tilde{L}}^o)^{-1}\cdot\cdot\delta\underset{\sim}{\tilde{\varepsilon}}^e)\cdot\cdot\delta\underset{\sim}{\tilde{\varepsilon}}^e + \underset{\sim}{\sigma}.\delta\underset{\sim}{F}.\delta\underset{\sim}{d}^e \tag{2.2.13}$$

after cancelling

$$\delta\underset{\sim}{\sigma}\,..\,\delta\tilde{\underset{\sim}{\varepsilon}}^{e} = \frac{1}{2}(\tilde{\underset{\sim}{L}}^{o}\,..\,\delta\underset{\sim}{\sigma})\,..\,\delta\underset{\sim}{\sigma} + \frac{1}{2}((\tilde{\underset{\sim}{L}}^{o})^{-1}\,..\,\delta\tilde{\underset{\sim}{\varepsilon}}^{e})\,..\,\delta\tilde{\underset{\sim}{\varepsilon}}^{e} \qquad (2.2.14)$$

Because of symmetry of $\delta\underset{\sim}{\sigma}$, $\delta\tilde{\underset{\sim}{\varepsilon}}^{e}$ and $\tilde{\underset{\sim}{L}}^{o}$ in the first two terms may be replaced by their symmetric parts:

$$\begin{aligned} \delta\varepsilon^{e} &= \frac{1}{2}(\underset{\sim}{F}^{T}\cdot\delta\underset{\sim}{\alpha}^{e} + \delta\underset{\sim}{\alpha}^{e\cdot T}\cdot\underset{\sim}{F}) \\ L^{o}_{MJNL} &= L^{o}_{JMNL} = L^{o}_{MJLN} \end{aligned} \qquad (2.2.15)$$

We remember that from assumption of hyperelastic behaviour of elastic part of deformation, symmetry $L^{o}_{MJNL} = L^{o}_{NLMJ}$ is assured because of $M^{o}_{iJkL} = M^{o}_{kLiJ}$. Considering in the third term of (2.2.14) only the symmetric part L^{o-1}_{MJNL} of tensor $\tilde{L}^{o-1}_{MJNL}$ we obtain the Legendre transformation

$$\delta\underset{\sim}{\sigma}\,..\,\delta\underset{\sim}{\varepsilon}^{e} = \frac{1}{2}\tilde{P}_{1}(\delta\underset{\sim}{\varepsilon}^{e}) + \frac{1}{2}\tilde{P}^{*}_{1}(\delta\underset{\sim}{\sigma}) \qquad (2.2.16)$$

with

$$\begin{aligned} \tilde{P}_{1}(\delta\underset{\sim}{\varepsilon}^{e}) &= \frac{1}{2}((\underset{\sim}{L}^{o})^{-1}\,..\,\delta\underset{\sim}{\varepsilon}^{e})\,..\,\delta\underset{\sim}{\varepsilon}^{e} \\ \tilde{P}^{*}_{1}(\delta\underset{\sim}{\sigma}) &= \frac{1}{2}(\underset{\sim}{L}^{o}\,..\,\delta\underset{\sim}{\sigma})\,..\,\delta\underset{\sim}{\sigma} \end{aligned} \qquad (2.2.17)$$

(2.2.16) represents the rotation-invariant material law for linear elastic behaviour for infinitesimal changes in the neighbourhood of reference state. Herein condition of equilibrium of moments is in advance assured by assumption of symmetry of $\underset{\sim}{\sigma}$ and $\delta\underset{\sim}{\sigma}$ if destributed moments in the volume vanish. Equivalently to (2.2.16) holds:

$$\begin{aligned} \delta\underset{\sim}{\varepsilon}^{e} &= \underset{\sim}{L}^{o}\,..\,\delta\underset{\sim}{\sigma} = \frac{\partial\tilde{P}^{*}_{1}(\delta\underset{\sim}{\sigma})}{\partial(\delta\underset{\sim}{\sigma})} \\ \delta\underset{\sim}{\sigma} &= (\underset{\sim}{L}^{o})^{-1}\,..\,\delta\varepsilon^{e} = \frac{\partial\tilde{P}_{1}(\delta\underset{\sim}{\varepsilon}^{e})}{\partial(\delta\underset{\sim}{\varepsilon}^{e})} \end{aligned} \qquad (2.2.18)$$

Because of positive definiteness of $\underset{\sim}{L}^{o}$ this transformation is of unrestricted validity. Then $\tilde{P}_1^*$ is complementary to $\tilde{P}_1$ over the entire range of definition of $\delta\underset{\sim}{\sigma}$. From (2.2.11-17) follows:

$$P_1^*(d\underset{\sim}{t}) - \tilde{P}_1^*(d\underset{\sim}{\sigma}) = \underset{\sim}{\sigma} \cdot d\underset{\sim}{F} \cdot d\underset{\sim}{d}^{e} \qquad (2.2.19)$$

2.3. Plastic behaviour

Assuming the existence of an elastic region C describing the set of all admissible stress states in space T of stresses $\underset{\sim}{t}$ of an infinitesimal subregion dV of the considered body. This constraint on the set of admissible stress states induced by assumption of a well defined yield-limit of material is not independant of deformation of the body if we use the first Piola stress tensor for description; in the definition enters the change of geometry from initial up to actual configuration. More adequate to local material behaviour is description of admissible stress states by using the second Piola-Kirchhoff stress tensor $\underset{\sim}{\sigma}$ or the Cauchy stress tensor. The herein chosen description, however, is more convenient to formulate and solve the boundary value problem because all quantities either for the rate or the initial boundary value problems are entirely referred to the initial configuration.

Starting from an equilibrium configuration we assume that additional stresses produce always positive work on a closed cycle of loading and unloading if plastic deformations occur and that for purely elastic behaviour this work is equal to zero. For infintesimal deformations this assumption is known as "Drucker's postulate" and had been used in [86] for finite deformations.

$$\int_{\tau_o}^{\tau_1}\int_{(V)} (\underset{\sim}{t} - \underset{\sim}{t}^{o}) \cdot\cdot \dot{\underset{\sim}{d}} \; dV d\tau \geq 0 \qquad (2.3.1)$$

with τ_o and τ_1 as time limits of the cycle, $(\dot{\ })$ as derivation with respect to time $\frac{d}{d\tau}$ and $(.)_o$ as indicator for initial state. Non hardening may be called any material for which region C remains constant during the entire history of deformation. In fig. 4 hardening is characterized by transition of C to C'.

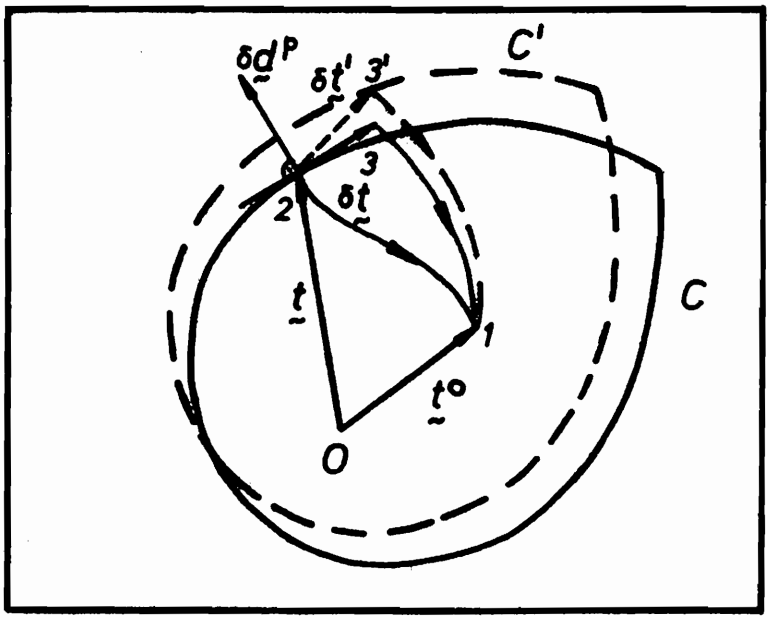


Fig. 4

Convexity of C and validity of normality rule are obtained in the following as necessary but not sufficient condition for this global stability-criterion [26]: In (2.3.1) part of the elastic work on the considered closed cycle is equal to zero because of assumption of strict convexity of $\psi(\underset{\sim}{d}^e)$:

$$\int_{\tau_0}^{\tau_1} \int_{(V)} (\underset{\sim}{t} - \underset{\sim}{t}^o) \cdot\cdot \dot{\underset{\sim}{d}}^e \, dV d\tau = 0 \tag{2.3.2}$$

Thus (2.3.1) becomes:

$$\int_{\tau_0}^{\tau_1} \int_{(V)} (\underset{\sim}{t} - \underset{\sim}{t}^o) \cdot\cdot \dot{\underset{\sim}{d}}^p \, dV d\tau \geq 0 \tag{2.3.3}$$

Sufficient condition for (2.3.3) is non-negativity of

$$(\underset{\sim}{t} - \underset{\sim}{t}^o) \cdot\cdot \dot{\underset{\sim}{d}}^p \, d\tau \geq 0 \tag{2.3.4}$$

If we expand (2.3.4) in the neighbourhood of an arbitrary stress state $\underset{\sim}{t}_B \in C$ into a Taylor series, then we obtain:

$$\begin{aligned} &(\underset{\sim}{t} - \underset{\sim}{t}^o)/_{\underset{\sim}{t}_B} \cdot\cdot \dot{\underset{\sim}{d}}^p d\tau + \dot{\underset{\sim}{t}}/_{\underset{\sim}{t}_B} \cdot\cdot \dot{\underset{\sim}{d}}^p (d\tau)^2 + \\ &(\underset{\sim}{t} - \underset{\sim}{t}^o)/_{\underset{\sim}{t}_B} \cdot\cdot \ddot{\underset{\sim}{d}}^p (d\tau)^2 + \ddot{\underset{\sim}{t}}/_{t_B} \cdot\cdot \dot{\underset{\sim}{d}}^p \frac{(d\tau)^3}{2} + \cdots \geq 0 \end{aligned} \qquad (2.3.5)$$

As we consider only quasistatic processes all terms of order $n > 2$ are equal to zero and we obtain for arbitrary $\underset{\sim}{t}_B$ conditions of convexity and normality as sufficient conditions for (2.3.3).

$$\begin{aligned} (\underset{\sim}{t} - \underset{\sim}{t}^o) \cdot\cdot \dot{\underset{\sim}{d}}^p d\tau &= (\underset{\sim}{t} - \underset{\sim}{t}^o) \cdot\cdot \delta \underset{\sim}{d}^p \geq 0 \\ \dot{\underset{\sim}{t}} \cdot\cdot \dot{\underset{\sim}{d}}^p (d\tau)^2 &= \delta \underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{d}^p \geq 0 \end{aligned} \qquad (2.3.6)$$

The influence of change of geometry during deformation in using $\underset{\sim}{t}$ has to be checked in every considered problem because material behaviour is usually described by the second Piola-Kirchhoff or by the Cauchy stress tensor. Analogously to the theory of elasticity where assumption of convexity of elastic strain energy in terms of displacement-gradient constrains validity of complementary variational principles [72], we shall exclude those cases where loss of convexity of C is caused by geometrical effects. In appendix A2 we investigate how geometrical effects enter in the case of von Kârmân plate theory.

Using this concept we can define a plastic potential $\varphi(\underset{\sim}{t})$ analogously to [36-42]:

$$\varphi(\underset{\sim}{t}) = \begin{cases} 0 & \text{if } \underset{\sim}{t} \in C \\ +\infty & \text{if } \underset{\sim}{t} \notin C \end{cases} \qquad (2.3.7)$$

From convexity of C follows immediately convexity of $\varphi(\underset{\sim}{t})$. $\varphi(\underset{\sim}{t})$ attains its minimum in the origin of T as C contains the origin. Using bilinear form $\underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{d}^p$ to put T and $\hat{D}^p$ into duality we may construct polar plastic potential $\varphi^*(\delta \underset{\sim}{d}^p)$ by Fenchel-transformation:

$$\varphi^*(\delta \underset{\sim}{d}^p) = \sup_{t \in T} [\underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{d}^p - \varphi(\underset{\sim}{t})] \tag{2.3.8}$$

Here $\hat{D}^p$ is the space of all rates of plastic deformations $\delta \underset{\sim}{d}^p$, tangent to reference state $(\ )_o$. Then equivalently to (2.3.8) holds:

$$\underset{\sim}{t} \in \partial \varphi^*(\delta \underset{\sim}{d}^p) \quad ; \quad \delta \underset{\sim}{d}^p \in \partial \varphi(\underset{\sim}{t}) \tag{2.3.9}$$

Taking

$$\underset{\sim}{\sigma} = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{t} \quad ; \quad \delta \underset{\sim}{\varepsilon}^p = (\underset{\sim}{F} \cdot \delta \underset{\sim}{d}^p)_s \tag{2.3.10}$$

into consideration we obtain:

$$\underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{d}^p = \underset{\sim}{\sigma} \cdot\cdot \delta \underset{\sim}{\varepsilon}^p \tag{2.3.11}$$

With $\underset{\sim}{t} = \underset{\sim}{F} \cdot \underset{\sim}{\sigma}$ and $\underset{\sim}{t}^o = \underset{\sim}{F}^o \cdot \underset{\sim}{\sigma}^o$ follows:

$$(\underset{\sim}{F} \cdot \underset{\sim}{\sigma} - \underset{\sim}{F}^o \cdot \underset{\sim}{\sigma}^o) \cdot\cdot \delta \underset{\sim}{d}^p \geq 0 \tag{2.3.12}$$

Only for $\underset{\sim}{F} - \underset{\sim}{F}^o = 0 + (0^2)$ follows immediately:

$$(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^o) \cdot\cdot (\underset{\sim}{F} \cdot\cdot \delta \underset{\sim}{d}^p) = (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^o) \cdot\cdot \delta \underset{\sim}{\varepsilon}^p \geq 0 \tag{2.3.13}$$

what expresses convexity of elastic region $\tilde{C} \subset S \ni \underset{\sim}{\sigma}$, $S \subset R^6$. Also normality rule does not hold automatically in terms of $\underset{\sim}{\sigma} \in S$: With

$$\delta \underset{\sim}{t} = (\underset{\sim}{F} \cdot \delta \underset{\sim}{\sigma} + \delta \underset{\sim}{F} \cdot \underset{\sim}{\sigma}) \tag{2.3.14}$$

follows

$$\begin{aligned} \delta \underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{d}^p &= (\underset{\sim}{F} \cdot \delta \underset{\sim}{\sigma} + \delta \underset{\sim}{F} \cdot \underset{\sim}{\sigma}) \cdot\cdot \delta \underset{\sim}{d}^p = \\ &\delta \underset{\sim}{\sigma} \cdot\cdot \delta \underset{\sim}{\varepsilon}^p + \underset{\sim}{\sigma} \cdot \delta \underset{\sim}{F} \cdot \delta \underset{\sim}{d}^p \geq 0 \end{aligned} \tag{2.3.15}$$

Questions of convexity and validity of normality rule using either $\underset{\sim}{t}$ or $\underset{\sim}{\sigma}$ are investigated for von Kármán plate theory in appendix A2.

If we now use bilinear form $\underset{\sim}{\sigma} .. \delta\underset{\sim}{\varepsilon}^p$ to put spaces S and $\hat{E}^p$ into duality, where $\underset{\sim}{\sigma} \in S$ and $\delta\underset{\sim}{\varepsilon}^p \in \hat{E}^p$, we can define polar plastic potential $\tilde{\varphi}^*(\delta\underset{\sim}{\varepsilon}^p)$:

$$\tilde{\varphi}^*(\delta\underset{\sim}{\varepsilon}^p) = \sup_{\sigma \in S} [\underset{\sim}{\sigma} .. \delta\underset{\sim}{\varepsilon}^p - \tilde{\varphi}(\underset{\sim}{\sigma})] \qquad (2.3.16)$$

with

$$\tilde{\varphi}(\underset{\sim}{\sigma}) = \begin{cases} 0 & \text{if } \underset{\sim}{\sigma} \in \tilde{C} \\ +\infty & \text{if } \underset{\sim}{\sigma} \notin \tilde{C} \end{cases} \qquad (2.3.17)$$

where $\tilde{C} \subset S$ expresses elastic region in terms of the second Piola-Kirchhoff stress measure $\underset{\sim}{\sigma}$.

2.4. Potentials of plastic rate quantities

We define indicator-function $P_2(\delta\underset{\sim}{d}^p)$ of subdifferential $\delta\varphi(t)$:

$$P_2(\delta\underset{\sim}{d}^p) = \begin{cases} 0 & \text{if } \delta\underset{\sim}{d}^p \in \partial\varphi(\underset{\sim}{t}) \\ +\infty & \text{if } \delta\underset{\sim}{d}^p \notin \partial\varphi(\underset{\sim}{t}) \end{cases} \qquad (2.4.1)$$

$P_2(\delta\underset{\sim}{d}^p)$ is convex and bounded from below under all assumptions made in chapter (2.3). Using the definition

$$\delta\underset{\sim}{\tilde{\varepsilon}}^p = \underset{\sim}{F} \cdot \delta\underset{\sim}{d}^p \qquad (2.4.2)$$

analogously to (2.2.12) we obtain equivalently:

$$\tilde{P}_2(\delta\underset{\sim}{\tilde{\varepsilon}}^p(\delta\underset{\sim}{d}^p)) = P_2(\delta\underset{\sim}{d}^p) \qquad (2.4.3)$$

Using bilinear forms $\delta\underset{\sim}{t} .. \delta\underset{\sim}{d}^p$ and $\delta\underset{\sim}{\sigma} .. \delta\underset{\sim}{\dot{\varepsilon}}^p$ we may determine those functions which are conjugate to $P_2(\delta\underset{\sim}{d}^p)$ and $\tilde{P}_2(\delta\underset{\sim}{\tilde{\varepsilon}}^p)$ by the Fenchel-transformation:

$$\begin{aligned} P_2^*(\delta\underset{\sim}{t}) &= \sup_{\delta\underset{\sim}{d}^p} [\delta\underset{\sim}{t} .. \delta\underset{\sim}{d}^p - P_2(\delta\underset{\sim}{d}^p)] \\ \tilde{P}_2^*(\delta\underset{\sim}{\sigma}) &= \sup_{\delta\tilde{\varepsilon}^p} [\delta\underset{\sim}{\sigma} .. \delta\underset{\sim}{\varepsilon}^p - \tilde{P}_2(\delta\underset{\sim}{\tilde{\varepsilon}}^p)] \end{aligned} \qquad (2.4.4)$$

From the definition of the Fenchel-transformation follows, that this is equivalent to:

$$\delta\underset{\sim}{t} \in \partial P_2(\delta\underset{\sim}{d}^p) \; ; \qquad \delta\underset{\sim}{d}^p \in \partial P_2^*(\delta\underset{\sim}{t})$$

$$\delta\underset{\sim}{\sigma} \in \partial \tilde{P}_2(\delta\tilde{\underset{\sim}{\varepsilon}}^p) \; ; \qquad \delta\tilde{\underset{\sim}{\varepsilon}}^p \in \partial \tilde{P}_2^*(\delta\underset{\sim}{\sigma}) \qquad (2.4.5)$$

If we compare (2.4.4a) with (2.4.4b), we obtain by using (2.3.14) the following expression:

$$P_2^*(\delta\underset{\sim}{t}) = \sup_{\delta d^p} [(\delta\underset{\sim}{F}.\underset{\sim}{\sigma} + \underset{\sim}{F}.\delta\underset{\sim}{\sigma})..\delta\underset{\sim}{d}^p - P_2(\delta\underset{\sim}{d}^p)] \leq$$

$$\leq \sup_{\delta\underset{\sim}{\varepsilon}^p} [\delta\underset{\sim}{\sigma}..\delta\underset{\sim}{\varepsilon}^p - P_2(\delta\underset{\sim}{\varepsilon}^p)] + \sup[\underset{\sim}{\sigma}.\delta\underset{\sim}{F}.\delta\underset{\sim}{d}^p] =$$

$$\doteq \tilde{P}_2^*(\delta\underset{\sim}{\sigma}) + \sup_{\delta\underset{\sim}{d}^p} [\underset{\sim}{\sigma}.\delta\underset{\sim}{F}.\delta d^p]$$

(2.4.6)

The difference

$$P_2^*(\delta\underset{\sim}{t}) - \tilde{P}_2^*(\delta\underset{\sim}{\sigma}) \leq \sup_{\delta\underset{\sim}{d}^p} [\sigma.\delta F.\delta\underset{\sim}{d}^p] \qquad (2.4.7)$$

corresponds to (2.2.19), where we compared the potentials of elastic rate quantities.

2.5. Formulation of the elasto - plastic rate boundary value problem

Assuming that all quantities which describe the mechanical state of the considered body are given at time τ_o, the following relations describe the rate boundary value problem for prescribed changes $\delta\underset{\sim}{b} \in V$, $\delta\underset{\sim}{u}^* \in B_k$, $\delta\underset{\sim}{f} \in B_s$:

$$\begin{aligned} Div\,\delta\underset{\sim}{t} + \delta\underset{\sim}{b} &= 0 \qquad \text{in } V \\ n.\delta\underset{\sim}{t} - \delta\underset{\sim}{f} &= 0 \qquad \text{on } B_s \end{aligned} \qquad (2.5.1)$$

$$\delta \underset{\sim}{d} = \mathrm{Grad}\, \delta \underset{\sim}{u} = \delta \underset{\sim}{F} \qquad \text{in V}$$

$$\delta \underset{\sim}{u} - \delta \underset{\sim}{u}^* = 0 \qquad \text{on } B_k \tag{2.5.2}$$

$$\delta \underset{\sim}{d} - \delta \underset{\sim}{d}^e - \delta \underset{\sim}{d}^p = 0 \tag{2.5.3}$$

$$\delta \underset{\sim}{d}^e - \frac{\partial P_1^*(\delta \underset{\sim}{t})}{\partial(\delta \underset{\sim}{t})} = 0 \tag{2.5.4}$$

$$\delta \underset{\sim}{d}^p \in \partial P_2^*(\delta \underset{\sim}{t}) \qquad \text{in V}$$

$$\delta \underset{\sim}{t} - \frac{\partial P_1(\delta \underset{\sim}{d}^e)}{\partial(\delta \underset{\sim}{d}^e)} = 0 \tag{2.5.5}$$

$$\delta \underset{\sim}{t} \in \partial P_2(\delta \underset{\sim}{d}^p)$$

If potentials $\tilde{P}_1(\delta \underset{\sim}{\varepsilon}^e)$, $\tilde{P}_1^*(\delta \underset{\sim}{\sigma})$, $\tilde{P}_2(\delta \underset{\sim}{\varepsilon}^p)$, $\tilde{P}_2^*(\delta \underset{\sim}{\sigma})$ are used, (2.4.1 - 2.4.5) are substituted by:

$$\mathrm{Div}\, \delta(\underset{\sim}{F} . \underset{\sim}{\sigma}) + \delta \underset{\sim}{b} \qquad \text{in V}$$

$$\underset{\sim}{n} . \delta(\underset{\sim}{F} . \underset{\sim}{\sigma}) - \delta \underset{\sim}{f} \qquad \text{on } B_s \tag{2.5.6}$$

$$\delta \underset{\sim}{\varepsilon} - \delta(\underset{\sim}{F}^T . \underset{\sim}{F}) \qquad \text{in V}$$

$$\delta \underset{\sim}{u} - \delta \underset{\sim}{u}^* \qquad \text{on } B_k \tag{2.5.7}$$

$$\delta \underset{\sim}{\varepsilon} - \delta \underset{\sim}{\varepsilon}^e - \delta \underset{\sim}{\varepsilon}^p \qquad \text{in V} \tag{2.5.8}$$

$$\delta \underset{\sim}{\varepsilon}^e - \frac{\partial \tilde{P}_1^*(\delta \underset{\sim}{\sigma})}{\partial(\delta \underset{\sim}{\sigma})}$$

$$\delta \underset{\sim}{\varepsilon}^p \in \partial \tilde{P}_2^*(\delta \underset{\sim}{\sigma}) \tag{2.5.9}$$

$$\delta \underset{\sim}{\sigma} - \frac{\partial \tilde{P}_1(\delta \underset{\sim}{\varepsilon}^e)}{\partial(\delta \underset{\sim}{\varepsilon}^e)} \qquad \text{in V}$$

$$\delta \underset{\sim}{\sigma} \in \partial \tilde{P}_2(\delta \underset{\sim}{\varepsilon}^p) \tag{2.5.10}$$

2.6. Solutions of the rate boundary value problem

Rate boundary value problem 1

Let us assume that reference state $(\ )_o$ of an elasto-plastic body K is given by all quantities which determine the mechanical state of the body. Moreover the reaction of an associated body K^o to prescribed external agencies $\delta \underset{\sim}{A} = (\delta \underset{\sim}{b}, \delta \underset{\sim}{f}, \delta \underset{\sim}{u}^*)$ is assumed to be known, where K^o differs from K only by the fact of purely elastic reaction to external agencies $\delta \underset{\sim}{A}$.

Definitions:

Kinematically admissible stress-rate $\delta \underset{\sim}{t}^{\mu}(\underset{\sim}{x})$ is every field $\delta \underset{\sim}{t}$ for which holds:

$$\begin{aligned} \underset{\sim}{M}_o \cdot\cdot\, \delta \underset{\sim}{t} &= \text{Grad } \delta \underset{\sim}{u} && \text{in V} \\ \delta \underset{\sim}{u} &= 0 && \text{on } B_k \end{aligned} \qquad (2.6.1)$$

Statically admissible stress-rate $\delta \underset{\sim}{t}^{\rho}(\underset{\sim}{x})$ is every field $\delta \underset{\sim}{t}$ for which holds:

$$\begin{aligned} \text{Div } \delta \underset{\sim}{t} &= 0 && \text{in V} \\ \underset{\sim}{n} \cdot \delta \underset{\sim}{t} &= 0 && \text{on } B_s \end{aligned} \qquad (2.6.2)$$

By partial integration and application of Green's formula orthogonality of spaces $T^{\mu} \subset T$, $\delta t^{\mu} \in T^{\mu}$ and $T^{\rho} \subset T$, $\delta t^{\rho} \in T^{\rho}$ with respect to scalar-product

$$(\underset{\sim}{\tau}^1, \underset{\sim}{\tau}^2)_{\underset{\sim}{M}_o} = \int_{(V)} \underset{\sim}{\tau}^1 \cdot\cdot\, \underset{\sim}{M}_o \cdot\cdot\, \underset{\sim}{\tau}^2 \, dV \qquad (2.6.3)$$

can be proved (appendix A3).

Elastic solution for associated body K^o

We assume that the solution $\delta \underset{\sim}{t}^o$ of the associated purely elastic boundary value problem is given with:

$$
\begin{aligned}
&Div\ \delta \underset{\sim}{t}^{o} + \delta \underset{\sim}{b} = 0 && \text{in V} \\
&\underset{\sim}{n} \cdot \delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{f} = 0 && \text{on } B_s \\
&\underset{\sim}{M}_o \cdot\cdot \delta \underset{\sim}{t}^{o} = Grad\ \delta \underset{\sim}{u}^{o} && \text{in V} \\
&\delta \underset{\sim}{u}^{o} - \delta \underset{\sim}{u}^{*} = 0 && \text{on } B_k
\end{aligned}
\qquad (2.6.4)
$$

$\delta \underset{\sim}{t}^{o}$ represents all external agencies $\delta \underset{\sim}{A}$.

Solution of boundary value problem 1

We use space C_9^{∞} of all smooth and bounded tensor fields , provided with scalarproduct $(\underset{\sim}{\tau}^1, \underset{\sim}{\tau}^2)_{\underset{\sim}{M}_o}$ as unitary space. Completion of this space with respect to the scalarproduct is Hilbert space H, provided with L^2-norm:

$$
\| \underset{\sim}{\tau} \|_{\underset{\sim}{M}_o} = \frac{1}{2} \int_{(V)} \underset{\sim}{\tau} \cdot\cdot \underset{\sim}{M}_o \cdot\cdot \underset{\sim}{\tau}\ dV \qquad (2.6.5)
$$

Subspaces $H^{\mu} \subset H$ of kinematically admissible fields $\delta \underset{\sim}{t}^{\mu}$ and $H^{\rho} \subset H$ of statically admissible fields $\delta \underset{\sim}{t}^{\rho}$ are constructed by completion of the unitary subspaces of kinematically and statically admissible fields with respect to scalarproduct (2.6.3). From (A3) follows the orthogonality of H^{μ} and H^{ρ}.

Plastic rate potential $P_2(\delta \underset{\sim}{d}^{p})$ and polar potential $P_2^{*}(\delta \underset{\sim}{t})$ had been defined in (2.4) for tensors $\delta \underset{\sim}{d}^{p} \in R^9$. These definitions have to be extended to elements of Hilbert space H if we want to solve boundary value problem 1. With

$$
\Phi(\underset{\sim}{t}) = \lim_{\alpha \to \infty} \int_{(V)} \varphi_{\alpha}(\underset{\sim}{t}(\underset{\sim}{x}))\ dV \ ; \qquad +\infty > \alpha > 0
$$

$$
P_2(\delta \underset{\sim}{d}^{p}) = \begin{cases} 0 & \text{if } \underset{\sim}{M}^{o} \cdot\cdot \delta \underset{\sim}{d}^{p} \in \partial \Phi(\underset{\sim}{t}) \\ +\infty & \text{if } \underset{\sim}{M}^{o} \cdot\cdot \delta \underset{\sim}{d}^{p} \notin \partial \Phi(\underset{\sim}{t}) \end{cases} \qquad (2.6.6)
$$

$$
\varphi_{\alpha}(\underset{\sim}{t}) = \begin{cases} 0 & \text{if} \quad \underset{\sim}{t} \in C \\ \alpha & \text{if} \quad \underset{\sim}{t} \notin C \end{cases}
$$

the problem can be formulated in the following way: Determine the tensor fields $\delta \underset{\sim}{t}^{\rho} \in H^{\rho}$ and $\delta \underset{\sim}{t}^{\mu} \in H^{\mu}$ such that:

$$\delta \underset{\sim}{t}^{\rho} + \delta \underset{\sim}{t}^{o} \in \partial P_2^{*}(\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}) \qquad (2.6.7)$$

for given $\delta \underset{\sim}{t}^{o}$. Equivalent condition for the solution $\delta \underset{\sim}{t}^{\rho} \in H^{\rho}$ and $\delta \underset{\sim}{t}^{\mu} \in H^{\mu}$ is:

$$P_2^{*}(\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}) - (\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}, \delta \underset{\sim}{t}^{\rho} + \delta \underset{\sim}{t}^{\mu})_{\underset{\sim}{M}^{o}} +$$

$$P_2(\delta \underset{\sim}{t}^{\mu} + \delta \underset{\sim}{t}^{\rho}) = 0 \qquad (2.6.8)$$

with

$$P_2^{*}(\delta \underset{\sim}{t}) = \sup_{\delta \underset{\sim}{d}^{\rho}} [\langle \delta \underset{\sim}{t}, \delta \underset{\sim}{d}^{\rho} \rangle - P_2(\delta \underset{\sim}{d}^{\rho})] \qquad (2.6.9)$$

using bilinear form

$$\langle \delta \underset{\sim}{t}, \delta \underset{\sim}{d}^{\rho} \rangle = \int_{(V)} \delta \underset{\sim}{t}(\underset{\sim}{x}) \cdot\cdot\, \delta \underset{\sim}{d}^{\rho}(\underset{\sim}{x})\, dV \qquad (2.6.10)$$

Analogously to [41] we introduce the functional

$$\mathcal{L}_1(\delta \underset{\sim}{t}^{\rho}, \delta \underset{\sim}{t}^{\mu}) = P_2^{*}(\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}) - (\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}, \delta \underset{\sim}{t}^{\rho} + \delta \underset{\sim}{t}^{\mu})_{\underset{\sim}{M}^{o}}$$

$$+ P_2(\delta \underset{\sim}{t}^{\rho} + \delta \underset{\sim}{t}^{\mu}) \qquad (2.6.11)$$

From the convexity of P_2, P_2^{*}, $(\delta \underset{\sim}{t}^{\rho}, \delta \underset{\sim}{t}^{\rho})_{\underset{\sim}{M}^{o}}$ follows convexity of L_1. From (2.6.8) follows that L_1 attains the value zero for the solution of the problem. Uniqueness of solution however is not assured as L_1 is not strictly convex.

If we resign from determination of $\delta \underset{\sim}{t}^{\mu}$, that means that plastic part of displacement gradient $\delta \underset{\sim}{d}^{p}$ cannot be evaluated, then we may define the functional

$$\mathcal{L}_{10}(\delta \underset{\sim}{t}^{\rho}) = P_{20}^{*}(\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}) - (\delta \underset{\sim}{t}^{o} - \delta \underset{\sim}{t}^{\rho}, \delta \underset{\sim}{t}^{\rho})_{\underset{\sim}{M}^{o}}$$

$$+ P_2(\delta \underset{\sim}{t}^{\rho}) \qquad (2.6.12)$$

with

$$P_{20}^{*}(\underset{\sim}{\tau}) = \begin{cases} P_2^{*}(\underset{\sim}{\tau}) & \text{if } \underset{\sim}{\tau} - \delta \underset{\sim}{t}^{o} \in \mathcal{H}^{\rho} \\ +\infty & \text{if } \underset{\sim}{\tau} - \delta \underset{\sim}{t}^{o} \notin \mathcal{H}^{\rho} \end{cases} \tag{2.6.13}$$

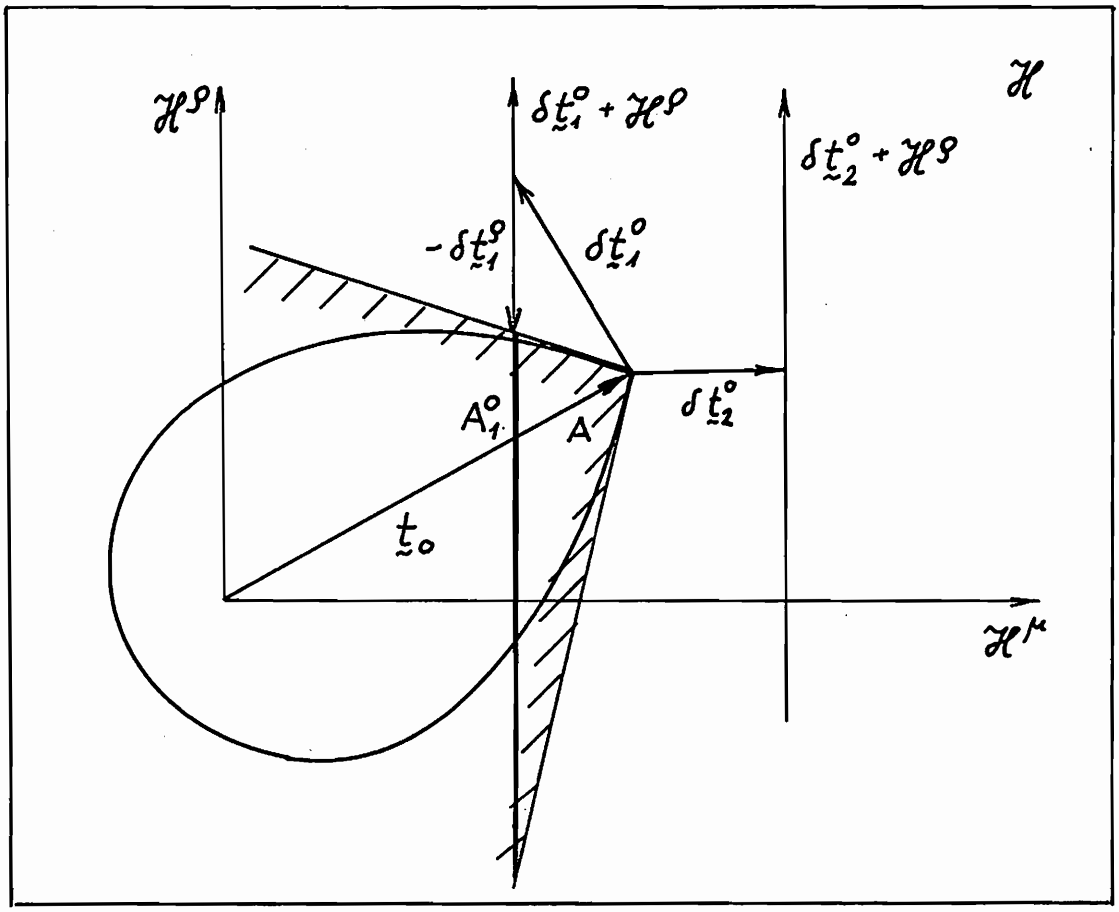


Fig. 5

In appendix A4 is proved that L_{10} is strictly convex, attains the minimum equal to zero for the solution $\delta \underset{\sim}{t}_1^{\rho} \in H^{\rho}$ and that $L_{10} \leq L_1$. So uniqueness of solution follows from strict convexity of L_{10}, existence of solution is assured if there exist $\delta \underset{\sim}{t}^{\rho} \in H^{\rho}$ so that $P_{20} < +\infty$.

For $(\delta \underset{\sim}{t}_1^o + H^{\rho}) \cap A = A_1^o \neq \emptyset$ exists a unique solution $\delta \underset{\sim}{t}_1^{\rho}$.

For $(\delta \underset{\sim}{t}_2^o + H^{\rho}) \cap A = A_2^o = \emptyset$ exists no solution.

Rate boundary value problem 2

If we use second Piola-Kirchhoff stress tensor $\underset{\sim}{\sigma}$ instead of first Piola stress tensor $\underset{\sim}{t}$ in order to solve the problem given in (2.6), from the relation

$$\delta \underset{\sim}{t} = \delta \underset{\sim}{F} \cdot \underset{\sim}{\sigma} + \underset{\sim}{F} \cdot \delta \underset{\sim}{\sigma} \tag{2.6.14}$$

the difficulty arises that not only stress tensor $\underset{\sim}{\sigma}$ and its rate $\delta\underset{\sim}{\sigma}$ but also $\underset{\sim}{F}$ and $\delta\underset{\sim}{F}$ enter conditions of equilibrium such that notion of statically admissible stress field in itself loses its sense. Therefore in this place we shall consider only the special case that after a finite deformation incremental change of deformation remains small enough to be neglected in equilibrium conditions. This procedure corresponds to the "theory of second order" in civil engineering [84].
Kinematically admissible stress-rates $\delta\underset{\sim}{\sigma}^{\mu}$ are those fields $\delta\underset{\sim}{\sigma} \in S$ for which holds

$$\begin{aligned} \underset{\sim}{L} \cdot\cdot \delta \underset{\sim}{\sigma} - (\underset{\sim}{F}^T \cdot \delta \underset{\sim}{F})_s &= 0 && \text{in } V \\ \delta \underset{\sim}{u} &= 0 && \text{on } B_k \end{aligned} \tag{2.6.15}$$

statically admissible stress-rate $\delta\underset{\sim}{\sigma}^{\rho}$ are those fields $\delta\underset{\sim}{\sigma} \in S$ for which holds:

$$\begin{aligned} \operatorname{Div} \underset{\sim}{F}_o \cdot \delta \underset{\sim}{\sigma} &= 0 && \text{in } V \\ \underset{\sim}{n} \cdot (\underset{\sim}{F}_o \cdot \delta \underset{\sim}{\sigma}) &= 0 && \text{on } B_s \end{aligned} \tag{2.6.16}$$

Analogously to (A3) orthogonality of $S^{\mu} \subset S$, $S^{\mu} \ni \delta\sigma^{\mu}$ and $S^{\rho} \subset S$, $S^{\rho} \ni \delta\underset{\sim}{\sigma}^{\rho}$ with respect to scalarproduct

$$(\delta \underset{\sim}{\sigma}^{\rho}, \delta \sigma^{\mu})_{\underset{\sim}{L}} = \int_{(V)} \delta \underset{\sim}{\sigma}^{\rho} \cdot\cdot \underset{\sim}{L} \cdot\cdot \delta \underset{\sim}{\sigma}^{\mu} \, dV \tag{2.6.17}$$

can be shown: (appendix A5)

Similarly to (1.5) we assume that purely elastic solution of an associated problem is given, satisfying the following relations:

$$Div\,(\underset{\sim}{F}_o \cdot \delta \underset{\sim}{\sigma}^o) + \delta \underset{\sim}{b} = 0 \qquad \text{in V}$$

$$\underset{\sim}{n} \cdot (\underset{\sim}{F}_o \cdot \delta \underset{\sim}{\sigma}^o) - \delta \underset{\sim}{f} = 0 \qquad \text{on } B_s \tag{2.6.18}$$

$$\delta \underset{\sim}{\varepsilon}^o - (\underset{\sim}{F}_o^T \cdot \delta \underset{\sim}{F})_s = 0 \qquad \text{in V}$$

$$\delta \underset{\sim}{u}^o - \delta \underset{\sim}{u}^* = 0 \qquad \text{on } B_k \tag{2.6.19}$$

$$\delta \underset{\sim}{\varepsilon}^o - \underset{\sim}{L} \cdot\cdot \delta \underset{\sim}{\sigma}^o = 0 \qquad \text{in V} \tag{2.6.20}$$

The rate boundary value problem 2 can then be formulated in the following way: Determine those rates of stresses $\delta\underset{\sim}{\sigma}$ and strains $\delta\underset{\sim}{\varepsilon}$ for given stress and strain states $\underset{\sim}{\sigma}_o$ and $\underset{\sim}{\varepsilon}_o$ of reference configuration $(\ )_o$ and known $\delta\underset{\sim}{\sigma}^o$ which satisfy the following conditions:

$$\begin{array}{ll} \delta \underset{\sim}{\sigma}^o - \delta \underset{\sim}{\sigma} & \text{is statically admissible} \\ \underset{\sim}{L}^{-1} \cdot\cdot \delta \underset{\sim}{\varepsilon} - \delta \underset{\sim}{\sigma}^o & \text{is kinematically admissible} \\ \left.\begin{array}{l} \delta \underset{\sim}{\varepsilon}^p = \delta \underset{\sim}{\varepsilon} - \underset{\sim}{L} \cdot\cdot \delta \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma},\ \delta \underset{\sim}{\sigma} \end{array}\right\} & \begin{array}{l} \text{satisfy normality rule and} \\ \text{condition of plastic admissi-} \\ \text{ble (yield-condition)} \end{array} \end{array} \tag{2.6.21}$$

Solution of boundary value problem 2

Analogously to (2.6.5) we use space C_6^∞, provided with scalarproduct (2.6.17) as unitary space, which defines by completion Hilbert space $\tilde{H}$ provided with L^2-norm:

$$\| \underset{\sim}{\tau} \|_{\underset{\sim}{L}}^2 = \int_{(V)} \underset{\sim}{\tau} \cdot\cdot \underset{\sim}{L} \cdot\cdot \underset{\sim}{\tau}\, dV \tag{2.6.22}$$

Subspaces $\tilde{H}^\mu \subset \tilde{H}$ and $\tilde{H}^\rho \subset \tilde{H}$ of kinematically and statically admissible fields $\delta\underset{\sim}{\sigma}^\mu$ and $\delta\underset{\sim}{\sigma}^\rho$ respectively are determined by completion of unitary spaces of smooth kinematically and statically admissible fields analogously to construction of H^μ and H^ρ. From (A5) follows orthogonality of spaces $\tilde{H}^\mu$ and $\tilde{H}^\rho$.

Plastic behaviour is now described by functionals $\tilde{P}_2(\delta\tilde{\underset{\sim}{\varepsilon}}^p)$ and $\tilde{P}_2^*(\delta\underset{\sim}{\sigma})$

$$\tilde{P}_2(\delta\tilde{\underset{\sim}{\varepsilon}}^p) = \begin{cases} 0 & \text{if } \delta\tilde{\underset{\sim}{\varepsilon}}^p \in \partial\tilde{\Phi}(\underset{\sim}{\sigma}) \\ +\infty & \text{if } \delta\tilde{\underset{\sim}{\varepsilon}}^p \notin \partial\tilde{\Phi}(\underset{\sim}{\sigma}) \end{cases} \tag{2.6.23}$$

$$\tilde{P}_2^*(\delta\underset{\sim}{\sigma}) = \sup_{\delta\tilde{\underset{\sim}{\varepsilon}}^p} \left[(\delta\underset{\sim}{\sigma} \cdot\cdot\, \delta\tilde{\underset{\sim}{\varepsilon}}^p) - \tilde{P}_2(\delta\tilde{\underset{\sim}{\varepsilon}}^p) \right] \tag{2.6.24}$$

with

$$\tilde{\Phi}(\underset{\sim}{\sigma}) = \lim_{\alpha\to\infty} \int_{(V)} \tilde{\varphi}_\alpha(\underset{\sim}{\sigma}(\underset{\sim}{x}))\, dV$$

$$\tilde{\varphi}_\alpha(\underset{\sim}{\sigma}) = \begin{cases} 0 & \text{if } \underset{\sim}{\sigma} \in \tilde{C} \\ \alpha & \text{if } \underset{\sim}{\sigma} \notin \tilde{C}\,, \qquad 0 < \alpha < +\infty \end{cases} \tag{2.6.25}$$

Analogously to (2.6.8) the solution of the rate boundary value problem 2 has to satisfy:

$$\delta\underset{\sim}{\sigma}^p + \delta\underset{\sim}{\sigma}^r \in \partial\tilde{P}_2^*(\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^p) \tag{2.6.26}$$

or, equivalently

$$\tilde{P}_2^*(\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^p) - (\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^p, \delta\underset{\sim}{\sigma}^p + \delta\underset{\sim}{\sigma}^r)_{\underset{\sim}{L}} +$$

$$\tilde{P}_2(\delta\underset{\sim}{\sigma}^p + \delta\underset{\sim}{\sigma}^r) = 0 \tag{2.6.27}$$

where $\delta\underset{\sim}{\sigma}^o$ is prescribed and $\delta\underset{\sim}{\sigma}^\mu$, $\delta\underset{\sim}{\sigma}^\rho$ are unkown. For von Kármán plate theory in appendix A2 the question has been investigated under which conditions from convexity of elastic region $C \subset T$ convexity of elastic region $\tilde{C} \subset S$ follows. In (2.3.6) normality of rate of plastic deformation $\delta\underset{\sim}{d}^p$ to yield surface was expressed; by transformations (2.3.11-13) is shown, that (2.6.13) does not violate normality rule.

Solution is constructed in the same way as in (2.6.11-13):
First we define a functional

$$\mathcal{L}_2(\delta\underset{\sim}{\sigma}^\rho, \delta\underset{\sim}{\sigma}^\mu) = \tilde{P}_2^*(\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^\rho) - (\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^\rho, \delta\underset{\sim}{\sigma}^\rho + \delta\underset{\sim}{\sigma}^\mu)_{\underset{\sim}{L}} + \tilde{P}_2(\delta\underset{\sim}{\sigma}^\rho + \delta\underset{\sim}{\sigma}^\mu) \qquad (2.6.28)$$

L_2 is convex and attains value zero for the solution $(\delta\underset{\sim}{\sigma}^\rho, \delta\underset{\sim}{\sigma}^\mu)$. However, L_2 is not strictly convex such that solution is not unique. At loss of any statement about rate of plastic deformation we define a new functional L_{20}, strictly convex (appendix A6) with $L_{20} \underset{\sim}{\leq} L_2$, $L_{20} = 0$ for the solution $\delta\underset{\sim}{\sigma}^\rho$. If any $\delta\underset{\sim}{\sigma}^\rho \in \tilde{H}^\rho$ exists such that $\tilde{P}^*_{20} < +\infty$ then existence and uniqueness of solution $\delta\underset{\sim}{\sigma}^\rho$ is assured.

$$\mathcal{L}_{20}(\delta\underset{\sim}{\sigma}^\rho) = \tilde{P}_{20}^*(\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^\rho) - (\delta\underset{\sim}{\sigma}^o - \delta\underset{\sim}{\sigma}^\rho, \delta\underset{\sim}{\sigma}^\rho)_{\underset{\sim}{L}} + \tilde{P}_2(\delta\underset{\sim}{\sigma}^\rho)$$

$$\tilde{P}_{20}^*(\underset{\sim}{\tau}) = \begin{cases} \tilde{P}_2^*(\underset{\sim}{\tau}) & \text{if } \underset{\sim}{\tau} \in \delta\underset{\sim}{\sigma}^o + \tilde{\mathcal{H}}^\rho \\ +\infty & \text{if } \underset{\sim}{\tau} \notin \delta\underset{\sim}{\sigma}^o + \tilde{\mathcal{H}}^\rho \end{cases} \qquad (2.6.29)$$

Rate boundary value problem 3

Basing on Hill's works, e.g. [45,46] a multitude of stationarity and extremum principles have been formulated [55-67], which are the foundation of broadly applied incremental methods in numerical calculations. In this paper we restrict our consideration to Mises-isotropic hardening material behaviour [1], where rate of plastic strain can be evaluated as gradient of a potential, though above mentioned principles are formulated, e.g. by Maier [61], for more general material behaviour. Again we assume that mechanical state of reference configuration $(\ )_o$ just as incremental changes of external agencies $\delta\underset{\sim}{A}$ are given. Then transition from reference state to actual state may be described by the following set of relations:

$$\begin{aligned} &Div(\underset{\sim}{F}\cdot\delta\underset{\sim}{\sigma} + \delta\underset{\sim}{F}\cdot\underset{\sim}{\sigma}) + \delta\underset{\sim}{b} = 0 && \text{in } V \\ &\underset{\sim}{n}\cdot(\underset{\sim}{F}\cdot\delta\underset{\sim}{\sigma} + \delta\underset{\sim}{F}\cdot\underset{\sim}{\sigma}) - \delta\underset{\sim}{f} = 0 && \text{on } B_s \end{aligned} \qquad (2.6.30)$$

$$\left.\begin{array}{ll} \delta \underset{\sim}{\varepsilon} - (\underset{\sim}{F}^T \cdot \delta \underset{\sim}{F})_s & = 0 \\ \delta \underset{\sim}{\varepsilon} - \delta \underset{\sim}{\varepsilon}^e - \delta \underset{\sim}{\varepsilon}^p & = 0 \\ \delta \underset{\sim}{\varepsilon}^e - (\underset{\sim}{F}^T \cdot \delta \underset{\sim}{d}^e)_s & = 0 \\ \delta \underset{\sim}{\varepsilon}^p - (\underset{\sim}{F}^T \cdot \delta \underset{\sim}{d}^p)_s & = 0 \end{array}\right\} \text{ in V} \qquad (2.6.31)$$

$$\delta \underset{\sim}{u} - \delta \underset{\sim}{u}^* = 0 \qquad \text{on } B_k \qquad (2.6.32)$$

$$\left.\begin{array}{ll} \delta \underset{\sim}{\varepsilon}^e - \dfrac{\partial \tilde{P}_1^*(\delta \underset{\sim}{\sigma})}{\partial(\delta \underset{\sim}{\sigma})} & = 0 \\ \delta \underset{\sim}{\varepsilon}^p - \dfrac{\partial \tilde{P}_2^*(\delta \underset{\sim}{\sigma})}{\partial(\delta \underset{\sim}{\sigma})} & = 0 \\ \delta \underset{\sim}{\sigma} - \dfrac{\partial[\tilde{P}_1(\delta \underset{\sim}{\varepsilon}) + \tilde{P}_2(\delta \underset{\sim}{\varepsilon})]}{\partial(\delta \underset{\sim}{\varepsilon})} & = 0 \end{array}\right\} \text{ in V} \qquad (2.6.33)$$

In [53,54] a method of systematical construction of stationarity and minimum principles for the rate boundary value problem basing on introduction of two linear spaces E and S with the elements

$$\begin{array}{ll} \underset{\sim}{\tau} = (\delta \underset{\sim}{t} \text{ in } V, \delta \underset{\sim}{t} \text{ on } B_s, \delta \underset{\sim}{t} \text{ on } B_k); & \delta \underset{\sim}{t} \in R^9 \\ \underset{\sim}{v} = (\delta \underset{\sim}{u} \text{ in } V, \delta \underset{\sim}{u} \text{ on } B_s, \delta \underset{\sim}{u} \text{ on } B_k); & \delta \underset{\sim}{u} \in R^3 \end{array} \qquad (2.6.34)$$

is given. E and S are provided with scalarproducts

$$\langle \delta \underset{\sim}{t}^1, \delta \underset{\sim}{t}^2 \rangle = \delta \underset{\sim}{t}^1 \cdot\cdot \delta \underset{\sim}{t}^2 dV + \delta \underset{\sim}{t}^1 \cdot\cdot \delta \underset{\sim}{t}^2 dS + \delta \underset{\sim}{t}^1 \cdot\cdot \delta \underset{\sim}{t}^2 dS$$

$$(\delta \underset{\sim}{u}^1, \delta \underset{\sim}{u}^2) = \delta \underset{\sim}{u}^1 \cdot \delta \underset{\sim}{u}^2 dV + \delta \underset{\sim}{u}^1 \cdot \delta \underset{\sim}{u}^2 dS + \delta \underset{\sim}{u}^1 \cdot \delta \underset{\sim}{u}^2 dS$$

(2.6.35)

and define by completion Hilbert spaces H_E and H_S with norms induced by (2.6.35). By use of adjoint operators T and T*, H_E and H_S are mapped onto each other.

$$T\delta \underset{\sim}{t} = \begin{pmatrix} -\mathrm{Div}\,\delta \underset{\sim}{t} \\ 0 \\ \underset{\sim}{n}\cdot\delta \underset{\sim}{t} \end{pmatrix} \; ; \quad T^*\delta \underset{\sim}{u} = \begin{pmatrix} \mathrm{Grad}\,\delta \underset{\sim}{u} \\ -\underset{\sim}{n}\cdot\delta \underset{\sim}{u} \\ 0 \end{pmatrix} \quad \begin{matrix} \text{in } V \\ \text{on } B_k \\ \text{on } B_s \end{matrix} \tag{2.6.36}$$

Adjointness of T and T* follows from

$$(\delta \underset{\sim}{t}, T^*\delta \underset{\sim}{u}) = \langle \delta \underset{\sim}{u}, T\delta \underset{\sim}{t} \rangle \tag{2.6.37}$$

Basing on the definition of a generating functional a Lagrangean functional $L(\delta \underset{\sim}{t}, \delta \underset{\sim}{\sigma}, \delta \underset{\sim}{u}, \delta \underset{\sim}{d})$

$$\begin{aligned} L = & \int_{(V)} \Big[\delta \underset{\sim}{t} \cdot\cdot \,\mathrm{Grad}\,\delta \underset{\sim}{u} - (\delta \underset{\sim}{t} - \underset{\sim}{F}\cdot\delta \underset{\sim}{\sigma})\cdot\cdot\,\delta \underset{\sim}{d} - \tilde{Q}^*(\delta \underset{\sim}{\sigma}) \\ & + \frac{1}{2}\, \underset{\sim}{\sigma}\cdot\delta \underset{\sim}{d}\cdot\delta \underset{\sim}{d} + \delta \underset{\sim}{b}\cdot\delta \underset{\sim}{u} \Big]\, dV + \\ & \int_{(B_k)} \underset{\sim}{n}\cdot\delta \underset{\sim}{t}\cdot(\delta \underset{\sim}{u} - \delta \underset{\sim}{u}^*)\, dS - \int_{(B_s)} \delta \underset{\sim}{f}\cdot\delta \underset{\sim}{u}\, dS \end{aligned} \tag{2.6.38}$$

is defined, which assumes a stationary value for the solution of boundary value problem 3. For details we refer to [54]. Starting from (2.6.38) a couple of variational principles may be derived systematically. If e.g. we restrict the set of $\delta \underset{\sim}{d}$ on those elements satisfying compatibility condition

$$\delta \underset{\sim}{d} = \mathrm{Grad}\,\delta \underset{\sim}{u} = \delta \underset{\sim}{F} \tag{2.6.39}$$

we obtain by use of

$$\delta \underset{\sim}{t} = \delta \underset{\sim}{F}\cdot \underset{\sim}{\sigma} + \underset{\sim}{F}\cdot\delta \underset{\sim}{\sigma} \tag{2.6.40}$$

from (2.6.38)

$$\tilde{J}_1(\delta \underset{\sim}{S}, \delta \underset{\sim}{u}) = \int_{(V)} [\delta \underset{\sim}{S} \cdot\cdot (\underset{\sim}{F}^T \cdot \delta \underset{\sim}{F})_s - \tilde{Q}^*(\delta \underset{\sim}{S}) + \frac{1}{2} \underset{\sim}{S} \cdot \delta \underset{\sim}{F} \cdot \delta \underset{\sim}{F}$$

$$+ \delta \underset{\sim}{b} \cdot \delta \underset{\sim}{u}]\, dV + \int_{(B_k)} (\delta \underset{\sim}{F} \cdot \underset{\sim}{S}) \cdot \underset{\sim}{n} \cdot (\delta \underset{\sim}{u}^* - \delta \underset{\sim}{u})\, dS$$

$$- \int_{(B_s)} \delta \underset{\sim}{f} \cdot \delta \underset{\sim}{u}\, dS \qquad (2.6.41)$$

With

$$\tilde{Q}^*(\delta \underset{\sim}{S}) = \tilde{P}_1^*(\delta \underset{\sim}{S}) + \tilde{P}_2^*(\delta \underset{\sim}{S}) = Q^*(\delta \underset{\sim}{t}) + \frac{1}{2} \underset{\sim}{S} \cdot \delta \underset{\sim}{F} \cdot \delta \underset{\sim}{F} \qquad (2.6.42)$$

(2.6.38) becomes

$$J_1(\delta \underset{\sim}{t}, \delta \underset{\sim}{u}) = \int_{(V)} [\delta \underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{F} - Q^*(\delta \underset{\sim}{t}) + \delta \underset{\sim}{b} \cdot \delta \underset{\sim}{u}]\, dV$$

$$+ \int_{(B_k)} \delta \underset{\sim}{t} \cdot \underset{\sim}{n} \cdot (\delta \underset{\sim}{u}^* - \delta \underset{\sim}{u})\, dS - \int_{(B_s)} \delta \underset{\sim}{f} \cdot \delta \underset{\sim}{u}\, dS \qquad (2.6.43)$$

with the Legendre-transformation

$$Q^*(\delta \underset{\sim}{t}) + Q(\delta \underset{\sim}{F}) = \delta \underset{\sim}{t} \cdot\cdot \delta \underset{\sim}{F} \qquad (2.6.44)$$

and the restriction to kinematically admissible functions $\delta \underset{\sim}{u} = \delta \underset{\sim}{u}^*$ on B_k we get from (2.6.38):

$$J_2(\delta \underset{\sim}{u}) = \int_{(V)} [Q(\delta \underset{\sim}{F}) + \delta \underset{\sim}{b} \cdot \delta \underset{\sim}{u}]\, dV - \int_{(B_s)} \delta \underset{\sim}{f} \cdot \delta \underset{\sim}{u}\, dS \qquad (2.6.45)$$

Restriction to statically admissible function in the sense of

$$\mathrm{Div}\, \delta \underset{\sim}{t} + \delta \underset{\sim}{b} = 0 \quad ; \quad \underset{\sim}{n} \cdot \delta \underset{\sim}{t} - \delta \underset{\sim}{f} = 0 \qquad (2.6.46)$$

leads to

$$J_3(\delta \underset{\sim}{t}) = -\int_{(V)} Q^*(\delta \underset{\sim}{t})\, dV + \int_{(B_k)} \delta \underset{\sim}{t} \cdot \underset{\sim}{n} \cdot \delta \underset{\sim}{u}^*\, dS \qquad (2.6.47)$$

In order to obtain a physical interpretation of (2.6.45) and (2.6.47) we transform $J_2(\delta \underset{\sim}{u})$ and $J_3(\delta \underset{\sim}{t})$ by use of generalized stress- and strain measures:

$$\underset{\sim}{e} = \begin{pmatrix} \delta \underset{\sim}{\varepsilon} \\ \delta \underset{\sim}{\omega} \end{pmatrix} \quad ; \qquad \underset{\sim}{s} = \begin{pmatrix} \delta \underset{\sim}{\sigma} \\ \delta \underset{\sim}{\pi} \end{pmatrix} \tag{2.6.48}$$

With the definitions $\delta \underset{\sim}{\omega} = \delta \underset{\sim}{F}$; $\delta \underset{\sim}{\pi} = \underset{\sim}{\sigma} \cdot \delta \underset{\sim}{\omega}$ we obtain:

$$\begin{aligned} \underset{\sim}{\sigma} \cdot\cdot (\delta \underset{\sim}{\omega} \cdot \delta \underset{\sim}{\omega}) = \sigma_{IJ} \delta \omega_{Ik} \delta \omega_{kJ} &= \sigma_{IJ}^{-1} \delta \pi_{Ik} \delta \pi_{kJ} \\ &= \sigma_{IJ} \delta F_{Ik} \delta F_{kJ} \end{aligned} \tag{2.6.49}$$

$$\underset{\sim}{s} \cdot\cdot \underset{\sim}{e} = W(\underset{\sim}{e}) + W^*(\underset{\sim}{s}) \tag{2.6.50}$$

$$W(\underset{\sim}{e}) = \frac{1}{2} \underset{\sim}{e} \cdot\cdot \underset{\sim}{\mathcal{L}}^{-1} \cdot\cdot \underset{\sim}{e} = \frac{1}{2} \delta \varepsilon_{IJ} L_{IJKL}^{-1} \delta \varepsilon_{KL} + \frac{1}{2} \sigma_{IJ} \delta \omega_{Ik} \delta \omega_{kJ}$$

$$W^*(\underset{\sim}{s}) = \frac{1}{2} \underset{\sim}{s} \cdot\cdot \underset{\sim}{\mathcal{L}} \cdot\cdot \underset{\sim}{s} = \frac{1}{2} \delta \sigma_{IJ} L_{IJKL} \delta \sigma_{KL} + \frac{1}{2} \sigma_{IJ}^{-1} \delta \pi_{Ik} \delta \pi_{kJ} \tag{2.6.51}$$

$$\underset{\sim}{\mathcal{L}} = \begin{pmatrix} L_{IJKL} & 0 \\ 0 & \sigma_{IJ}^{-1} \end{pmatrix} \tag{2.6.52}$$

With these definitions we obtain finally

$$\begin{aligned} J_2(\underset{\sim}{e}, \delta \underset{\sim}{u}) &= \int_{(V)} [W(\underset{\sim}{e}) + \delta \underset{\sim}{b} \cdot \delta \underset{\sim}{u}]\, dV - \int_{(B_s)} \delta \underset{\sim}{f} \cdot \delta \underset{\sim}{u}\, dS \\ J_3(\underset{\sim}{s}, \delta \underset{\sim}{t}) &= -\int_{(V)} W^*(\underset{\sim}{s})\, dV + \int_{(B_K)} \delta \underset{\sim}{t} \cdot \underset{\sim}{n} \cdot \delta \underset{\sim}{u}^*\, dS \end{aligned} \tag{2.6.53}$$

We emphasize that Legendre-transformation (2.6.50) is only possible for strictly convex W(e), i.e. for positive definite $\underset{\sim}{L}$. In the special case $\underset{\sim}{\sigma} \equiv \underset{\sim}{0}$, $\underset{\sim}{L}$ is restricted to $\underset{\sim}{L}$ (e.g. in the first step of incremental calculation starting from unloaded state). If in any special case under consideration elements L vanish systematically (e.g. in case of beam where $L_{ijkl} \equiv 0$ for $i,j,k,l \neq 1$, $\sigma_{ij} \equiv 0$ for $i,j \neq 1$) we reduce dimension of $\underset{\sim}{L}$ in such a way that $\underset{\sim}{L}$ becomes positiv definite and no subdeterminante of $\underset{\sim}{\sigma}$ vanishes identically.

Requirement of positive definiteness of $\underset{\sim}{L}$ requires positive definite $\underset{\sim}{\sigma}$, i.e. positive principle stresses as necessary condition for extremum principles in context of this chapter. This has not been taken into consideration in [47]. Stationarity properties of (2.6.41) are not affected by these requirements. We see that the functionals derived in this chapter by means of [54] correspond to those functionals derived in [47].

A simple description for a class of hardening material behaviour

An example for material laws as used in formulation of boundary value problem 3 is obtained by assumption of isotropic hardening and existence of a Mises-dissipation potential $P_2^*(\delta\sigma)$ [47]. Either elastic and plastic part of strain-rate are then derivable from a potential:

$$\delta\underset{\sim}{\varepsilon} = \delta\underset{\sim}{\varepsilon}^e + \delta\underset{\sim}{\varepsilon}^p ; \quad \delta\underset{\sim}{\varepsilon}^e = \frac{\partial\tilde{P}_2^*(\delta\underset{\sim}{\sigma})}{\partial(\delta\underset{\sim}{\sigma})} ; \quad \delta\underset{\sim}{\varepsilon}^p = \frac{\partial\tilde{P}_1^*(\delta\underset{\sim}{\sigma})}{\partial(\delta\underset{\sim}{\sigma})}$$

$$\delta\underset{\sim}{\sigma} = \frac{\partial[\tilde{P}_1(\delta\underset{\sim}{\varepsilon}^e) + \tilde{P}_2(\delta\underset{\sim}{\varepsilon}^p)]}{\partial(\delta\underset{\sim}{\varepsilon})} = \frac{\partial\tilde{Q}(\delta\underset{\sim}{\varepsilon})}{\partial(\delta\underset{\sim}{\varepsilon})} \qquad (2.6.54)$$

$$\delta\underset{\sim}{\varepsilon} = \frac{\partial[\tilde{P}_1^*(\delta\underset{\sim}{\sigma}) + \tilde{P}_2^*(\delta\underset{\sim}{\sigma})]}{\partial(\delta\underset{\sim}{\sigma})} = \frac{\partial\tilde{Q}^*(\delta\underset{\sim}{\sigma})}{\partial(\delta\underset{\sim}{\sigma})}$$

With

$$\tilde{P}_1^*(\delta\underset{\sim}{\sigma}) = \frac{1}{2} L_{IJKL}\,\delta\sigma_{IJ}\,\delta\sigma_{KL} ; \quad L_{IJKL} : \text{elastic coefficients}$$

$$\tilde{P}_2^*(\delta\underset{\sim}{\sigma}) = \frac{1}{2} A_{IJKL}\,\delta\sigma_{IJ}\,\delta\sigma_{KL}\cdot k ; \quad A_{IJKL} = \sigma'_{IJ}\,\sigma'_{KL} \qquad (2.6.55)$$

$$\tilde{Q}(\delta\underset{\sim}{\varepsilon}) + \tilde{Q}^*(\delta\underset{\sim}{\sigma}) = \delta\underset{\sim}{\varepsilon}\cdot\cdot\,\delta\underset{\sim}{\sigma} = \delta\varepsilon_{IJ}\,\delta\sigma_{IJ} ; \quad \sigma'_{IJ} = \sigma_{IJ} - \delta_{IJ}\sigma_{KK}$$

and following definitions:

$$\kappa = \begin{cases} G & \text{if } \delta I_2 = \sigma'_{IJ}\,\delta\sigma'_{IJ} > 0 \;; \quad I_2 = \frac{1}{2}\sigma'_{IJ}\sigma'_{IJ} \\ 0 & \text{if } \delta I_2 \leq 0 \end{cases}$$

$$G = \frac{3}{4 I_2}\left[\frac{1}{E_T} - \frac{1}{E}\right] \tag{2.6.56}$$

$$\frac{1}{E_T} = \frac{d\varepsilon_{11}}{d\sigma_{11}} \;; \qquad \frac{1}{E} = \frac{d\varepsilon^e_{11}}{d\sigma_{11}}$$

E_T is called "tangent modulus" determined by uniaxial tension test and E is usual modulus of elasticity. Assuming that such unique tension test describes material behaviour adequately, in [25] expression

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \alpha\left(\frac{\sigma_{11}}{E_0}\right)^K \tag{2.6.57}$$

with parameters E, E_0, K which have to be adapted to experimental results is given as possibility of analytical description of material behaviour. Herein definitions

$$\alpha = \begin{cases} 0 & \text{if } \sigma_{11} < \sigma_S \\ 1 & \text{if } \sigma_{11} \geq \sigma_S \end{cases} \qquad \sigma_S : \text{yield-limit} \tag{2.6.58}$$

are used. We then obtain E_T by

$$\frac{1}{E_T} = \frac{1}{E} + \alpha\,\frac{K}{E_0^K}\,\sigma_{11}^{K-1} \tag{2.6.59}$$

From this G can be determined for every state of the body.

Though this description is rather practical real material behaviour is only approximated by assumption that one unique uniaxeral tension-test is sufficient to describe material for more complex situations. Moreover this material law allows only for hardening material and elastic-idealplastic behaviour is only included as limit case.

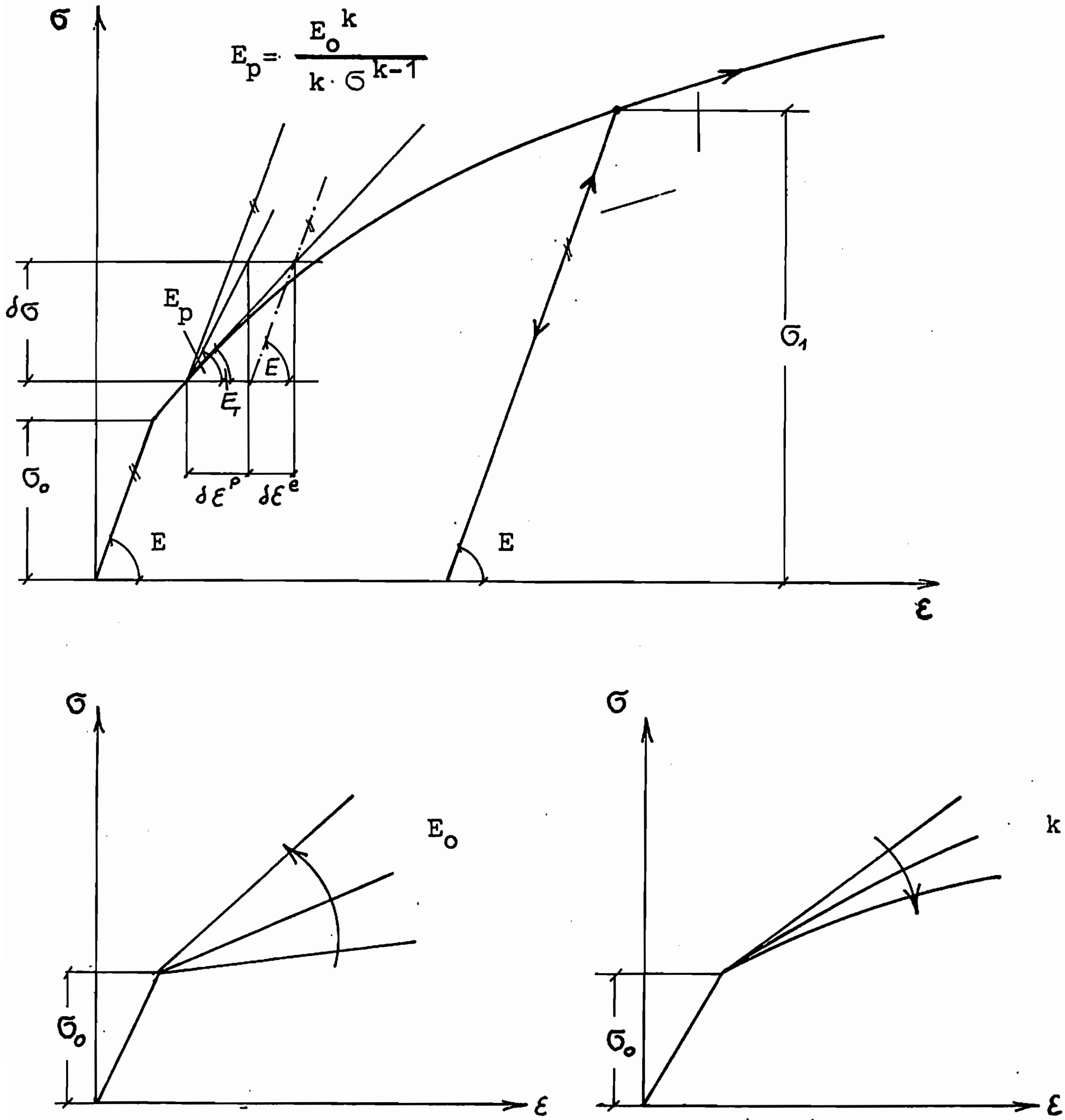
With (2.6.54 - 59) material coefficients describing behaviour in an infinitesimal neighbourhood of reference state $(\;)_0$ can be defined:

$$\underset{\sim}{L}_o = \underset{\sim}{L} + K \underset{\sim}{A} \quad ; \qquad L_{oIJKL} = L_{IJKL} + K \sigma'_{IJ} \sigma'_{KL}$$

$$\underset{\sim}{L}_o^{-1} = (\underset{\sim}{L} + K \underset{\sim}{A})^{-1} \tag{2.6.60}$$

Again we emphasize that inversion of $\underset{\sim}{L}_o$ is only possible if Det $\underset{\sim}{L}_o$ does not vanish.

Representation of hardening material behaviour following [25]



Changing of σ – ε diagram for increasing E_o

Changing of σ – ε diagram for increasing k

Fig. 6

2.7. Initial boundary value problem for finite displacements and small strains

In general the initial boundary value problem is treated by incremental methods. Starting from some given reference state a linear system of algebraic equations is solved either directly or by means of minimal or stationarity principles. Solution vector of this step of calculation is then used to determine a new reference-configuration leading to a modified system of linear equations. This procedure is repeated as long as the final state of loading is not yet reached. However these numerical methods as derived e.g. in (2.6) are basing on rate-principles and are consequently only valid for infinitesimal change of all field quantities. As in numerical calculation finite steps have to be performed, some systematical errors cannot be avoided:

- The linearized system of equations approximates by neglection of nonlinear terms in case of finite steps the original rate boundary value problem and may lead to bad mistakes in the neighbourhood of geometrical instabilities.

- Use of linearized rate equations for finite steps numerical errors are introduced into calculation of the following reference states which propagate in uncontrolled manner.

- Use of finite-dimensional trial-functions (polynomials finite elements) generates errors which do not only influence the solution of the rate problem but the entire calculation via determination of reference states.

We see that rate principles are well defined in theory of elasto-plastic media, however that application to initial boundary value problems is problematic in spite of many successful numerical applications. An extended listing and description of methods in order to avoid at least non-equilibrium states of the body can be found in [60,82].
These methods are formulated either for infinitesimal and for finite displacements and applied to numerous examples [56-59].

A qualitative progress in dealing with the initial boundary value problem for infinitesimal deformations has been made by a functional-analytic formulation by use of convex analysis [33-38]. Moreover in

[39] RAFALSKI introduced minimal principles for residual stresses in elasto-plastic bodies; for a special class of material, "regular material" [21], corresponding plastic strains can be evaluated [40-42].

In the following we use the concept of [39] for infinitesimal deformations and not necessarily quadratic elastic strain energy density in order to formulate, basing on derivations (2.1 - 2.4), minimum principles for elasto-plastic bodies for finite displacements and small strains. Subsequently we deal with the special case of infinitesimal displacements with extension to hardening material as foundation for treatment of the Kirchhoff-plate in chapter (3.3).

The initial boundary value problem at finite displacements

In dealing with this problem we base on derivations (2.1 - 2.4). Our proposition consists essentially in an extension of [42], where infinitesimal displacements where discussed.

In opposition to the rate problem the herein treated body is considered as a region in four-dimensional space-time continuum including his mechanical history with volume $\mathbb{V} = V \times T$, parts $\mathbb{B}_k = B_k \times T$ and $\mathbb{B}_s = B_s \times T$ of entire surface, where $T \in [0,\infty)$. All field quantities are although defined in $\mathbb{V}$:

$$\underset{\sim}{t}(\underset{\sim}{x},\tau)\ ,\ \underset{\sim}{d}(\underset{\sim}{x},\tau)\ ; \qquad x = (x_1, x_2, x_3) \tag{2.7.1}$$

etc..
We assume that entire rate of energy

$$W = \underset{\sim}{t} \cdot\cdot \dot{\underset{\sim}{d}} \tag{2.7.2}$$

may be split up additively in purely elastic and purely plastic parts:

$$W = W^e + W^p \tag{2.7.3}$$

with

$$W^e = \underset{\sim}{t} \cdot\cdot \dot{\underset{\sim}{d}}^e \ ; \qquad W^p = \underset{\sim}{t} \cdot\cdot \dot{\underset{\sim}{d}}^p \tag{2.7.4}$$

This assumption corresponds to assumption of additivity of elastic and plastic part of rate of strain (2.4.4). With the assumption of existence of an elastic strain energy density $\psi(\underset{\sim}{d}^e)$ according to (2.1.5) follows:

$$W^e = \frac{d}{d\tau}\,\psi(\underset{\sim}{d}^e) \tag{2.7.5}$$

and

$$\underset{\sim}{t} \in \partial\psi(\underset{\sim}{d}^e) \tag{2.7.6}$$

respectively for differentiable $\psi(\underset{\sim}{d}^e)$:

$$\underset{\sim}{t} = \frac{\partial\psi(\underset{\sim}{d}^e)}{\partial \underset{\sim}{d}^e} \tag{2.7.7}$$

From assumption of a convex region of admissible stresses $\underset{\sim}{t}$ and definition of convex plastic potential (2.3.7) we obtain plastic rate of energy (dissipation) W^p:

$$W^p = \underset{\sim}{t}\cdot\cdot\,\dot{\underset{\sim}{d}}^p = \varphi(\underset{\sim}{t}) + \varphi^*(\dot{\underset{\sim}{d}}^p) \tag{2.7.8}$$

with:

$$\varphi^*(\dot{\underset{\sim}{d}}^p) = \sup_{\underset{\sim}{t}\in\mathcal{T}} \left[\underset{\sim}{t}\cdot\cdot\,\dot{\underset{\sim}{d}}^p - \varphi(\underset{\sim}{t})\right] \tag{2.7.9}$$

so that equivalently holds:

$$\dot{\underset{\sim}{d}}^p \in \partial\varphi(\underset{\sim}{t}) \quad ; \qquad \underset{\sim}{t} \in \partial\varphi^*(\dot{\underset{\sim}{d}}^p) \tag{2.7.10}$$

Global formulation of the problem

Be H the space of all stress-tensor fields $\underset{\sim}{t}(\underset{\sim}{x},\tau)$, constructed as completion of C_9^∞ with respect to the norm $\|\underset{\sim}{t}\|$, induced by scalar-product

$$(\underset{\sim}{t}^1,\underset{\sim}{t}^2) = \int_{(V)} \underset{\sim}{t}^1\cdot\cdot\,\underset{\sim}{t}^2\, e^{-\tau}\, dV \tag{2.7.11}$$

Be H^* dual to H with $\underset{\sim}{d}(\underset{\sim}{x},\underset{\sim}{\tau}) \in H^*$. $\underset{\sim}{d}$ is called kinematically admissible if there exists a displacement vector $\underset{\sim}{u}(\underset{\sim}{x},\underset{\sim}{\tau}) \in V$ with $\underset{\sim}{u} = 0$ on B_k so that

$$\underset{\sim}{d} - \operatorname{Grad} \underset{\sim}{u} = 0 \tag{2.7.12}$$

Kinematically admissible tensor fields $\underset{\sim}{d}$ will be denoted by $\underset{\sim}{d}^{\mu}$. Hereby subspace $H^{*\mu} \subset H$, $d^{\mu} \in H^{*\mu}$ is defined. Statically admissible stress fields $\underset{\sim}{t}^{\rho}$ will be called such tensor field $\underset{\sim}{t} \in H$ for which

$$(\underset{\sim}{t}, \underset{\sim}{d}) = 0 \quad ; \quad \underset{\sim}{d} \in \mathcal{H}^{*\mu} \quad , \quad \underset{\sim}{u}\,/_{B_K} = 0 \tag{2.7.13}$$

holds. (2.7.12 - 13) are equivalent with the satisfaction of compatibility condition, equilibrium condition and kinematical and statical boundary conditions.

Just like for the rate problem we assume that an associated elastic solution, i.e. the solution of the initial boundary value problem of a body K^{o} with the same load history and geometry as the herein treated body K, different only by the fact of purely elastic behaviour, is given. This solution (index $(\ )^{o}$) statisfies following conditions:

$$\begin{aligned} \operatorname{Div} \underset{\sim}{t}^{o} + \underset{\sim}{b} &= 0 && \text{in } V \\ \underset{\sim}{n}\cdot\underset{\sim}{t}^{o} - \underset{\sim}{f} &= 0 && \text{on } B_s \\ \underset{\sim}{u}^{o} - \underset{\sim}{u}^{*} &= 0 && \text{on } B_k \\ \underset{\sim}{t}^{o} &\in \partial \Psi(\underset{\sim}{d}^{e}) && \text{in } V \end{aligned} \tag{2.7.14}$$

Following [42] global potentials $\Psi(d^{e})$ and $\Phi(t)$ and their polar potentials may be defined by local potentials $\psi(d^{e})$, $\psi^*(t)$, $\varphi(t)$, $\varphi^*(d^{p})$:

$$\begin{aligned} \Psi(\underset{\sim}{d}^{e}) &= \int_{(V)} \psi(\underset{\sim}{d}^{e})\, e^{-\tau}\, dV \\ \Phi(\underset{\sim}{t}) &= \int_{(V)} \varphi(\underset{\sim}{t})\, e^{-\tau}\, dV \end{aligned} \tag{2.7.15}$$

Ψ and Φ are lower-semi-continuous, convex and attain their minimum in the origin [41].

New formulation of the problem

Determine stress-field $\underset{\sim}{t} \in H$ and strain-field $\underset{\sim}{d} \in H^*$, with

$$\underset{\sim}{d}(\underset{\sim}{x}, \tau = 0) = 0 , \qquad \underset{\sim}{t}(\underset{\sim}{x}, \tau = 0) = 0 \tag{2.7.16}$$

such that

$$\begin{aligned} &\underset{\sim}{t}^{\rho} = \underset{\sim}{t}^{o} - \underset{\sim}{t} \in \mathcal{H}^{\rho} \\ &\underset{\sim}{d}^{\mu} = \underset{\sim}{d} - \underset{\sim}{d}^{o} \in \mathcal{H}^{*\mu} \\ &\underset{\sim}{t}^{o} \in \partial \Psi(\underset{\sim}{d}^{e}) \\ &\dot{\underset{\sim}{d}}^{\rho} \in \partial \Phi(\underset{\sim}{t}) \end{aligned} \tag{2.7.17}$$

Minimum principles

Above formulation leads directly to minimal principles if we use Fenchel-transformation. We define two functionals F_1 and F_2:

$$\begin{aligned} F_1(\underset{\sim}{t}, \underset{\sim}{d}^{e}) &= \Psi(\underset{\sim}{d}^{e}) + \Psi^*(\underset{\sim}{t}) - (\underset{\sim}{t}, \underset{\sim}{d}^{e}) \geq 0 \\ F_2(\underset{\sim}{t}, \dot{\underset{\sim}{d}}^{\rho}) &= \Phi(\underset{\sim}{t}) + \Phi^*(\dot{\underset{\sim}{d}}^{\rho}) - (\underset{\sim}{t}, \dot{\underset{\sim}{d}}^{\rho}) \geq 0 \end{aligned} \tag{2.7.18}$$

Using definition (2.7.17) we get:

$$\begin{aligned} F_1(\underset{\sim}{t}^{\rho}, \underset{\sim}{d}^{e}) &= \Psi(\underset{\sim}{d}^{e}) + \Psi^*(\underset{\sim}{t}^{o} - \underset{\sim}{t}^{\rho}) - (\underset{\sim}{t}^{o} - \underset{\sim}{t}^{\rho}, \underset{\sim}{d}^{e}) \geq 0 \\ F_2(\underset{\sim}{t}^{\rho}, \dot{\underset{\sim}{d}}^{\mu}, \dot{\underset{\sim}{d}}^{e}) &= \Phi(\underset{\sim}{t}^{o} - \underset{\sim}{t}^{\rho}) + \Phi^*(\dot{\underset{\sim}{d}}^{\mu} + \dot{\underset{\sim}{d}}^{o} - \dot{\underset{\sim}{d}}^{e}) - (\underset{\sim}{t}^{o} - \underset{\sim}{t}^{\rho}, \dot{\underset{\sim}{d}}^{\mu} + \dot{\underset{\sim}{d}}^{o} - \dot{\underset{\sim}{d}}^{e}) \geq 0 \end{aligned} \tag{2.7.19}$$

The solution of the problem is then obtained by minimizing $F = F_1 + F_2$ for $\underset{\sim}{t}^{\rho} \in H^{\rho}$, $\underset{\sim}{d}^{e} \in H$ and $\dot{\underset{\sim}{d}}^{\mu} \in H^{*\mu}$. Vanishing of F is necessary condition for the solution of the problem, as it indicates satisfaction of the constitutive relations.

Analogously to (2.4. - 2.5) we consider now the region A_o, defined by:

$$A_o : \{ \underset{\sim}{t} : \underset{\sim}{t}^o + \mathcal{H}^\rho \cap \operatorname{dom} \Phi \cap \operatorname{dom} \Psi \} \tag{2.7.20}$$

Herein dom(.) indicates: (.) $< +\infty$.

Be I_{Ao} indicatorfunction of A_o with:

$$I_{Ao}(\underset{\sim}{t}) = \begin{cases} 0 & \text{if } \underset{\sim}{t} \in A_o \\ +\infty & \text{if } \underset{\sim}{t} \notin A_o \end{cases} \tag{2.7.21}$$

If A_o is not empty (otherwise solution of the problem does not exist), we define:

$$\begin{aligned} \Psi_o^* &= \Psi^*(\underset{\sim}{t}) + I_{Ao}(\underset{\sim}{t}) \\ \Phi_o &= \Phi(\underset{\sim}{t}) + I_{Ao}(\underset{\sim}{t}) \end{aligned} \tag{2.7.22}$$

and Ψ_o, Φ_o as corresponding polar functionals, obtained by Fenchel-transformation.

From orthogonality of $\underset{\sim}{t}^\rho$ and $\underset{\sim}{d}^\mu$ and general properties of polar functionals follows:

$$\begin{aligned} F_{1o}(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) &\le F_1(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) \\ F_{2o}(\underset{\sim}{t}^\rho, \dot{d}^e) &\le F_2(\underset{\sim}{t}^\rho, \dot{\underset{\sim}{d}}^\mu, \dot{\underset{\sim}{d}}^e) \end{aligned} \tag{2.7.23}$$

for every $\dot{\underset{\sim}{d}}^\mu \in H^{*\mu}$, $\underset{\sim}{t}^\rho \in \underset{\sim}{t}^o - A_o$, $\underset{\sim}{d}^e \in H^*$ with

$$\begin{aligned} F_{1o} &= \Psi_o^*(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) - (\underset{\sim}{t}^o - \underset{\sim}{t}^\rho, \underset{\sim}{d}^e) + \Psi_o(\underset{\sim}{d}^e) \ge 0 \\ F_{2o} &= \Phi_o(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) - (\underset{\sim}{t}^o - \underset{\sim}{t}^\rho, \dot{\underset{\sim}{d}}^o - \dot{\underset{\sim}{d}}^e) + \Phi^*(\dot{\underset{\sim}{d}}^o - \dot{\underset{\sim}{d}}^e) \ge 0 \end{aligned} \tag{2.7.24}$$

(see appendix A7)

If there exists a solution of the problem, then $F_o = F_{10} + F_{20}$ reaches for this solution minimum equal to zero. If $\underset{\sim}{t}^\rho$ and $\underset{\sim}{d}^e$ minimize F_o uniquely, then state of stress and elastic part of deformation are uniquely determined for the initial boundary value problem.

2.8. The initial boundary value problem at infinitesimal deformations

The initial boundary value problem at infinitesimal deformations is formulated by following relations:

$$\mathrm{Div}\,\underset{\sim}{\sigma} + \underset{\sim}{b} = 0 \qquad \text{in } V$$
$$\underset{\sim}{n}\cdot\underset{\sim}{\sigma} - \underset{\sim}{f} = 0 \qquad \text{on } B_s \tag{2.8.1}$$

$$\underset{\sim}{\varepsilon} - (\mathrm{Grad}\,\underset{\sim}{u})_s = 0 \qquad \text{in } V$$
$$\underset{\sim}{u} - \underset{\sim}{u}^* = 0 \qquad \text{on } B_k \tag{2.8.2}$$

constitutive relations in V (2.8.3)

$\underset{\sim}{\sigma}(\underset{\sim}{x},\tau)$ and $\underset{\sim}{\varepsilon}(\underset{\sim}{x},\tau)$ denote herein usual measures of stress and strain in geometrically linear theory. Linearly hardening material behaviour is described in this chapter by internal parameters [21]: $\underset{\sim}{e}^e = [\underset{\sim}{\varepsilon}^e, \underset{\sim}{\omega}] \hat{=} [\underset{\sim}{\varepsilon}^e_{ij}, \underset{\sim}{\omega}_n]$, $\underset{\sim}{e}^p = [\underset{\sim}{\varepsilon}^p, \underset{\sim}{\kappa}] \hat{=} [\underset{\sim}{\varepsilon}^p_{ij}, \underset{\sim}{\kappa}_n]$, $\underset{\sim}{s} = [\underset{\sim}{\sigma}, \underset{\sim}{\tau}] \hat{=} [\underset{\sim}{\sigma}_{ij}, \underset{\sim}{\pi}_n]$ denote generalized elastic and plastic strain and generalized stress, respectively. Here, assumption of convex strain energy density $\psi(\underset{\sim}{d}^e)$ in (2.1) is replaced by assumption of existence of a quadratic, convex strain energy density $\psi(\underset{\sim}{e}^e)$:

$$\psi(\underset{\sim}{e}^e) = \underset{\sim}{e}^e \cdot\cdot\, \underset{\sim}{G} \cdot\cdot\, \underset{\sim}{e}^e = \varepsilon^e_{ij}\, L^{-1}_{ijkl}\, \varepsilon^e_{kl} + \omega_n\, Z^{-1}_{nm}\, \omega_m \tag{2.8.4}$$

$\underset{\sim}{\omega}$, $\underset{\sim}{\kappa}$ and $\underset{\sim}{\pi}$ are internal kinematical and statical parameters [21,41], with n,m = 1,...,h, where h denotes the number of components of internal parameters. $\underset{\approx}{L} \hat{=} L_{ijkl}$ and $\underset{\approx}{Z} \hat{=} Z_{mn}$ are positive definite matrices with constant coefficients.

Polar energy density $\psi^*(\underset{\sim}{s})$ is defined by the Legendre-transformation:

$$\psi^*(\underset{\sim}{s}) + \psi(\underset{\sim}{e}) = \underset{\sim}{s} \cdot\cdot\, \underset{\sim}{e}^e = \sigma_{ij}\, \varepsilon^e_{ij} + \pi_n\, \omega_n \tag{2.8.5}$$

From quadratic form of $\psi(\underset{\sim}{e}^e)$ follows generalized the Hooke's law:

$$\underset{\sim}{s} = \frac{\partial \Psi(\underset{\sim}{e}^e)}{\partial \underset{\sim}{e}^e} = \underset{\sim}{L}^{-1} \cdot\cdot \underset{\sim}{\varepsilon}^e + \underset{\sim}{Z}^{-1} \cdot \underset{\sim}{\omega} = L^{-1}_{ijkl} \varepsilon^e_{kl} + Z^{-1}_{mn} \omega_n$$

$$\underset{\sim}{e}^e = \frac{\partial \Psi^*(\underset{\sim}{s})}{\partial \underset{\sim}{s}} = \underset{\sim}{L} \cdot\cdot \underset{\sim}{\sigma} + \underset{\sim}{Z} \cdot \underset{\sim}{\Pi} = L_{ijkl} \sigma_{kl} + Z_{mn} \Pi_n \tag{2.8.6}$$

Plastic behaviour is taken into account by generalized plastic potential $\varphi(\underset{\sim}{s})$:

$$\varphi(\underset{\sim}{s}) = \begin{cases} 0 & \text{if } \underset{\sim}{s} \in E \\ +\infty & \text{if } \underset{\sim}{s} \notin E \end{cases} \qquad \underset{\sim}{s} \in S \tag{2.8.7}$$

Here E denotes the region of admissible generalized stresses $\underset{\sim}{s}$ which is assumed to be convex. $\varphi(\underset{\sim}{s})$ is analogously to (2.3.7) convex, lower semi-continuous and attains minimum equal to zero in the origin of space S of all stress tensors $\underset{\sim}{s}(\underset{\sim}{x},\tau)$, $S \subset R^{6+h}$ [41]. Validity of the normality rule can then be expressed also for non-differentiable $\varphi(\underset{\sim}{s})$ by

$$\dot{\underset{\sim}{e}}^p \in \partial \varphi(\underset{\sim}{s}) \tag{2.8.8}$$

Polar plastic potential $\varphi^*(\dot{\underset{\sim}{e}}^p)$ is constructed just like in (2.3) by Fenchel-transformation:

$$\varphi^*(\dot{\underset{\sim}{e}}^p) = \sup_{\underset{\sim}{s} \in S} [\dot{\underset{\sim}{e}}^p \cdot\cdot \underset{\sim}{s} - \varphi(\underset{\sim}{s})] \tag{2.8.9}$$

Equivalently following relations hold:

$$\dot{\underset{\sim}{e}}^p \in \partial \varphi(\underset{\sim}{s}) \; ; \qquad \underset{\sim}{s} \in \partial \varphi^*(\dot{\underset{\sim}{e}}^p) \tag{2.8.10}$$

If we assume according to [41] that total rate of energy density is given by $\underset{\sim}{s} \cdot\cdot \dot{\underset{\sim}{e}}$ with $\dot{\underset{\sim}{e}} = \dot{\underset{\sim}{e}}^e + \dot{\underset{\sim}{e}}^p$ and that rate of energy density can only be produced by external forces acting on the volume element so we have:

$$\underset{\sim}{s} \cdot\cdot \dot{\underset{\sim}{e}} = \underset{\sim}{\sigma} \cdot\cdot \dot{\underset{\sim}{\varepsilon}} \tag{2.8.11}$$

If internal parameters vanish for $\tau = 0$ this relation follows from

$$\underset{\sim}{\omega} + \underset{\sim}{\kappa} = \omega_n + \kappa_n = 0 \tag{2.8.12}$$

because of

$$\underset{\sim}{s} \cdot\cdot \dot{\underset{\sim}{e}} = \underset{\sim}{s} \cdot\cdot (\dot{\underset{\sim}{e}}^e + \dot{\underset{\sim}{e}}^p) = \underset{\sim}{\sigma} \cdot\cdot (\dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p) + \underset{\sim}{\pi} \cdot (\dot{\underset{\sim}{\omega}} + \dot{\underset{\sim}{\kappa}}) \tag{2.8.13}$$

Global formulation of the problem

We use the space of all smooth and bounded functions from C^{∞}_{6+n} as unitary space and construct Hilbert space H by completion with respect to norm $\|\cdot\|_G$, induced by scalarproduct

$$\langle \underset{\sim}{a}, \underset{\sim}{b} \rangle = \int_{(V)} (a_{ij} L_{ijkl} b_{kl} + \alpha_n Z_{nm} \beta_m) e^{-\tau} dV \quad ,$$

$$\underset{\sim}{a} = [a_{ij}, \alpha_n] \qquad \underset{\sim}{b} = [b_{ij}, \beta_n] \tag{2.8.14}$$

Statically admissible stress fields $\underset{\sim}{s}^{\rho}$ are then defined by:

$$\underset{\sim}{s}^{\rho} = [\underset{\sim}{\sigma}^{\rho}, \underset{\sim}{\pi}^{\rho}] := \{\underset{\sim}{s} \in \mathcal{H} : \mathrm{Div}\,\underset{\sim}{\sigma} = 0 \text{ in } V, \ \underset{\sim}{n} \cdot \underset{\sim}{\sigma} = 0 \text{ on } B_s\} \tag{2.8.15}$$

Kinematically admissible stress fields $\underset{\sim}{s}^{\mu}$ are defined by:

$$\underset{\sim}{s}^{\mu} = [\underset{\sim}{\sigma}^{\mu}, 0] := \{\underset{\sim}{s} \in \mathcal{H} : \underset{\sim}{\sigma} = \underset{\sim}{L}^{-1} \cdot\cdot \mathrm{Grad}\,\underset{\sim}{u} \text{ in } V,$$

$$\underset{\sim}{u} = 0 \text{ on } B_K\} \tag{2.8.16}$$

By application of divergence theorem orthogonality of every $\underset{\sim}{s}^{\rho}(\underset{\sim}{x},\tau)$ to every $\underset{\sim}{s}^{\mu}(\underset{\sim}{x},\tau)$ with respect to scalarproduct $\langle .,. \rangle$ can be proved. If we denote subspaces of all statically admissible fields $\underset{\sim}{s}^{\rho}$ and kinematically admissible fields $\underset{\sim}{s}^{\mu}$ by H^{ρ} and H^{μ} respectively, then orthogonality of $\underset{\sim}{s}^{\rho}$ and $\underset{\sim}{s}^{\mu}$ induces that H^{ρ} and H^{μ} are orthogonal subspaces of H.

Global plastic potential

By the definition

$$\Phi(\underset{\sim}{s}) = \lim_{\alpha\to\infty} \lim_{n\to\infty} \int_{(V)} \varphi_\alpha (\underset{\sim}{s}_{(n)}(\underset{\sim}{x},\tau))\, e^{-\tau} dV \qquad (2.8.17)$$

a global plastic potential is defined, satisfying conditions of convexity and lower semi-continuity. It assume the value zero in the origin of H [41]. According to (1.3.7) φ_α is defined by

$$\varphi_\alpha(\underset{\sim}{s}) = \begin{cases} 0 & \text{if } \varphi(\underset{\sim}{s}) = 0 \\ \alpha & \text{if } \varphi(\underset{\sim}{s}) = +\infty \quad ; \quad 0<\alpha<+\infty \end{cases} \qquad (2.8.18)$$

Plastic flow-law is then globally formulated by:

$$\underset{\sim}{G}^{-1} \cdot\cdot\, \dot{\underset{\sim}{e}}^{p} \in \partial \Phi(\underset{\sim}{s}) \qquad (2.8.19)$$

Equivalently holds

$$\Phi(\underset{\sim}{s}) + \Phi^*(\underset{\sim}{G}^{-1} \cdot\cdot\, \dot{\underset{\sim}{e}}^{p}) = \langle \underset{\sim}{G}^{-1} \cdot\cdot\, \dot{\underset{\sim}{e}}^{p}, \underset{\sim}{s} \rangle \qquad (2.8.20)$$

with

$$\Phi^*(\underset{\sim}{G}^{-1} \cdot\cdot\, \dot{\underset{\sim}{e}}^{p}) = \sup_{\underset{\sim}{s}\in\mathcal{H}} [\langle \underset{\sim}{G}^{-1} \cdot\cdot\, \dot{\underset{\sim}{e}}^{p}, \underset{\sim}{s} \rangle - \Phi(\underset{\sim}{s})] \qquad (2.8.21)$$

The initial boundary value problem may then be reformulated: Determine those fields $\underset{\sim}{s}^{\rho} \in H^{\rho\prime}$ and $\underset{\sim}{s}^{\mu} \in H^{\mu\prime}$ for which

$$\dot{\underset{\sim}{s}}^{\rho} + \dot{\underset{\sim}{s}}^{\mu} \in \partial \Phi(\underset{\sim}{s}^{0} - \underset{\sim}{s}^{\rho}) \qquad (2.8.22)$$

is satisfied. Here (.)' denotes restriction of subspaces H^{ρ} and H^{μ} to time-differentiable fields because in the relevant relations functions themselves and their differentiation with respect to time occur. $\underset{\sim}{s}^{0}(\underset{\sim}{x},\tau)$ describes herein the solution of an associated initial boundary value problem for purely elastic material behaviour, assumed to be given. $\underset{\sim}{s}^{0}$ then satisfies the following conditions: $\underset{\sim}{s}^{0} = [\underset{\sim}{\sigma}^{0}, 0]$;

$$\begin{aligned}
&\mathrm{Div}\,\underset{\sim}{\sigma}^{o} - \underset{\sim}{b} &&= 0 &&\text{in } V\\
&\underset{\sim}{n}.\underset{\sim}{\sigma}^{o} - \underset{\sim}{f} &&= 0 &&\text{on } B_s\\
&\underset{\sim}{L}..\underset{\sim}{\sigma}^{o} - (\mathrm{Grad}\,\underset{\sim}{u})_s &&= 0 &&\text{in } V\\
&\underset{\sim}{u}^{o} - \underset{\sim}{u}^{*} &&= 0 &&\text{on } B_k
\end{aligned} \tag{2.8.23}$$

By application of definition of subdifferential formulation (2.8.22) may be equivalently replaced by the condition:

$$\Lambda(\underset{\sim}{s}^{\rho}, \underset{\sim}{s}^{\mu}) = 0 \tag{2.8.24}$$

with

$$\Lambda(\underset{\sim}{s}^{\rho}, \underset{\sim}{s}^{\mu}) = \Phi(\underset{\sim}{s}^{o} - \underset{\sim}{s}^{\rho}) + \Phi^{*}(\dot{\underset{\sim}{s}}^{\rho} + \dot{\underset{\sim}{s}}^{\mu}) - \langle \underset{\sim}{s}^{o} - \underset{\sim}{s}^{\rho}, \dot{\underset{\sim}{s}}^{\rho} + \dot{\underset{\sim}{s}}^{\mu} \rangle$$

$$\underset{\sim}{s}^{\rho} \in \mathcal{H}^{\rho'}, \qquad \underset{\sim}{s}^{\mu} \in \mathcal{H}^{\mu'} \tag{2.8.25}$$

$\Lambda(\underset{\sim}{s}^{\rho}, \underset{\sim}{s}^{\mu})$ is convex and attains minimum equal to zero only if plastic flow-law is satisfied [39].

Λ, however, is not strictly convex such that minimization of Λ does not deliver a unique solution of the problem. If we introduce the functional

$$\Lambda_o(\underset{\sim}{s}^{\rho}) = \Phi_o(\underset{\sim}{s}^{o} - \underset{\sim}{s}^{\rho}) + \Phi_o^{*}(\dot{\underset{\sim}{s}}^{\rho}) - \langle \underset{\sim}{s}^{o} - \underset{\sim}{s}^{\rho}, \dot{\underset{\sim}{s}}^{\rho} \rangle \tag{2.8.26}$$

with

$$\Phi_o(\underset{\sim}{s}) = \begin{cases} \Phi(\underset{\sim}{s}) & \text{if } \underset{\sim}{s} \in \underset{\sim}{s}^{o} + \mathcal{H}^{\rho'} \\ +\infty & \text{if } \underset{\sim}{s} \notin \underset{\sim}{s}^{o} + \mathcal{H}^{\rho'} \end{cases} \tag{2.8.27}$$

and

$$\Phi_o^{*}(\dot{\underset{\sim}{s}}^{\rho}) = \sup_{\underset{\sim}{s} \in \mathcal{H}^{\rho'}} \left[\langle \underset{\sim}{s}^{o} - \underset{\sim}{s}, \dot{\underset{\sim}{s}}^{\rho} \rangle - \Phi_o(\underset{\sim}{s}^{o} - \underset{\sim}{s}) \right] \tag{2.8.28}$$

then it is proved that Λ_o is strictly convex. If there exists a solution of the problem, then Λ_o attains uniquely for this solution the minimum equal to zero. Moreover in case of "regular material" plastic deformation can be uniquely determined [41].

Special case of proportional loading and elasto-idealplastic material behaviour

First of all we remember that following [21] region of admissible stress states $\underset{\sim}{s}$ does not change during any arbitrary loading process. By this fact (2.2.26) can be replaced by:

$$\Lambda_o(\underset{\sim}{s}^\rho) = \sup_{\underset{\sim}{s}^\rho - \underset{\sim}{s}^{\rho *} \in E} \langle \underset{\sim}{s}^\rho - \underset{\sim}{s}^{\rho *}, \dot{\underset{\sim}{s}}^\rho \rangle \qquad (2.8.29)$$

$\Lambda_o(\underset{\sim}{s}^\rho)$ is then minimized on the set $\underset{\sim}{s}^o - E$ (see appendix A8). In case of proportional loading, i.e. for proportional increase of external loads and all field-quantities, $\dot{\underset{\sim}{s}}(\tau)$ may be replaced by $\underset{\sim}{s}/\tau$ as material behaviour itself is time-independent. History of loading is then prescribed and factor $e^{-\tau}$ in (2.8.14,17) may be replaced by unity without loss of strict convexity of $\langle \underset{\sim}{s}^\rho, \dot{\underset{\sim}{s}}^\rho \rangle$.

With this modified functional $\Lambda_o(\underset{\sim}{s}^\rho)$ in (3.2) the problem of Kirchhoff plate is dealt with. In numerical calculation elasto-idealplastic behaviour is assumed such that internal parameters vanish and $\underset{\sim}{s}$ reduces to $\underset{\sim}{\sigma}$ with only six independent components.

3. APPLICATION TO PLATE PROBLEMS

Definitions

We consider a three-dimensional body of volume V in undeformed state, given by a flat midsurface F, with boundary $Z = Z_k \cup Z_s$ sufficiently smooth and constant thickness 2h. Part B_s of surface B of the body where forces P_i are prescribed, may consist of flat surfaces F^+ and F^-, parallel to F and the vertical surface, determined by boundary line Z_s and limitations +h and -h. Part B_k of B where displacements are prescribed is reduced to that vertical surface which is determined by boundary line Z_k and limitations +h and -h. Vertical components of P_i are treated as if they act on midsurface F, components of P_i lying in x_1, x_2-plane are treated as volume forces, continuously distributed over the thickness of the body. Such a body will be called "plate" in the following chapters.

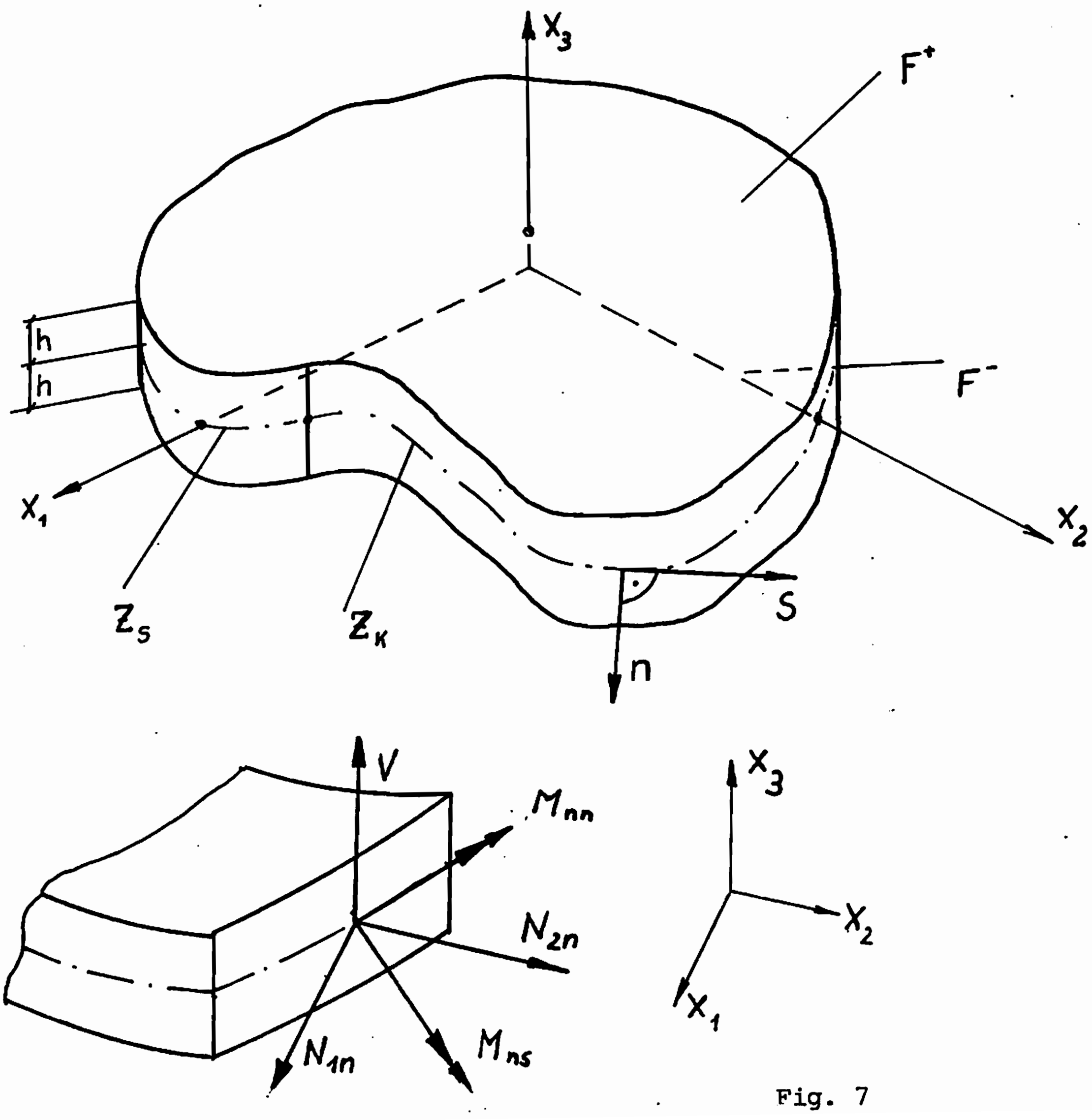


Fig. 7

3.1. Two-dimensional representation of three-dimensional quantities

As plates and shells made of non-hyperelastic material have to be regarded as three-dimensional bodies if the general initial boundary value problem should be solved, we derive three methods for the systematical representation of three-dimensional functions by two-dimensional functions in this chapter. Allowing for the special properties of plates and shells these representations help us to apply the results for three-dimensional bodies in a mathematically consistent manner.

(i) Polynomial representation by Taylor-expansion

We consider three-dimensional functions

$$\underset{\sim}{a} = \underset{\sim}{a}(x_1, x_2, x_3) \qquad \underset{\sim}{a} \in \mathcal{A}_3 \subset C_3^{\infty} \tag{3.1.1}$$

with A_3 as bounded region in C_3^{∞}, where these functions may be scalar, vectors or tensors. By Taylor-expansion of $\underset{\sim}{a}$ with respect to the plane $(x_1, x_2, 0)$ we obtain a set $\underset{\sim}{A}^p(x_1, x_2)$ of two-dimensional functions $\underset{\sim}{A}^{(k)}$:

$$\underset{\sim}{A}^p(x_1, x_2) := \{ \underset{\sim}{A}^{(1)}, \underset{\sim}{A}^{(2)}, \cdots \underset{\sim}{A}^{(k)}, \cdots \underset{\sim}{A}^{(p)} \} \qquad k \leq p \geq 1 \tag{3.1.2}$$

with

$$\underset{\sim}{A}^{(k)}(x_1, x_2) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \underset{\sim}{a}(x_1, x_2, x_3)}{(\partial x_3)^{k-1}} \tag{3.1.3}$$

Three-dimensional function $\underset{\sim}{a} \in A_3$ is then represented by the set $\underset{\sim}{A}^p$ of two-dimensional functions $\underset{\sim}{A}^{(k)}$.

The inverse relation is:

$$\underset{\sim}{a}(x_1, x_2, x_3) = \sum_{k=1}^{p} \left(\underset{\sim}{A}^{(k)}(x_1, x_2) \cdot x_3^{k-1} \right) + R^p(x_1, x_2, x_3) \tag{3.1.4}$$

with the remainder $\underset{\sim}{R}^p$. In operator notation we say for brevity that operator

$$D^{p} := \left\{ \frac{1}{(\kappa-1)!} \, \frac{\partial^{\kappa-1}(\cdot)}{(\partial x_3)^{\kappa-1}} \bigg|_0 \right\} ; \qquad \kappa = 1,2,\cdots p \qquad (3.1.5)$$

defined by (2.1.3) maps uniquely every function $\underset{\sim}{a}(x_1,x_2,x_3) \in A_3$ on a two-dimensional region $A_2 \subset C_2$. Operator $\tilde{D}^p(A_3)$,

$$\tilde{D}^{p}(\underset{\sim}{A}^{p}) := \sum_{\kappa=1}^{p} \underset{\sim}{A}^{(\kappa)} x_3^{\kappa-1} \qquad \kappa = 1,2,\cdots p \qquad (3.1.6)$$

maps every set A^p only on a subregion $A_3' \subset A_3$ only, so that he can only be denoted as inverse operator to D^p if we restrict A_3 to A_3' from the beginning, i.e. we admit only such functions for which the remainder vanishes. In this case D^p and $\tilde{D}^p$ define a one-to-one mapping of A_3' onto A_2 and all operations defined for three-dimensional functions may be expressed as two-dimensional operations.

(ii) Polynomial representation by integrals

Instead of operator D^p an integral operator J^p may be used to get a unique mapping of three-dimensional functions onto two-dimensional space. Analogously to (3.1.2) we define the set $B^p(x_1,x_2)$ of two-dimensional representatives of a three-dimensional function $\underset{\sim}{b}(x_1,x_2,x_3)$ by:

$$\underset{\sim}{B}^{(\kappa)} = \int_{\alpha}^{\beta} \underset{\sim}{b}(x_1,x_2,x_3)\, x_3^{\kappa-1}\, dx_3 \; ; \qquad \kappa = 1,2,\cdots p \qquad (3.1.7)$$

where α and β denote the bounds of coordinate x_3 in region A_3, where function $\underset{\sim}{b}(x)$ is defined. This way of representation is more general then the previous one as only the assumption of integrability of function $\underset{\sim}{b}$ is necessary. The set:

$$\underset{\sim}{B}^{p}(x_1,x_2) := \{ \underset{\sim}{B}^{(1)}, \underset{\sim}{B}^{(2)}, \cdots \underset{\sim}{B}^{(\kappa)}, \cdots \underset{\sim}{B}^{(p)} \} \qquad (3.1.8)$$

is then called "integral representative of $\underset{\sim}{b}$".

The inverse relation is:

$$\underset{\sim}{b}(x_1,x_2,x_3) = \sum_{\kappa=1}^{p} \sum_{l=1}^{p} (\underset{\sim}{B}^{(\kappa)} m_{\kappa l}\, x_3^{l-1}) + \tau^{R}(x_1,x_2,x_3) \qquad (3.1.9)$$

with

$$m_{kl}^{-1} = \int_{\alpha}^{\beta} x_3^{k+l-2}\, dx_3 \qquad -\alpha = \beta = h \qquad (3.1.10)$$

Here h denotes half of the constant thickness of the considered region. $\underset{\sim}{\tau}^R(x_1,x_2,x_3)$ satisfies:

$$\int_{\alpha}^{\beta} \tau^R(x_1,x_2,x_3)\, x_3^{k-1}\, dx_3 = 0 \qquad (3.1.11)$$

This way of representation had been used in [79].

Inverse relation (3.1.9), however, in this form is only valid for p = 2.

(iii) "Finitely valued function" representation [69]

Besides the possibility to represent three-dimensional functions equivalently by a set of two-dimensional polynomial representatives we may cut the considered region into a finite number of subregions and assign to each region a constant finite value. In order to represent a three-dimensional function $\underset{\sim}{\tau}(x_1,x_2,x_3)$ we may write:

$$\underset{\sim}{T}^{(k)}(x_1,x_2) = \int_{-h}^{+h} \alpha\, \underset{\sim}{\tau}(x_1,x_2,x_3)\, dx_3 \quad , \quad k = 1,2,\cdots p$$

$$\alpha = \begin{cases} 1 & \text{if } x_3 \in [g_k, g_{k+1}] \\ 0 & \text{if } x_3 \notin [g_k, g_{k+1}] \end{cases} \qquad (3.1.12)$$

$$g_1 = -h \ , \quad g_{p+1} = h \ , \quad g_{k-1} < g_k < g_{k+1}$$

The set

$$\underset{\sim}{T}^{p} := \{ \underset{\sim}{T}^{(1)}, \underset{\sim}{T}^{(2)}, \cdots \underset{\sim}{T}^{(k)}, \cdots \underset{\sim}{T}^{(p)} \} \qquad (3.1.13)$$

is then called "finitely valued function"-representant of order p of three-dimensional function $\underset{\sim}{\tau}(x_1,x_2,x_3)$" [69]. Inversion of (3.1.12) is defined by:

$$\underset{\sim}{\tau}(x_1, x_2, x_3) = \beta(x_3)\cdot\underset{\sim}{T}^{(k)}(x_1,x_2) + \underset{\sim}{\tau}^R(x_1,x_2,x_3)$$

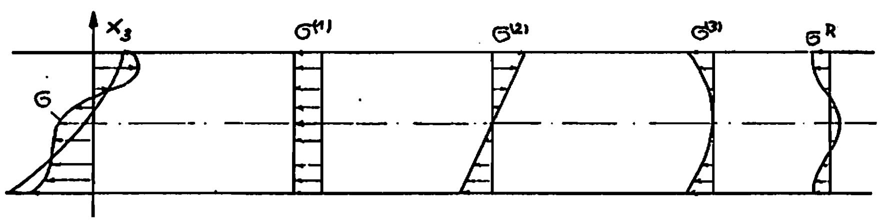
$$\beta(x_3) = \begin{cases} \dfrac{1}{g_{k+1}-g_k} & \text{if } x_3 \in [g_k, g_{k+1}] \\ 0 & \text{if } x_3 \notin [g_k, g_{k+1}] \end{cases} \qquad (3.1.14)$$

Here $\underset{\sim}{\tau}^R(\underset{\sim}{x})$ satisfies:

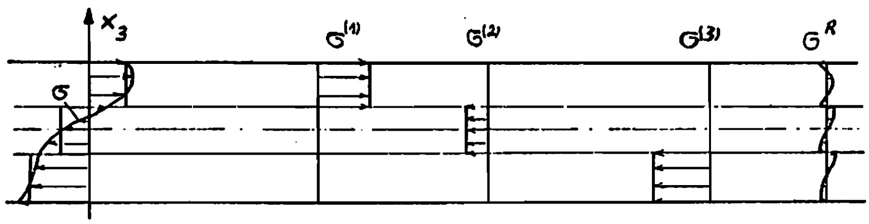
$$\int_{-h}^{+h} \alpha\, \underset{\sim}{\tau}^R(x_1, x_2, x_3)\, dx_3 = 0 \qquad (3.1.15)$$

This method of replacing three-dimensional functions by sets of two-dimensional functions is often applied to solve elasto-plastic plate and shell problems, justified only be evidence [80,82,48,49]. We shall use it in chapters (3.4, 4.3) in order to represent stress distribution in connection with polynomial representation of strain- and displacement-functions.

Illustration of (3.1.4) and (3.1.14) for a one-dimensional example:



Polynomial representatives



"finitely-valued"-representatives

Fig. 8

3.2. Application to elasto-plastic plates at infinitesimal deformations and linear hardening

We substitute in (2.8) generalized tensor fields $\underset{\sim}{s} = [\underset{\sim}{\sigma}, \underset{\sim}{\pi}]$ and $\underset{\sim}{e}^e = [\underset{\sim}{\varepsilon}^e, \underset{\sim}{\omega}]$ by their two-dimensional representatives $\underset{\sim}{n}(x_1, x_2, \tau)$ and $\underset{\sim}{q}(x_1, x_2, \tau)$ with:

$$\underset{\sim}{n} := [\underset{\sim}{N}^p, \underset{\sim}{\Pi}^p] \quad ; \qquad \underset{\sim}{q} := [\underset{\sim}{Q}^p, \underset{\sim}{\Omega}^p]$$

$$\begin{aligned}
\underset{\sim}{N}^p &:= \{ N_{ij}^{(1)}, N_{ij}^{(2)}, \cdots N_{ij}^{(p)} \} && i,j \in [1,2,3] \\
\underset{\sim}{\Pi}^p &:= \{ \Pi_n^{(1)}, \Pi_n^{(2)}, \cdots \Pi_n^{(p)} \} && n \in [1,2,\cdots r] \\
\underset{\sim}{Q}^p &:= \{ Q_{ij}^{(1)}, Q_{ij}^{(2)}, \cdots Q_{ij}^{(p)} \} && p \geq 1 \\
\underset{\sim}{\Omega}^p &:= \{ \Omega_n^{(1)}, \Omega_n^{(2)}, \cdots \Omega_n^{(p)} \}
\end{aligned} \tag{3.2.1}$$

Here r denotes the number of internal parameters (3.1) and p the order of two-dimensional representatives. In the region, restricted according to (3.1.6) by the requirement of vanishing remainder of Taylor-expansion (3.1.3) we obtain the following representation of scalarproduct, equivalent to (2.8.15):

$$\langle\langle \underset{\sim}{n}, \underset{\sim}{q} \rangle\rangle = \int_{(F)} (N_{ij}^p \underset{\sim}{m} Q_{ij}^p + \Pi_n^p \underset{\sim}{m} \Omega_n^p)\, e^{-\tau} dx_1\, dx_2\, d\tau \tag{3.2.2}$$

with

$$\begin{aligned}
N_{ij}^p \underset{\sim}{m} Q_{ij}^p &= \sum_{k=1}^{p} \sum_{l=1}^{p} N_{ij}^{(k)} m_{kl}\, Q_{ij}^{(l)} \\
\Pi_n^p \underset{\sim}{m} \Omega_n^p &= \sum_{k=1}^{p} \sum_{l=1}^{p} \Pi_n^{(k)} m_{kl}\, \Omega_n^{(l)}
\end{aligned} \tag{3.2.3}$$

$$m_{kl} = \int_{-h}^{+h} x_3^{\,k+l-2}\, dx_3 \tag{3.2.4}$$

By this transformation the original three-dimensional problem is reduced to a two-dimensional problem in the restricted region defined by (3.1.4).

Relations for plates and thin plates

If we split up expression (3.2.2) into those parts which contain only tensor-components in direction of x_1,x_2-plane and those containing only tensor-components in vertical direction then we obtain:

$$\langle\langle \underset{\sim}{n}, \underset{\sim}{q} \rangle\rangle = \langle\langle n_{\alpha\beta}, q_{\alpha\beta} \rangle\rangle + 2\langle\langle n_{\alpha 3}, q_{\alpha 3} \rangle\rangle + \langle\langle n_{33}, q_{33} \rangle\rangle \tag{3.2.5}$$

Following [83] we assume for plates:

$$\langle\langle n_{33}, q_{33} \rangle\rangle \ll 2\langle\langle n_{\alpha 3}, q_{\alpha 3} \rangle\rangle + \langle\langle n_{\alpha\beta}, q_{\alpha\beta} \rangle\rangle \tag{3.2.6}$$

For thin plates we assume additionally:

$$2\langle\langle n_{\alpha 3}, q_{\alpha 3} \rangle\rangle \ll \langle\langle n_{\alpha\beta}, q_{\alpha\beta} \rangle\rangle \tag{3.2.7}$$

Both assumptions are compatible with the assumption of linear theory of thin plates that all components of stress in x_3-direction are neglectible small [83]. In our procedure, however, well known contradiction of the assumption that either stresses and strains in x_3-direction vanish is avoided. - Now three-dimensional theory developed in (2.8) can be applied without any additional restriction to plate problems.

Kinematically and statically admissible stress-representatives

If we want to apply relations fo chapter (2.8), definition of statically and kinematically admissible stress representatives $\underset{\sim}{n}^{\rho}$ and $\underset{\sim}{n}^{\mu}$ is needed. Following (2.8.15 - 16) it holds that $\underset{\sim}{s}^{\mu} = [\underset{\sim}{\sigma}^{\mu}, \underset{\sim}{\omega}]$ is kinematically admissible if:

$$\begin{aligned} &\underset{\sim}{\sigma}^{\mu} = \underset{\sim}{L}^{-1} \cdot\cdot (\operatorname{Grad} \underset{\sim}{u})_s && \text{in } V \\ &\underset{\sim}{u} = 0 && \text{on } B_k \\ &\underset{\sim}{\omega} = 0 && \text{in } V \end{aligned} \tag{3.2.8}$$

$\underset{\sim}{s}^{\rho} = [\underset{\sim}{\sigma}^{\rho}, \underset{\sim}{\pi}]$ is statically admissible if:

$$\langle \underset{\sim}{s}^{\mu}, \underset{\sim}{s}^{\rho} \rangle = 0 \tag{3.2.9}$$

This condition of orthogonality is equivalent with satisfying homogeneous equilibrium conditions

$$\begin{aligned} \mathrm{Div}\ \underset{\sim}{\sigma}^{\rho} &= 0 && \text{in } V \\ n \cdot \underset{\sim}{\sigma}^{\rho} &= 0 && \text{on } B_s \end{aligned} \tag{3.2.10}$$

and condition that internal parameters do not contribute to mechanical work.

In order to apply this condition in such a way that it can be used for calculations of plates, including Kirchhoff plate as special case, we define according to (3.1.2) the set $\underset{\sim}{U}^{p}$ of representatives of displacement-functions:

$$\underset{\sim}{U}^{p} := \{ \tilde{U}_{\alpha}^{(k)} \} \qquad k = 1, 2, \cdots p \tag{3.2.11}$$

To this corresponds the set $\underset{\sim}{E}^{p}$ of representatives of strain tensors:

$$\underset{\sim}{E}^{p} := \{ (\tilde{U}_{\alpha,\beta}^{(1)})_s, (\tilde{U}_{\alpha,\beta}^{(2)})_s, \cdots (\tilde{U}_{\alpha,\beta}^{(p)})_s \} \tag{3.2.12}$$

with the inverse relation:

$$\underset{\sim}{\varepsilon} = \varepsilon_{\alpha\beta} = (\tilde{U}_{\alpha,\beta}^{(1)})_s + (\tilde{U}_{\alpha,\beta}^{(2)})_s X_3 + \cdots (\tilde{U}_{\alpha,\beta}^{(k)})_s X_3^{k-1} + \cdots (\tilde{U}_{\alpha,\beta}^{p-1})_s X_3^{p-1} \tag{3.2.13}$$

according to (2.1.4). In addition we assume that:

$$\tilde{U}_{\alpha,\beta}^{(2)} = U_{,\alpha\beta}^{(2)} \quad ; \qquad \tilde{U}_{\alpha}^{(i)} = U_{\alpha}^{(i)} \qquad i = 1, 3, \cdots p; \;\; i \neq 2 \tag{3.2.14}$$

$\underset{\sim}{E}^{p}$ satisfies condition of compatibility in V and $U^{(2)}$ can be interpreted as deflection w in special case of Kirchhoff plate.

If we insert (3.2.1) and (3.2.3) into condition of orthogonality (3.2.9) then we obtain equilibrium condition in V and kinematical and statical boundary conditions, orthogonal to each other:

$$\langle\!\langle n_{\alpha\beta}, q_{\alpha\beta} \rangle\!\rangle = \int_{(F)} N^{p}_{\alpha\beta}\, \underset{\sim}{m}\, E^{p}_{\alpha\beta}\, e^{-\tau}\, dx_1\, dx_2\, d\tau \; +$$

$$+ \int_{(F)} \Pi^{p}_{n}\, \underset{\sim}{m}\, \Omega^{p}_{n}\, e^{-\tau}\, dx_1\, dx_2\, d\tau = 0 \qquad (3.2.15)$$

Because of (3.2.8 - 9) second term vanishes. Matrix $\underset{\sim}{m}$ is defined by:

$$m_{kl} = \begin{array}{c|cccccccc} k \backslash l & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & p \\ \hline 1 & 2h & 0 & \frac{2h^3}{3} & 0 & \frac{2h^5}{5} & 0 & \cdots & \cdots \\ 2 & 0 & \frac{2h^3}{3} & 0 & \frac{2h^5}{5} & 0 & \frac{2h^7}{7} & \cdots & \cdots \\ 3 & \frac{2h^3}{3} & 0 & 2h^5 & 0 & \frac{2h^7}{7} & 0 & \cdots & \cdots \\ 4 & 0 & \frac{2h^5}{5} & 0 & \frac{2h^7}{7} & 0 & \frac{2h^9}{9} & \cdots & \cdots \\ 5 & \frac{2h^5}{5} & 0 & \frac{2h^7}{7} & 0 & \frac{2h^9}{9} & 0 & \cdots & \cdots \\ 6 & 0 & \frac{2h^7}{7} & 0 & \frac{2h^9}{9} & 0 & \frac{2h^{11}}{11} & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{2h^{(2p-1)}}{2p-1} \end{array} \qquad (3.2.16)$$

Using relations (3.2.3 - 7 , 15 - 16) we obtain e.g. for p = 4:

$$\langle\!\langle \underset{\sim}{n}, \underset{\sim}{q} \rangle\!\rangle = \int_{(F)} \Big[2h \big(N^{(1)}_{\alpha\beta} U^{(1)}_{\alpha,\beta} \big) + \frac{2h^3}{3} \big(N^{(2)}_{\alpha\beta} U^{(2)}_{,\alpha\beta} + N^{(1)}_{\alpha\beta} U^{(3)}_{\alpha,\beta} + N^{(3)}_{\alpha\beta} U^{(1)}_{\alpha,\beta} \big) +$$

$$+ \frac{2h^5}{5} \big(N^{(3)}_{\alpha\beta} U^{(3)}_{\alpha,\beta} + N^{(2)}_{\alpha\beta} U^{(4)}_{\alpha,\beta} + N^{(4)}_{\alpha\beta} U^{(2)}_{\alpha,\beta} \big) + \frac{2h^7}{7} \big(N^{(4)}_{\alpha\beta} U^{(4)}_{\alpha,\beta} \big) \Big] e^{-\tau}\, dx_1\, dx_2\, d\tau =$$

$$= \int_{(F)} \Big[\big(2h N^{(1)}_{\alpha\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta} \big), \big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta} \big), \big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta} \big),$$

$$\big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta} \big) \Big] \Big[U^{(1)}_{\alpha,\beta}, U^{(2)}_{,\alpha\beta}, U^{(3)}_{\alpha,\beta}, U^{(4)}_{\alpha,\beta} \Big]^{T} e^{-\tau}\, dx_1\, dx_2\, d\tau$$

(3.2.17)

Application of divergence-theorem leads to:

$$\langle\langle \underset{\sim}{n}, \underset{\sim}{q} \rangle\rangle = -\int_{(F)} \Big[\Big(2h\, N^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta,\beta}\Big), \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta,\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta,\beta}\Big),$$

$$\Big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta,\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta,\beta}\Big) \Big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta,\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta,\beta}\Big)\Big] \Big[U^{(1)}_{\alpha}, U^{(2)}_{,\alpha}\; U^{(3)}_{\alpha}, U^{(4)}_{\alpha}\Big]^T e^{-\tau} dx_1 dx_2 d\tau$$

$$+ \int_{(Z)} \Big[\Big(2h\, N^{(1)}_{\alpha\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta}\Big) n_\beta, \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta}\Big) n_\beta,$$

$$\Big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta}\Big) n_\beta, \Big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta}\Big) n_\beta\Big] \cdot$$

$$\Big[U^{(1)}_{\alpha}, U^{(2)}_{,\alpha}, U^{(3)}_{\alpha}, U^{(4)}_{\alpha}\Big]^T e^{-\tau} dx_1\, dx_2\, d\tau$$

(3.2.18)

With

$$\frac{\partial}{\partial x_1} = n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial s} \quad ; \qquad \frac{\partial}{\partial x_2} = n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial s}$$

$$n_1 = \cos(x_1, \underset{\sim}{n}) \qquad\qquad n_2 = \cos(x_2, \underset{\sim}{n})$$

$$V = n_\alpha \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta}\Big)_{,\beta} \quad ; \qquad N^{(i)}_{\alpha n} = N^{(i)}_{\alpha\beta} n_\beta$$

$$M_{ns} = -n_2 n_\alpha \Big(\frac{2h^3}{3} N^{(2)}_{\alpha 1} + \frac{2h^5}{5} N^{(4)}_{\alpha 1}\Big) - n_1 n_\alpha \Big(\frac{2h^3}{3} N^{(2)}_{\alpha 2} + \frac{2h^5}{5} N^{(4)}_{\alpha 2}\Big)$$

$$M_{nn} = n_\beta n_\alpha \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta}\Big)$$

(3.2.19)

we obtain

$$\langle\langle \underset{\sim}{n}, \underset{\sim}{q} \rangle\rangle = -\int_{(F)} \Big[\big(2h\, N^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta,\beta}\big),\ \big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta,\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta,\alpha\beta}\big),$$

$$\big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta,\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta,\beta}\big),\ \big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta,\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta,\beta}\big)\Big] \cdot$$

$$\big[U^{(1)}_{\alpha}, U^{(2)}, U^{(3)}_{\alpha}, U^{(4)}_{\alpha}\big]^T e^{-\tau} dx_1\, dx_2\, d\tau \ -$$

$$\int_{(Z)} \Big[\big(2h\, N^{(1)}_{\alpha n} + \frac{2h^3}{3} N^{(3)}_{\alpha n}\big),\ \big(V - M_{ns,s}\big),\ \big(M_{nn}\big),$$

$$\big(\frac{2h^3}{3} N^{(1)}_{\alpha n} + \frac{2h^5}{5} N^{(3)}_{\alpha n}\big),\ \big(\frac{2h^5}{5} N^{(2)}_{\alpha n} + \frac{2h^7}{7} N^{(4)}_{\alpha n}\big)\Big] \cdot$$

$$\big[U^{(1)}_{\alpha}, U^{(2)}, U^{(2)}_{,n}, U^{(3)}_{\alpha}, U^{(4)}_{\alpha}\big]^T e^{-\tau} dx_1\, dx_2\, d\tau \tag{3.2.20}$$

Angular brackets denote super vectors, upper index "T" stands for "transposed".

From (3.2.20) conditions for statically admissible stress-representatives $\underset{\sim}{N}^{\rho}$ follow immediately: As in the interior F of the plate integral over F must vanish,it must hold:

$$\left.\begin{aligned} N^{(1)}_{\alpha\beta,\beta} &+ \frac{h^2}{3} N^{(3)}_{\alpha\beta,\beta} = 0 \\ N^{(2)}_{\alpha\beta,\alpha\beta} &+ \frac{3h^2}{5} N^{(4)}_{\alpha\beta,\alpha\beta} = 0 \\ N^{(1)}_{\alpha\beta,\beta} &+ \frac{3h^2}{5} N^{(3)}_{\alpha\beta,\beta} = 0 \\ N^{(2)}_{\alpha\beta,\beta} &+ \frac{5h^2}{7} N^{(4)}_{\alpha\beta,\beta} = 0 \end{aligned}\right\} \quad \text{in } F \tag{3.2.21}$$

On the boundary Z of the plate conditions of statical admissibility of $\underset{\sim}{N}^{\rho}$ depend on the support of the plate. Requirement of orthogonality of statical and kinematical quantities induces that either the respective kinematical quantity or the adjoint statical quantity becomes zero. The first happens on Z_k, the second on Z_s. As in (3.2.20) stress-representatives of different order appear as statical boundary quantities, independent from each other, the case of combined statical and kinematical boundary conditions (e.g. simply supported boundary of a plate) is included in this theory.

On a free boundary e.g. we obtain the following requirement for statical admissible stress representatives.

$$\begin{aligned}
&N_{\alpha n}^{(1)} + \frac{h^2}{3} N_{\alpha n}^{(3)} = 0\\
&V - M_{ns,s} = 0\\
&M_{nn} = 0\\
&N_{\alpha n}^{(1)} + \frac{2h^2}{5} N_{\alpha n}^{(3)} = 0\\
&N_{\alpha n}^{(2)} - \frac{5h^2}{7} N_{\alpha n}^{(4)} = 0
\end{aligned} \qquad (3.2.22)$$

<u>Three minimum principles for the Kirchhoff plate</u>

Be A_o the region of admissible stress states according to (2.8.29) of the unrestricted three-dimensional problem, defined by yield-condition. According to (2.8.29) solution of problem (3.2) can be constructed by minimization of functional

$$\Lambda_o(\underset{\sim}{\rho}) = \sup_{\underset{\sim}{\rho}^* \in \underset{\sim}{\rho}^o - A_o} \langle \underset{\sim}{\rho} - \underset{\sim}{\rho}^*, \dot{\underset{\sim}{\rho}} \rangle = \sup_{\underset{\sim}{\rho}^* \in \underset{\sim}{\rho}^o - A_o} \int_{(V)} (\underset{\sim}{\rho} - \underset{\sim}{\rho}^*) \underset{\sim}{G} \dot{\underset{\sim}{\rho}} e^{-\tau} dV d\tau \qquad (3.2.23)$$

for $\underset{\sim}{\rho} \in H^{\rho'}$.

(i) three-dimensional method:

Here we choose

$$\begin{aligned}
\underset{\sim}{\rho} &= [\sigma_{\alpha\beta}^{\rho}(x_1,x_2,x_3,\tau), \pi_m(x_1,x_2,x_3,\tau)] \in \underset{\sim}{S}^o - A_o\\
\underset{\sim}{\rho}^* &= [\sigma_{\alpha\beta}^{\rho *}(x_1,x_2,x_3,\tau), \pi_m^*(x_1,x_2,x_3,\tau)] \in \underset{\sim}{S}^o - A_o
\end{aligned} \qquad (3.2.24)$$

Λ_o assumes absolute minimum of value zero for the solution $\underset{\sim}{\sigma}^{\rho}(x_1,x_2,x_3,\tau) \in H^{\rho\prime}$.

(ii) two-dimensional method:
here we choose:

$$\begin{aligned} \underset{\sim}{\rho} &= [N^{\rho}_{\alpha\beta}(x_1,x_2,\tau),\ \Pi_m(x_1,x_2,\tau)] \in \underset{\sim}{\Omega}^{o} - A_{o_t} \\ \underset{\sim}{\rho}^{*} &= [N^{\rho *}_{\alpha\beta}(x_1,x_2,\tau),\ \Pi^{*}_m(x_1,x_2,\tau)] \in \underset{\sim}{\Omega}^{o} - A_{o_t} \end{aligned} \tag{3.2.25}$$

With $N^{\rho *}_{\alpha\beta}, N^{\rho}_{\alpha\beta}, N^{\rho}_{\alpha\beta}$ as two-dimensional represesentatives according to (3.1). Λ_o assumes absolute minimum of value zero for the solution $N^{\rho}_{\alpha\beta}(x_1,x_2,\tau) \in H^{\rho\prime}_t$. A_{o_t} and $H^{\rho\prime}_t$ are the region of admissible stress states and Hilbert space of statically admissible functions in two-dimensional representation resp..

(iii) mixed method:
Here we choose:

$$\begin{aligned} \underset{\sim}{\rho} &= [N^{\rho}_{\alpha\beta}(x_1,x_2,\tau),\ \Pi_m(x_1,x_2,\tau)] \in \underset{\sim}{\Omega}^{o} - A_{o_t} \\ \underset{\sim}{\rho}^{*} &= [(\sigma^{\rho *}_{\alpha\beta}(x_1,x_2,x_3,\tau),\ \pi^{*}_m(x_1,x_2,x_3,\tau)] \in \underset{\sim}{S}^{o} - A_o \end{aligned} \tag{3.2.26}$$

Λ_o attains absolute minimum for the solution $N^{\rho}_{\alpha\beta}(x_1,x_2,\tau) \in H^{\rho\prime}_t$. Its value is not necessarily equal to zero.

3.3. <u>The rate boundary value problem of the plate according to von Kârmân plate theory for elasto-plastic material behaviour</u>

Analogously to (3.2) in case of the initial boundary value problem of the Kirchhoff plate we derive the conditions for statical admissibility of two-dimensional stress rate representatives $\underset{\sim}{T}(x_1,x_2)$ by orthogonality condition for kinematically and statically admissible strain and stress rate quantities $\delta\underset{\sim}{d}(x_1,x_2,x_3)$ and $\delta\underset{\sim}{t}(x_1,x_2,x_3)$, resp.. For this purpose we define two-dimensional representatives $\underset{\sim}{D}$ and $\underset{\sim}{T}$ of three-dimensional tensors $\delta\underset{\sim}{d}$ and $\delta\underset{\sim}{t}$ according to (3.1.2 - 4):

$$D_{ij}^{p}(x_1,x_2) := \{ D_{ij}^{(1)}, D_{ij}^{(2)}, \dots D_{ij}^{(p)} \} \qquad i,j \in [1,2,3]$$

$$T_{ji}^{p}(x_1,x_2) := \{ T_{ji}^{(1)}, T_{ji}^{(2)}, \dots T_{ji}^{(p)} \} \qquad 1 < p < +\infty \tag{3.3.1}$$

The scalarproduct for the three-dimensional case is then defined according to (3.2.2) by:

$$\langle\langle \underset{\sim}{T}^p, \underset{\sim}{D}^p \rangle\rangle = \int_{(F)} \left(\sum_{k=1}^{p} \sum_{l=1}^{p} T_{ji}^{(k)} m_{kl} D_{ij}^{(l)} \right) dx_1 \, dx_2 \tag{3.3.2}$$

Just like in (3.1) we restrict our considerations to those elements $\delta\underset{\sim}{d}$ and $\delta\underset{\sim}{t}$ for which remainder of Taylor expansion according to (3.1.4 - 6) vanishes, so that analogously to (3.1.6) a unique relation exists between three-dimensional quantities $\delta\underset{\sim}{d}$ and $\delta\underset{\sim}{t}$ and two-dimensional representatives $\underset{\sim}{D}$ and $\underset{\sim}{T}$.

Splitting up (3.3.2) analogously to our procedure in case of the Kirchhoff plate, we obtain:

$$\langle\langle \underset{\sim}{T}, \underset{\sim}{D} \rangle\rangle = \langle\langle T_{\alpha\beta}, D_{\beta\alpha} \rangle\rangle + \langle\langle T_{\beta 3}, D_{3\beta} \rangle\rangle + \langle\langle T_{3\beta}, D_{\beta 3} \rangle\rangle + \langle\langle T_{33}, D_{33} \rangle\rangle \tag{3.3.3}$$

In case of the von Kármán plate we now assume that third and fourth term is small in comparison with $\langle\langle \underset{\sim}{T}, \underset{\sim}{D} \rangle\rangle$.

<u>Kinematically admissible strain tensor representatives</u>

Kinematically admissible strain tensor representatives are defined by:

$$\underset{\sim}{D}^{p\mu} := \{ D_{\alpha\beta}^{p\mu}, D_{3\beta}^{p\mu} \} = \{ U_{\alpha,\beta}^{(1)}, U_{,\alpha\beta}^{(2)}, U_{\alpha,\beta}^{(3)}, \dots U_{\alpha,\beta}^{(p)}; U_{,\alpha}^{(2)}, U_{,\alpha}^{(2)}, \dots U_{,\alpha}^{(2)} \} \tag{3.3.4}$$

Here $U_\alpha^{(i)}$ are two-dimensional displacement vectors where $U^{(2)}$ is an exception according to (3.2.14). $U^{(2)}$ may again be interpreted as deflection of the plate in x_3-direction.

Because of assumption of moderate rotations and small strains and the requirement of fulfilling the condition of balance of momentum according to [72,74] holds:

$$T_{\beta 3}^{(n)} = T_{\alpha\beta}^{(n)}\big|_0 \, U_{,\alpha}^{(2)} + T_{\alpha\beta}^{(n)} \, U_{,\alpha}^{(2)}\big|_0 \quad ; \quad T_{\alpha\beta}^{(n)} = T_{\beta\alpha}^{(n)} \tag{3.3.5}$$

Then the scalarproduct (3.3.3) becomes:

$$\langle\langle \underset{\sim}{T}^p, \underset{\sim}{D}^p \rangle\rangle = \int\limits_{(F)} \sum_{K=1}^{p} \sum_{L=1}^{p} \left[T_{\alpha\beta}^{(K)} m_{KL} D_{\beta\alpha}^{(L)} + T_{\beta 3}^{(K)} m_{KL} D_{3\beta}^{(L)} \right] dx_1 \, dx_2 \tag{3.3.6}$$

with

$$m_{KL} = \int\limits_{-h}^{+h} x_3^{K-1} \, dx_3 \tag{3.3.7}$$

<u>Example for p = 4</u>

Explicitely we obtain for p = 4

$$\langle\langle \underset{\sim}{T}^4, D^4 \rangle\rangle = \int\limits_{(F)} \Big\{ \Big[\Big(2h\, T_{\alpha\beta}^{(1)} + \frac{2h^3}{3} T_{\alpha\beta}^{(3)}\Big), \Big(\frac{2h^3}{3} T_{\alpha\beta}^{(2)} + \frac{2h^5}{5} T_{\alpha\beta}^{(4)}\Big),$$

$$\Big(\frac{2h^3}{3} T_{\alpha\beta}^{(1)} + \frac{2h^5}{5} T_{\alpha\beta}^{(3)}\Big), \Big(\frac{2h^5}{5} T_{\alpha\beta}^{(2)} + \frac{2h^7}{7} T_{\alpha\beta}^{(4)}\Big), \Big(2h\, T_{\alpha\beta}^{(1)} + \frac{2h^3}{3} T_{\alpha\beta}^{(3)}\Big)\Big|_0 U_{,\alpha}^{(2)}$$

$$\Big(2h\, T_{\alpha\beta}^{(1)} + \frac{2h^3}{3} T_{\alpha\beta}^{(3)}\Big) U_{,\alpha}^{(2)}\Big|_0 \Big] \cdot \Big[U_{\alpha,\beta}^{(1)}, U_{,\alpha\beta}^{(2)}, U_{\alpha,\beta}^{(3)}, U_{\alpha,\beta}^{(4)}, U_{,\beta}^{(2)} \Big]^T \Big\} dx_1 \, dx_2$$

$$= \int\limits_{(Z)} \Big\{ \Big[\Big(2h\, T_{\alpha n}^{(1)} + \frac{2h^3}{3} T_{\alpha n}^{(3)}\Big), \Big(\frac{2h^3}{3} T_{\alpha n}^{(2)} + \frac{2h^5}{5} T_{\alpha n}^{(4)}\Big), \Big(\frac{2h^3}{3} T_{\alpha n}^{(1)} + \frac{2h^5}{5} T_{\alpha n}^{(3)}\Big),$$

$$\Big(\frac{2h^5}{5} T_{\alpha n}^{(2)} + \frac{2h^7}{7} T_{\alpha n}^{(4)}\Big), \Big(2h\, T_{\alpha n}^{(1)} + \frac{2h^5}{5} T_{\alpha n}^{(4)}\Big)\Big|_0 U_{,\alpha}^{(2)} + \Big(2h T_{\alpha n}^{(1)} + \frac{2h^5}{5} T_{\alpha n}^{(4)}\Big) U_{,\alpha}^{(2)}\Big|_0 \Big] \cdot$$

$$\Big[U_\alpha^{(1)}; U_{,\alpha}^{(2)}, U_\alpha^{(3)}, U_\alpha^{(4)}, U^{(2)} \Big]^T \Big\} dx_1 \, dx_2 -$$

$$- \int\limits_{(F)} \Big\{ \Big[\Big(2h\, T_{\alpha\beta,\beta}^{(1)} + \frac{2h^3}{3} T_{\alpha\beta,\beta}^{(3)}\Big), \Big(\frac{2h^3}{3} T_{\alpha\beta,\beta}^{(2)} + \frac{2h^5}{5} T_{\alpha\beta,\beta}^{(4)}\Big), \Big(\frac{2h^3}{3} T_{\alpha\beta,\beta}^{(1)} + \frac{2h^5}{5} T_{\alpha\beta,\beta}^{(3)}\Big),$$

$$\left(\frac{2h^5}{5}T^{(2)}_{\alpha\beta,\beta}+\frac{2h^7}{7}T^{(4)}_{\alpha\beta,\beta}\right),\ \left(2hT^{(1)}_{\alpha\beta,\beta}+\frac{2h^3}{3}T^{(3)}_{\alpha\beta,\beta}\right)\Big|_0 U^{(2)}_{,\alpha}+\left(2hT^{(1)}_{\alpha\beta,\beta}+\frac{2h^3}{3}T^{(3)}_{\alpha\beta,\beta}\right)U^{(2)}_{,\alpha}\Big|_0\Big]\cdot$$

$$\left[U^{(1)}_{\alpha},\ U^{(2)}_{,\alpha},\ U^{(3)}_{\alpha},\ U^{(4)}_{\alpha},\ U^{(2)}\right]^T\Big\}\,dx_1\,dx_2 \tag{3.3.8}$$

Here $T^{(i)}_{\alpha n}$ is defined by

$$T^{(i)}_{\alpha n}=T^{(i)}_{\alpha\beta}\,n_\beta \tag{3.3.9}$$

With: (3.3.9)

$$\int_{(Z)}\left(\frac{2h^3}{3}T^{(2)}_{\alpha n}+\frac{2h^5}{5}T^{(4)}_{\alpha n}\right)U^{(2)}_{,\alpha}\,dx_1dx_2-\int_{(F)}\left(\frac{2h^3}{3}T^{(2)}_{\alpha\beta,\beta}+\frac{2h^5}{5}T^{(4)}_{\alpha\beta,\beta}\right)U^{(2)}_{,\alpha}\,dx_1dx_2=$$

$$=\int_{(Z)}\left[\left(\frac{2h^3}{3}T^{(2)}_{\alpha n}+\frac{2h^5}{5}T^{(4)}_{\alpha n}\right)U^{(2)}_{,\alpha}-\left(\frac{2h^3}{3}T^{(2)}_{\alpha\beta,\beta}+\frac{2h^5}{5}T^{(4)}_{\alpha\beta,\beta}\right)n_\alpha\,U^{(2)}\right]dx_1dx_2+$$

$$+\int\left(\frac{2h^3}{3}T^{(2)}_{\alpha\beta,\beta\alpha}+\frac{2h^5}{5}T^{(4)}_{\alpha\beta,\beta\alpha}\right)U^{(2)}\,dx_1dx_2 \tag{3.3.10}$$

we finally obtain

$$\langle\langle I^4,D^4\rangle\rangle=\int_{(Z)}\Big\{\left(2hT^{(1)}_{\alpha n}+\frac{2h^3}{3}T^{(3)}_{\alpha n}\right),\ \Big[\left(2hT^{(1)}_{\alpha n}+\frac{2h^3}{3}T^{(3)}_{\alpha n}\right)\Big|_0 U^{(2)}_{,\alpha}+$$

$$\left(2hT^{(1)}_{\alpha n}+\frac{2h^3}{3}T^{(3)}_{\alpha n}\right)U^{(2)}_{,\alpha}\Big|_0-\left(\frac{2h^3}{3}T^{(2)}_{\alpha\beta,\beta}+\frac{2h^5}{5}T^{(4)}_{\alpha\beta,\beta}\right)n_\alpha\Big],\left(\frac{2h^3}{3}T^{(1)}_{\alpha n}+\frac{2h^5}{5}T^{(3)}_{\alpha n}\right),$$

$$\left(\frac{2h^5}{5}T^{(2)}_{\alpha n}+\frac{2h^7}{7}T^{(4)}_{\alpha n}\right),\left(\frac{2h^3}{3}T^{(2)}_{\alpha n}+\frac{2h^5}{5}T^{(4)}_{\alpha n}\right)\Big\}\left\{U^{(1)}_{\alpha},U^{(2)},U^{(3)}_{\alpha},U^{(4)}_{\alpha},U^{(2)}_{,\alpha}\right\}^T dx_1dx_2$$

$$-\int_{(F)}\Big\{\left(2hT^{(1)}_{\alpha\beta,\beta}+\frac{2h^3}{3}T^{(3)}_{\alpha\beta,\beta}\right),\ \Big[\left(2hT^{(1)}_{\alpha\beta,\beta}+\frac{2h^3}{3}T^{(3)}_{\alpha\beta,\beta}\right)\Big|_0 U^{(2)}_{,\alpha}+\left(2hT^{(1)}_{\alpha\beta,\beta}+\frac{2h^3}{3}T^{3}_{\alpha\beta,\beta}\right)U^{(2)}_{,\alpha}\Big|_0$$

$$-\left(\frac{2h^3}{3}T^{(2)}_{\alpha\beta,\beta\alpha}+\frac{2h^5}{5}T^{(4)}_{\alpha\beta,\beta\alpha}\right)\Big],\left(\frac{2h^3}{3}T^{(1)}_{\alpha\beta,\beta}+\frac{2h^5}{5}T^{(3)}_{\alpha\beta,\beta}\right),\left(\frac{2h^5}{5}T^{(2)}_{\alpha\beta,\beta}+\frac{2h^7}{7}T^{(4)}_{\alpha\beta,\beta}\right)\Big\}$$

$$\left\{U^{(1)}_{\alpha},\ U^{(2)},\ U^{(3)}_{\alpha},\ U^{(4)}_{\alpha}\right\}^T dx_1dx_2 \tag{3.3.11}$$

Using the following abreviations we reformulate the boundary integral and obtain an expression similar to (2.2.19):

$$V_1 = -\left(\frac{2h^3}{3} T^{(2)}_{\alpha\beta,\beta} + \frac{2h^5}{5} T^{(4)}_{\alpha\beta,\beta}\right) n_\alpha$$

$$V_2 = \left(2h\, T^{(1)}_{\alpha n} + \frac{2h^3}{3} T^{(3)}_{\alpha n}\right)\Big|_o\, U^{(2)}_{,\alpha} + \left(2h\, T^{(1)}_{\alpha n} + \frac{2h^3}{3} T^{(3)}_{\alpha n}\right) U^{(2)}_{,\alpha}\Big|_o$$

$$M_{ns} = -n_2 n_\alpha \left(\frac{2h^3}{3} T^{(2)}_{\alpha 1} + \frac{2h^5}{5} T^{(4)}_{\alpha 1}\right) - n_1 n_\alpha \left(\frac{2h^3}{3} T^{(2)}_{\alpha 2} + \frac{2h^5}{5} T^{(4)}_{\alpha 2}\right)$$

$$M_{nn} = n_\beta n_\alpha \left(\frac{2h^3}{3} T^{(2)}_{\alpha\beta} + \frac{2h^5}{5} T^{(4)}_{\alpha\beta}\right)$$

(3.3.12)

Then boundary integral becomes:

$$\int_{(Z)}\Big\{\Big[\left(2h\, T^{(1)}_{\alpha n} + \frac{2h^3}{3} T^{(3)}_{\alpha n}\right),\ \left(V_1 + V_2 + M_{ns,s}\right),\ M_{nn},\ \left(\frac{2h^3}{3} T^{(1)}_{\alpha n} + \frac{2h^5}{5} T^{(3)}_{\alpha n}\right),$$
$$\left(\frac{2h^5}{5} T^{(2)}_{\alpha n} + \frac{2h^7}{7} T^{(4)}_{\alpha n}\right)\Big]\cdot\Big[U^{(1)}_\alpha, U^{(2)}, U^{(2)}_{,n}, U^{(3)}_\alpha, U^{(4)}_\alpha\Big]^T\Big\}\, dx_1\, dx_2$$

(3.3.13)

Either in (3.3.11) and in (3.3.13) first derivations of $T^{(i)}_{\alpha\beta}$ are contained as factors. However if reference state $(\ )_o$ is given, $T_{\alpha\beta}|_o$ and $T_{\alpha n}|_o$ are known, satisfying homogeneous equilibrium conditions in V and homogeneous statical boundary conditions on Z_s so that expressions

$$\left(2h\, T^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} T^{(3)}_{\alpha\beta,\beta}\right)\Big|_o\, U^{(2)}_{,\alpha}$$
$$\left(2h\, T^{(1)}_{\alpha n} + \frac{2h^3}{3} T^{(3)}_{\alpha n}\right)\Big|_o\, U^{(2)}_{,\alpha}$$

(3.3.14)

vanish identically.

In this way we obtain analogously to (2.2.21) conditions for statical admissibility of stress-representatives $\underset{\sim}{T}^{(i)}$:

$$2h\, T^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} T^{(3)}_{\alpha\beta,\beta} = 0$$
$$\left(2h\, T^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} T^{(3)}_{\alpha\beta,\beta}\right) U^{(2)}_{,\alpha}\Big|_o - \left(\frac{2h^3}{3} T^{(2)}_{\alpha\beta,\beta\alpha} + \frac{2h^5}{5} T^{(4)}_{\alpha\beta,\beta\alpha}\right) = 0$$
$$\frac{2h^3}{3} T^{(1)}_{\alpha\beta,\beta} + \frac{2h^5}{5} T^{(3)}_{\alpha\beta,\beta} = 0$$
$$\frac{2h^5}{5} T^{(2)}_{\alpha\beta,\beta} + \frac{2h^7}{7} T^{(4)}_{\alpha\beta,\beta} = 0$$

(3.3.15)

Boundary conditions on Z_s are like in (3.2) dependent on support of the plate, so that requirements for $\underset{\sim}{T}^{(i)}$ on Z_s are obtained by condition of orthogonality of adjoint kinematical and statical quantities allowing (3.3.14).

All stress rate representatives $\underset{\sim}{T}^{(i)}$ satisfying above conditions in F and boundary conditions on Z_s according to the considered problem, are called two-dimensional stress-rate representatives $\underset{\sim}{T}^{p}$ of order $p = 4$.

Now it is possible to use the relations of three-dimensional theory according to (2.5 - 6) in order to deal with the rate boundary value problem for the von Kármán plate without additional assumptions.

3.4. Three functionals for the solution of the rate boundary value problem of the plate according to the von Kârmân theory for hardening material

In opposition to (3.2 - 3) in this chapter we start from the Kirchhoff-Love-hypothesis of plane cross sections of the plate during the entire loading process. Be $v_i(x_1, x_2, x_3)$ the displacement vector of an arbitrary point of the plate, so that Green's strain tensor and its derivation with respect to time $\underset{\sim}{\dot{\varepsilon}}$ are defined by:

$$
\begin{aligned}
2\varepsilon_{IJ} &= v_{I,J} + v_{J,I} + v_{k,I} v_{k,J} \\
2\dot{\varepsilon}_{IJ} &= \dot{v}_{I,J} + \dot{v}_{J,I} + \dot{v}_{k,I} v_{k,J} + v_{k,I} \dot{v}_{k,J} \\
2\varepsilon_{\alpha\beta} &= u_{\alpha,\beta} + u_{\beta,\alpha} - X_3 u_{3,\alpha\beta} + u_{3,\alpha} u_{3,\beta} \\
2\dot{\varepsilon}_{\alpha\beta} &= \dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} - X_3 \dot{u}_{3,\alpha\beta} + \dot{u}_{3,\alpha} u_{3,\beta} + u_{3,\alpha} \dot{u}_{3,\beta}
\end{aligned}
\qquad (3.4.1)
$$

where u_α denotes displacements of points of midsurface of the plate in x_1, x_2-direction and u_3 denotes deflection in x_3-direction. Deformation gradient $\underset{\sim}{F}$ is defined by:

$$
F_{iJ} = \left(\begin{array}{c|c|c}
1 + u_{1,1} - X_3 u_{3,11} & u_{1,2} - X_3 u_{3,12} & -u_{3,1} \\
\hline
u_{2,1} - X_3 u_{3,21} & 1 + u_{2,2} - X_3 u_{3,22} & -u_{3,2} \\
\hline
u_{3,1} & u_{3,2} & 1
\end{array}\right)
\qquad (3.4.2)
$$

its derivation with respect to time accordingly.

Stresses and stress rates are not, as in case of purely elastic processes, proportional to $\underset{\sim}{\varepsilon}$ and $\dot{\underset{\sim}{\varepsilon}}$. In order to describe shape of stress distribution in x_3-direction according to (3.1) we may:

(i) cut the plates into sheets and assign to each sheet a stress-tensor depending only on x_1, x_2-coordinate.

(ii) choose polynomial test-functions in x_3-direction for the stress tensor where its free parameters decide for the shape of stress-distribution in x_3-direction.

As in (3.2 - 3.3) polynomial methods have been treated broadly we choose the first (i) possibility. Then we represent stress and stress-rate by the following sets:

$$N^{r}_{\alpha\beta}(x_1,x_2) = \{ N^{(1)}_{\alpha\beta}, N^{(2)}_{\alpha\beta}, \dots N^{(r)}_{\alpha\beta} \} \quad ; \quad N^{(i)}_{\alpha\beta} = N^{(i)}_{\beta\alpha}$$

and (3.4.3)

$$\dot{N}^{r}_{\alpha\beta}(x_1,x_2) = \{ \dot{N}^{(1)}_{\alpha\beta}, \dot{N}^{(2)}_{\alpha\beta}, \dots \dot{N}^{(r)}_{\alpha\beta} \} \quad ; \quad \dot{N}^{(i)}_{\alpha\beta} = \dot{N}^{(i)}_{\beta\alpha}$$

(see fig. 8). If we introduce these quantities into functional (2.6.41), we obtain:

$$\begin{aligned} J_1 = \int_{(F)} \frac{1}{2} \Big\{ & \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha\beta} (\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} + \dot{u}_{3,\alpha} u_{3,\beta} + u_{3,\alpha} \dot{u}_{3,\beta}) + \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha\beta} X_3 (\dot{u}_{3,\alpha\beta} + \dot{u}_{3,\beta\alpha}) \\ & + \sum_{i=1}^{r} N^{(i)}_{\alpha\beta} \dot{u}_{3,\alpha} \dot{u}_{3,\beta} + \dot{b}_\alpha \dot{u}_\alpha + \dot{b}_3 \dot{u}_3 - \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha\beta} L^{(i)}_{\alpha\beta\gamma\delta} \dot{N}^{(i)}_{\gamma\delta} \Big\} \, dX_1 \, dX_2 \\ & - \int_{(Z_s)} \Big\{ \sum_{i=1}^{r} \dot{f}^{(i)*}_{\alpha} \dot{u}_\alpha - \sum_{i=1}^{r} \dot{f}^{(i)*}_{\alpha} X^{(i)}_3 \dot{u}_{3,\alpha} + \dot{f}^{*}_3 \dot{u}_3 \Big\} \, dX_1 \, dX_2 \\ & + \int_{(Z_u)} \Big\{ \sum_{i=1}^{r} \dot{f}^{(i)}_{\alpha} (\dot{u}_\alpha - \dot{u}^{*}_\alpha) - \sum_{i=1}^{r} \dot{f}^{(i)}_{\alpha} X^{(i)}_3 (\dot{u}_{3,\alpha} - \dot{u}^{*}_{3,\alpha}) + \dot{f}_3 (\dot{u}_3 - \dot{u}^{*}_3) \Big\} \, dX_1 \, dX_2 \end{aligned}$$

(3.4.4)

$$\dot{N}^{r}_{\alpha\beta} = \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha\beta} \quad ; \quad \dot{M}^{r}_{\alpha\beta} = \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha\beta} X_3^{(i)}$$

$$\dot{N}^{r}_{\alpha n} = \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha n} \quad ; \quad \dot{M}^{r}_{\alpha n} = \sum_{i=1}^{r} \dot{N}^{(i)}_{\alpha n} X_3^{(i)} . \tag{3.4.5}$$

After transformation we obtain

$$\begin{aligned} J_1 = \frac{1}{2} \int_{(F)} \Big\{ & \dot{N}^{r}_{\alpha\beta} (\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} + \dot{u}_{3,\alpha} u_{3,\beta} + \dot{u}_{3,\beta} u_{3,\alpha}) + \dot{M}_{\alpha\beta} (\dot{u}_{3,\alpha\beta} + \dot{u}_{3,\beta\alpha}) \\ & + N^{r}_{\alpha\beta} \dot{u}_{3,\alpha} \dot{u}_{3,\beta} + \dot{b}_{\alpha} \dot{u}_{\alpha} + \dot{b}_3 \dot{u}_3 - \sum_{i=1}^{r} L^{(i)}_{\alpha\beta\gamma\delta} \dot{N}_{\alpha\beta} \dot{N}_{\gamma\delta} \Big\} \, dX_1 \, dX_2 \\ & - \int_{(Z_s)} (\dot{N}_{\alpha n} \dot{u}_{\alpha} - \dot{M}_{\alpha n} \dot{u}_{3,\alpha} + \dot{V} \dot{u}_3) \, dX_1 \, dX_2 \\ & + \int_{(Z_k)} [\dot{N}_{\alpha n} (\dot{u}_{\alpha} - \dot{u}^{*}_{\alpha}) - \dot{M}_{\alpha n} (\dot{u}_{3,\alpha} - \dot{u}^{*}_{3,\alpha}) + \dot{V} (\dot{u}_3 - \dot{u}^{*}_3)] \, dX_1 \, dX_2 \end{aligned} \tag{3.4.6}$$

This mixed functional does not have extremum properties (2.6), but is very convenient in application because test functions for rates of stresses and strains may be chosen indepedently from each other. A numerical example will be given in (4.3).

If we choose e.g. in (2.6.57,59) k = 3, we obtain for $\dot{N}^{(i)}_{\alpha\beta} L^{(i)}_{\gamma\beta\lambda\delta} \dot{N}^{(i)}_{\lambda\delta}$:

$$\begin{pmatrix} \dot{N}^{(i)}_{11} \\ \dot{N}^{(i)}_{22} \\ \dot{N}^{(i)}_{12} \end{pmatrix}^{T} \cdot \left(\begin{array}{c:c:c} \frac{1}{E} - \frac{27}{4E_0^3} \big[N^{(i)}_{11} - \frac{1}{2} (N^{(i)}_{11} - N^{(i)}_{22}) \big]^2 & -\frac{\nu}{E} - \frac{27}{4E_0^3} \big[N^{(i)}_{11} - \frac{1}{2} (N^{(i)}_{11} + N^{(i)}_{22}) \big] \cdot \big[N^{(i)}_{22} - \frac{1}{2} (N^{(i)}_{11} - N^{(i)}_{22}) \big] & \frac{27}{2E_0^3} \big[N^{(i)}_{11} - \frac{1}{2} (N^{(i)}_{11} + N^{(i)}_{22}) \big] \cdot N^{(i)}_{12} \\ \hdashline * & \frac{1}{E} - \frac{27}{4E_0^3} \big[N^{(i)}_{11} - \frac{1}{2} (N^{(i)}_{11} - N^{(i)}_{22}) \big]^2 & \frac{27}{2E_0^3} \big[N^{(i)}_{22} - \frac{1}{2} (N^{(i)}_{11} - N^{(i)}_{22}) \big] \cdot N^{(i)}_{12} \\ \hdashline * & * & \frac{4(1+\nu)}{E} + \frac{27}{E_0^3} N^{(i)\,2}_{12} \end{array} \right) \cdot \begin{pmatrix} \dot{N}_{11} \\ \dot{N}_{22} \\ \dot{N}_{12} \end{pmatrix}$$

(3.4.7)

Index ( )* denotes here and in the following prescribed quantities on the boundary of the plate, $f_\alpha^{(i)}$ denotes forces distributed along the boundary of the plate in x_1, x_2-direction, V denotes the resulting force on the boundary of the plate in x_3-direction, $x_3^{(i)}$ is the distance between component $N_{\alpha\beta}^{(i)}$ of stress representative and midspan of the plate.

If we use functional (2.6.53a) having extremum properties under conditions derived in (2.6), we obtain with (3.2.19) and with definitions of boundary quantities M_{nn}, M_{ns}, V the following expression:

$$\begin{aligned}
J_2 = \frac{1}{2}\int_{(F)} \Big\{ & \frac{1}{4} K^1_{\alpha\beta\gamma\delta} (\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} + \dot{u}_{3,\alpha} u_{3,\beta} + \dot{u}_{3,\beta} u_{3,\alpha})(\dot{u}_{\gamma,\delta} + \dot{u}_{\delta,\gamma} + \dot{u}_{3,\gamma} u_{3,\delta} + \dot{u}_{3,\delta} u_{3,\gamma}) \\
& + \frac{1}{4} K^2_{\alpha\beta\gamma\delta} (\dot{u}_{3,\alpha\beta} + \dot{u}_{3,\beta\alpha})(\dot{u}_{3,\gamma\delta} + \dot{u}_{3,\delta\gamma}) + \\
& + \frac{1}{2} K^3_{\alpha\beta\gamma\delta} (\dot{u}_{3,\alpha\beta} + \dot{u}_{3,\beta\alpha})(\dot{u}_{\gamma,\delta} + \dot{u}_{\delta,\gamma} + \dot{u}_{3,\gamma} u_{3,\delta} + \dot{u}_{3,\delta} u_{3,\gamma}) \\
& + \frac{1}{2} N^r_{\alpha\beta}\, \dot{u}_{3,\alpha} \dot{u}_{3,\beta} + \dot{f}_\alpha \dot{u}_\alpha + \dot{f}_3 \dot{u}_3 \Big\}\, dx_1\, dx_2 + \\
& + \int_{(Z_3)} [\dot{M}^*_{nn} \dot{u}_{3,n} + \dot{N}^*_\alpha \dot{u}_\alpha + (\dot{M}^*_{ns,s} + \dot{V}^*) \dot{u}_3]\, dx_1\, dx_2 - \dot{M}^r_{ns} \dot{u}^*_3 \Big|_{Z_K}
\end{aligned} \tag{3.4.8}$$

With

$$K^1_{\alpha\beta\gamma\delta} = \sum_{i=1}^{r} L^{(i)}_{\alpha\beta\gamma\delta}; \quad K^2_{\alpha\beta\gamma\delta} = \sum_{i=1}^{r} L^{(i)}_{\alpha\beta\gamma\delta} X_3^{(i)2}; \quad K^3_{\alpha\beta\gamma\delta} = \sum L_{\alpha\beta\gamma\delta} X_3^{(i)} \tag{3.4.9}$$

$L^{(i)}_{\alpha\beta\gamma\delta}$ is defined in (3.4.7). Convexity of (3.4.8) is assured if, analogously to (2.2.60) in three-dimensional analysis, $L^{(i)}_{\alpha\beta\gamma\delta}$ and $N^r_{\alpha\beta}$ are positive definite. Necessary condition is that $N^r_{\alpha\alpha}$ and $N^r_{11} - N^{r2}_{12}$ are positive. This requirement corresponds to the requirement in [72] of positiv definit membrane stress-tensor. Interesting is also the difference to [72]: Whilst in this chapter conditions of convexity concern only the given reference state, in [72] the corresponding requirement concerns unknown state of solution of the problem and is in this way an assumption. This difference however, is essentially connected with linearization of the problem in the neighbourhood of the reference state $(\ )_o$. If nonlinear terms in the rates would be considered, conditions equivalent to those in [72] should be posed as assumptions on the herein treated rate problem.

Functional (3.4.8) is treated in (4.3) numerically. We shall see that analytical effort in using (3.4.8) is much bigger than in using (3.4.6), however for comparable degree of approximation number of unknown scalar quantities entering the linear system of equations is much smaller so that the needed time of calculation is much shorter and numerical errors are less important.

If we use functional (2.6.35) we obtain:

$$
\begin{aligned}
J_3 = &- \int_{(F)} \Big\{ \frac{1}{2} \sum_{i=1}^{r} \dot N^{(i)}_{\alpha\beta} L^{(i)}_{\alpha\beta\gamma\delta} \dot N^{(i)}_{\gamma\delta} + \frac{1}{2} N^{r}_{\alpha\beta} \dot u_{3,\alpha} \dot u_{3,\beta} \Big\} dX_1 dX_2 \\
&+ \int_{(Z_\kappa)} \Big\{ \big[(\dot N^{r}_{\alpha\beta} n_\beta u_{3,\alpha} + N^{r}_{\alpha\beta} n_\beta \dot u_{3,\alpha}) (\dot u_3^{*} - \dot u_3) + \\
&+ \sum_{i=1}^{r} N^{(i)}_{\alpha\beta} n_\beta \big[\dot u_\alpha - \dot u^{*}_\alpha - X_3^{(i)} (\dot u_{3,\alpha} - \dot u^{*}_{3,\alpha}) \big] \big] \Big\} dX_1 dX_2
\end{aligned}
\qquad (3.4.10)
$$

Difficulties in use of this functional arise from the fact that displacement gradients must be kinematically and statically admissible as follows from (2.6.46, 2.08 - 13). This has been discussed in [8] for the rate problem of rigid plastic material behaviour. However in order to obtain global and pointwise error bounds analogously to theory of elasticity, investigations in this direction, up to now not succesful, should be continued in using functional (3.4.10).

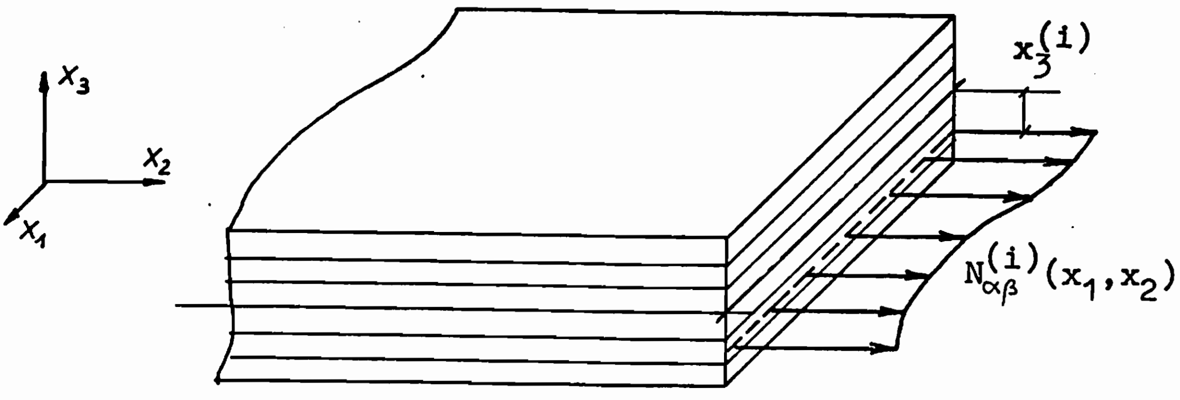


Fig. 9

4. NUMERICAL RESULTS

4.1. Calculation of stress state in a quadratic plate according to (3.2) without hardening

A quadratic, homogeneous, on boundary Z simply supported plate is loaded proportionally by a distributed load q orthogonal to the midspan of the plate assuming for plane vertical boundary surface. Material behaviour be elasto-idealplastic. Assuming for infinitesimal displacements we apply the method of approximation of stress state in the plate derived in chapter (3.2). Tresca's and von Mises' yield-conditions are used parallely. Load is symmetrically and sinusoidally distributed over the plate with:

$$q(x_1,x_2) = q_o \cdot \cos\left(\frac{\pi}{2a} x_1\right) \cdot \cos\left(\frac{\pi}{2a} x_2\right)$$

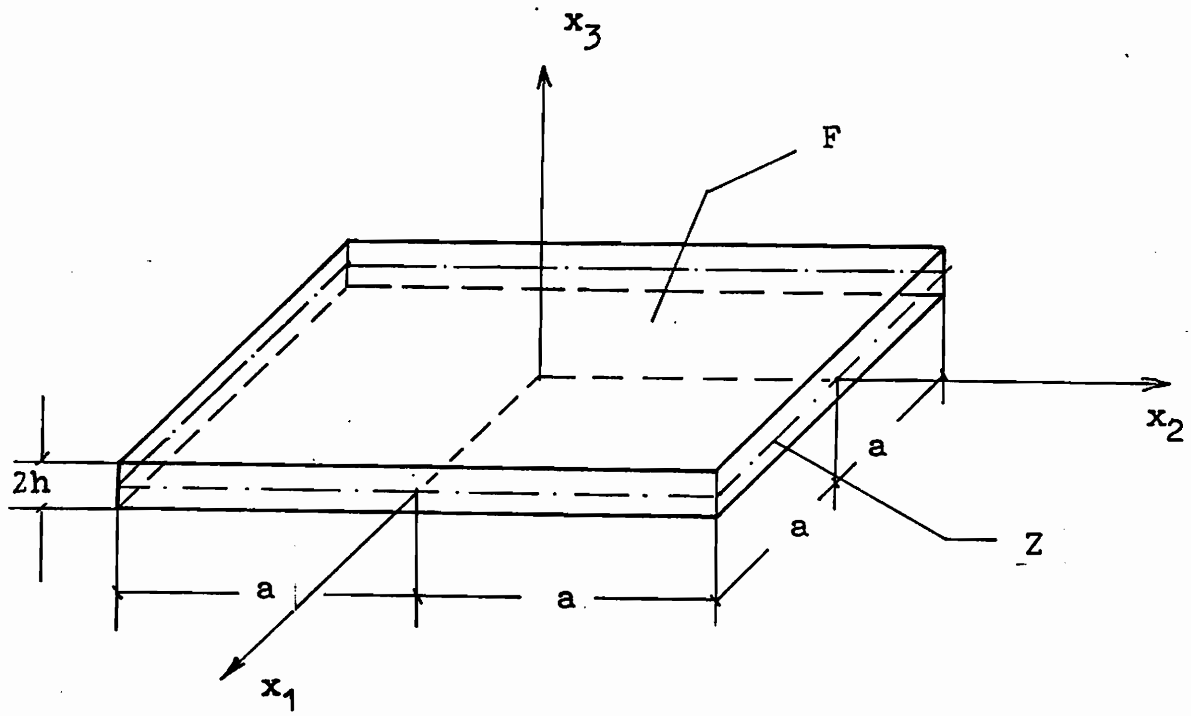


Fig. 10

Following (3.1.7 - 10) we introduce two-dimensional representatives $N^n_{\alpha\beta}$ and $E^n_{\alpha\beta}$ for stress- and strain tensors. Here we restrict on representatives up to order two.

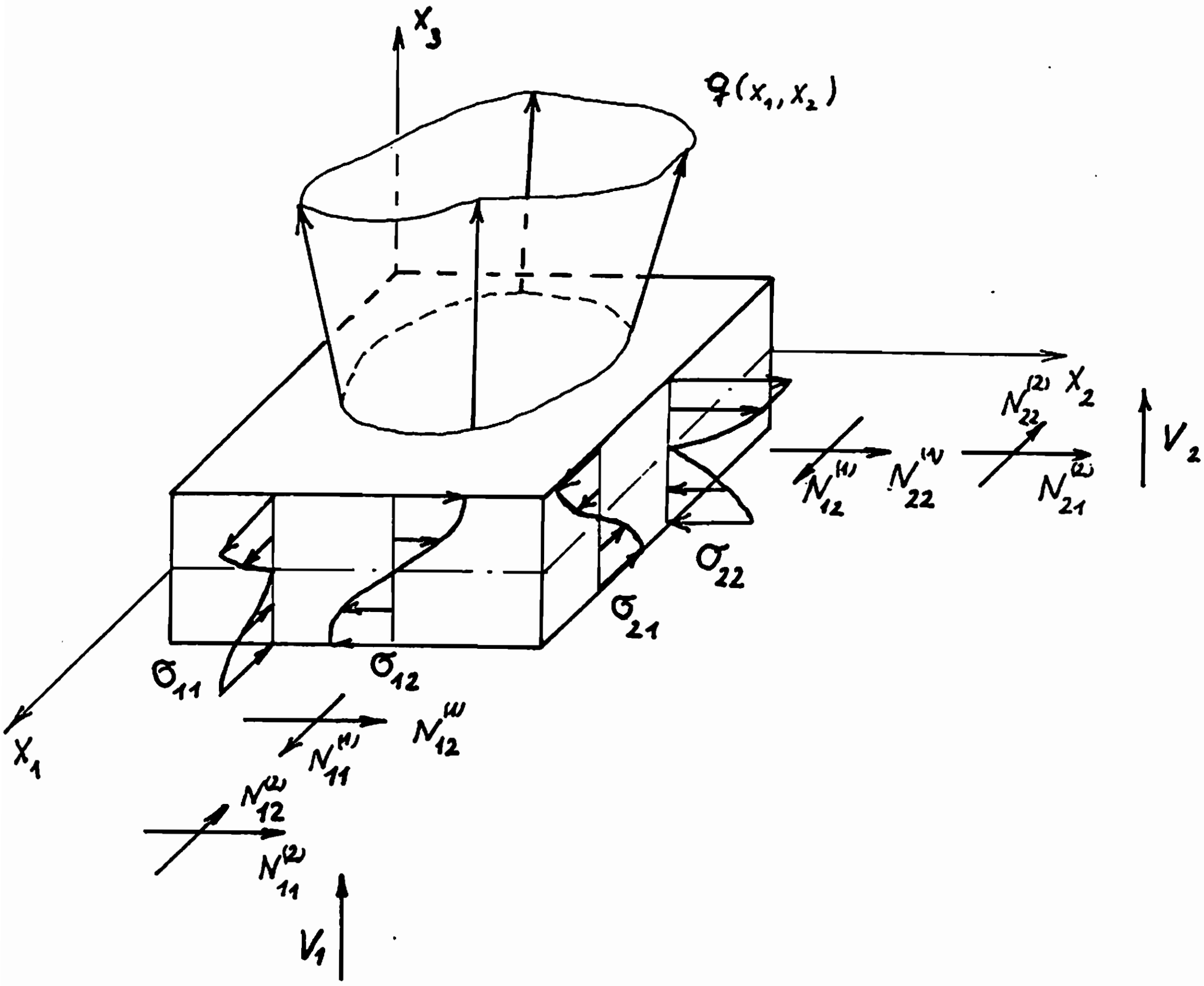


Fig. 11

So the scalarproduct (3.2.2) becomes:

$$\langle \underset{\sim}{N}^2, \underset{\sim}{E}^2 \rangle = \int_{(F)} [N_{\alpha\beta}^{(1)} \frac{1}{2h} E_{\alpha\beta}^{(1)} + N_{\alpha\beta}^{(2)} \frac{3}{2h^3} E_{\alpha\beta}^{(2)}] \, dx_1 \, dx_2 \qquad (4.1.1)$$

Kinematically admissible strain representatives $\underset{\sim}{E}^{\mu}$ are given according to (3.2.12 - 14) by:

$$E_{\alpha\beta}^{(1)\mu} = \frac{1}{2} (U_{\alpha,\beta}^{(1)} + U_{\beta,\alpha}^{(1)})$$

$$E_{\alpha\beta}^{(2)\mu} = \frac{1}{2} (U_{\alpha,\beta}^{(2)} + U_{\beta,\alpha}^{(2)}) \qquad (4.1.2)$$

With

$$U_{\alpha}^{(1)} = 2h \cdot u_{\alpha}$$

$$U_{\alpha}^{(2)} = - \frac{2h^3}{3} u_{3,\alpha}$$

$$u_{\alpha} = u_3 = 0 \qquad \text{on } Z_k$$

Condition of orthogonality (3.2.15) then defines statically admissible stress-representatives:

$$\langle \underset{\sim}{N}^{2\rho}, \underset{\approx}{E}^{2\mu} \rangle = \int_{(F)} \left[\frac{1}{2h} N^{(1)\rho}_{\alpha\beta}, \frac{3}{2h^3} N^{(2)\rho}_{\alpha\beta} \right] \left[U^{(1)}_{\alpha,\beta}, U^{(2)}_{,\alpha\beta} \right]^T dx_1\, dx_2$$

$$= -\int_{(F)} \left[\frac{1}{2h} N^{(1)\rho}_{\alpha\beta,\beta}, \frac{3}{2h^3} N^{(2)\rho}_{\alpha\beta,\beta} \right] \left[U_\alpha, U^{(2)} \right]^T dx_1\, dx_2$$

$$+ \int_{(Z)} \left[\frac{1}{2h} N^{(1)\rho}_{\alpha n}, (V + M_{ns,s})^\rho, M^\rho_{nn} \right] \left[U^{(1)}_\alpha, U^{(2)}, U^{(2)}_{,n} \right]^T dx_1\, dx_2$$

$$+ \frac{3}{2h^3} N^{(2)\rho}_{\alpha\beta} n_\alpha n_\beta \, U^{(2)} \Big|_{C_-}^{C_+} = 0 \qquad (4.1.3)$$

For plates simply supported on the entire boundary, u_α and u_3 are identically zero on the boundary. Only $u_{3,n}$ is not prescribed such that statically admissible stress representatives must satisfy:

$$\left.\begin{aligned} N^{(1)\rho}_{\alpha\beta,\alpha} &= 0 \\ N^{(2)\rho}_{\alpha\beta,\beta\alpha} &= 0 \end{aligned}\right\} \quad \text{in F}$$

$$M^\rho_{nn} = 0 \qquad \text{on Z} \qquad (4.1.4)$$

Because of assumptions of infinitesimal displacements and vertical loading, in our case $N^{(1)\rho}_{\alpha\beta}$ vanishes identically in F and on Z.

Definition of the region of admissible stress-states

In the following we shall use Tresca's and von Mises' yield condition to describe the region of plastically admissible stress states. Using principal stresses σ_1, σ_2, σ_3 we obtain for the general three-dimensional case:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \leq 2\,\sigma_s^2 \qquad (4.1.5)$$

$$|\sigma_2-\sigma_3| \leq \sigma_s \qquad \sigma_2>\sigma_1>\sigma_3$$
$$|\sigma_3-\sigma_1| \leq \sigma_s \qquad \sigma_3>\sigma_2>\sigma_1 \tag{4.1.6}$$
$$|\sigma_1-\sigma_2| \leq \sigma_s \qquad \sigma_1>\sigma_2>\sigma_3$$

With (4.1.5) and (4.1.6) as yield conditions following von Mises and Tresca, respectively. σ_s denotes yield limit from uniaxial tension test. For plates we obtain [1]:

$$\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2 \leq \sigma_s^2 \tag{4.1.7}$$

for condition (4.1.5) and

$$\begin{aligned} &|\sigma_1-\sigma_2| \leq \sigma_s && \text{if } \sigma_1\sigma_2 \leq 0 \\ &\left.\begin{aligned} |\sigma_1| &\leq \sigma_s \\ |\sigma_2| &\leq \sigma_s \end{aligned}\right\} && \text{if } \sigma_1\sigma_2 > 0 \end{aligned} \tag{4.1.8}$$

instead of (4.1.6). With

$$\sigma_{1,2} = \frac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11}-\sigma_{22})^2+\sigma_{12}^2} \tag{4.1.10}$$

(4.1.7) may be transformed to

$$\sigma_{11}^2+\sigma_{22}^2-\sigma_{11}\sigma_{22}+3\sigma_{12}^2 \leq \sigma_s^2 \tag{4.1.11}$$

and (4.1.8) becomes

$$\begin{aligned} &(\sigma_{11}-\sigma_{22})^2+4\sigma_{12}^2 \leq \sigma_s^2 && \text{if } \sigma_{11}\sigma_{22} \leq \sigma_{12}^2 \\ &\frac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11}-\sigma_{22})^2+\sigma_{12}^2} \leq \sigma_s && \text{if } \sigma_{11}\sigma_{22} > \sigma_{12}^2 \end{aligned} \tag{4.1.12}$$

Using three-dimensional approach, stress state is defined by $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_1,x_2,x_3)$ such that region of admissible stress state is also defined in three-dimensional space.

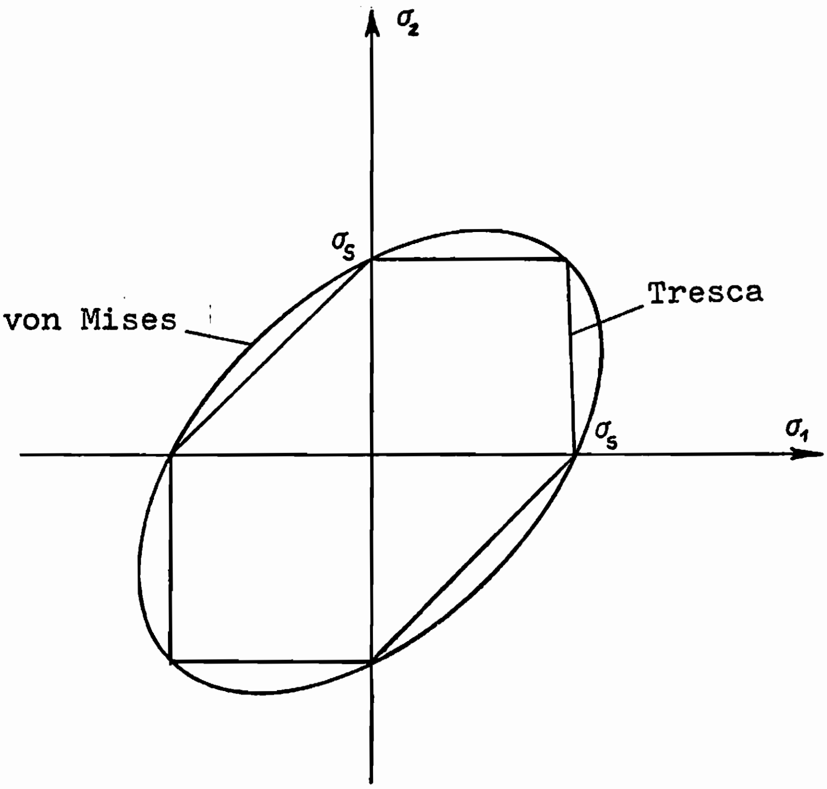


Fig. 12: Plane representation of yield-conditions (three-dimensional problem)

Using two-dimensional method (3.2.25) stress state $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(x_1,x_2,x_3)$ is represented by $\underset{\sim}{N}^2$ with the inverse relation (3.1.9):

$$\sigma_{\alpha\beta}(x_1,x_2,x_3) = \frac{3}{2h^3} N^{(2)}_{\alpha\beta}(x_1,x_2)\cdot x_3 + \frac{1}{2h} N^{(1)}_{\alpha\beta}(x_1,x_2) \qquad (4.1.13)$$

For example 1 stress representative $\underset{\sim}{N}^2$ is in the admissible region if $\overset{+}{\sigma}_{\alpha\beta}(x_1,x_2) \equiv \sigma_{\alpha\beta}(x_1,x_2,h)$ is inside the admissible region according to (4.1.7), (4.1.8) resp., with:

$$\overset{+}{\sigma}_{\alpha\beta}(x_1,x_2) = \frac{3}{2h^2} N^{(2)}_{\alpha\beta}(x_1,x_2) \qquad (4.1.14)$$

Then we have for von Mises' yield condition

$$N_{11}^{(2)} + N_{22}^{(2)} - N_{11}^{(2)} N_{22}^{(2)} + 3 N_{12}^{(2)2} \leqslant N_s^{(2)2}$$

$$(4.1.15)$$

with $\quad N_s^{(2)} = \frac{2h^2}{3}\sigma_s$

and for Tresca's yield condition

$$
\begin{aligned}
&(N_{11}^{(2)} - N_{22}^{(2)})^2 + 4N_{12}^{(2)2} \leq N_s^{(2)2} && \text{if } N_{11}^{(2)}N_{22}^{(2)} \leq N_{12}^{(2)2} \\
&\tfrac{1}{2}(N_{11}^{(2)} + N_{22}^{(2)})^2 \pm \sqrt{\tfrac{1}{4}(N_{11}^{(2)} - N_{22}^{(2)})^2 + N_{12}^{(2)2}} \leq N_s^{(2)} && \text{if } N_{11}^{(2)}N_{22}^{(2)} > N_{12}^{(2)2}
\end{aligned}
\qquad (4.1.16)
$$

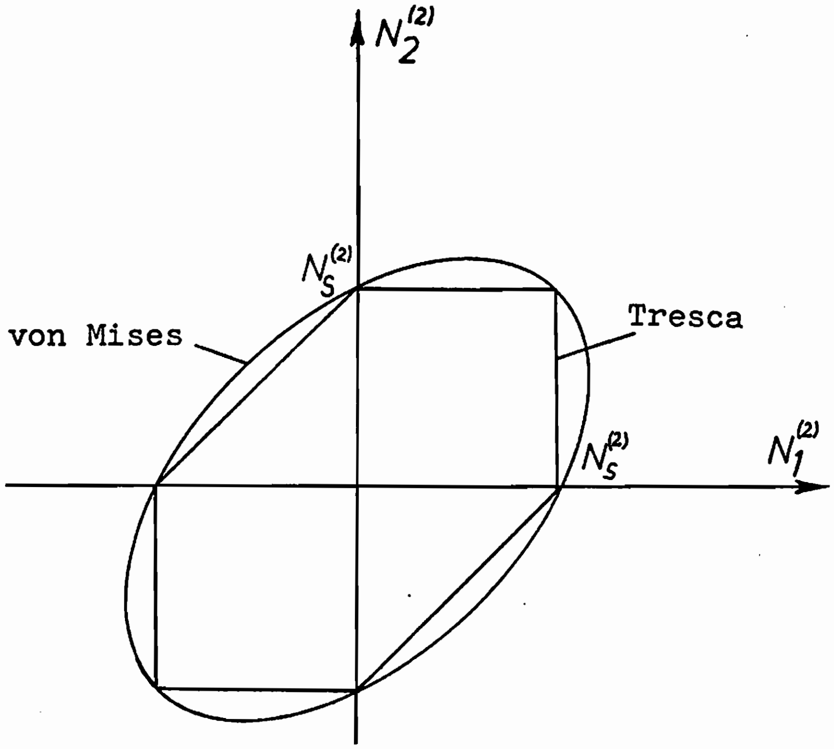
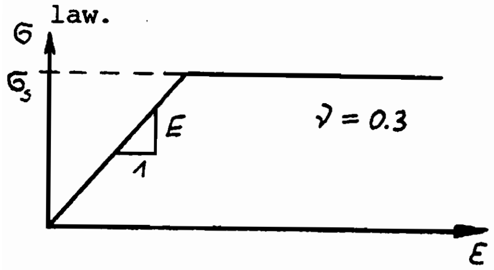


Fig. 13: Plane representation of yield-condition
(two-dimensional problem)

Assumed material behaviour

For our examples 1 and 2 we use the steel with designation of U.S.-standard 1025 carbon steel (NACA technical note N° 902) [25]. This material can be well approximated by an elasto-idealplastic material law.



with yield limit $\sigma_s = 3.792 \cdot 10^4 \frac{N}{cm^2}$

modulus of elasticity $E = 2.0167 \cdot 10^7 \frac{N}{cm^2}$

so that $\sigma_s/E = 1.80 \cdot 10^{-3}$

Fig. 14

Purely elastic solution

For example 1 purely elastic solution is analytically known [85]:

$$N_{\alpha\alpha}^{o(2)} = \frac{a^2 q_o}{\pi^2}(1+\nu)\sin\left[\frac{\pi}{2}\left(\frac{x_1}{a}+1\right)\right]\cdot\sin\left[\frac{\pi}{2}\left(\frac{x_2}{a}+1\right)\right] \qquad \text{no summation}$$

$$N_{\alpha\beta}^{o(2)} = -\frac{a^2 q_o}{\pi}(1-\nu)\cos\left[\frac{\pi}{2}\left(\frac{x_1}{a}+1\right)\right]\cdot\cos\left[\frac{\pi}{2}\left(\frac{x_2}{a}+1\right)\right] \qquad \alpha \neq \beta$$

(4.1.17)

With ν as Poisson's ratio and Q_o as load parameter. Following (4.1.13) three-dimensional stress-distribution is therefore given by:

$$\sigma_{\alpha\beta}^{o} = \frac{3}{2h^3} N_{\alpha\beta}^{o(2)} x_3 \qquad (4.1.18)$$

Three-dimensional approach

A very simple class of shape functions is characterized by the requirement of vanishing of residual moments $N_{\alpha\beta}^{\rho(2)}$ induced by plasticity.

$$N_{\alpha\beta}^{\rho(2)} = 0 \qquad (4.1.19)$$

So we may use the following approximating test functions over the cross section of the plate:

$$\sigma_{\alpha\beta}^{\rho}(x_1, x_2, x_3) = \sum_{i=1}^{4} b_{\alpha\beta}^{(i)} x_3^{i-1} \qquad (4.1.20)$$

with $b_{\alpha\beta}^{(i)}(x_1, x_2)$ as unknown coordinate-functions. As q_o acts only in x_3-direction in example 1 we have:

$$b_{\alpha\beta}^{(1)} = b_{\alpha\beta}^{(3)} = b_{\alpha\beta}^{(5)} = 0 \qquad (4.1.21)$$

Written in detail condition (4.1.19) becomes:

$$\int_{-h}^{+h} \left(b_{\alpha\beta}^{(2)} x_3 + b_{\alpha\beta}^{(4)} x_3^3\right) x_3 \, dx_3 = 0 \;\Rightarrow\; b_{\alpha\beta}^{(4)} = -\frac{5}{3h^2} b_{\alpha\beta}^{(2)} \qquad (4.1.22)$$

If we use test-functions

$$\begin{aligned} b^{(2)}_{\alpha\alpha}(X_1,X_2) &= c_{\alpha\alpha}\,(a^2 - X_\alpha^2) && \text{no summation} \\ b^{(2)}_{\alpha\beta}(X_1,X_2) &= c_{\alpha\beta}\,X_1 X_2 && \alpha \neq \beta \end{aligned} \tag{4.1.23}$$

then statical boundary conditions are identically fulfilled. If we allow for symmetry of the problem, we obtain:

$$\begin{aligned} c_{11} &= c_{22} = c_1 \\ c_{12} &= c_{21} = c_2 \end{aligned} \tag{4.1.24}$$

This means that we introduce into functional (3.2.23) test-functions with only two free parameters:

$$\underset{\sim}{\rho} = \begin{cases} \rho_{\alpha\alpha} = c_1\left(X_3 - \frac{5}{3h^2}\right)(a^2 - X_\alpha^2)\,; & \text{no summation} \\ \rho_{\alpha\beta} = c_2\left(X_3 - \frac{5}{3h^2}\right)X_\alpha X_\beta & \alpha \neq \beta \end{cases} \tag{4.1.25}$$

With (4.1.25) and A_o as region of admissible functions $\underset{\sim}{\sigma}^o - \underset{\sim}{\rho}$ we obtain the functional:

$$\Lambda_o = \sup_{\underset{\sim}{\rho}^* \in \sigma^o - A_o} \iint_{(F)} \int_{-h}^{+h} \Big\{ \frac{1}{E}\left[(\rho_{11}+\rho_{22})^2 + 2(1+\nu)(\rho_{12}^2 - \rho_{11}\rho_{22}) - \right.$$

$$\left. \frac{1}{E}\Big[(\rho_{11}+\rho_{22})(\rho^*_{11}+\rho_{22}) + 2(1+\nu)(\rho_{12}\rho^*_{12} - \frac{1}{2}(\rho_{11}\rho^*_{22} + \rho^*_{11}\rho_{22})\Big]\right\} dX_3\, dX_1\, dX_2 \tag{4.1.26}$$

After inserting (4.1.25) we obtain for (3.2.25) the following expression:

$$\frac{\Lambda_o}{Ea^2 2h} = \sup_{\underset{\sim}{\tilde\rho}^* \in \underset{\sim}{\tilde\sigma}^o - \tilde A_o} \left\{ \frac{4}{63}\left[\frac{4}{5}(\tilde c_1^2 - \tilde c_1 \tilde c_2^*) + \frac{13}{45}(\tilde c_2^2 - \tilde c_2 \tilde c_2^*)\right]\right\} \tag{4.1.27}$$

here $(\tilde{\ })$ denotes dimensionless quantities with:

$$\begin{aligned} \tilde c_{1,2} &= \frac{c_{1,2}\,a^2\,2h}{E} & \tilde X_\alpha &= \frac{X_\alpha}{a} \\ \tilde\rho_{\alpha\beta} &= \frac{\rho_{\alpha\beta}}{E}\left(\frac{a}{2h}\right)^2 & \tilde X_3 &= \frac{X_3}{2h} \\ \tilde\sigma_{\alpha\beta} &= \frac{\sigma_{\alpha\beta}}{E}\left(\frac{a}{2h}\right)^2 & \tilde q_o &= \frac{q_o}{E}\left(\frac{a}{2h}\right)^4 \end{aligned} \tag{4.1.28}$$

Numerical results

With $\nu = 0.3$ we obtain purely elastic solution as:

$$\tilde{\sigma}^{o}_{\alpha\alpha} = \frac{A_1}{\pi^2} \sin\left[\frac{\pi}{2}(\tilde{X}_1+1)\right]\cdot \sin\left[\frac{\pi}{2}(\tilde{X}_2+1)\right]\cdot \tilde{X}_3 \quad \text{no summation}$$

$$\tilde{\sigma}^{o}_{\alpha\beta} = \frac{A_2}{\pi^2} \cos\left[\frac{\pi}{2}(\tilde{X}_1+1)\right]\cdot \cos\left[\frac{\pi}{2}(\tilde{X}_2+1)\right]\cdot \tilde{X}_3 \quad \alpha \neq \beta \tag{4.1.29}$$

So additional stresses induced by plastification are defined by

$$\tilde{\sigma}^{g}_{\alpha\alpha} = B_1\left(\tilde{X}_3 - \frac{20}{3}\tilde{X}_3^{3}\right)\left(1 - \tilde{X}_\alpha^{2}\right) \quad \text{no summation}$$

$$\tilde{\sigma}^{g}_{\alpha\beta} = B_2\left(\tilde{X}_3 - \frac{20}{3}\tilde{X}_3^{3}\right) \tilde{X}_\alpha \tilde{X}_\beta \quad \alpha \neq \beta \tag{4.1.30}$$

such that entire three-dimensional stress-distribution in the plate is given by:

$$\tilde{\sigma}_{\alpha\alpha} = \left\{\frac{A_1}{\pi^2} \sin\left[\frac{\pi}{2}(\tilde{X}_1+1)\right]\sin\left[\frac{\pi}{2}(\tilde{X}_2+1)\right] - B_1\left(1 - \frac{20}{3}\tilde{X}_3^{2}\right)\left(1-\tilde{X}_\alpha^{2}\right)\right\}\tilde{X}_3$$

$$\tilde{\sigma}_{\alpha\beta} = \left\{\frac{A_2}{\pi^2} \cos\left[\frac{\pi}{2}(\tilde{X}_1+1)\right]\cos\left[\frac{\pi}{2}(\tilde{X}_2+1)\right] - B_2\left(1 - \frac{20}{3}\tilde{X}_3^{2}\right)\tilde{X}_\alpha \tilde{X}_\beta\right\}\tilde{X}_3 \tag{4.1.31}$$

Numerical values

$\tilde{q}_o$	A_1	A_2	B_1	B_2
1.0	15.6	8.4	- 0.2109	- 0.0299
1.15	17.9	9.7	- 0.5655	- 0.2221

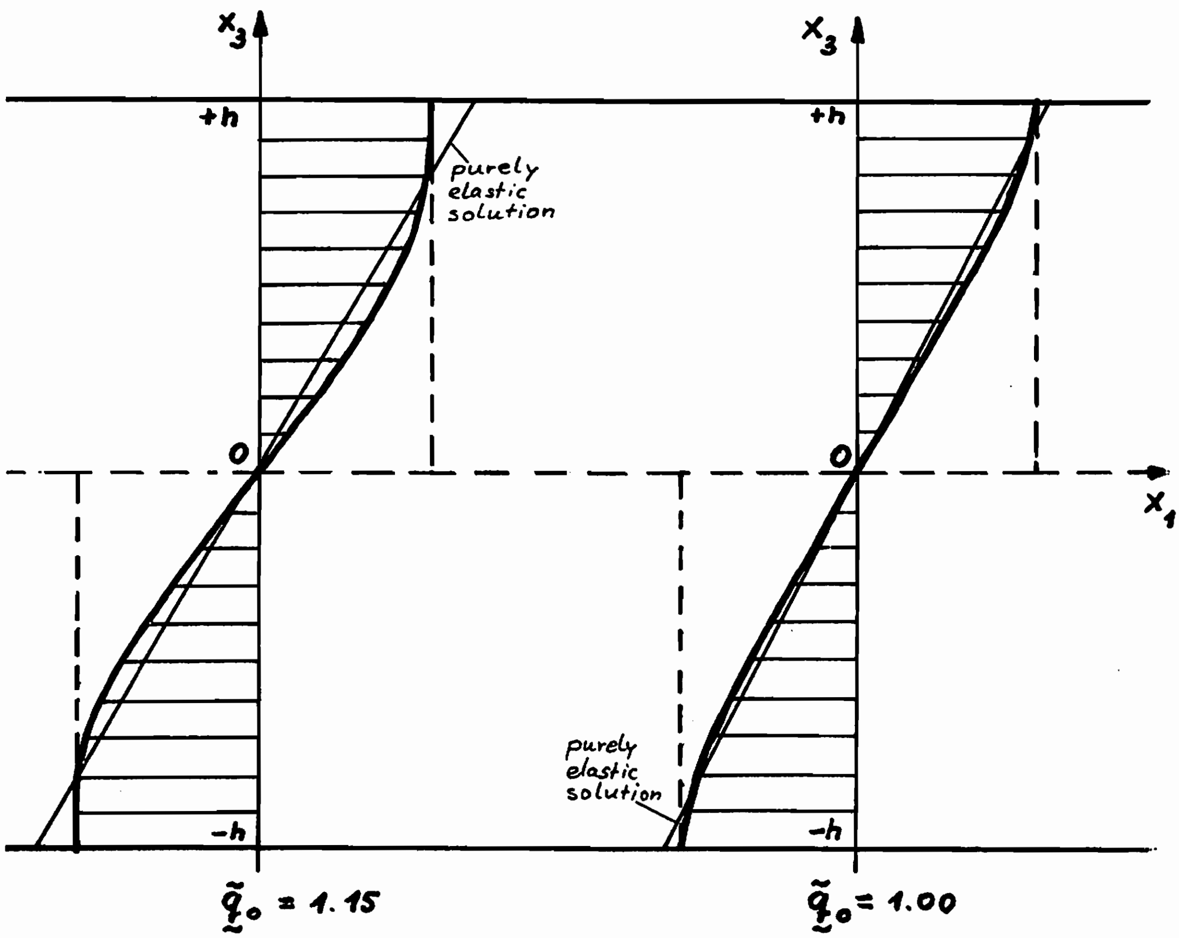


Fig. 15: stress-distribution over the cross section of the plate in the middle of the plate in x_1-direction

Two-dimensional approach

Here we use test-functions $N^{\rho(2)}_{\alpha\beta}$, satisfying equilibrium conditions in F and on Z.

$$N^{\rho(2)}_{11} = c_1(1-\tilde{X}_1^2)(1-\tilde{X}_2^2) + c_2(1-\tilde{X}_1^2)(1-\tilde{X}_2^4)$$

$$N^{\rho(2)}_{22} = c_3(1-\tilde{X}_2^2)(1-\tilde{X}_1^2) + c_4(1-\tilde{X}_2^2)(1-\tilde{X}_1^4) \qquad (4.1.32)$$

$$N^{\rho(2)}_{12} = c_5[2\tilde{X}_1\tilde{X}_2 - \tfrac{1}{3}(\tilde{X}_1^3\tilde{X}_2 + \tilde{X}_2^3\tilde{X}_1)] + c_6[2\tilde{X}_1\tilde{X}_2 - \tfrac{1}{5}(\tilde{X}_1^5\tilde{X}_2 + \tilde{X}_2^5\tilde{X}_1)]$$

with

$$N^{\rho(2)}_{11}(\tilde{X}_2 = \pm 1) = N^{\rho(2)}_{22}(\tilde{X}_1 = \pm 1) = N^{\rho(2)}_{12}(\tilde{X}_1 = \tilde{X}_2 = 0) = 0 \qquad (4.1.33)$$

From symmetry of loading and geometry of the considered plate follows:

$$c_1 = c_3 \quad ; \quad c_2 = c_4 \tag{4.1.34}$$

Equilibrium condition in the interior is given by:

$$\begin{aligned} N^{\rho(2)}_{11,11} + N^{\rho(2)}_{22,22} + 2N^{\rho(2)}_{12,12} &= -2\left[c_1(1-x_2^2) + c_2(1-x_2^4) + \right. \\ &\left. c_1(1-x_1^2) + c_2(1-x_1^4)\right] + 2\left[c_5(2-x_1^2-x_2^2) + c_6(2-x_1^4-x_2^4)\right] \end{aligned} \tag{4.1.35}$$

From this follows: $c_5 = -c_1$, $c_6 = -c_2$.

So statical admissible stress representatives $\underset{\sim}{\rho}$ are given by:

$$\underset{\sim}{\rho} = \begin{cases} \rho_{\alpha\alpha} = c_1(1-x_\alpha^2)(1-x_\beta^2) - c_2(1-x_\alpha^2)(1-x_\beta^4) & \text{no summation} \\ \rho_{\alpha\beta} = c_1(x_\alpha - \frac{1}{3}x_\alpha^3)(x_\beta - \frac{1}{3}x_\beta^3) + c_2(x_\alpha - \frac{1}{5}x_\alpha^5)(x_\beta - \frac{1}{5}x_\beta^5) & \alpha \neq \beta \end{cases} \tag{4.1.36}$$

Functional (3.3.35)

$$\Lambda_0(\underset{\sim}{\rho}) = \sup_{\underset{\sim}{\rho}^* \in \underset{\sim}{N}^0 - A_{0_t}} (\underset{\sim}{\rho} - \underset{\sim}{\rho}^*, \underset{\sim}{\rho})_{\underset{\sim}{L}}$$

after inserting (4.1.33) becomes:

$$\begin{aligned} \Lambda_0(c_1, c_2) = \left[(c_1^2 - c_1 c_1^*) \cdot 4{,}01468 + (c_2^2 - c_2 c_2^*) \cdot 6{,}01351 \right. \\ \left. + (2c_1 c_2 - c_1 c_2^* - c_1^* c_2) \cdot 5{,}21133\right] \end{aligned} \tag{4.1.37}$$

Numerical values

$\tilde{q}_0$	c_1	c_2	yield-conditions
1.5	0.1933	-0.2708	v. Mises
2.5	-0.3327	0.1234	
1.5	0.1575	-0.2351	Tresca
2.5	0.3673	0.1580	

Interesting herein is the change of sign of coefficients for load parameter $\tilde{q}_o$ increasing from 1.5 to 2.5. This may be interpreted by the different position of regions of admissible stress representatives (appendix A10). Result (4.1.31) as well as result (4.1.37) are rather rough approximations:

1. in three-dimensional case the shape of test-functions in x_1, x_2-direction was prescribed, only distribution in x_1-direction was object of optimization.

2. in both cases we restricted our test-functions to order 6 (taking into account the vanishing of coefficients in (4.1.31) representing vanishing of membrane stresses), where symmetry and equilibrium condition reduced the number of unknown parameters in the functionals (4.1.26) and (4.1.37) to two.

By increasing the number of coefficients, use of representatives of higher order than two and by combination of three-dimensional and two-dimensional method a technically useful application of the herein developed methods seems to be possible. Our examples on the contrary have illustrative character.

In general in every calculation under assumption of infinitesimal deformations validity of this assumption has to be estimated for the considered problem; especially for thin plates and shells deflections may assume easily large values. To get some idea of the obtained numerical results we imagine the plate from example 1 having 2 dm length of the edges and a constant thickness of 5 mm. For the modulus of elasticity $E = 2106 \cdot 7 \cdot 10^4 \frac{N}{cm^2}$ of the chosen material dimensionless laods $\tilde{q}_{01} = 1.0$ and $\tilde{q}_{02} = 1.15$ correspond with physical loads $q_{01} = 131.6 \frac{N}{cm^2}$ and $q_{02} = 151.4 \frac{N}{cm^2}$. Deflection for purely elastic material behaviour with the same modulus of elasticity in the middle of the plate would be $w_{max_1} = 2.242$ mm and $w_{max_2} = 2.578$ mm. Dimensionless relations w/2h would then assume the values $w_{max_1}/2h = 0.4484$ and $w_{max_2}/2h = 0.5156$ such that use of theory of infinitesimal deformations is only justified, if we assume that the unknown plastic deformations are small with respect to the deformations in the (hypothetical) case of purely elastic behaviour.

4.2. Calculation of the stress state in a circular plate according to (3.2) without hardening (example 2)

A homogeneous circular plate, clamped on the entire boundary is proportionally loaded by a vertical distributed load q, constant over the entire plate. Material is assumed to behave according to (4.1) elastic-idealplastic. Assuming for infinitesimal deformations we calculate approximately the stress state by the method derived in (3.2) and already used for example 1 again using yield conditions of von Mises and Tresca

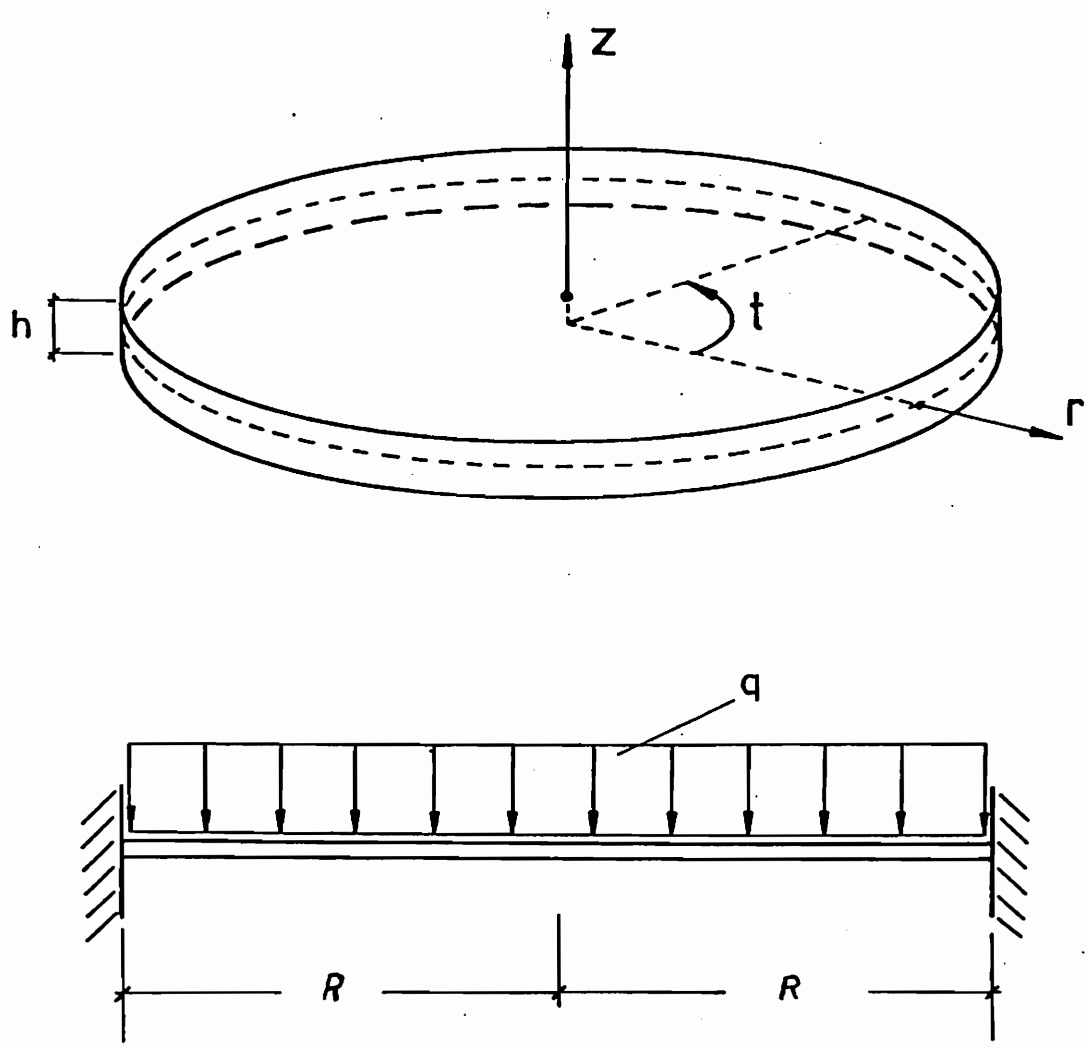


Fig. 16

Calculation of this example is analogous to the procedure in example 1. For details we therefore refer to [85].

Purely elastic solution

Also for this example analytical solution for hypothetical purely elastic behaviour is known:

$$\tilde{N}_r^{o(2)}(\tilde{r}) = \frac{\tilde{q}}{16}\left((1+\nu) - \tilde{r}^2(3+\nu)\right)$$

$$\tilde{N}_t^{o(2)}(\tilde{r}) = \frac{\tilde{q}}{16}\left((1+\nu) - \tilde{r}^2(1+3\nu)\right)$$

$$\tilde{\sigma}_r^{o}(\tilde{r},\tilde{x}_3) = \frac{3}{8}\tilde{q}\left((1+\nu) - \tilde{r}^2(3+\nu)\right)\tilde{x}_3 \tag{4.2.1}$$

$$\tilde{\sigma}_t^{o}(\tilde{r},\tilde{x}_3) = \frac{3}{8}\tilde{q}\left((1+\nu) - \tilde{r}^2(1+3\nu)\right)\tilde{x}_3$$

Quantities denoted by $(\tilde{\ })$ are dimensionless with:

$$\tilde{N}_r^{o(2)} = \frac{N_r^{o(2)}}{E(2h)^2}\left(\frac{R}{2h}\right)^2 \quad ; \qquad \tilde{x}_3 = \frac{x_3}{2h}$$

$$\tilde{r} = \frac{r}{R} \tag{4.2.2}$$

$$\tilde{N}_t^{o(2)} = \frac{N_t^{o(2)}}{E(2h)^2}\left(\frac{R}{2h}\right)^2 \quad ; \qquad \tilde{q} = \frac{q}{E}\left(\frac{R}{2h}\right)^4$$

Here $N_r^{(2)}$ and $N_t^{(2)}$ represent stress representatives of second order, expressed in cylindrical coordinates.

Region of plastically admissible stresses are described according to (4.1.10 - 14) by yield-condition according to von Mises and Tresca. Elastic limit load is $\tilde{q}^{e}_{lim} = \tilde{\sigma}^{s} \cdot \frac{4}{3} \; \frac{1}{\sqrt{1+\nu^2-\nu}}$.

Two-dimensional approach

As in (4.1) we use representatives of order 2 and choose the following test-functions

$$\tilde{N}_r^{\rho(2)}(\tilde{r}) = \sum_{i=1}^{6} a_i \tilde{r}^{i-1}$$

$$\tilde{N}_t^{\rho(2)}(\tilde{r}) = \sum_{i=1}^{6} b_i \tilde{r}^{i-1} \qquad (4.2.3)$$

for stress representatives of residual stresses. Taking into account symmetry and satisfaction of equilibrium conditions (4.2.3) we obtain:

$$\underset{\sim}{\rho} = \begin{cases} \tilde{N}_r^{\rho(2)} = a_1 + a_3\tilde{r}^2 + a_4\tilde{r}^3 + a_5\tilde{r}^4 + a_6\tilde{r}^5 \\ \tilde{N}_t^{\rho(2)} = a_1 + 3a_3\tilde{r}^2 + 4a_4\tilde{r}^3 + 5a_5\tilde{r}^4 + 6a_6\tilde{r}^5 \end{cases} \qquad (4.2.4)$$

These functions are introduced into functional

$$\Lambda_0 = (\underset{\sim}{\rho}, \underset{\sim}{\rho})_{\underset{\sim}{L}} \qquad (4.2.5)$$

which is (appendix A8) special case of (2.8.29) in case of proportional loading. $\tilde{\Lambda}_o$ is then minimized on $\underset{\sim}{\tilde{N}}^o - \tilde{A}_o$ where analogously to (4.1.15-16) $\tilde{A}_o$ expresses region of admissible two-dimensional representatives, described by von Mises and Tresca's yield-conditions.

Numerical results

$\tilde{q}$	a_1	a_3	a_4	a_5	a_6	yield-condition
2,650	0.0547	0,1410	-0,3945	0,4020	-0,1419	Tresca
2,675	0,0525	0,2489	-0,6911	0,7003	-0,2461	Tresca
2,700	0,0318	-0,0461	0,0730	0,0215	-0,0515	von Mises
3,000	0,0262	0,0708	0,0324	0,0029	-0,0687	von Mises
3,300	0,0017	0,1394	0,0850	0,0192	-0,1427	von Mises

For purely elastic material behaviour maximal dimensionless deflections would become 0.614 so that use of infinitesimal theory is some extend justified also in example 2. Already with our very simple test-functions we obtain an increase of limit load of 36 %, relative to elastic limit load $\tilde{q}^{e}_{lim} = 2.4301$, though redistribution of stresses in x_3-direction remained unregarded.

Three-dimensional approach

As in (4.1) we prescribe in r-direction a shape-function for residual stresses, here proportional to the distribution of elastic stresses. In x_3-direction we choose:

$$\begin{aligned} \tilde{\sigma}_r(\tilde{x}_3) &= a_1 + a_2 \tilde{x}_3 + a_3 \tilde{x}_3^2 + a_4 \tilde{x}_3^3 \\ \tilde{\sigma}_t(\tilde{x}_3) &= b_1 + b_2 \tilde{x}_3 + b_3 \tilde{x}_3^2 + b_4 \tilde{x}_3^3 \end{aligned} \tag{4.2.6}$$

Analogously to (4.1.20) we satisfy equilibrium condition by satisfaction of

$$\begin{aligned} \int_{-1}^{+1} \tilde{\sigma}_r^{\rho}(\tilde{x}_3)\, \tilde{x}_3 \, d\tilde{x}_3 &= 0 \\ \int_{-1}^{+1} \tilde{\sigma}_t^{\rho}(\tilde{x}_3)\, \tilde{x}_3 \, d\tilde{x}_3 &= 0 \end{aligned} \tag{4.2.7}$$

Introducing of (4.2.6) into (4.2.7) leads to the following test-functions:

$$\underset{\sim}{\rho} = \begin{cases} \tilde{\sigma}_r^{\rho}(\tilde{x}_3) = (a_2 \tilde{x}_3 - \frac{20}{3} a_2 \tilde{x}_3^3)(1{,}65\, \tilde{r}^2 - 0{,}65) \\ \tilde{\sigma}_t^{\rho}(\tilde{x}_3) = (b_2 \tilde{x}_3 \quad \frac{20}{3} b_2 \tilde{x}_3^3)(0{,}95\, \tilde{r}^2 - 0{,}65) \end{cases} \tag{4.2.8}$$

If we insert these functions into the functional according to (4.2.5) for three-dimensional quantities, we obtain after minimization in the region of admissible function the following numerical values:

$\tilde{q}$	a_2	b_2	yield-conditions
2,5	-0,7651	-0,2169	Tresca
2,5	-0,1475	-0,0176	von Mises
2,6	-0,9901	-0,2807	Tresca
2,6	-0,3593	-0,0457	von Mises
2,9	-1,6651	-0,4720	Tresca
2,9	-0,9998	-0,1500	von Mises
3,0	-1,2148	-0,1914	von Mises
3,3	-1,8642	-0,3349	von Mises

For loads $\tilde{q}$ > 2.9 for the used test-functions no result could be obtained for the Tresca yield-condition.

On the numerical procedure in example 1 and example 2

In example 1 the functional

$$\Lambda_o(\underset{\sim}{\rho}) = \sup_{\underset{\sim}{\rho}^* \in \underset{\sim}{\rho}^o - A_o} (\underset{\sim}{\rho} - \underset{\sim}{\rho}^*, \underset{\sim}{\rho})_{\underset{\sim}{L}}$$

was minimized by a double optimization process: first for fixed $\underset{\sim}{\rho}$ the set of suprema $\sup_{\underset{\sim}{\rho}^* \in \underset{\sim}{\rho}^o - A_o} (\underset{\sim}{\rho} - \underset{\sim}{\rho}^*, \underset{\sim}{\rho})_L$ was determined from which that vector $\underset{\sim}{\rho}$ is obtained as solution, for which the supremum attains minimum value equal to zero. As shown in (A8) proportional loading solution may be determined from minimizing $\tilde{\Lambda}_o(\underset{\sim}{\rho}) = (\underset{\sim}{\rho}, \underset{\sim}{\rho})_L$, $\underset{\sim}{\rho} \in \underset{\sim}{\rho}^o - A_o$. This numerically more convenient method, corresponding to the method of MOREAU [33] had been used in example 2. In this case, however, we have to take into account that $\tilde{\Lambda}_o$ does not attain the value zero for the minimum at which the solution $\underset{\sim}{\rho}$ is found. Both examples have been calculated on TR 440 of Ruhr-University Bochum by a direct optimization method of Box [90].

4.3. Calculation of an elasto-plastic beam with hardening according to (3.4) (example 3)

A symmetrical simply supported homongeneous thin beam, with material behaviour according to (2.6.54 - 60), is loaded by a proportionally increasing load, constantly distributed over the length of the beam. Using the functionals (3.4.6,8), derived in (3.4), we determine the history of the beam.

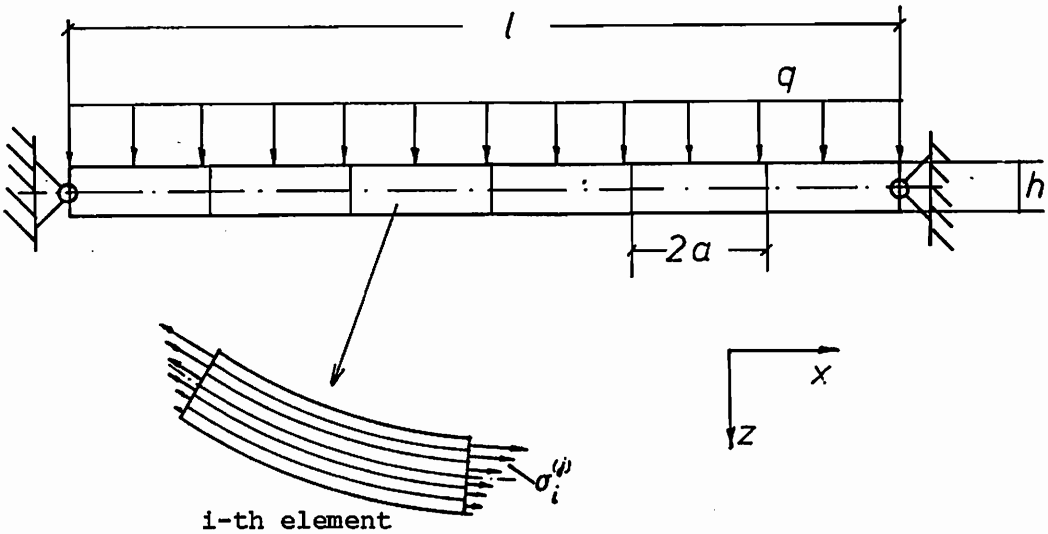


Fig. 17

physical quantities		dimensionless quantities
local coordinates	x, z	$\tilde{x} = \frac{x}{a}$, $\tilde{z} = \frac{z}{h}$
displacement in $\tilde{x}$-direction	U	$\tilde{u} = \frac{U}{h}\frac{a}{h}$
displacement in $\tilde{z}$-direction	W	$\tilde{w} = \frac{W}{h}$ (4.3.1)
stress	σ	$\tilde{\sigma} = \frac{\sigma}{E}\left(\frac{a}{h}\right)^2$
distributed load	q	$\tilde{q} = \frac{q}{E}\left(\frac{a}{h}\right)^4$

The problem will be solved by three subsequent discretisations:

1) In x-direction the beam is cut into n elements
2) Each element is cut into m sheets in z-direction
3) Load is applied incrementally.

If we use functional (3.4.8) then we obtain:

$$\mathcal{J}_2 = \sum_{i=1}^{n} \int_{-1}^{+1} \Big\{ \sum_{j=1}^{m} \left(\delta\varepsilon_i^{(j)} L_{0_i}^{(j)} \delta\varepsilon_i^{(j)} \right) + \frac{1}{2} \sum_{j=1}^{m} \sigma_i^{(j)} \delta w_{,x_i} \delta w_{,x_i} - \delta q_i \delta w_i \Big\} dx_i \tag{4.3.2}$$

Here $\delta\varepsilon_i$ have to fulfill the condition of compatibility and the rates of displacements of the middle-line have to fulfill geometrical boundary conditions. In order to take plastification into account, the stress state in the middle of every sheet j in every element i is taken as measure to determine the tangent modulus. For this purpose material law according to (2.6.57) is used. Here (2.6.60) reduces to the scalar expression

$$L_{0_i}^{(j)} = \frac{1}{E_i} + \frac{3\sigma_i^{(j)2}}{E_{0_i}^{3}}$$

<u>Test-functions for displacements of the middle-line of the beam</u>

We introduce in every element test-functions for the displacement-field:

$$w(\tilde{x}) = \frac{1}{4}\begin{pmatrix} 2+\tilde{x}(3-\tilde{x}^2) \\ 2-\tilde{x}(3-\tilde{x}^2) \\ \tilde{x}(\tilde{x}^2-1)+(\tilde{x}^2-1) \\ \tilde{x}(\tilde{x}^2-1)-(\tilde{x}^2-1) \end{pmatrix}^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} ; \quad u(\tilde{x}) = \frac{1}{2}\begin{pmatrix} 1+\tilde{x} \\ 1-\tilde{x} \end{pmatrix}^T \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{4.3.3}$$

Here, α_1, α_2 denote incremental node-displacements in z-direction, β_1, β_2 as incremental node-deflection and γ_1, γ_2 as incremental node displacements in x-direction. With these test-functions we calculate incremental strains $\delta\underset{\sim}{\varepsilon}$:

$$\delta\varepsilon = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \left\{ \frac{1}{4}\begin{pmatrix} 3-3\tilde{x}^2 \\ -3+3\tilde{x}^2 \\ 3\tilde{x}^2-2\tilde{x}-1 \\ 3\tilde{x}+2\tilde{x}-1 \end{pmatrix}^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} \right\} .$$

$$\left\{ \frac{1}{4} \begin{pmatrix} 3-3\tilde{x}^2 \\ -3+3\tilde{x}^2 \\ 3\tilde{x}^2-2\tilde{x}-1 \\ 3\tilde{x}^2+2\tilde{x}-1 \end{pmatrix}^T \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} \right\} - \tilde{z} \left\{ \frac{1}{4} \begin{pmatrix} -6\tilde{x} \\ 6\tilde{x} \\ 6\tilde{x}+2 \\ 6\tilde{x}-2 \end{pmatrix}^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} \right\} \tag{4.3.4}$$

Here A_1, A_2, B_1, B_2 are node displacements and -deflections of reference state according to the rate quantities. Rate of stress $\delta\sigma$ can be determined by use of tangent modulus (2.6.59).

In our example geometrical boundary conditions restrict to the requirement of vanishing of displacements at the ends of the beam. This will be satisfied by attributing value zero to the relevant parameters of test-functions. If we enter these quantities into functional (4.3.2) then we obtain the solution of the incremental problem using Ritz method after integration over the length of the beam and introduction of global system parameters as those free parameters α_i, β_i, γ_i for which functional J_2 assumes a stationary value. If we assign in (4.3.2) the value zero to

$$\sum_{j=1}^{m} \tilde{\sigma}^{(j)} \delta \tilde{w}_{,x_i} \delta \tilde{w}_{,x_i} \tag{4.3.5}$$

so equilibrium conditions are not fulfilled in the neighbour-state but in the reference state by the such transformed functional. This corresponds to the proceeding in (2.6.14 - 29) and is comparable with the so-called "second order approximation" [84]. The problems in using this method are evident: As matrix L, here degenerated to a scalar, is always positiv definit, instable states cannot be noticed. In the herein treated cases, however, instable states do not occur. In order to get an impression of the differences occuring in using one or the other functional we compare the solutions for purely elastic behaviour using both functionals. We state that the difference between them relative to the solutions obtained by geometrical linear theory (first-order-theory) is not essential. (fig. appendix A 17).

Stiffness matrix of the problem is in both cases symmetrical and has band structure (appendix A9), The elements of the matrix are complicated polynomial expressions.

Functional (2.6.41) degenerates in the case of our example 3 to:

$$\mathcal{J}_1 = \sum_{i=1}^{n} \int_{-1}^{+1} \Big\{ \Big[-\sum_{j=1}^{m} \big(\delta\tilde{\sigma}_i^{(j)} L_{o_i}^{(j)} \delta\tilde{\sigma}_i^{(j)}\big) + \sum_{j=1}^{m} \big(\delta\tilde{\sigma}_i^{(j)} \delta\tilde{\varepsilon}_i^{(j)}\big) \Big]$$

$$- \frac{1}{2} \sum_{j=1}^{m} \tilde{\sigma}_i^{(j)} \delta\tilde{w}_{,x_i} \, \delta\tilde{w}_{,x_i} \Big\} d\tilde{x}_i \tag{4.3.6}$$

With dimensionless quantities according to (4.3.1) we introduce test-functions for the incremental quantities in each element:

$$\begin{aligned} \delta\tilde{\sigma}^{(j)} &= n_1^{(j)} + n_2^{(j)} \cdot \tilde{x} + n_3^{(j)} \tilde{x}^2 \\ \delta\tilde{u} &= \gamma_1 + \gamma_2 \tilde{x} \\ \delta\tilde{w} &= \alpha_1 + \alpha_2 \tilde{x} + \alpha_3 \tilde{x}^2 \end{aligned} \tag{4.3.7}$$

The according functions of the reference state are obtained by

$$\begin{aligned} \tilde{\sigma}^{(j)} &= N_1^{(j)} + N_2^{(j)} \tilde{x} + N_3^{(j)} \tilde{x}^2 \\ \tilde{u} &= \Gamma_1 + \Gamma_2 \tilde{x} \\ \tilde{w} &= A_1 + A_2 \tilde{x} + A_3 \tilde{x}^2 \end{aligned} \tag{4.3.8}$$

Where the parameter $N_1^{(j)}$, $N_2^{(j)}$, $N_3^{(j)}$, Γ_1, Γ_2, A_1, A_2, A_3 are obtained from summation over time steps from the incremental parameters. As $\delta\varepsilon_i$ are computed from displacement-test-functions the compatibility conditions is fulfilled in each element by definition. In opposition to (4.3.4 - 5) here we use Lagrange multipliers to formulate jump-conditions between neighbouring elements and the boundary conditions [51].

Calculation of stiffness matrix is in this case easier but the number of unknown quantities in the considered system of equations is larger than in using (4.3.2): In both cases we used 6 elements and 6 sheets in every element; in using functional (4.3.6) we get 167 unknown parameters, in using (4.3.2) we have only 21 unknown quantities. Although in using (4.3.6) the stiffness matrix has band structure, however, the bandwidth is much bigger than in the previous example.

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Appendix A1

a) $$\delta(F_{iM}\sigma_{MJ})\delta d^e_{ij} = \delta F_{iM}\sigma_{MJ}\delta d^e_{ij} + F_{iM}\delta d^e_{ij}\sigma_{MJ}$$

b) $$[\underset{\sim}{M}^o \, .. \, \delta(\underset{\sim}{F}.\underset{\sim}{\sigma})] \, .. \, \delta(\underset{\sim}{F}.\underset{\sim}{\sigma}) =$$

$$\frac{1}{2} M^o_{iJkL}\,\delta F_{iM}\,\delta F_{kN}\,\sigma_{MJ}\,\sigma_{NL} + \frac{1}{2} M^o_{iJkL}\,F_{iM}\,F_{kN}\,\delta\sigma_{MJ}\,\delta\sigma_{NL}$$

$$+ \frac{1}{2} M^o_{iJkL}\,\delta F_{iM}\,\sigma_{MJ}\,F_{kN}\,\delta\sigma_{NL} + \frac{1}{2} M^o_{iJkL}\,F_{iM}\,\delta\sigma_{MJ}\,\delta F_{kN}\,\sigma_{NL} =$$

$$\frac{1}{2}\delta^2(\overbrace{M^o_{iJkL}\,F_{iM}\,F_{kL}}^{L^o_{MJNL}})\,\sigma_{MJ}\,\sigma_{NL} + \frac{1}{2} M^o_{iJkL}\,F_{iM}\,F_{kL}\,\delta\sigma_{MJ}\,\delta\sigma_{NL} +$$

$$M^o_{iJkL}\,\delta F_{iM}\,\sigma_{MJ}\,F_{kN}\,\delta\sigma_{NL} =$$

$$\frac{1}{2}\delta^2 L^o_{MJNL}\,\sigma_{MJ}\,\sigma_{NL} + \frac{1}{2} L^o_{MJNL}\,\delta\sigma_{MJ}\,\delta\sigma_{NL} + \underbrace{M^o_{iJrL}\,F_{iM}\,F_{rs}\,F^{-1}_{St}\,\delta\sigma_{MJ}\,\delta F_{tN}\,\sigma_{NL}}_{\underbrace{L^o_{MJSL}\,\delta\sigma_{MJ}}_{\delta\varepsilon^e_{SL} = F_{is}\delta d^e_{ij}}\,\delta F_{tN}\,F^{-1}_{St}\,\sigma_{NL}} =$$

$$\rightarrow = F_{is}\,F^{-1}_{St}\,\sigma_{NL}\,\delta F_{tN}\,\delta d^e_{iL} = \sigma_{NL}\,\delta F_{iN}\,\delta d^e_{iL} = \sigma_{MJ}\,\delta F_{iM}\,\delta d^e_{ij}$$

$$\rightarrow = \frac{1}{2} L^o_{MJNL}\,\delta\sigma_{MJ}\,\delta\sigma_{NL} + \frac{1}{2}\delta^2(\overbrace{L^o_{MJNL}}^{\alpha_{MJNL}})\,\sigma_{MJ}\,\sigma_{NL} + \sigma_{MJ}\,\delta F_{iM}\,\delta d^e_{ij}$$

c) $$\frac{1}{2} M^o_{iJkL}\,\delta d^e_{ij}\,\delta d^e_{kL} = \frac{1}{2} M^o_{rJtL}\,F_{iM}\,F^{-1}_{Mr}\,\delta d^e_{ij}\,F_{kN}\,F^{-1}_{Nt}\,\delta d^e_{kL}$$

$$= \frac{1}{2} L^o_{MJNL}\,F_{iM}\,\delta d^e_{ij}\,F_{kN}\,\delta d^e_{kL}$$

$$[a, b, c] \rightarrow \delta\sigma_{MJ}\,F_{iM}\,\delta d^e_{ij} + \sigma_{MJ}\,\delta F_{iM}\,\delta d^e_{ij} = \frac{1}{2} L^o_{MJNL}\,\delta\sigma_{MJ}\,\delta\sigma_{NL}$$

$$+ \frac{1}{2} L^{o-1}_{MJNL}\,F_{iM}\,\delta d^e_{ij}\,F_{kN}\,\delta d^e_{kL} + \sigma_{MJ}\,\delta F_{iM}\,\delta d^e_{ij} \Rightarrow$$

$$\delta\sigma_{MJ}\,F_{iM}\,\delta d^e_{ij} = \frac{1}{2} L^o_{MJNL}\,\delta\sigma_{MJ}\,\delta\sigma_{NL} + \frac{1}{2} L^{o-1}_{MJNL}\,F_{iM}\,\delta d^e_{ij}\,F_{kN}\,\delta d^e_{kL}$$

Appendix A2

Convexity of the elastic region for the von Kármán-plate using von Mises' yield condition for the first and second Piola stress-tensor $\underset{\sim}{t}$ and $\underset{\sim}{\sigma}$, resp.

In (2.3) we showed that the change of geometry has an influence on the description of elastic region if we use the first Piola stress tensor. Here we investigate if this effect has under assumptions of chapters (2.1 - 2.3) an essential influence of convexity of elastic region C and validity of normality rule in transition from $\tilde{C}$ to C.

If we use $\underset{\sim}{\sigma}$, von Mises' yield condition in three dimensions is given by

$$\sigma'_{IJ}\sigma'_{IJ} - k^2 \leq 0 \quad ; \qquad \sigma'_{IJ} = \sigma_{IJ} - \tfrac{1}{3}\delta_{IJ}\sigma_{KK} \qquad \text{(A2-1)}$$

If we regard the plate as plane object, (A2-1) reduces to:

$$\sigma'_{\alpha\beta}\sigma'_{\alpha\beta} - k^2 \leq 0 \quad ; \qquad \sigma'_{\alpha\beta} = \sigma_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta}\sigma_{\alpha\alpha}$$

$$\sigma'_{\alpha\beta} = \begin{pmatrix} \frac{1}{2}(\sigma_{11}-\sigma_{22}) & \sigma_{12} \\ \sigma_{21}=\sigma_{12} & -\frac{1}{2}(\sigma_{11}-\sigma_{22}) \end{pmatrix} \qquad \text{(A2-2)}$$

If we assume validity of Mises' yield-condition in using $\underset{\sim}{t}$ and we take into account relations for von Kármán plate derived in [74]

$$t_{\alpha\beta} = \sigma_{\alpha\beta} \quad ; \quad t_{3\beta} = \sigma_{\alpha\beta}\varphi_\alpha \qquad \text{(A2-3)}$$

with φ_α as deflection of the midspace of the plate in α-direction, so we obtain:

$$t'_{ji}t'_{ji} - k^2 \leq 0 \quad ; \quad t'_{\alpha\beta}t'_{\alpha\beta} + t'_{3\beta}t'_{3\beta} - k^2 \leq 0 \qquad \text{(A2-4)}$$

Because of $t_{\alpha\alpha} = \sigma_{\alpha\alpha}$ and $t'_{\alpha\beta} = t_{\alpha\beta}$ for $\alpha \neq \beta$ we have:

$$t'_{\alpha\beta}\,t'_{\alpha\beta} = \sigma'_{\alpha\beta}\sigma'_{\alpha\beta} \quad ; \quad t'_{3\beta}\,t'_{3\beta} = \sigma_{\gamma\beta}\varphi_\gamma\sigma_{\eta\beta}\varphi_\eta \qquad \text{(A2-5)}$$

With (A2-2) we obtain:

$$t'_{\alpha\beta} t'_{\alpha\beta} = (\sigma_{\alpha\beta} - \delta_{\alpha\beta} \tfrac{1}{2} \sigma_{\kappa\kappa})(\sigma_{\alpha\beta} - \delta_{\alpha\beta} \tfrac{1}{2} \sigma_{\lambda\lambda}) = \sigma_{\alpha\beta}\sigma_{\alpha\beta} - \tfrac{3}{4}\sigma^2_{\lambda\lambda} \qquad \text{(A2-6)}$$

so that (A2-4) becomes

$$\sigma_{\gamma\beta}\sigma_{\eta\beta}[\delta_{\gamma\eta} + \varphi_\gamma \varphi_\eta] - \tfrac{3}{4}\sigma_\lambda\sigma_\lambda - k^2 \leq 0 \qquad \text{(A2-7)}$$

If we write (A2-2) in the same form, we obtain:

$$\sigma_{\gamma\beta}\sigma_{\eta\beta}[\delta_{\gamma\eta}] - \tfrac{3}{4}\sigma_\lambda\sigma_\lambda - k^2 \leq 0 \qquad \text{(A2-8)}$$

Written in detail $\delta_{\gamma\eta} + \varphi_\gamma\varphi_\eta$ has the following form:

$$\delta_{\gamma\eta} + \varphi_\gamma\varphi_\eta = \begin{pmatrix} 1+\varphi_1^2 & \varphi_1\varphi_2 \\ \varphi_2\varphi_1 & 1+\varphi_2^2 \end{pmatrix} \qquad \text{(A2-9)}$$

Because of general assumption of small strains squares of deflection must be small in comparison with unity. So we have:

$$\delta_{\gamma\eta} + \varphi_\gamma\varphi_\eta = \delta_{\gamma\eta} + (0^2)$$

This means that convexity of region C is assured for convex regions $\tilde{C}$ in the case of the von Kármán plate.

<u>Validity of normality-rule</u>

If we compare the formulations of normality rule for elasto-plastic material using the first Piola stress tensor $\underset{\sim}{t}$ and second Piola-Kirchhoff stress tensor $\underset{\sim}{\sigma}$

$$\delta\underset{\sim}{t} \cdot\cdot\, \delta\underset{\sim}{d}^p = 0 \quad ; \quad \delta\underset{\sim}{\sigma} \cdot\cdot\, \delta\underset{\sim}{\varepsilon}^p = 0 \qquad \text{(A2-10)}$$

it follows (1.3.15) that they differ by the scalar value

$$\underset{\sim}{\sigma} \cdot \delta\underset{\sim}{F} \cdot \delta\underset{\sim}{d}^p = \sigma_{JM}\,\delta F_{iM}\,\delta d^p_{iJ} \qquad \text{(A2-11)}$$

This difference does not vanish in general. In order to investigate the influence of the assumptions of (1.1 - 1.3) in the case of the von Kármán plate assuming the validity of the Kirchhoff-Love hypothesis, we write (A2-11) in detail: With

$$\dot{F}_{iM} = \begin{pmatrix} \dot{u}_{1,1} - x_3 \dot{\varphi}_{1,1} & \dot{u}_{1,2} - x_3 \dot{\varphi}_{1,2} & -\dot{\varphi}_1 \\ \dot{u}_{2,1} - x_3 \dot{\varphi}_{2,1} & \dot{u}_{2,2} - x_3 \dot{\varphi}_{2,2} & -\dot{\varphi}_2 \\ \dot{\varphi}_1 & \dot{\varphi}_2 & 0 \end{pmatrix} ; \quad \dot{d}^p_{ij} ;\ i,j \in [1,2,3] \tag{A2-12}$$

we obtain

$$\sigma_{JM}\, \delta F_{iM}\, \delta d^p_{ij} = \sigma_{\alpha\gamma} \left[(\delta u_{\alpha,\beta} - x_3 \delta\varphi_{\alpha,\beta}) \delta d^p_{\beta\gamma} - \delta\varphi_\alpha\, \delta d^p_{3\gamma} \right] =$$

$$\begin{aligned} &\sigma_{11} \left[(\delta u_{1,1} - x_3 \delta\varphi_{1,1}) \delta d^p_{11} + (\delta u_{12} - x_3 \delta\varphi_{1,2}) \delta d^p_{21} - \delta\varphi_1\, \delta d^p_{31} \right] + \\ &\sigma_{12} \left[(\delta u_{1,1} - x_3 \delta\varphi_{1,1}) \delta d^p_{12} + (\delta u_{1,2} - x_3 \delta\varphi_{1,2}) \delta d^p_{22} - \delta\varphi_1\, \delta d^p_{32} \right] + \\ &\sigma_{21} \left[(\delta u_{2,1} - x_3 \delta\varphi_{2,1}) \delta d^p_{11} + (\delta u_{2,2} - x_3 \delta\varphi_{2,2}) \delta d^p_{21} - \delta\varphi_2\, \delta d^p_{31} \right] + \\ &\sigma_{22} \left[(\delta u_{2,1} - x_3 \delta\varphi_{2,1}) \delta d^p_{12} + (\delta u_{2,2} - x_3 \delta\varphi_{2,2}) \delta d^p_{22} - \delta\varphi_2\, \delta d^p_{32} \right] \end{aligned} \tag{A2-13}$$

In order to get an impression of the size of (A2-11) we consider the one-dimensional case of the nonlinear beam with:

$$\dot{\underset{\sim}{F}} = \begin{pmatrix} \dot{u}_{1,1} - x_3 \dot{\varphi}_{1,1} & -\dot{\varphi}_1 \\ \dot{\varphi}_1 & 0 \end{pmatrix} , \quad \underset{\sim}{\sigma} = \sigma_{11} \tag{A2-14}$$

Then (A2-13) becomes:

$$\sigma_{JM}\, \delta F_{iM}\, \delta d^p_{ij} = \sigma_{11} \left[(\delta u_{1,1} - x_3 \delta\varphi_{1,1}) \delta d^p_{11} - \delta\varphi_1\, \delta d^p_{31} \right] \tag{A2-15}$$

If we consider the limit case that all notes are purely plastic, we have: $\delta d^p_{11} = \delta u_{1,1} - x_3\, \delta\varphi_{1,1}$, $\delta d^p_{31} = \delta\varphi_1$

(A2-16)

In comparison we consider the expression:

$$\frac{1}{2}\delta\varepsilon_{IJ}\, L^{-1}_{IJKL}\,\delta\varepsilon_{KL} = \frac{1}{2}\left(\delta u_{1,1} - x_3\delta\varphi_{1,1} + \varphi_1\delta\varphi_1\right)^2 \cdot E \tag{A2-17}$$

with modulus of elasticity E. (A2-17) may be interpreted as rate of potential of the second Piola-Kirchhoff stress tensor for vanishing plastic rates of deformation.

For a hypothetical case we compare (A2-16) with (A2-17): A beam may have maximal deflection $\varphi_1 = \frac{1}{10} \equiv 6^o$, maximal rate of deflection may be $\frac{1}{20} \cdot \varphi_{1max}$, stress of reference state may be limit stress σ_s for uniaxial tension test. The material may be the same steel as in example (4.3) with $E = 2.1 \cdot 10^7 \frac{N}{cm^2}$ and $\sigma_s = 3.8 \cdot 10^4 \frac{N}{cm^2}$. If we assume according to the von Kármán plate theory φ_1^2 and $u_{1,1} - x_3\varphi_{1,1}$ to be of the order of magnitude, we finally obtain:

$$\sigma_{JM}\,\delta F_{iM}\,\delta d^p_{ij} \leqslant 3.8 \cdot 10^4 \left(\frac{1}{20}\cdot\frac{1}{10}\right)^2\left[4\left(\frac{1}{10}\right)^2 - 1\right] = -0.912 \approx -1\,\frac{N}{cm^2}$$

$$\frac{1}{2}\delta\varepsilon_{IJ}\, L^{-1}_{IJKL}\,\delta\varepsilon_{KL} = \frac{1}{2}\cdot 4\left(\frac{1}{10}\right)^2\cdot\left(\frac{1}{20}\cdot\frac{1}{10}\right)^2\cdot 2.1\cdot 10^7 = 10.5 \approx 10\,\frac{N}{cm^2}$$

This means that for this example replacement of (A2-10a) by (A2-10b) would induce a numerical deviation of 10 % relative to (A2-17).

Appendix A3

$$(\delta \underset{\sim}{t}^r, \delta \underset{\sim}{t}^\rho)_{\underset{\sim}{M}_o} = \int_{(V)} \delta t^r_{Ji}\, M_{oiJkL}\, \delta t^\rho_{Lk}\, dx_1 dx_2 dx_3 =$$

$$\int_{(V)} \delta u_{k,L}\, \delta t^\rho_{Lk}\, dx_1 dx_2 dx_3 =$$

$$\int_{B_s \cup B_K} \overbrace{\delta t^\rho_{Lk}}^{=0 \text{ on } B_s}\, n_L\, \underbrace{\delta u_k}_{=0 \text{ on } B_K}\, dx_1 dx_2 dx_3 - \int_{(V)} \overbrace{\delta t^\rho_{Lk,L}}^{=0 \text{ in } V}\, \delta u_k\, dx_1 dx_2 dx_3 = 0$$

Appendix A4

$$\mathcal{L}_1(\delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\mu) = P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho) - \langle \delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\rho + \delta\underset{\sim}{t}^\mu \rangle +$$

$$\sup_{\substack{\delta\underset{\sim}{t}^{\rho*} \in \mathcal{H}^\rho \\ \delta\underset{\sim}{t}^{\mu*} \in \mathcal{H}^\mu}} \left[(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*} + \delta\underset{\sim}{t}^{\mu*}, \delta\underset{\sim}{t}^\rho + \delta\underset{\sim}{t}^\mu)_{\underset{\sim}{M}_0} - P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*} + \delta\underset{\sim}{t}^{\mu*}) \right] =$$

$$P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho) - \langle \delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\rho \rangle - \langle \delta\underset{\sim}{t}^0, \delta\underset{\sim}{t}^\mu \rangle + \langle \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\mu \rangle +$$

$$\sup_{\substack{\delta\underset{\sim}{t}^{\rho*} \in \mathcal{H}^\rho \\ \delta\underset{\sim}{t}^{\mu*} \in \mathcal{H}^\mu}} \left[(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}, \delta\underset{\sim}{t}^\rho)_{\underset{\sim}{M}_0} + (\delta\underset{\sim}{t}^0, \delta\underset{\sim}{t}^\mu)_{\underset{\sim}{M}_0} - (\delta\underset{\sim}{t}^{\rho*}, \delta\underset{\sim}{t}^\mu)_{\underset{\sim}{M}_0} + \right.$$

$$\left. (\delta\underset{\sim}{t}^{\mu*}, \delta\underset{\sim}{t}^\rho)_{\underset{\sim}{M}_0} + (\delta\underset{\sim}{t}^{\mu*}, \delta\underset{\sim}{t}^\mu)_{\underset{\sim}{M}_0} - P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*} + \delta\underset{\sim}{t}^{\mu*}) \right] =$$

$$= P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho) - \langle \delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\rho \rangle + \sup_{\substack{\delta\underset{\sim}{t}^{\rho*} \in \mathcal{H}^\rho \\ \delta\underset{\sim}{t}^{\mu*} \in \mathcal{H}^\mu}} \left[(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}, \delta\underset{\sim}{t}^\rho)_{\underset{\sim}{M}_0} \right.$$

$$\left. + (\delta\underset{\sim}{t}^{\mu*}, \delta\underset{\sim}{t}^\mu)_{\underset{\sim}{M}_0} - P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*} + \delta\underset{\sim}{t}^{\mu*}) \right] \geq$$

$$P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho) - \langle \delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\rho \rangle + \sup_{\delta\underset{\sim}{t}^{\rho*} \in \mathcal{H}^\rho} \left[(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}, \delta\underset{\sim}{t}^\rho)_{\underset{\sim}{M}_0} - \right.$$

$$\left. P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}) \right]$$

With

$$P_{20}(\delta\underset{\sim}{t}^\rho) = \sup_{\delta\underset{\sim}{t}^{\rho*} \in \mathcal{H}^\rho} \left[(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}, \delta\underset{\sim}{t}^\rho)_{\underset{\sim}{M}_0} - P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^{\rho*}) \right]$$

We have

$$\mathcal{L}_{10}(\delta\underset{\sim}{t}^\rho) = P_2^*(\delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho) - \langle \delta\underset{\sim}{t}^0 - \delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\rho \rangle + P_{20}(\delta\underset{\sim}{t}^\rho)$$

$$\mathcal{L}_{10}(\delta\underset{\sim}{t}^\rho) \leq \mathcal{L}_1(\delta\underset{\sim}{t}^\rho, \delta\underset{\sim}{t}^\mu) .$$

P_{20}, P_2^* are convex, $<\delta \underset{\sim}{t}^o, \delta \underset{\sim}{t}^\rho>$ is linear, $<\delta \underset{\sim}{t}^\rho, \delta \underset{\sim}{t}^\rho>$ is strictly convex for positive definite $\underset{\sim}{M}_o$, as we had assumed. By this $L_{10}(\delta \underset{\sim}{t}^\rho)$ is strictly convex as the sum of convex, strictly convex and linear functions is always strictly convex.

Appendix A5, A6

Proofs are analogous to A_3 and A_4. $\delta \underset{\sim}{t}^\rho, \delta \underset{\sim}{t}^\rho$, $\underset{\sim}{M}_o$ are replaced by $\delta \underset{\sim}{\sigma}^\rho$, $\delta \underset{\sim}{\sigma}^\mu$, $\underset{\sim}{L}$, respectively.

Appendix A7

From definition of polar functionals follows that F_1 and F_{10} can be written in the following way:

$$F_1(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) = \sup_{\underset{\sim}{t}^{\rho *} \in \mathcal{H}^\rho} \left[(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}, \underset{\sim}{d}^e) - \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}) \right] + \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) - (\underset{\sim}{t}^o - \underset{\sim}{t}^\rho, \underset{\sim}{d}^e)$$

$$F_{10}(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) = \sup_{\underset{\sim}{t}^{\rho *} \in \mathcal{H}^\rho} \left[(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}, \underset{\sim}{d}^e) - \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}) - I_{A_o} \right] + \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) + \underbrace{\cancel{I_{A_o}}}_{= 0,\ \text{as}\ \underset{\sim}{t}^\rho \in \underset{\sim}{t}^o - A_o} - (\underset{\sim}{t}^o - \underset{\sim}{t}^\rho, \underset{\sim}{d}^e) \quad .$$

From subtraction follows immediately

$$F_{10}(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) - F_1(\underset{\sim}{t}^\rho, \underset{\sim}{d}^e) = \sup_{\underset{\sim}{t}^{\rho *} \in \mathcal{H}^\rho} \left[(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}, \underset{\sim}{d}^e) - \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}) - I_{A_o} \right]$$

$$- \sup_{\underset{\sim}{t}^{\rho *} \in \mathcal{H}^\rho} \left[(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}, \underset{\sim}{d}^e) - \Psi^*(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho *}) \right] \leq 0 \quad .$$

We proceed analogously in regarding F_2 and F_{20} :

$$F_2(\underset{\sim}{t}^\rho, \underset{\sim}{\dot{d}}^e, \underset{\sim}{\dot{d}}^\mu) = \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) + \sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e + \underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) \right]$$

$$- (\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^\rho) - (\underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^o - \underset{\sim}{t}^\rho)$$

$$F_{20}(\underset{\sim}{t}^\rho, \underset{\sim}{\dot{d}}^e) = \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^\rho) + \underbrace{\cancel{I_{A_o}}}_{=0, \text{ as } \underset{\sim}{t}^\rho \in \underset{\sim}{t}^o - A_o} + \sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - I_{A_o} \right]$$

$$- (\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^\rho)$$

Because $H^{*\mu} \perp H^\rho$ we have: $(\underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^{\rho*}) = (\underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^\rho) = 0$

By this fact and by taking into account that supremum is researched for elements $\underset{\sim}{t}^{\rho*} \in H^\rho$ it follows:

$$\sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e + \underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) \right] =$$

$$\sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) \right] + (\underset{\sim}{\dot{d}}^\mu, \underset{\sim}{t}^o)$$

If we compute $F_{20} - F_2$, we obtain:

$$F_{20}(\underset{\sim}{t}^\rho, \underset{\sim}{\dot{d}}^e, \underset{\sim}{\dot{d}}^\mu) - F_2(\underset{\sim}{t}^\rho, \underset{\sim}{\dot{d}}^e) = \sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - I_{A_o} \right]$$

$$- \sup_{\underset{\sim}{t}^{\rho*} \in \mathcal{H}^{\rho'}} \left[(\underset{\sim}{\dot{d}}^o - \underset{\sim}{\dot{d}}^e, \underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) - \Phi(\underset{\sim}{t}^o - \underset{\sim}{t}^{\rho*}) \right] \leq 0 \ .$$

Appendix A8

From definition we have

$$\Lambda_o(\underset{\sim}{s}^\rho) = \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{s}^\rho) + \Phi_o^*(\underset{\sim}{\dot{s}}^\rho) - \langle \underset{\sim}{s}^o - \underset{\sim}{s}^\rho, \underset{\sim}{\dot{s}}^\rho \rangle =$$

$$\Phi_o(\underset{\sim}{s}^o - \underset{\sim}{s}^\rho) + \sup_{\underset{\sim}{s}^{\rho*} \in \mathcal{H}^{\rho'}} \left[\langle \underset{\sim}{s}^o - \underset{\sim}{s}^{\rho*}, \underset{\sim}{\dot{s}}^\rho \rangle - \Phi_o(\underset{\sim}{s}^o - \underset{\sim}{s}^{\rho*}) \right] - \langle \underset{\sim}{s}^o - \underset{\sim}{s}^\rho, \underset{\sim}{\dot{s}}^\rho \rangle$$

for elasto-plastic material we have:

$$\phi_o(\underset{\approx}{s}^o - \underset{\approx}{s}^\rho) = 0 \qquad \text{for } \underset{\approx}{s}^o - \underset{\approx}{s}^\rho \in E$$

If we restrict to $\underset{\approx}{s}^{\rho *}, \underset{\approx}{s}^{\rho} \in \underset{\approx}{s}^o - E$, $\underset{\approx}{s}^{\rho *}, \underset{\approx}{s}^{\rho *} \in H^{\rho\,\prime}$, we have:

$$\Lambda_o(\underset{\approx}{s}^\rho) = \sup_{\underset{\approx}{s}^{\rho *} \in \underset{\approx}{s}^o - E} [\langle \underset{\approx}{s}^o - \underset{\approx}{s}^{\rho *}, \dot{\underset{\approx}{s}}^\rho \rangle] - \langle \underset{\approx}{s}^o - \underset{\approx}{s}^\rho, \dot{\underset{\approx}{s}}^\rho \rangle$$

As the supremum is searched for elements $\underset{\approx}{s}^{\rho *} \in \underset{\approx}{s}^o - E$ it follows immediately:

$$\Lambda_o(\underset{\approx}{s}^\rho) = \sup_{\underset{\approx}{s}^{\rho *} \in \underset{\approx}{s}^o - E} \langle \underset{\approx}{s}^\rho - \underset{\approx}{s}^{\rho *}, \dot{\underset{\approx}{s}}^\rho \rangle$$

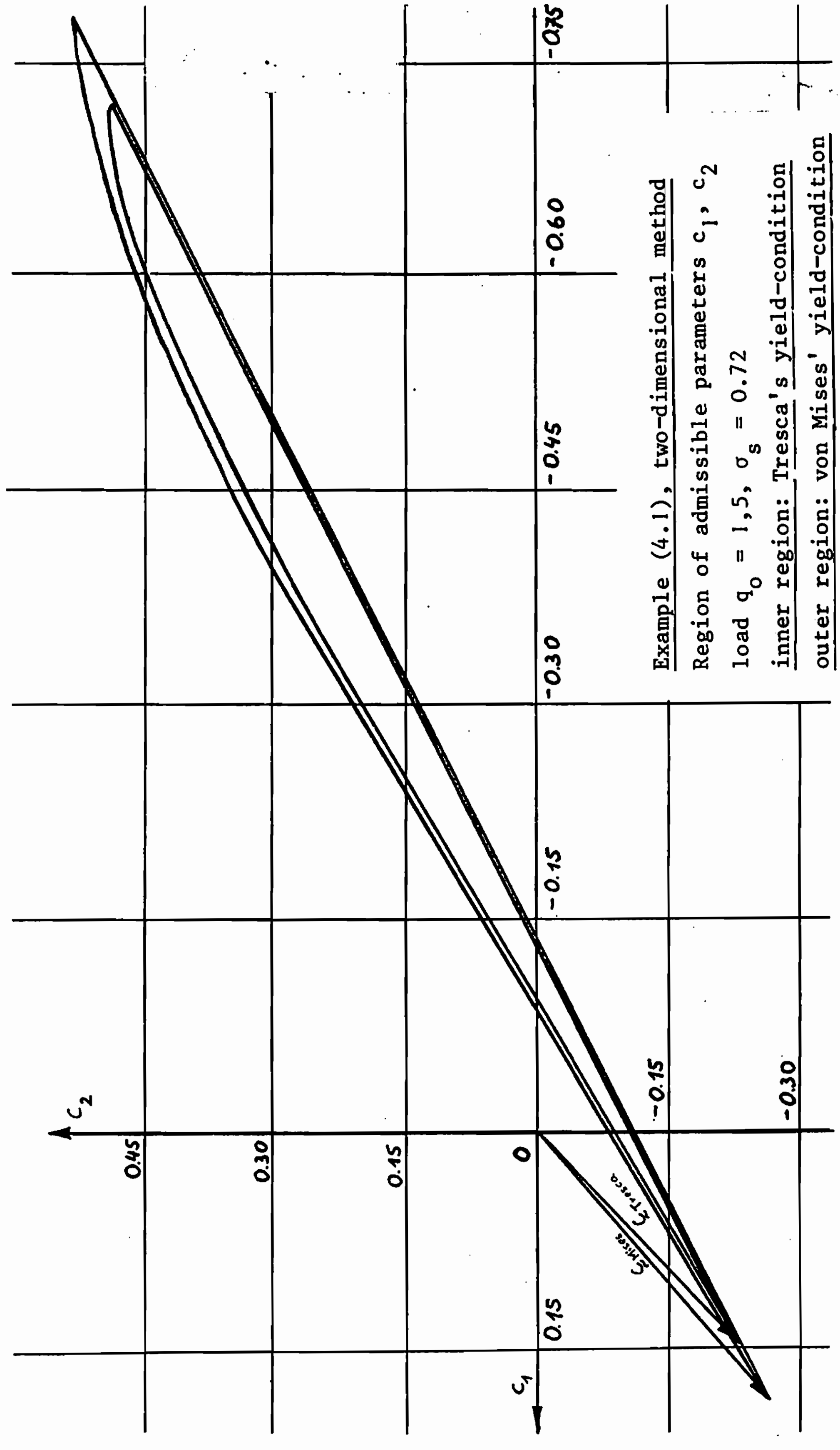
Appendix A9

Structure of the stiffness matrix for functional (4.3.2) from example (4.3).

$$
\begin{array}{c|ccccccccccccccccccccc|c}
 & \gamma_1 & \alpha_1 & \beta_1 & \gamma_2 & \alpha_2 & \beta_2 & \cdot & \cdot & \cdot & & & & & & & & & & & & & \\
\hline
 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & \gamma_1\\
 & *&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & \alpha_1\\
 & *&*&P&P&P&P&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 & \beta_1\\
 & *&*&*&P&P&P&P&P&P&0&0&0&0&0&0&0&0&0&0&0&0 & \gamma_2\\
 & *&*&*&*&P&P&P&P&P&0&0&0&0&0&0&0&0&0&0&0&0 & \alpha_2\\
 & *&*&*&*&*&P&P&P&P&0&0&0&0&0&0&0&0&0&0&0&0 & \beta_2\\
 & *&*&*&*&*&*&P&P&P&P&P&P&0&0&0&0&0&0&0&0&0 & \cdot\\
 & *&*&*&*&*&*&*&P&P&P&P&P&0&0&0&0&0&0&0&0&0 & \cdot\\
 & *&*&*&*&*&*&*&*&P&P&P&P&0&0&0&0&0&0&0&0&0 & \cdot\\
 & *&*&*&*&*&*&*&*&*&P&P&P&P&P&P&0&0&0&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&P&P&P&P&P&0&0&0&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&P&P&P&P&0&0&0&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&P&P&P&P&P&P&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&P&P&P&P&P&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&P&P&P&P&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&*&P&P&P&0&0&P & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&P&P&0&0&P & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&P&0&0&P & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&0&0&0 & \\
 & *&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&0&0 & \\
 & *&P & \\
\end{array}
$$

Symmetrical elements are denoted by (*), elements denoted by (P) are polynomials of a certain complexity depending on the geometry and material state of the reference state.

Appendix A10



Example (4.1), two-dimensional method

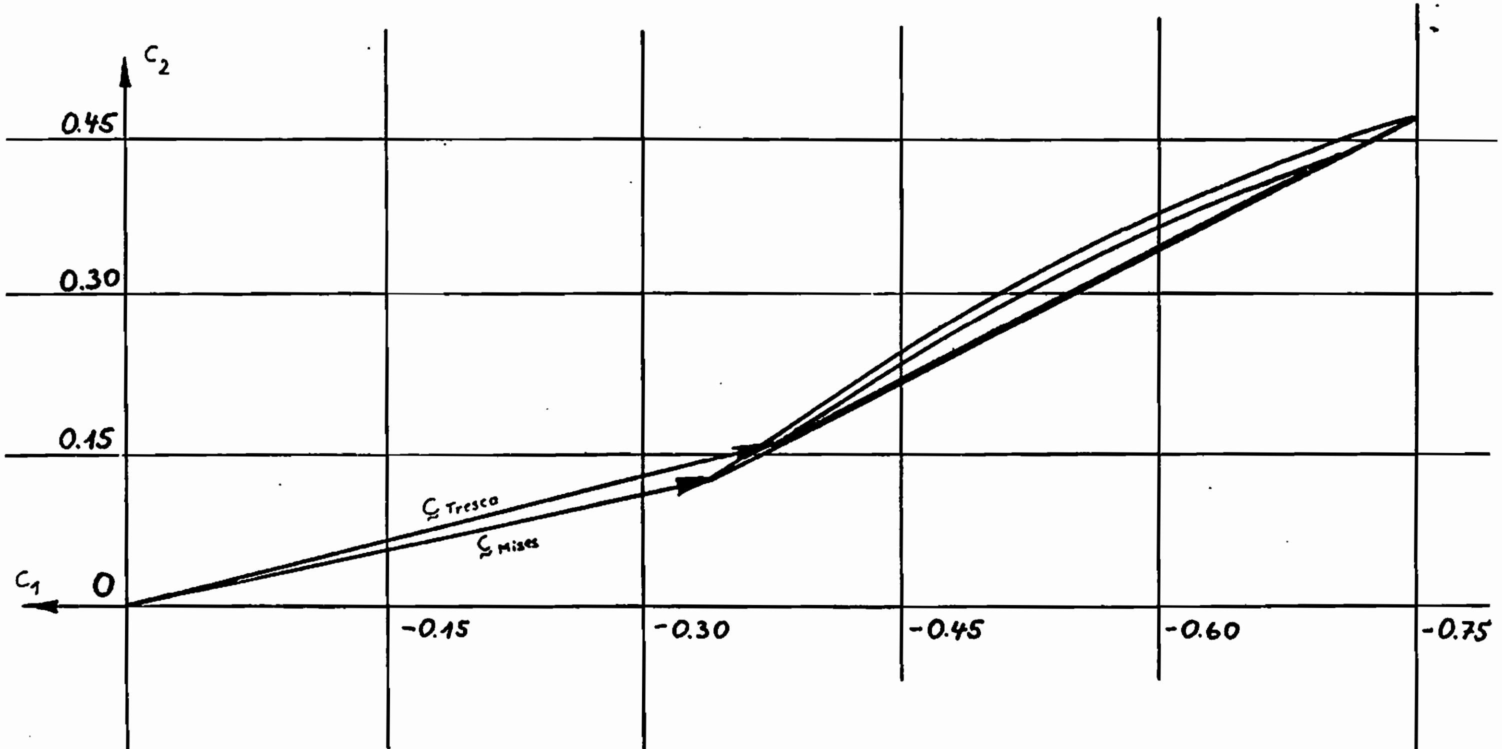
Region of admissible parameters c_1, c_2

load $q_o = 1{,}5$, $\sigma_s = 0.72$

inner region: Tresca's yield-condition

outer region: von Mises' yield-condition

$\underset{\sim}{C}$: solution-vectors.



<u>Example (4.1), two-dimensional method</u>

Region of admissible parameters c_1, c_2

Load parameter $q_o = 2.5$, $\sigma_s = 0.72$

<u>inner region: Tresca's yield-condition</u>

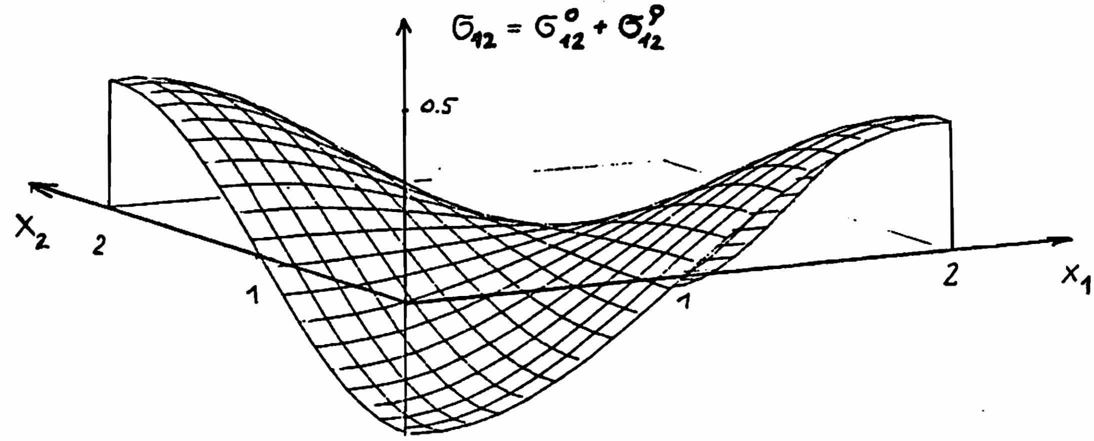
<u>outer region: von Mises' yield-condition</u>

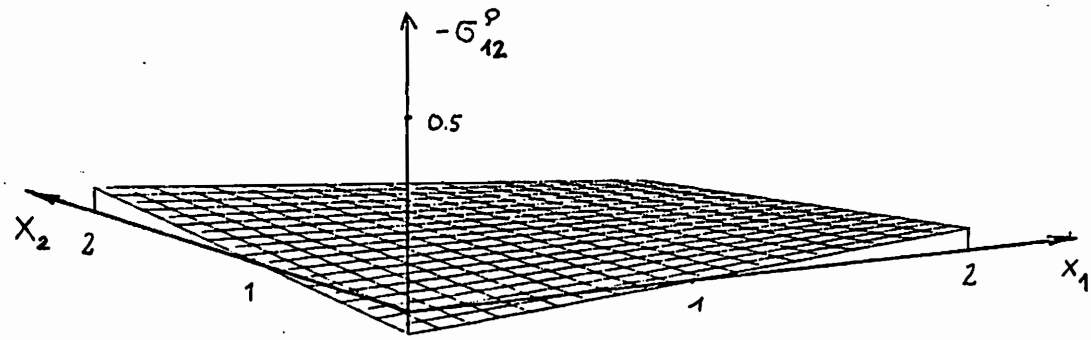
$\underset{\sim}{C}$ = solution-vectors

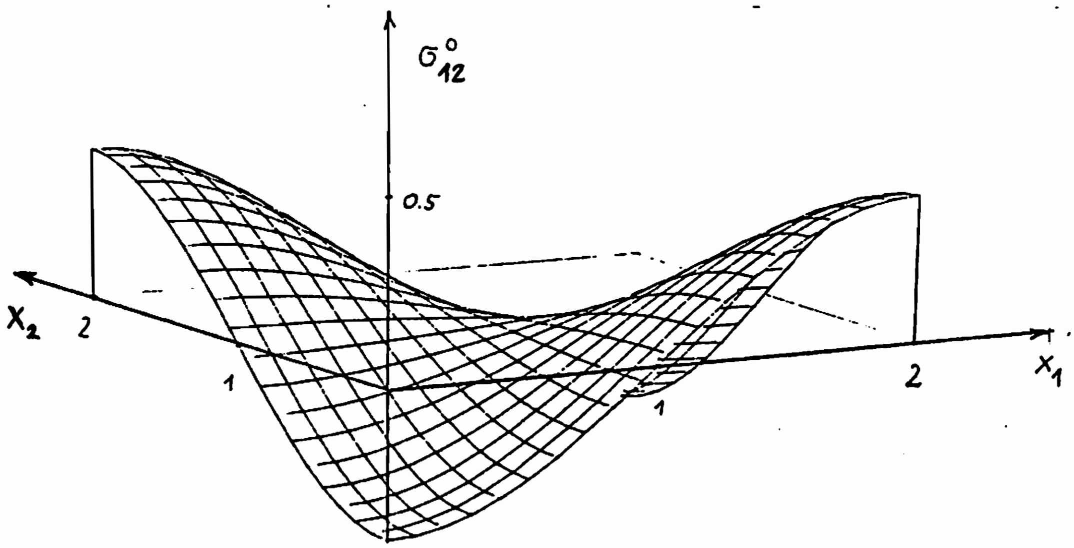
Appendix A11

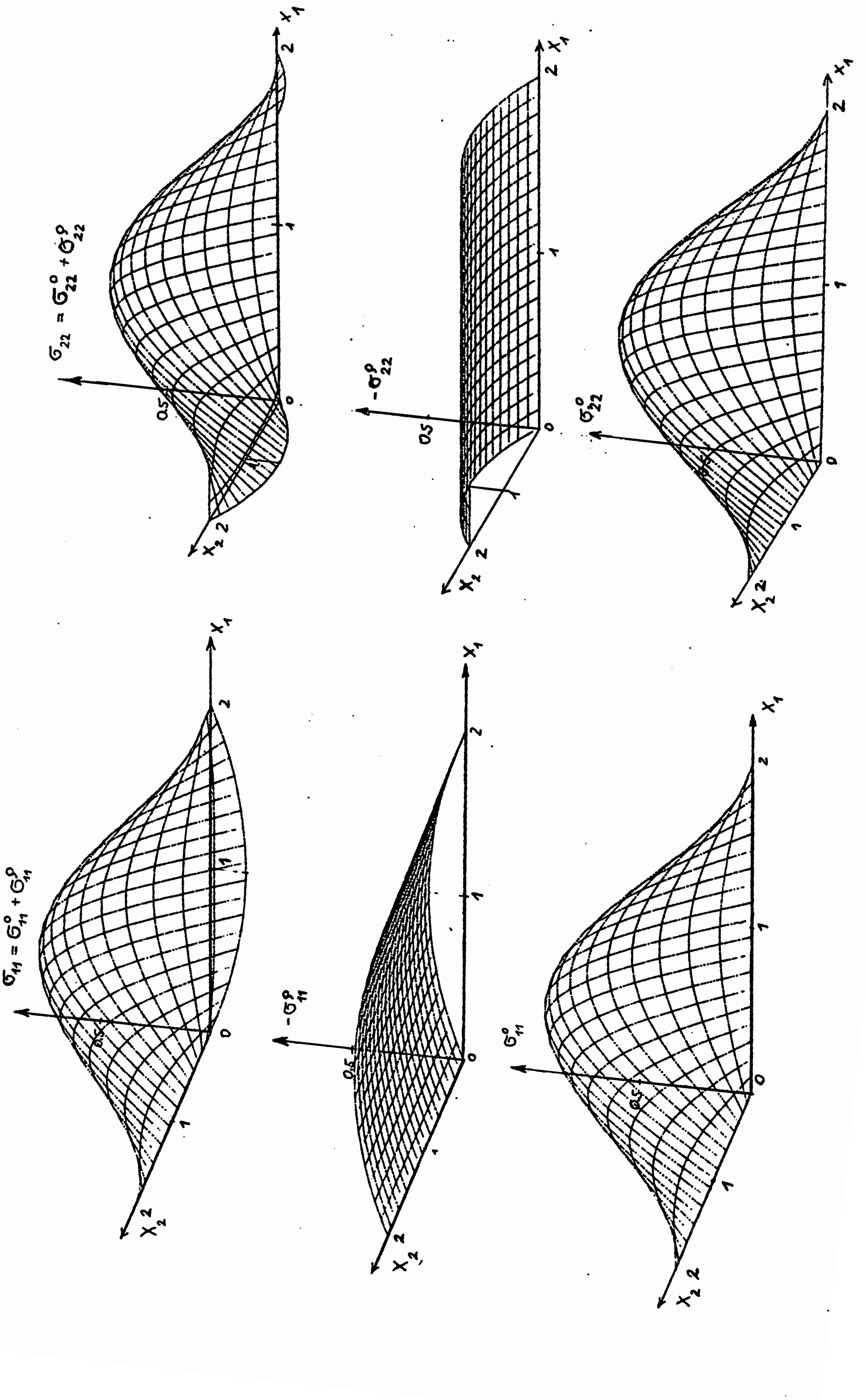
Graphical representation of redistribution of stresses for example (4.1), three-dimensional method, q_o = 1,15, σ_S = 0.72, von Mises' yield-condition.

Scale: σ = 1.0 $\hat{=}$ 5.095 cm







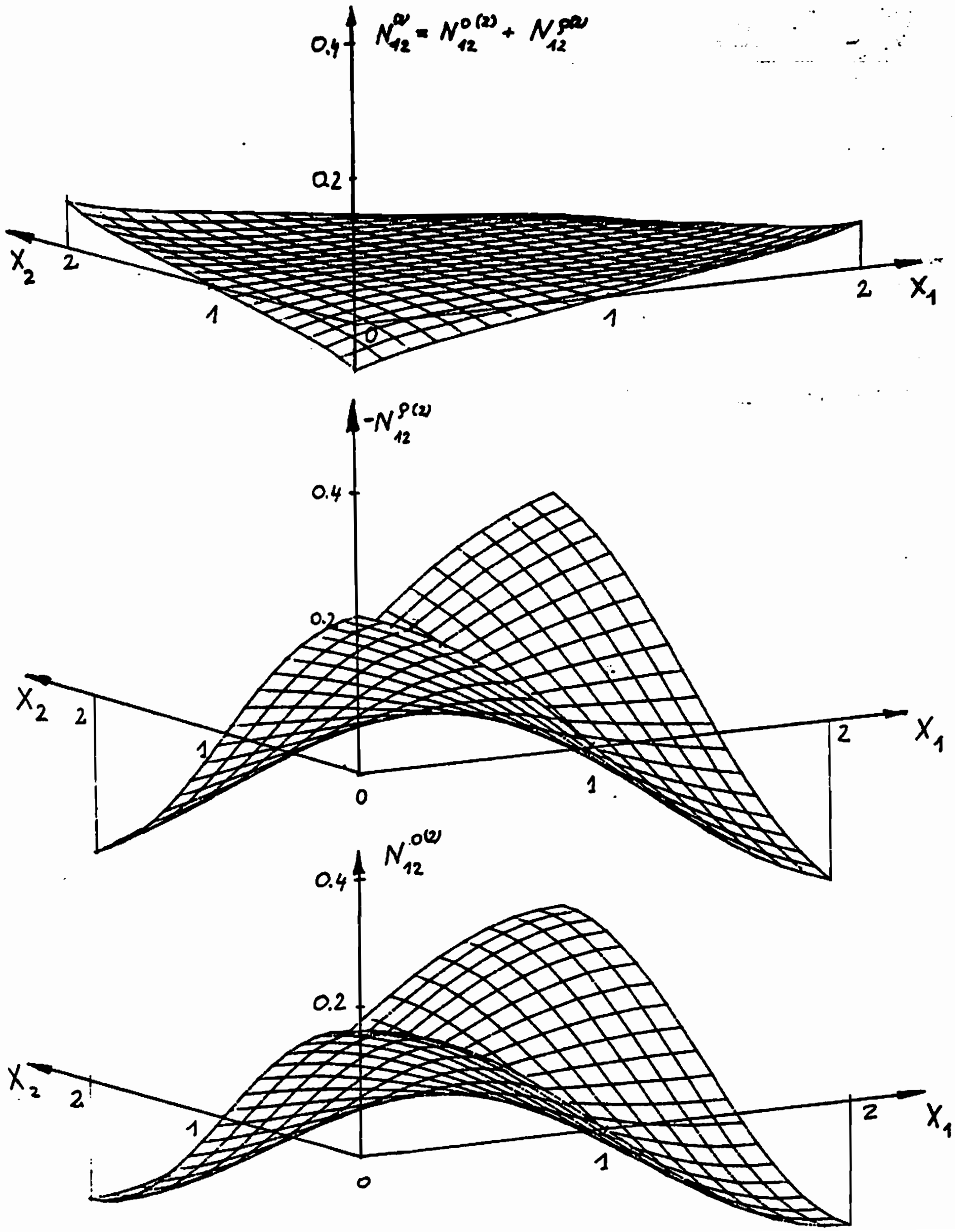

$\sigma_{22} = \sigma_{22}^o + \sigma_{22}^p$
$-\sigma_{22}^p$
σ_{22}^o
$\sigma_{11} = \sigma_{11}^o + \sigma_{11}^p$
$-\sigma_{11}^p$
σ_{11}^o
x_1
x_2

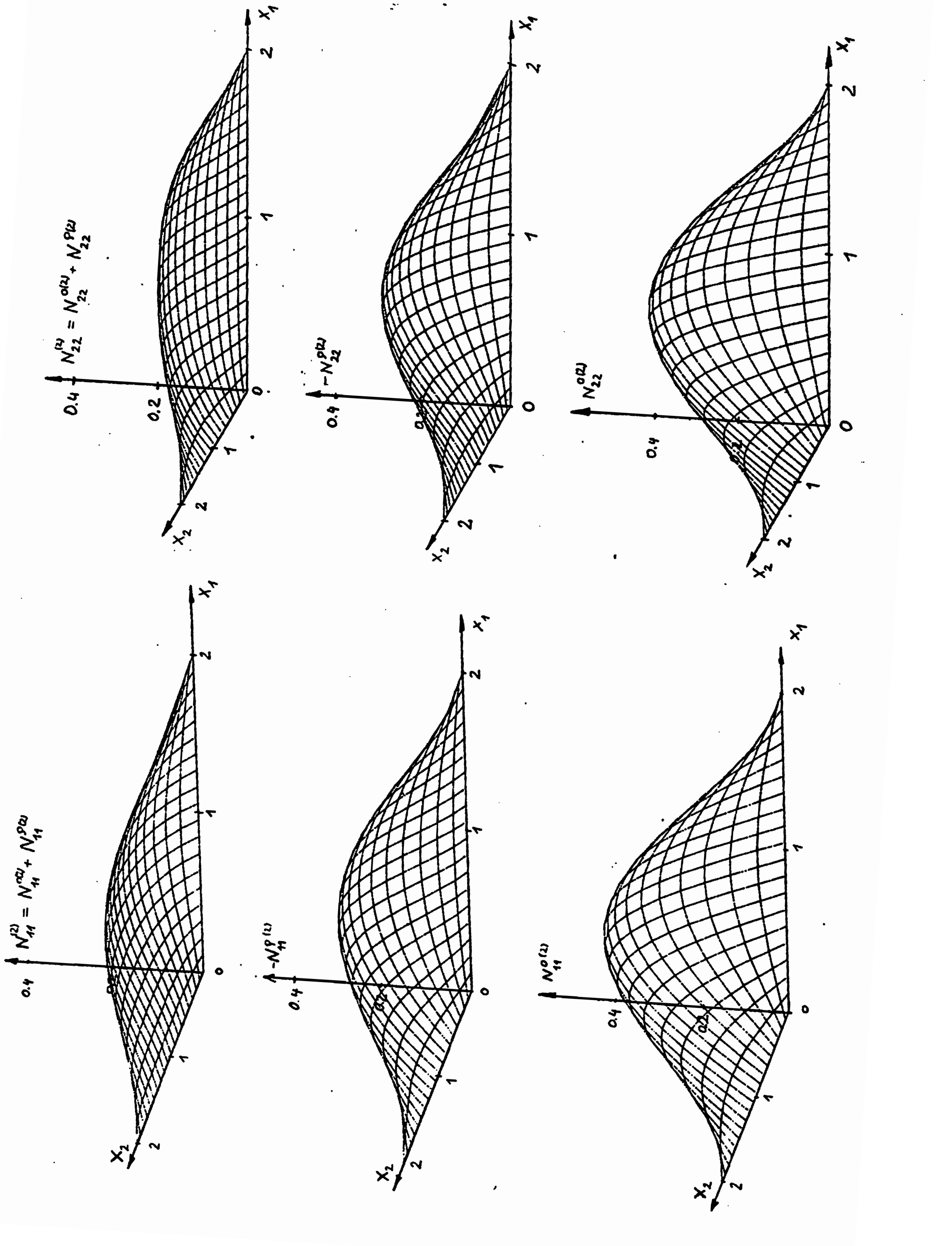
Appendix A 12

Graphical representation of redistribution of stresses for example (4.1), two-dimensional method,

$q_o = 2.5$, $\sigma_s = 0.72$, von Mises' yield-condition.

Scale: $N^{(2)} = 0.2 \mathrel{\hat{=}} 2.075$ cm.

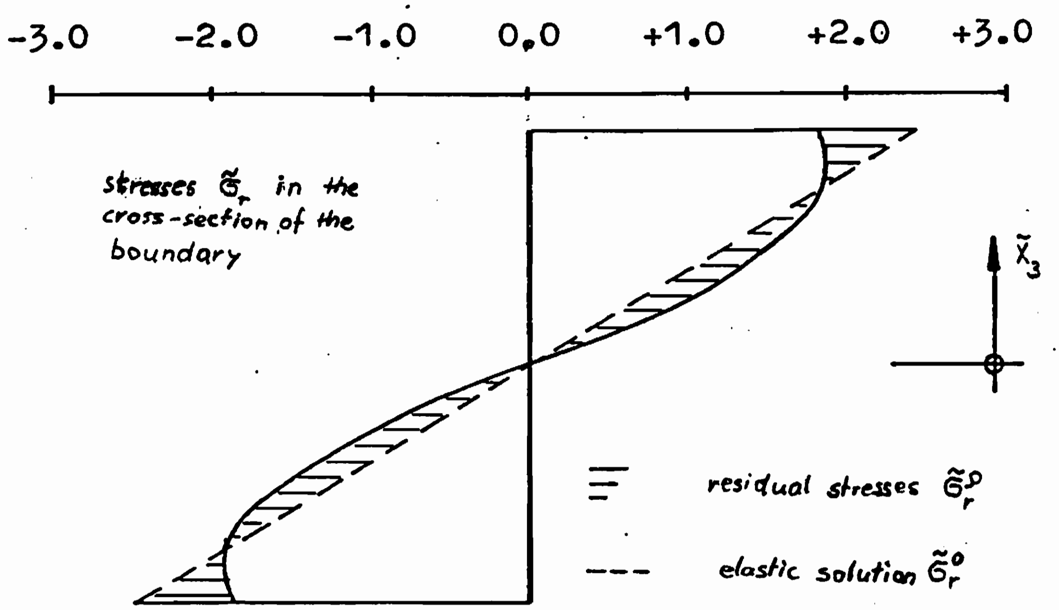


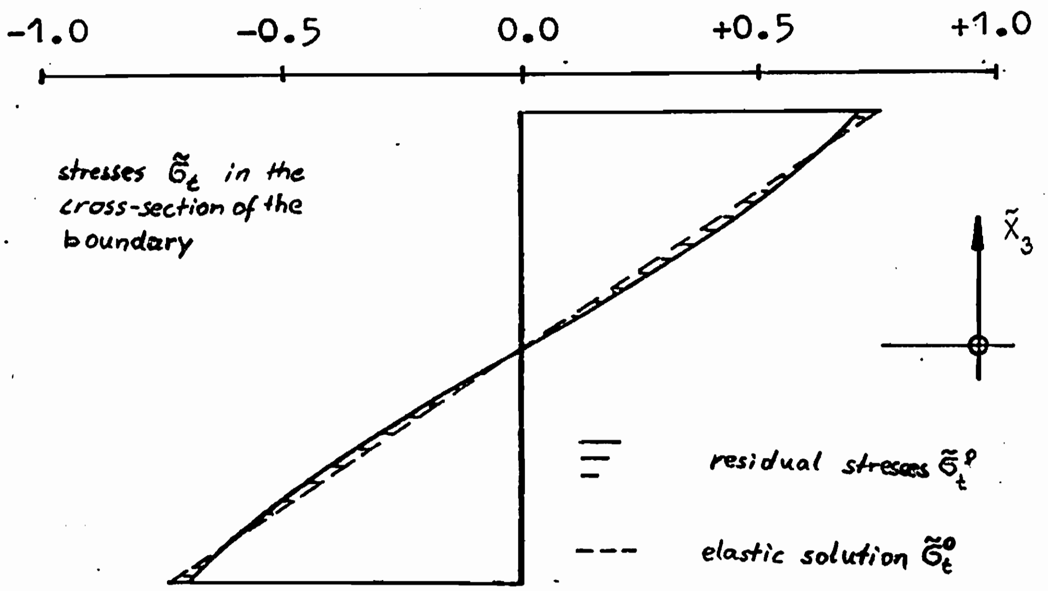


$N_{22}^{(2)} = N_{22}^{o(2)} + N_{22}^{\rho(2)}$
$-N_{22}^{\rho(2)}$
$N_{22}^{o(2)}$
$N_{11}^{(2)} = N_{11}^{o(2)} + N_{11}^{\rho(2)}$
$-N_{11}^{\rho(2)}$
$N_{11}^{o(2)}$
x_1
x_2
0.4
0.2
0
1
2

Appendix A13

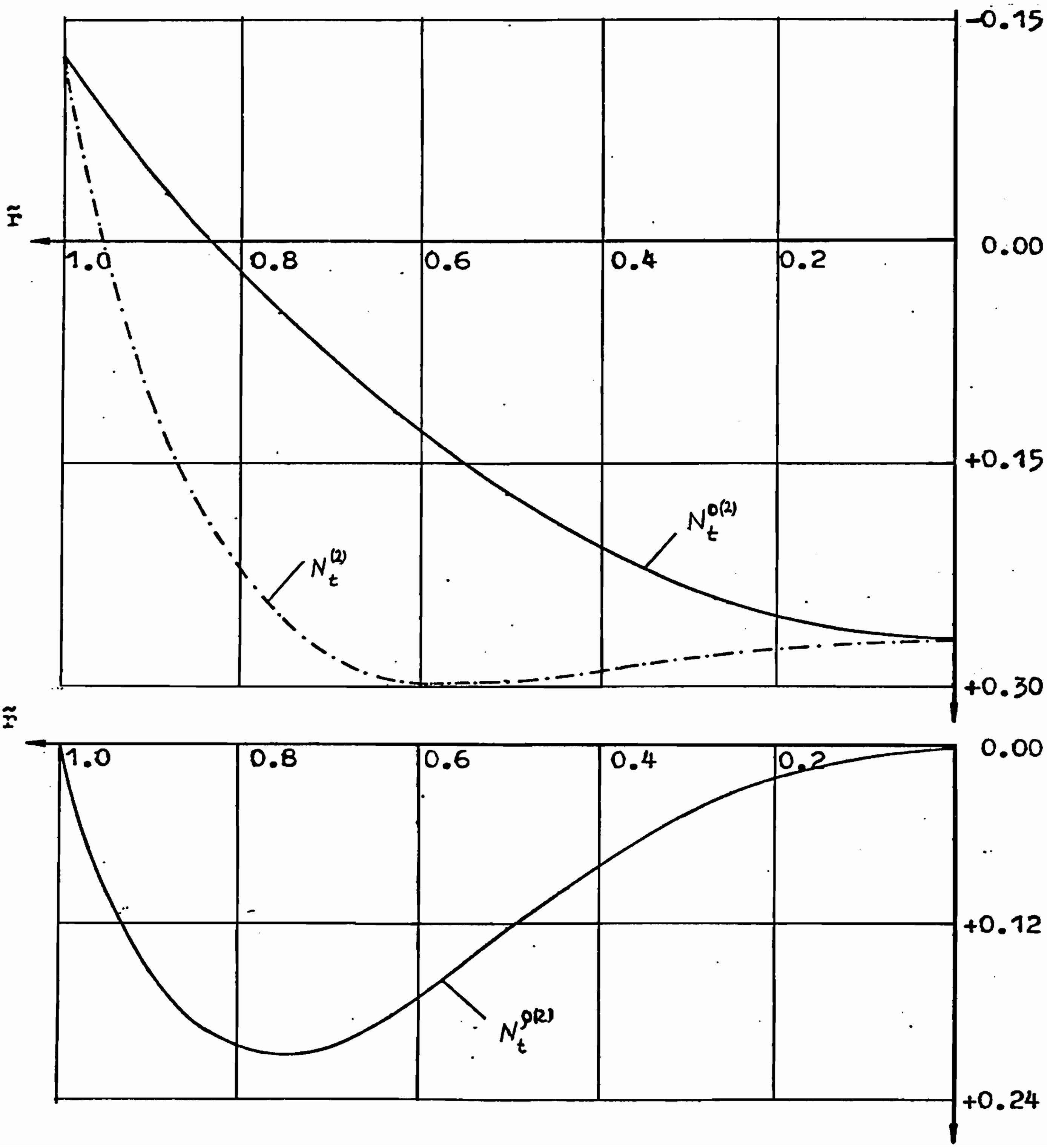
Graphical representation of redistribution of stress for example (4.2), three-dimensional method, $q_o = 3.3$, $\sigma_s = 1.62$, von Mises' yield-condition

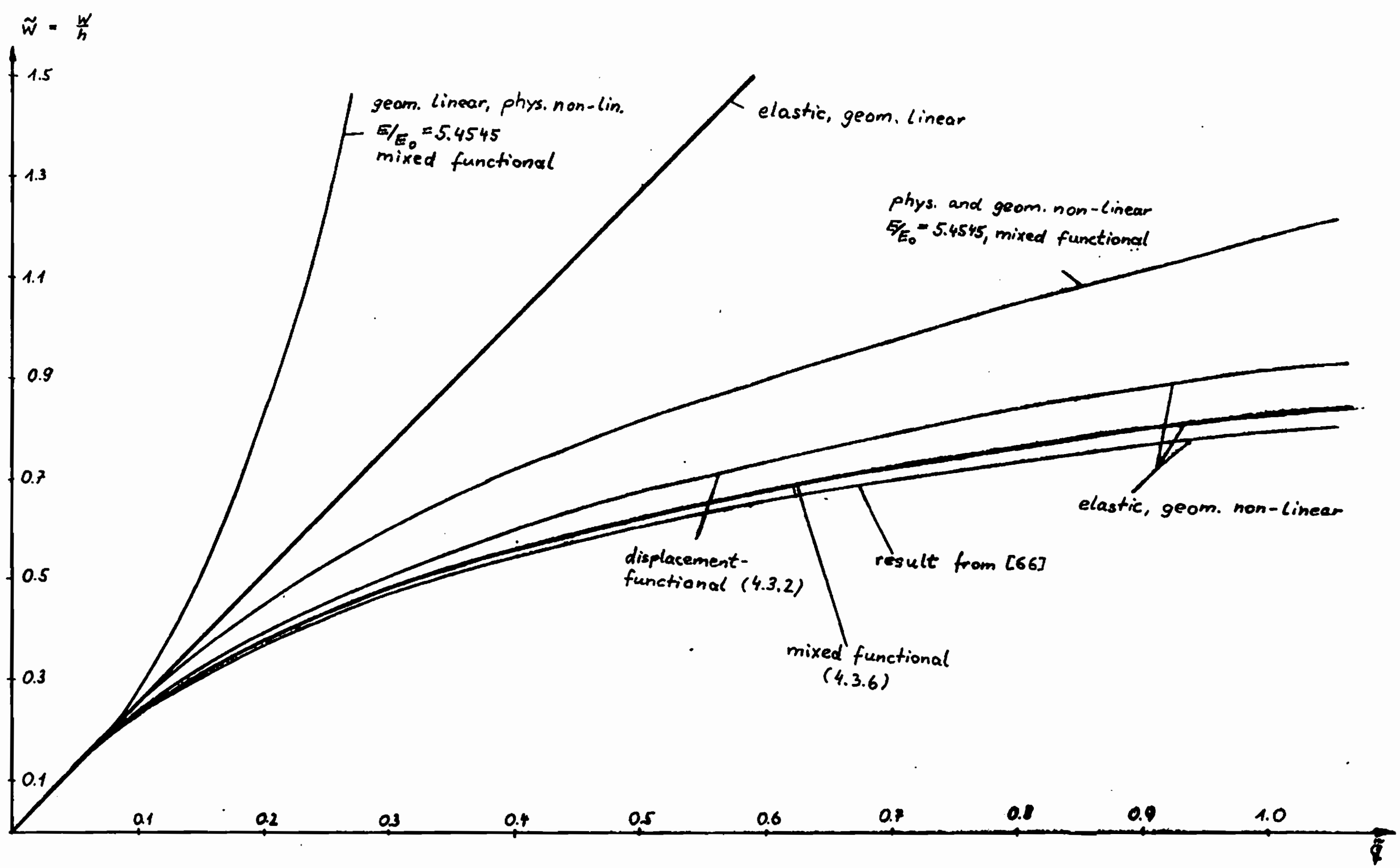




Appendix A14

Graphical representation of redistribution of stress for example (4.2), two-dimensional method, $q_o = 3.3$, $\sigma_s = 1.62$, von Mises' yield-condition.

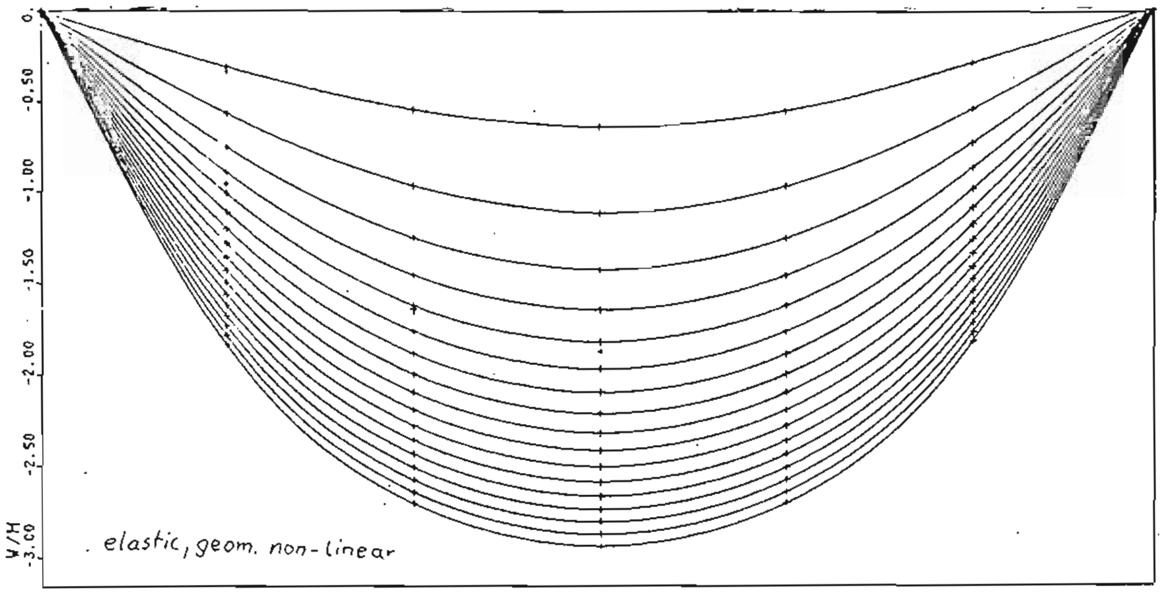


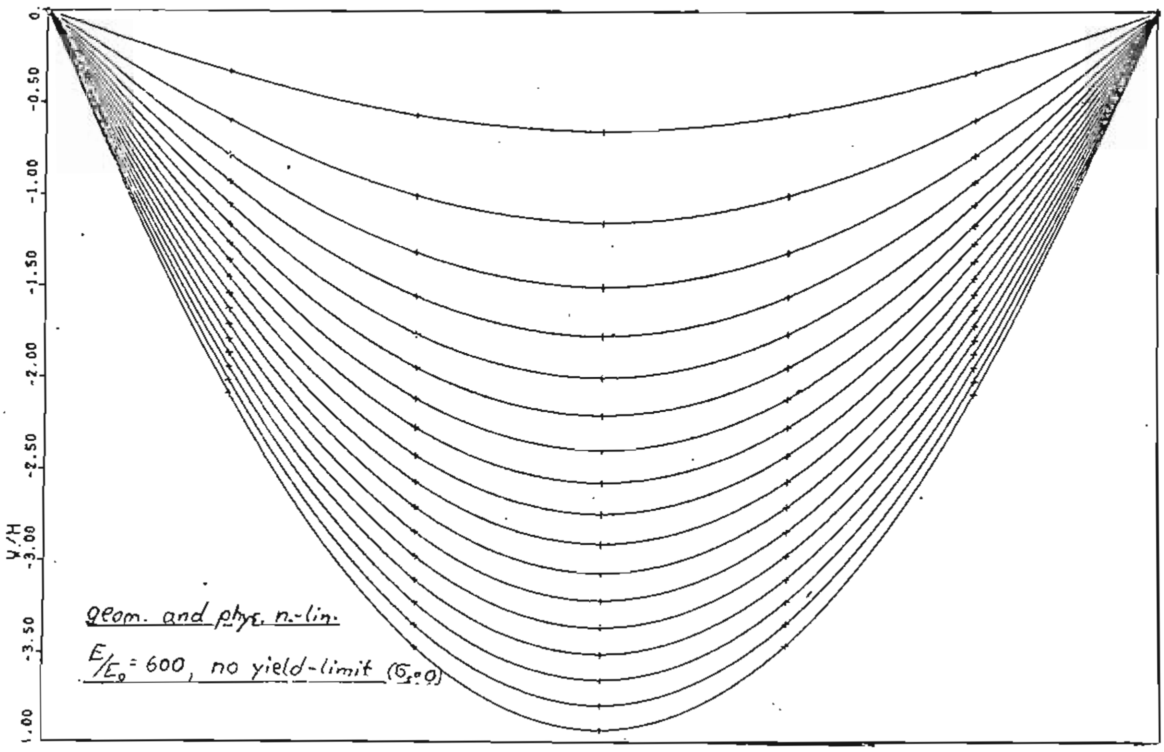


Appendix A15: Test-computations with functionals (4.3.2) and (4.3.6)

Appendix A16

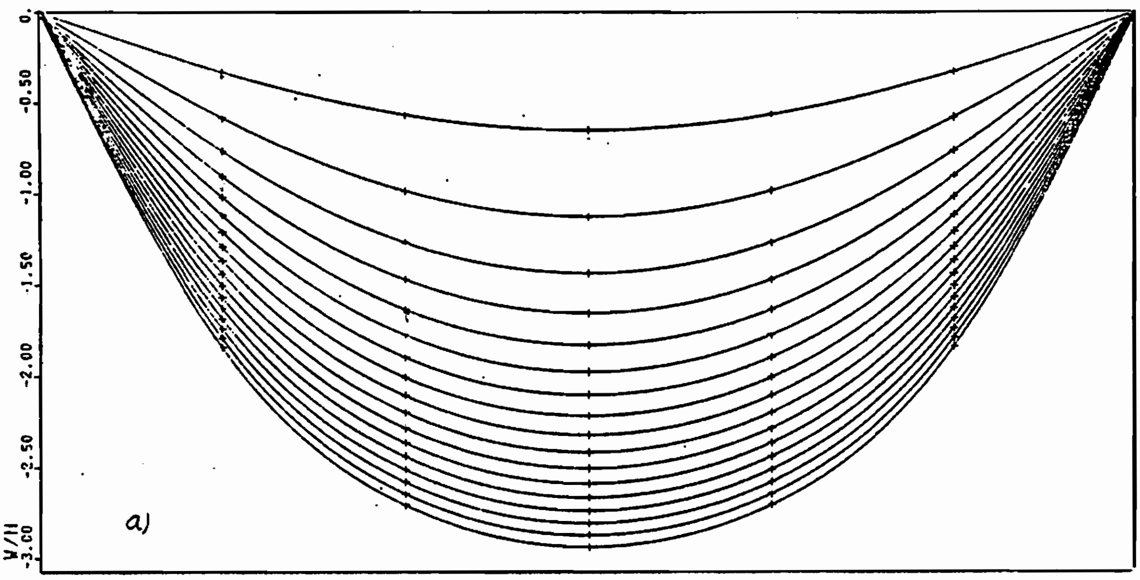
Qualitative dependance of increments of displacement z-direction from the quotient E/E_o. Material according to (2.6), no yield-condition (σ_s = 0), load-increments Δq = 0.025.

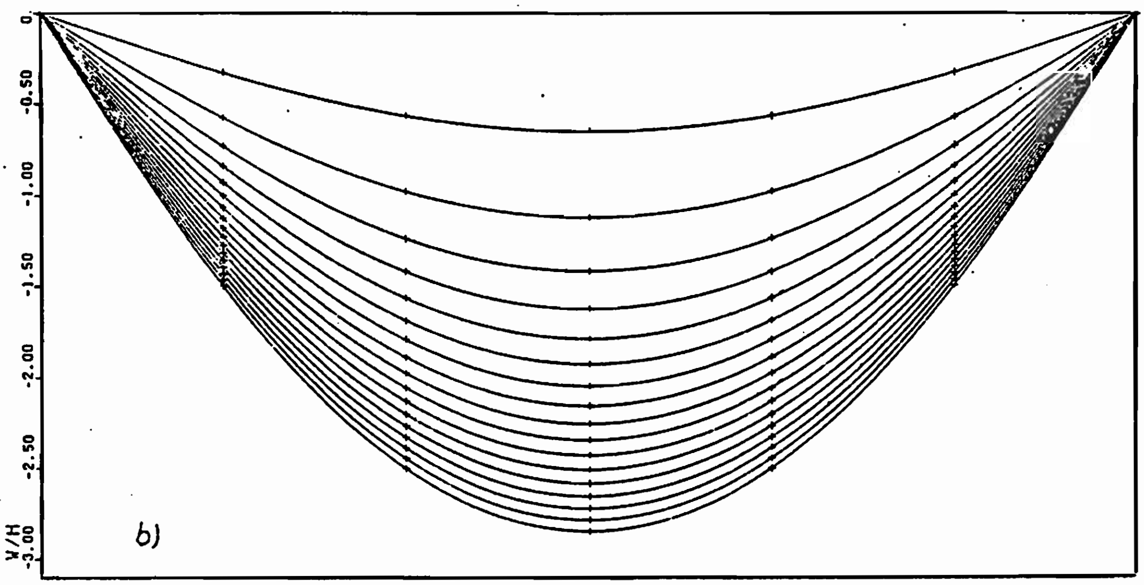




Appendix A17

Comparison of displacement-functions for geometrically nonlinear, elastic behaviour using functional (3.4.2) for the same load-increments
a) allowing for all terms
b) cancelling expression (4.3.5)





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