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SUMMARY

For the formulation and solution of many engineering problems considered nowaday there is a need to introduce additionally rotational degrees of freedom. Continuum models with internal structures of this type are connected with the names of E. & F. Cosserat, but it is a cumbersome task to read their original papers and understand all results given therein. The aim of this paper is a reexamination of the original work of E. & F. Cosserat and to show that connections and analogies of their results with modern continuum mechanics and field theory are deeper than commonly recognized.

Contents

§ 1. Introduction

During the past few years there had been an increasing interest in continuum models with additional degrees of freedom. Examples are the theory of thin shells and the theory of plasticity. The widely used Kirchhoff-Love type shell theories are based on a 3-parameter description with the middle surface displacement vector as primary unknown. This is an adequate description for many shell problems under the assumption of small strains. If we want to drop this restriction we can construct higher order shell models by introducing additional rotational degrees of freedom. To consider the behavior of material with block structure in the framework of a continuum theory of plasticity, a constitutive model can be constructed taking into account the influence of relative rotations between individual blocks.

These examples show that for many problems of continuum mechanics there is a need for models with additional rotational degrees of freedom, which leads directly to the concept of E. and F. Cosserat, who defined a generalized continuum with three translational degrees of freedom and three rotational degrees of freedom: It is assumed that to each material point of the continuum is attached a local rigid coordinate cross. During the deformation the rigid crosses are displaced and rotated with respect to any fixed coordinate system. However, while the translational movement is widely understood the analysis of rotation is difficult and still poorly known.

Unfortunately the basic results in the papers of E. and F. Cosserat are presented in a description and by using notations no more familiar nowadays. Therefore these results are difficult to understand even for experts of mechanics.

The aim of this paper is a reexamination of the original work of E. and F. Cosserat from the point of view of modern continuum theory. The paper contains a brief exposition of the basic Cosserat concept in the language and spirit of continuum mechanics or, more generally, of modern field theory. It is shown that the Cosserat model lies at the very foundation of so-called generalized models of continuum mechanics and physics.

§ 2. Prehistory of rotational kinematics

Among the first things we notice about the physical world are two kinds of motion: the first is associated with a translational change of place with respect to a stationary reference frame, and the second is associated with rotation. Numerous mathematical models derived for the description of physical phenomena often used some obvious and some not so obvious consequences of translational and rotational symmetry. These two kinds of freedom of motion, observed in physics of atoms, molecules, fluids and solids are fundamental to modern visualizing and computing. They are sometimes very complex problems of theoretical and applied physics. However, whereas the translational movement is widely understood, the analysis of rotation is difficult and still poorly known and often reserved for experts in non-Abelian group theory or mechanics (see ARGYRIS [1982]).

The modern study of rotation in 3-dimensional space had been opened by HAMILTON [1848] in his fundamental work devoted to the quaternion theory. In this theory the subject of a momental quaternion as a set of fourth order was introduced

$$A = A_0 + A_1 i + A_2 j + A_3 k = (A_0, \mathbf{A})$$

together with the fundamental multiplication rule for $A = (A_0, \mathbf{A})$ and $B = (B_0, \mathbf{B})$

$$AB = (A_0, \mathbf{A})(B_0, \mathbf{B}) = (A_0 B_0 - \mathbf{AB},\ \mathbf{A} \times \mathbf{B} + A_0 \mathbf{B} + B_0 \mathbf{A}) \ . \tag{1}$$

The quaternion $1 = (1, \mathbf{0})$ now is the identity quaternion, $1\,A = A\,1 = A$, in the algebra of quaternions where the quaternions $i = (0,1,0,0)$, $j = (0,0,1,0)$ and $k = (0,0,0,1)$ span the base in a vector space. The multiplication rule applied to the base $1, i, j, k$ may conveniently be written as

$$\begin{gathered} ij = -ji = k \ , \quad jk = -kj = i \ , \quad ki = -ik = j \\ i^2 = j^2 = k^2 = ijk = -1 \ . \end{gathered} \tag{2}$$

The quaternion $\bar{A} = (A_0, -\mathbf{A})$ is called conjugate to A, so that $\|A\| = \sqrt{(A\bar{A})} = (A_0^2 + \mathbf{A}^2)^{1/2}$ is the quaternion norm.

From the point of view of 3-dimensional rotations only quaternions with the norm equal to 1 are important. The set of such quaternions are elements of the 3-parameter group Sp(1) with (1) as group multiplication. Physically, it means that we will replace the vector field theory subject to the constraint condition $A_\mu A^\mu = 1$, $\mu = 0,1,2,3$ by a three rotational parameter field theory which identically satisfies the constraint $A_\mu A^\mu = 1$. With respect to the isomorphism between Sp(1) and SU(2), the universal covering group of SO(3), we can obtain a useful description of the quaternion $A \in Sp(1)$ by introducing the 2-dimensional representation of the algebra su(2) (Pauli matrices σ^0, σ^i i=1,2,3)

$$R(A) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} A_o+A_3 i & A_1 i+A_2 \\ A_1 i-A_2 & A_0-A_3 i \end{pmatrix} = A_0\sigma^0+(A_1\sigma^1+A_2\sigma^2+A_3\sigma^3)i$$

where (3)

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and a,b are complex numbers representing the 2-dimensional rotation matrix $\mathbf{R} \in SU(2)$. Since SU(2) is locally isomorphic to SO(3) it is easy to see that $\mathbf{R}(A)$ may also be represented by three Euler angles α,β,γ (fig. 1) (EULER [1862]) or by a rotation vector $\boldsymbol{\lambda} = \lambda\mathbf{e}$ whose parameters are traditionally used to describe the special orthogonal group SO(3). The final result of these transformations is the relation

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & -e^{i(\gamma-\alpha)/2}\sin\beta/2 \\ e^{i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{pmatrix}$$

or

$$R(\boldsymbol{\lambda}) = \cos\lambda/2\,\sigma^0 - i\sin\lambda/2\,(\lambda_1\sigma^1 + \lambda_2\sigma^2 + \lambda_3\sigma^3)\ .$$

At present we are aware of the fact that Hamilton's quaternions so much related to Pauli's spinors play a significant role in the history of rotation theory. Hamilton also hoped that quaternions would have a significance in 3-dimensional space analogous to that of complex numbers in the plane. Hamilton gave the following description of his discovery of quaternions in a letter to his son Archibald:

In October, 1843, having recently returned from a meeting of the British Association in York, the desire to discover the laws of the multiplication of

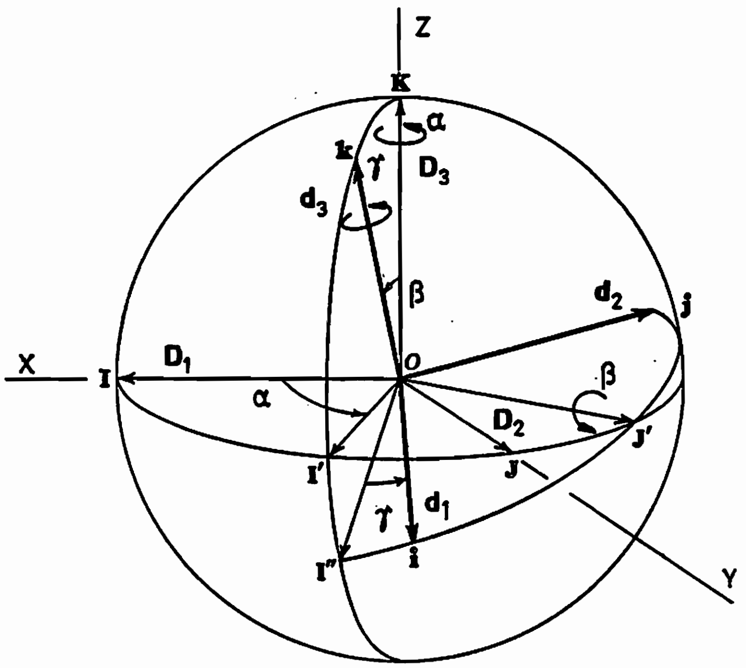

FIG. 1 The Eulerian angles.

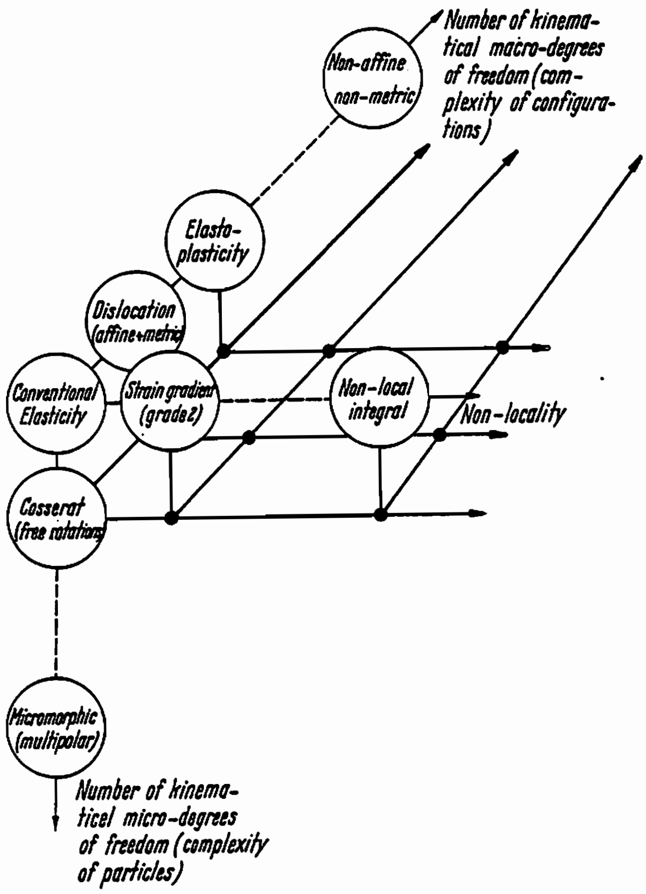

Fig. 2. Geometric classification of continuum mechanics.

triplets regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your brother William Edwin and yourself used to ask me, "Well, Papa, can you multiply tripletes ?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them". But on the 16th day of the same month - which happened to be a Monday and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the parts of others, if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i,j,k ;

$$i^2 = j^2 = k^2 = ijk = -1 ,$$

which contains the solution of the Problem, but of course, as an inscription, has long since mouldered away [Instead a plaque on the bridge now commemorates this event].

G. DARBOUX [1887] first extended the Hamilton's approach to the case of the construction of a spinor by a 3-dimensional vector satisfying constraint condition. He has rewritten the equation of motion of a rigid body attached at one fixed point

$$\dot{\alpha} = \beta r - \gamma q \quad , \quad \dot{\beta} = \gamma p - \alpha r \quad , \quad \dot{\gamma} = \alpha q - \beta p$$

(here (α,β,γ) denotes - in Darboux notation - the position vector of a point on a unit sphere fixed in the body with $\alpha^2+ \beta^2+ \gamma^2=1$ and (p,q,r) is the angular velocity of rotation of the body) in terms of a spinor ψ_A

$$\dot{\psi}_A = \frac{i}{2}(p\ \sigma^1_{AB} + q\ \sigma^2_{AB} + r\ \sigma^3_{AB})\ \psi_B\ , \qquad A,B = 1,2 \tag{4}$$

where σ^i is defined by (3). The rigid-body kinematics governed by the position vector (α,β,γ) is now described by two component of the spinor ψ_A and two spinor equations (4). It is certainly remarkable that Darboux formulated this problem many years before the discovery of Pauli's spinor and quantum spin.

At the same time, long before Cosserat, correctness and meaning of the controversial paper of MacCULLAGH [1839] had been discussed. Let us recall that the paper was devoted to the construction of a model of elastic medium which can simultaneously describe observed reflection and refraction.

The energy of deformation in MacCullagh continuum depends on the rotational components of deformation. Numerous works, especially by Fresnel, Neumann, Thomson, Boussinesq (see WHITTAKER [1910]) who extensively criticized the rotational continuum gave a substantial contribution to the development of the mathematical theory of elasticity. As a result, the fundamental equations of classical 3-dimensional Cauchy continuum had been developed and examined in the books by MOSSOTTI [1851], CLEBSCH [1862], KIRCHHOFF [1874], THOMSON & TAIT [1879], DUHEM [1891] and HERTZ [1894]. First of all, constitutive relations for isotropic and anisotropic continua had been discussed. Geometrically as well as physically nonlinear formulations of the state of stress and strain in finite deformed Cauchy medium were successfully applied by HELMHOLTZ [1897], APPELL [1903] and LORENTZ [1903], who worked also on formulations in Eulerian and Lagrangian description, on fundamental conservation laws in rotational fluid flow, etc.

However in the theory of rods and plates, respectively, attention had been directed to the explanation of flexural and bending properties of very thin shell-like bodies. But the problem of some kind of rotational energy of deformation returned into interest and conceptions, first discussed by CLEBSCH [1860], [1862], KIRCHHOFF [1874] and DUHEM [1891], [1893]. A. Clebsch, adopting from Kirchhoff the concept of "stress-resultants" and "stress-couple" and using this hypothesis, formulated the energetically conjugate couple for "rotational energy". The rotational measure of deformation, similar as in Duhem's work, has been rewritten as a function of motion of so-called "hidden rigid triad". For this reason, we can say that E. and F. Cosserat generalized and developed Kirchhoff's, Clebsch's and Duhem's works.

§ 3. The main results of E. and F. Cosserat

This chapter is devoted to the interpretation of the results of the outstanding work of E. and F. COSSERAT [1909] in the light of modern continuum theory. It turns out that connections and analogies of the Cosserat model with the classical field theory are deeper then has commonly been recognized.

Let us discuss now several aspects of Cosserat's monograph. In particular, the development of two most important aspects of Cosserat's approach will be discussed independently in the following two chapters entiteled **"the unification sector"** and **"the conservation sector"**. Now we present briefly the fundamental results of Cosserat.

3.1 n-dimensional continua

In axiomatic continuum mechanics the deformation is defined as a mapping α of a Borel set Ω describing a n-dimensional submanifold into the Euclidean space $\mathbb{R}^p$ (BETOUNES [1987]). The notion of deformation is appropriate for submanifolds Ω which idealize strings, membranes or solids. More physically speaking, we consider the n-dimensional body manifold immersed in p-dimensional space. Interesting cases exist for $n \leq p$ only. The method of immersion is frequently used also in field theory, particularly in the generalized Kaluza-Klein theory (see also LAGOUDAS [19]), where one considers the suitable immersion of four-dimensional Riemannian submanifold in five-dimensional space-time.

The starting point of the work of E. and F. Cosserat is quite similar. They assume that: *"A deformable line is a one-parameter ensemble of triads; a deformable surface is an ensemble of two parameters; a deformable medium is an ensemble with three paramaters ρ_i, i=1,2,3. When there is motion, the time* t *must be added to these geometric parameters ρ_i"* (COSSERAT [1909], p. 2).

Thus they consider four cases of immersions in $\mathbb{R}^p$, p = 6

$$n = 1 \quad - \text{statics of deformable line} \qquad \rho_\mu \equiv \rho_1 = s \; ; \quad \mu = 1$$

n = 2 - statics of deformable surface $\rho_\mu \equiv \rho_1, \rho_2$; $\mu = 1,2$
- dynamics of deformable line $\rho_\mu \equiv t,\ \rho_1$; $\mu = 0,1$

n = 3 - statics of deformable medium $\rho_\mu \equiv \rho_1, \rho_2, \rho_3$; $\mu = 1,2,3$
- dynamics of deformable surface $\rho_\mu \equiv t, \rho_1, \rho_2$; $\mu = 0,1,2$

n = 4 - dynamics of deformable medium $\rho_\mu \equiv t, \rho_1, \rho_2, \rho_3$; $\mu = 0,1,2,3$

The main feature of such approach is that the structure of the theory remains independent of the dimension of the submanifold.

All measures of deformation are indexed via index μ which is connected with a differentation with respect to the holonomic parameters ρ_μ. However, the quantity of these measures is strictly dependent on the dimension of $\mathbb{R}^p$. In the Cosserat theory the dimension $p = 6$ is equal to the dimension of a group $T(3) \triangleright SO(3)$.

The measures for $p = 1,2,3$ are called translational measures of deformation or geometrical rates of translation defined as follows (COSSERAT [1909], p. 155).

$$
\begin{aligned}
p = 1 \qquad & \xi_\mu = \alpha \frac{\partial x}{\partial \rho_\mu} + \alpha' \frac{\partial y}{\partial \rho_\mu} + \alpha'' \frac{\partial z}{\partial \rho_\mu} , \\
p = 2 \qquad & \eta_\mu = \beta \frac{\partial x}{\partial \rho_\mu} + \beta' \frac{\partial y}{\partial \rho_\mu} + \beta'' \frac{\partial z}{\partial \rho_\mu} , \\
p = 3 \qquad & \zeta_\mu = \gamma \frac{\partial x}{\partial \rho_\mu} + \gamma' \frac{\partial y}{\partial \rho_\mu} + \gamma'' \frac{\partial z}{\partial \rho_\mu} , \quad \mu = 0,1,2,3
\end{aligned}
\tag{5}
$$

and for $p = 4,5,6$ they are called rotational measures of deformation or geometrical rates of rotations

$$
\begin{aligned}
p = 4 \qquad & p_\mu = \gamma \frac{\partial \beta}{\partial \rho_\mu} + \gamma' \frac{\partial \beta'}{\partial \rho_\mu} + \gamma'' \frac{\partial \beta''}{\partial \rho_\mu} , \\
p = 5 \qquad & q_\mu = \alpha \frac{\partial \gamma}{\partial \rho_\mu} + \alpha' \frac{\partial \gamma'}{\partial \rho_\mu} + \alpha'' \frac{\partial \gamma''}{\partial \rho_\mu} , \\
p = 6 \qquad & r_\mu = \beta \frac{\partial \alpha}{\partial \rho_\mu} + \beta' \frac{\partial \alpha'}{\partial \rho_\mu} + \beta'' \frac{\partial \alpha''}{\partial \rho_\mu} , \quad \mu = 0,1,2,3
\end{aligned}
\tag{6}
$$

where $\mathbf{x} = xD_1 + yD_2 + zD_3$ describes the position vector and

$$
\mathbf{R} = \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{pmatrix}
\tag{7}
$$

is the rotation matrix connecting the unrotated rigid triad $\mathbf{D}_a$ and the rotated one $\mathbf{d}_a$ with $\mathbf{d}_a = \mathbf{R}\,\mathbf{D}_a$, $a = 1,2,3$ (fig. 1).

From the following constitutive equations (COSSERAT [1909], p. 160)

$$A'_\mu = \frac{\partial W}{\partial \xi_\mu} , \quad B'_\mu = \frac{\partial W}{\partial \eta_\mu} , \quad C'_\mu = \frac{\partial W}{\partial \zeta_\mu} ,$$
$$P'_\mu = \frac{\partial W}{\partial p_\mu} , \quad Q'_\mu = \frac{\partial W}{\partial q_\mu} , \quad R'_\mu = \frac{\partial W}{\partial r_\mu} \tag{8}$$

with the total energy W one can obtain the Cosserat measures of stresses for $\mu = 1,2,3$ and the Cosserat momentum and angular momentum for $\mu = 0$:

$$\textbf{translational :} \quad \mathbf{N}^\mu = A'_\mu \mathbf{d}_1 + B'_\mu \mathbf{d}_2 + C'_\mu \mathbf{d}_3$$
$$\textbf{rotational :} \quad \mathbf{M}^\mu = P'_\mu \mathbf{d}_1 + Q'_\mu \mathbf{d}_2 + R'_\mu \mathbf{d}_3 . \tag{9}$$

Also the form of the equations of motion is independent of the dimension of the body submanifold. Using (9) we write all Cosserat equations of motion in compact form (COSSERAT [1909], p. 161)

$$\mathbf{N}^\mu{}_{|\mu} + \mathbf{p} = 0$$
$$\mathbf{M}^\mu{}_{|\mu} + \mathbf{x},_\mu \times \mathbf{N}^\mu + \mathbf{m} = 0 \tag{10}$$

In these equations of motion the external sources $\mathbf{p}$ and $\mathbf{m}$ and the position vector $\mathbf{x} = x'\mathbf{d}_1 + y'\mathbf{d}_2 + z'\mathbf{d}_3$ are assumed to be referred to the corotational base $\mathbf{d}_a$.

It is essential for a deeper understanding of the n-dimensional approach that the time plays the role of a geometric coordinate parametrizing the submanifold under consideration. It means, for instance, that a moving string is treated as a surface (world sheet) swept out by the line during any motion. The sheet is, similar to a material surface, parametrized by two internal coordinates, say τ and s, where τ represents the internal time and s a length parameter. The formal treatment of the internal time as a geometrical coordinate is similar to that in modern string theory where the discussed approach is recognized as the basic mathematical formalism.

3.2 $T(3)\triangleright SO(3)$ unification

We frequently analyze a physical system which is naturally and conveniently divided into individual parts called extended objects. The objects of dimensionality higher than of a point, among them strings, membranes and bags, may idealize any classical medium only if they do not possess an internal structure or internal degree of freedom. However, the presence of an internal geometry such as "*Zitterbewegung*" in a theory of electron, leads to new generalized models of field theory.

From the point of view of continuum mechanics the simplest internal structure is the existence of rotational degrees of freedom. The question arises: how to connect two, physically different fields in a compatible model which can describe the displacement and rotation fields independently but also their mutual interaction. The first idea which comes into mind is to consider only first gradients of the fields in the Lagrangean action energy W . Candidates for such measure of strength of field are the deformation gradient $\mathbf{F}$ = Grad($\mathbf{X}$ + $\mathbf{u}$) and the gradient of rotation tensor $\mathbf{F}'$ = Grad $\mathbf{R}$ or the gradient of rotation vector $\mathbf{F}''$ = Grad $\boldsymbol{\omega}$. The question how the fields interact between themself and how many independent measures must be defined to describe completely the model is, for obvious reasons, rather difficult to answer, especially when the reader is not familiar with the fundamentals of symmetry groups and affine spaces as presented, for instance, in BETOUNES [1987] or LAGOUDAS [1989].

In their treatment of unification of displacement and rotation fields E. and F. Cosserat used a formalism which had been applied earlier in dynamics of rigid body motion. The rigid body, in general, possesses six degrees of freedom of motion and very often it is convenient to describe the motion in a moving frame connected with the own axis of the moving body.

We recall that the motion described with respect to the moving reference frame is characterized by a translation vector and a rotation vector or rotation parameters

$$\mathbf{x}(t) = x^a(t)\mathbf{d}_a(t) = x'd_1 + y'd_2 + z'd_3 , \qquad \mathbf{R}(t) = \mathbf{R}(\ \alpha(t),\beta(t),\gamma(t)\ ) .$$

The rates of change of position and orientation are measured in the moving frame as well and determined by

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t) = \dot{x}^a \mathbf{d}_a + x^a \dot{\mathbf{d}}_a = (\dot{x}^a + \epsilon^{abc} x_b \omega_c)\mathbf{d}_a = \xi \mathbf{d}_1 + \eta \mathbf{d}_2 + \zeta \mathbf{d}_3$$
$$\boldsymbol{\omega} = \tfrac{1}{2}\boldsymbol{\varepsilon}\cdot(\dot{\mathbf{R}}\mathbf{R}^{-1}) = p\,\mathbf{d}_1 + q\,\mathbf{d}_2 + r\,\mathbf{d}_3 = \omega_a \mathbf{d}_a \qquad (11)$$

with the permutation tensor $\boldsymbol{\varepsilon}$ and with

$$\xi = \dot{x}' + qz' - ry' \;,\; \eta = \dot{y}' + rx' - pz' \;,\; \zeta = \dot{z}' + py' - qx'$$
$$p = \dot{\beta}\sin\gamma - \dot{\alpha}\sin\beta\cos\gamma \quad q = \dot{\beta}\cos\gamma + \dot{\alpha}\sin\beta\sin\gamma \quad r = \dot{\gamma} + \dot{\alpha}\cos\beta$$

where x', y', z' denote the components of the position vector in the moving frame. These are nothing else but the translational and rotational velocities referred to the moving frame. A formal exchange of the time parameter t and the geometric parameters ρ_i $i=1,2,3$ in (11) leads to the definition of translational and rotational measures of deformation as geometrical rates calculated in this same moving frame. The quantities ξ_j, η_j, ζ_j and p_j, q_j, r_j, $j=1,2,3$ are functions of displacements and rotations and given by (5) and (6).

The reason for the difference between the definition of translational measures in (5) and (11) is due to the fact that in eqns. (5) the components x,y,z of the position vector are expressed in the fixed reference base $\mathbf{D}_a = \mathbf{d}_a(t=0)$. However, with the components x', y', z' of the position vector in the moving frame, the translational measure of deformation (5) will have the following form similar to (11), (COSSERAT [1909], p. 123)

$$\xi_i = \frac{\partial x'}{\partial \rho_i} + q_i z' - r_i y' \;,$$
$$\eta_i = \frac{\partial y'}{\partial \rho_i} + r_i x' - p_i z' \;, \qquad (12)_1$$
$$\zeta_i = \frac{\partial z'}{\partial \rho_i} + p_i y' - q_i x' \;,$$

$$p_i = \frac{\partial \beta}{\partial \rho_i}\sin\gamma - \frac{\partial \alpha}{\partial \rho_i}\sin\beta\cos\gamma \;,$$
$$q_i = \frac{\partial \beta}{\partial \rho_i}\cos\gamma + \frac{\partial \alpha}{\partial \rho_i}\sin\beta\sin\gamma \;, \qquad (12)_2$$
$$r_i = \frac{\partial \gamma}{\partial \rho_i} + \frac{\partial \alpha}{\partial \rho_i}\cos\beta \;.$$

In a modern Cosserat notation $(12)_1$ corresponds to strain measures $\gamma_{\alpha\beta}$ and

$(12)_2$ to curvature measures $\kappa_{\alpha\beta}$ (see also KAFADAR, ERINGEN [1971]).

Although we actually know many ways to derive the above results, the construction of a unified field-theoretic model is, in some sense, a still open problem and a good field for research intuition.

3.3 Local symmetry

The time dependent (local in time) special orthogonal group SO(3) was well-known long time before Cosserat. Nonlinearity of equations of motion which follows from the non-Abelity of SO(3) was also known in the mechanics of rigid body.

In 1909, E. and F. Cosserat extended the conception of time locality on simultaneous, space-time locality for the semi-simple multiplication of the translation group T(3) and rotation group SO(3). In this manner, a new type of nonlinearity has been introduced to the mechanics of deformable continuum. Together with the geometrical nonlinearity associated with the Abelian group T(3), the Cosserat continuum possesses an additional nonlinearity connected with the fully local non-Abelian group SO(3).

These results have been obtained in 1909, many years before YANG and MILLS [1954] discovered the non-Abelity as a new kind of nonlinearity in theoretical physics.

3.4 Least action principle

The least action principle given by W. R. Hamilton (1834) establishes an application of the calculus of variation to the Lagrange equation of second kind

$$\frac{d}{dt}\frac{\partial W}{\partial q_n} - \frac{\partial W}{\partial q_n} = 0 \, . \qquad (13)$$

Hamilton has shown that the equations (13) are equivalent to the variational principle

$$\delta \int_{t_1}^{t_2} W(q_n,\dot{q}_n)\, dt = 0 \;. \tag{14}$$

This is the least action principle applied to functions $q_n = q_n(t)$ which are time-dependent only. The principle states that if the functions $q_n(t)$ are the solution of (13) then the action integral (14) achieves an extremum for a variation restricted to $\delta q_n(t_1) = \delta q_n(t_2) = 0$, whatever the chosen end times t_1, t_2 may be.

Hamilton's principle can of course be extended to the case of functions depending on variables. Then the principle states that the integral action

$$I = \int dx^{\mu}\, W(q_i,\partial_{\mu} q_i)$$

attains an extremum leading to the Euler-Lagrange equations of motion

$$\partial^{\mu} \frac{\partial W}{\partial q_{i,\mu}} - \frac{\partial W}{\partial q_i} = 0 \;. \tag{15}$$

E. and F. COSSERAT [1909] had been the first to extend the Lagrange formalism to the mechanic of deformable continua. The main Cosserat question had been how to find equations of motion for the rotational parameters when they are unknown functions not only of time but of space coordinates as well. Stimulated by the papers of MacCULLAGH [1839], J. LARMOR and G. KIRCHHOFF [1874] E. and F. Cosserat have applied the least action principle to the following functional

$$I = \int_{t_1}^{t_2} \iiint W(x,y,z,\dot{\lambda}_x,\dot{\lambda}_y,\dot{\lambda}_z,\xi_{\mu},\eta_{\mu},\zeta_{\mu},p_{\mu},q_{\mu},r_{\mu})\, dxdydzdt \tag{16}$$

where λ_x, λ_y, λ_z are the components of the Gibb's rotation vector.

3.5 Conservation principles

The Hamilton principle of least action connected with the theorems of NOETHER [1918] yields a relation between conservation laws and symmetry groups. Such relation was independently discovered by E. and F. Cosserat. They required that the action density W must be invariant in the group of Euclidean displacements. As a result they obtained the following strong conservation law (COSSERAT [1909], p. 126).

$$\frac{\partial W}{\partial x} = 0 \quad , \quad \frac{\partial W}{\partial y} = 0 \quad , \quad \frac{\partial W}{\partial z} = 0$$

$$\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_\mu}} \frac{\partial z}{\partial \rho_\mu} - \frac{\partial W}{\partial \frac{\partial z}{\partial \rho_\mu}} \frac{\partial y}{\partial \rho_\mu} = 0 \tag{17}$$

$$\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_\mu}} \frac{\partial x}{\partial \rho_\mu} - \frac{\partial W}{\partial \frac{\partial x}{\partial \rho_\mu}} \frac{\partial z}{\partial \rho_\mu} = 0 \quad , \quad \frac{\partial W}{\partial \frac{\partial x}{\partial \rho_\mu}} \frac{\partial y}{\partial \rho_\mu} - \frac{\partial W}{\partial \frac{\partial y}{\partial \rho_\mu}} \frac{\partial x}{\partial \rho_\mu} = 0$$

The first three equations show that W is independent of a displacement, the latter ones indicate that W does not depend in an arbitrary manner on the first derivatives of the displacements with respect to the holonomic coordinates ρ_μ, where $\mu = 0,1,2,3$.

3.6 Simplifications of the model

Let us consider now special cases of the Cosserat continuum model. The first is well-known as the inextensible medium or "exotic medium". The concept of inextensible continuum based on the Appell's principle of solidification was first applied to an inextensible one dimensional string, THOMSON & TAIT [1879]. The inextension is identical with the assumption that the energy of translational movement is of higher order small than the torsional and bending energy. Therefore the string may be considered to be rigid along the tangential direction and that its total length is constant. For static deformations of a thin rod it means that the translational measure of deformation tends to zero. The simplest example considered by Cosserat is the "Euler elastica"

$$A\,\beta'' + R_z \sin\beta = 0 \quad , \qquad \frac{d}{ds}(\cdot) = (\cdot)' \tag{18}$$

where A is the flexural rigidity. Eqn. (18) describes the inextensible bending deformation of a plane, isotropic, homogeneous, linearly elastic rod with constant circular cross section under the action of a force R_z applied to the end of the rod from which s is measured. We recall here that this elastica equation, expressed in terms of Euler angles α,β,γ (originaly denoted by ϕ,φ,ψ) written down for this same, but spatially deformed rod has the form of the rotational equation of motion of a rigid body (CLEBCH [1862], KIRCHHOFF [1874]):

$$\begin{aligned} & C(\gamma' + \alpha'\cos\beta)' = 0 \quad , \\ & [A\alpha'\sin^2\beta + C(\gamma' + \alpha'\cos\beta)\cos\beta]' - R_x\sin\beta\cos\alpha - R_y\sin\beta\sin\alpha = 0 \ , \\ & A(\beta'' - \alpha'^2\sin\beta\cos\beta) + C\alpha'(\gamma' + \alpha'\cos\beta)\sin\beta - R_x\cos\beta\sin\alpha + \\ & \quad + R_y\cos\beta\cos\alpha + R_z\sin\beta = 0 \ . \end{aligned} \tag{19}$$

Here A,C are coefficients describing the flexural and torsional rigidity, respectively. Additionally, the Eulerian angles are measured in a way that the Z axis of fig. 1 coincides with the straight central axis of the rod. The three equations of equilibrium (19) are presented in Navier's form under the assumption that the translational deformations ξ_1, η_1, ζ_1 are equal to zero.

Starting from the analogy between the equations of motion of a heavy rigid body rotating about a fixed point and the equation of equlibrium of an inextensible string, KIRCHHOFF [1874] has proved the validity of the additional conservation law

$$\frac{d}{ds}\left[T + \frac{1}{2}\left(A\kappa^2 + B\kappa_\bullet^2 + C\tau^2\right)\right] = 0$$

which means that the following quantity

$$T + \frac{1}{2}\left(A\kappa^2 + B\kappa_\bullet^2 + C\tau^2\right) = \text{const} \tag{20}$$

is conserved during an inextensible motion. The conserved quantity (20) consists of a tension force T and the rotational energy expressed by curvatures κ , $\kappa_\bullet$ and twist τ

$$\kappa = \beta'\sin\gamma - \alpha'\sin\beta\cos\gamma \quad , \qquad \kappa_\bullet = \beta'\cos\gamma + \alpha'\sin\beta\sin\gamma$$
$$\tau = \gamma' + \alpha'\cos\beta \ . \tag{21}$$

Note that formulae (21) are in agreement with the eqns. (12) when we identify

$(\kappa,\kappa^{\bullet},\tau)$ with the components of the curvature vector (p_1,q_1,r_1) in $(12)_2$. For the Euler *elastica* considered above the conserved quantity (20) has the simple form

$$-R_z\cos\beta + \frac{1}{2}A(\beta'')^2 = \text{const}$$

and is the first integral of the Euler equation (18). From this we conclude that the conserved quantity is constant along the rod axis and can be interpreted as an unknown tension (compression) force.

Of course, speaking about conservation and conserved quantities, we mean here the conservation in space, what is a generalization of the notion "conservation". We note also that from the mathematical point of view, in the Kirchhoff's kinetic analogue we identify the time coordinate with the space coordinate in a way that the logical structure of the theory remains without any change. We agree with suggestions, that the Kirchhoff's kinetic analogue was some kind of pattern for the geometrization of space-time in Cosserat's n-dimensional approach.

The 3-dimensional counterpart of an inextensible line is difficult to imagine. Nevertheless, one can find the mechanical analogue of an inextensionable string in the hydrodynamics of nematic liquid crystal where the internal energy is a function of rotational changes of a unit vector n correlated with the mean orientation of molecular axies (ERICKSEN [1961]). A similar behavior can be found in the theory of elasticity of amorphic materials as metallic glass, window glass, etc.

Another example in which the "exotic" Cosserat continuum has been used to describe systems with non-mechanical energy are the non-equilibrium states of disordered magnets (DZYALOSHINSKII & VOLOVIK [1980]). In such magnets as, for instance, spin glasses the internal energy of spin systems is a function of gradients (in space and time) of the local rotation. These gradients, denoted by ω_k and ω are used in Cosserat medium as (p_i,q_i,r_i) according to $(12)_2$ and (p,q,r) according to (11). The thermodynamically conjugated variables $s = \frac{\partial W}{\partial \omega}$, $\gamma^k = \frac{\partial W}{\partial \omega_k}$ play the role of vectors of angular momentum and moments. As has been shown by Dzyaloshinskii and Volovik these dynamical variables satisfy the following set of governing equations

$$\nabla_k \omega_l - \nabla_l \omega_k + \omega_l \times \omega_k = \rho_{kl} ,$$

$$\nabla_t \omega_k - \nabla_k \omega + \omega \times \omega_k = j_k , \qquad (22)$$

$$\nabla_t s = \omega \times s + \nabla_k \gamma^k + \gamma^k \times \omega_k$$

$$\omega_i = p_i d_1 + q_i d_2 + r_i d_3 , \qquad \omega = p d_1 + q d_2 + r d_3 ,$$

where $\rho_{kl} = -\rho_{lk}$ and j_k are known densities. Let us note that the first two equations of (22) correspond to the equations of the space-time compatibility for strains and the last one corresponds to the equation of motion (10) satisfying the inextensibility conditions $N^\mu \equiv 0$. It should be pointed out that in the disordered magnet continuum the Cosserat assumption about nonseparability of the action integral into purely spatial and purely temporal parts is valid.

The next particular model of continuum, derived by Cosserat, is based on the concept of **hidden triad** and **concealed action integral** W. E. and F. Cosserat considered a different reduced form of the action functional in the hope that some relation between their model and the classical Cauchy model can be found. In fact, if one supposes that the independent micro-rotation of the Cosserat model become identical with a macro-rotation of the material neighbourhood of particle then the action functional $W(\xi_\mu, \eta_\mu, \zeta_\mu, p_\mu, q_\mu, r_\mu) \equiv W(\mathbf{u}_{|\mu}, \mathbf{u}_{|\mu\nu})$ becomes a function of the displacement **u** and its derivatives only. Such "hidden" rotations lead to the model of a second-order continuum with the internal energy expressed by second derivatives of displacement as well.

It is obvious that even a successive reduction of the action functional to the form $W(\xi_\mu, \eta_\mu, \zeta_\mu, 0, 0, 0)$ does not lead directly to the Cauchy model. Because of this reason Cosserat considered the action functional $W(\varepsilon_i, \gamma_i, \varphi_i, v^2, 0, 0, 0)$ in which the translational measures $\xi_\mu, \eta_\mu, \zeta_\mu$ (12 functions) are present in the action functional only via ten expressions $\varepsilon_i, \gamma_i, \varphi_i, v^2$ defined by the following formulae (COSSERAT [1909], p. 176)

$$
\begin{aligned}
\varepsilon_1 &= \tfrac{1}{2}(\xi_1^2 + \eta_1^2 + \zeta_1^2 - 1)\,, & \varphi_1 &= \xi_1\xi + \eta_1\eta + \zeta_1\zeta\,,\\
\varepsilon_2 &= \tfrac{1}{2}(\xi_2^2 + \eta_2^2 + \zeta_2^2 - 1)\,, & \varphi_2 &= \xi_2\xi + \eta_2\eta + \zeta_2\zeta\,,\\
\varepsilon_3 &= \tfrac{1}{2}(\xi_3^2 + \eta_3^2 + \zeta_3^2 - 1)\,, & \varphi_3 &= \xi_3\xi + \eta_3\eta + \zeta_3\zeta\,,\\
\gamma_1 &= \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3\,, & \upsilon^2 &= \xi^2 + \eta^2 + \zeta^2\,.\\
\gamma_2 &= \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1\,, & &\\
\gamma_3 &= \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2\,. & &
\end{aligned}
\tag{23}
$$

The expressions $(\varepsilon_i, 2\gamma_i, 2\varphi_i, 2\upsilon^2) \equiv E_{\mu\nu}$, $\mu,\nu = 0,1,2,3$ are nothing else but the components of the Green strain tensor defined in 4-dimensional Cartesian coordinate system. From this we note that the concealed action functional $W(E_{\mu\nu})$ cannot be directly obtained from $W(\xi_\mu,\eta_\mu,\zeta_\mu,0,0,0)$ by a simple exchange of $\xi_\mu,\eta_\mu,\zeta_\mu$, to $\xi_\mu^2,\eta_\mu^2,\zeta_\mu^2$. As a first consequence of these assumptions Cosserat obtained the result that the moment stress measure is equal to zero. The next one was that stress measures (A'_μ, B'_μ, C'_μ) $\mu = 0, i$ energetically coupled with $(\xi_\mu,\eta_\mu,\zeta_\mu)$ are connected with the second Piola-Kirchhoff stress tensor in the following manner (COSSERAT [1909], p. 176).

$$
\begin{aligned}
A'_i &= \xi_i\frac{\partial W}{\partial \varepsilon_i} + \xi_k\frac{\partial W}{\partial \gamma_j} + \xi_j\frac{\partial W}{\partial \gamma_k} + \xi\,\frac{\partial W}{\partial \varphi_i}\,,\\
B'_i &= \eta_i\frac{\partial W}{\partial \varepsilon_i} + \eta_k\frac{\partial W}{\partial \gamma_j} + \eta_j\frac{\partial W}{\partial \gamma_k} + \eta\,\frac{\partial W}{\partial \varphi_i}\,,\\
C'_i &= \zeta_i\frac{\partial W}{\partial \varepsilon_i} + \zeta_k\frac{\partial W}{\partial \gamma_j} + \zeta_j\frac{\partial W}{\partial \gamma_k} + \zeta\,\frac{\partial W}{\partial \varphi_i}\,,\\
A' &= \frac{1}{\upsilon}\frac{\partial W}{\partial \upsilon}\,\xi + \xi_i\frac{\partial W}{\partial \varphi_i}\,,\\
B' &= \frac{1}{\upsilon}\frac{\partial W}{\partial \upsilon}\,\eta + \eta_i\frac{\partial W}{\partial \varphi_i}\,, \qquad (i,j,k=1,2,3,\ \text{cyclic indices})\\
C' &= \frac{1}{\upsilon}\frac{\partial W}{\partial \upsilon}\,\zeta + \zeta_i\frac{\partial W}{\partial \varphi_i}\,,
\end{aligned}
\tag{24}
$$

with (ξ,η,ζ) according to (11) and (ξ_i,η_i,ζ_i) according to $(12)_1$. From the point of view of the so-called corotational description, which frequently is used in contemporary computational mechanics, the equations (23) and (24) lead to basic relations between the total Lagrangian description and the co-rotational description. The relations inform us that during passing from one description to the other an exchange of both strain and stress measures follows. From conceptual standpoint the exchange is rather serious - the metric conjugate tensors {Piola-Kirchhoff - Green} are replaced by semi-metric conjugate measures { A'_μ, B'_μ, C'_μ - $\xi_\mu, \eta_\mu, \zeta_\mu$ } . Most applications of such Cosserat continuum with hidden triad and concealed W deal with the models of thin, shell-like bodies, where the local rotations remain large even if the strains are small.

Another consequence of the hidden triad assumption can be observed by considering the equations of motion for the rotation parameters. E. and F. Cosserat have shown, after some additional derivations, that from six basic equations of motion three differential equations from six basic equations of motion reduce to an algebraic identity for the components of a Cauchy type stress tensor and velocity (COSSERAT [1909], p. 175)

$$
\begin{aligned}
\mathfrak{D}(p_{yz} - p_{zy}) &= B\frac{dz}{dt} - C\frac{dy}{dt} , \\
\mathfrak{D}(p_{zx} - p_{xz}) &= C\frac{dx}{dt} - A\frac{dz}{dt} , \\
\mathfrak{D}(p_{xy} - p_{yx}) &= A\frac{dy}{dt} - B\frac{dx}{dt} , \qquad \mathfrak{D} = \det \mathbf{F}
\end{aligned}
\tag{25}
$$

where (A,B,C) are the zero components of $(9)_1$. In the co-rotational description of the Cauchy continuum the equations (25) play an important role, because during the computational process they represent sufficient conditions between the micro- and macro-rotations.

§ 4. The unification sector

H. WEYL [1917] had been the first to introduce the electromagnetic interaction as a special vector field of the group of phase transformations in a charged field. Following the same method YANG and MILLS [1954] formulated their theory by considering the non-Abelian group of rotation in iso-space. Such an approach was based on two possibilities either by means of an additional internal group space or by means of a localization of the group space-time symmetry. As a result UTIYAMA [1956] has formulated a generalization of these two approaches. He considered conditions for a full invariance of the Lagrangian written in terms of a set of matter fields and a n-parameter global Lie group, representing some internal symmetry. During localization of the global group, from invariance condition, he obtained 4xn new compensating fields interacting with matter. It appeared that the new fields, arising form space-time symmetries have a univeral significance and this gave rise to a gravitational theory more general then general relativity. We have to point out that this gravitation theory taking into account the spin has similarities with the Cosserat continuum (HEHL [1968]). KRÖNER [1965] had been the first to consider the geometrical analogy between the localization of T(3)⊳SO(3) in Cosserat non-relativistic continuum and the localization of the Poincare 10-parameter group. We can find a full explanation of this analogy in EDELEN [1985].

It is worth to note that many works published on this subject show that the first translational measure of deformation in the Cosserat model plays a role identical as the vielbein fields in gravitation model based on Riemann-Cartan geometry. It means that the same formalism of anholonomic reference frame can be applied for the description of Cosserat continuum. Such possibilities are widely used in shell models based on Cosserat surface (REISSNER [1974], FERRARESE [1976] and by MAKOWSKI & STUMPF [1988], [1989] and by FERRARESE [1976] in three-dimensional continuum).

The advantage of the formalism introduced by Utiyama lies in its generality. Most of the results obtained so far can immediately be used for other hypothetical space-time symmetry. The largest one, known up to now, is represented by the 15-parameter restricted conformal group, generating the most general transformations of coordinates under which electrodynamics are covariant. Among the Lie groups most application have found those which describe a space of constant curvature. A generalization of a simple Poincaré group is De Sitter group SO(3,2) or, next in order, the conformal group SO(4,2).

The same geometrical procedure based on replacing a contracted group by a single larger group of inner symmetry has been applied to construct generalized models of mechanical continua. In 1958 ERICKSEN & TRUESDELL generalized the concept of T(3)⊳SO(3) continua to T(3)⊳GL(n,ℝ) by requiring the set of n-Cosserat directors to be deformable. In the particular case, when n = 3 and the directors become undeformable, the T(3)⊳GL(3,ℝ) model is reduced to the Cosserat continuum with the displacement and rotation fields as independent primary unknowns. In Ericksen-Truesdell continuum, a deformation between the undeformed $\mathbf{D}_a$ and the deformed directors $\mathbf{d}_a$ is described by means of an element of the 9-parameter group GL(3,ℝ) represented (in Ericksen & Truesdell notation) by $\mathbf{A} = \mathbf{d}_a \otimes \mathbf{D}^a = A^K{}_N G_K \otimes G^N$. The action of any element of GL(3,ℝ) is realized by three pure micro-stretches in the main directions of the micro-deformation which are oriented in space by three parameters and, finally, by three parameters of the global rotation identical with the

Cosserat rotation field.

Analogous to the Cosserat approach the set of moving and deforming directors is a natural frame to describe completely the mechanical behavior of a generalized continuum. Because the deformed directors are not connected with any coordinate system, the differentiation of a geometrical object, given in such anholonomic base, with respect to Lagrangian coordinates is possible only after pull-back the object to a Lagrangian holonomic frame. Of course, the result of differentiation must next be pushed-forward to the anholonomic base. For instance, applying this procedure to $\mathbf{d}_a(X^N)$ Ericksen and Truesdell obtained the fundamental measure of micro-deformation [1958, (6.4)]

$$\mathbf{F}_M = (\partial_M \mathbf{A})\, \mathbf{A}^{-1} + \mathbf{A}\, \mathbf{W}_M \mathbf{A}^{-1} \ , \ M = 1,2,3 \tag{26}$$

which, in general, is represented by 27 director gradients and the known formula $\partial_M \mathbf{D}_a = \mathbf{W}_M \mathbf{D}_a$ for undeformed triad $\mathbf{D}_a$ (ERICKSEN & TRUESDELL [1958]). When, however, the directors form a rigid triad with a constant metric with only 9 independent director gradients in (26), it follows that $\mathbf{F}_M = -\mathbf{F}_M^T$ and $\mathbf{F}_M$ corresponds to the Cosserat rotation measures of deformation. From the point of view of modern geometry the formula (26) describes the rule of change of connection under co-ordinate transformation or the transformation of gauge potential fields under a gauge transformation.

Additionally, Ericksen and Truesdell completed the above measure (26) by a translation-like measure $\boldsymbol{\chi} = \mathbf{F}\, \mathbf{A}^{-1}$ with $\mathbf{F}$ = Grad $\mathbf{x}$ and a metric measure $\mathbf{g}_{ab} = \mathbf{d}_a \cdot \mathbf{d}_b$ of the deformed directors. As a result they obtained the complete state of deformation described altogether by 27+9+6 = 42 functions of generalized strain for a $T(3) \triangleright GL(3,\mathbb{R})$ deformable continuum. From the geometrical point of view also other works , concerning a generalization of microstructure, are a continuation of Cosserat's approach too (see also: TOUPIN [1962], MINDLIN [1964], KOITER [1964], MAUGIN & ERINGEN [1972]).

While ERICKSEN & TRUESDELL introduce a kinematical concept of deformation analogous to the one of Cosserat, they use a different concept for stresses. First of all the stress measures (only two not three) are of Cauchy type, what means that they are defined in Eulerian description and specified in a holonomic base only. It would be interesting to define stress measures of Cauchy type also referred to either anholonomic $\mathbf{d}_a$ or $\mathbf{D}_a$ base and next to connect these quantities with appropriate Eulerian measures of deformation.

Secondly, in contrast to Cosserat, the energetically conjugate couples of Lagrangean measures of strain and stress do not occur, so the problem of Lagrangian stresses in Ericksen-Truesdell continuum remains still open.

An interesting feature is the fact, that most papers devoted to generalized continua, for example ERICKSEN [1961], MINDLIN [1964], TOUPIN [1964], precisely investigate the Lagrangean theory of deformation and an Eulerian theory of stresses. It results from the opinion, still yet expressed in McLELLAN [1984], that the Cauchy stress tensor even in the infinitesimal strain theory is not conjugate to any strain measure. This opinion created difficulties in the construction of constitutive relations between appropriate measures of stress and strain, what would finaly close the theory.

The Ericksen-Truesdell model of $T(3) \rhd GL(n,\mathbb{R})$ deformable continuum has initiated a new branch of research connected with the unification of various approaches with the aim of entering into the microscopic world of matter by means of field theories. Numerous papers have been devoted to investigations of continuum mechanical theories of materials with structure. Main research directions were stimulated mainly by the papers of TRUESDELL & TOUPIN [1960], ERICKSEN [1961], [1982], TOUPIN [1962], [1964], MINDLIN [1964], GREEN, NAGHDI & WAINWRIGHT [1965], COHEN & DeSILVA [1966], WOZNIAK [1968], ERINGEN [1972], [1976], ANTMAN [1972], CAPRIZ & PODIO GUIDUGLI [1983], BETOUNES [1987] and LAGOUDAS [1989]. As a result more or less complete nonlinear models of mechanical continua had been obtained in which the deformation is described not only by the usual displacement vector field, but also by other scalar, vector, spinor or tensor fields as well. The common language for such unification of physically different fields was and still is the geometry language, especially the language of modern differential geometry.

An important turning point in the application of models of mechanical continua had been the joint GAMM/IUTAM symposium held in Freudenstadt and Stuttgart and devoted to the mechanics of generalized continua, KRÖNER [1968]. There Kröner has presented a spatial diagram with a logical structure of relations between various branches of mechanical activities (fig. 2). It was confirmed that many elements of a general model need a deeper physical interpretation and more intensive relation to applications. From this time the model of generalized mechanical continua has been extended and adopted to describe very different and, sometimes, very complex physical phenomena.

We would like to draw the attention of the reader to the fact that the problem of constructing a unified model is more complicated then the geometrical problem of finding a common covering group for phenomena which can be described by two, three or more simple groups. The unification, accepted from physical point of view, still remains the most difficult problem of field theory. Thus some ideas were developed independently or parallely in theoretical physics as well as in continuum mechanics.

Yet a new branch of generalized continuum mechanics had been opened with the investigations of EDELEN [1985], LAGOUDAS & EDELEN [1989] and LAGOUDAS [1989] considering mechanical structures of solids with a dense distribiution of defects. They started with the concept of broken translational and rotational symmetry connected with Utiyama's procedure of the minimal coupling and minimal replacement. The physical meaning of the minimal replacement operation and the requirement of invariance of Lagrangean led to a generalization of the Frame Indifference Principle.

§ 5. The conservation sector

The symmetries which occur in Lagrangian field theory can generally be classified as discrete or continuous. The well-known discrete symmetries include the Time Reversal Invariance, CP or CPT Invariance. However, among the continuous symmetries are well-known the Galilean, Lorentz, Poincaré, DeSitter groups connected with space-time and the groups U(1), SU(2), SU(3), SO(32), $E_8 \times E_8$, etc. connected rather by internal structure of fundamental interactions then by space-time. From the formal point of view the continuous symmetries can be divided into two kinds: the symmetry under continuous coordinate-independent transformation (global symmetry) and the symmetry under coordinate-dependent transformation (local symmetry). It is commonly accepted that the Frame Indifference Principle, for instance, is based on an invariance requirement with repect to transformations which are global in space and local in time (ROHRLICH [1965], WOZNIAK [1968], OLVER [1986]).

Generally speaking, every transformation is specified by a finite set of numerical parameters. In particular, the elements of Lie groups are characterized via n parameters which play the role of co-ordinates on the group manifold and are the partners of their generators in an appropriate Lie algebra. In the special case, which had been used in Cosserat's approach, the n parameters of the local group may be treated as fundamental primary fields which are continuous functions of the coordinates.

The invariance of a theory with respect to the group transformation is usually understood as invariance of equation of motions or invariance of the Euler-Lagrange equations. It is convenient to formulate and explore the symmetry requirement in an equivalent manner, i.e. by considering a Lagrangian formulation of the field theory where basic dynamical equations ("equations of motion") are derived from an action integral defined on suitable Lagrangian densities. Consequences of the action integral invariance with respect to the Lie group of transformation had been studied before Cosserat by HERTZ [1894], APPELL [1903], LORENTZ [1903], and afterwards by HERGLOTZ [1911], MIE [1912], [1913], H. REISSNER [1916] and WEYL [1917]. Of course, most works in field theory, especially after the discoveries of Hilbert, Klein, Einstein and Weyl were stimulated by the rapid development of relativity theory rather then by the development of models of mechanical continua.

In 1918 Emmy Noether generalized the Cosserat result concerning the action integral invariance for the case of a n-parameter continuous groups of transformations. The frist fundamental result had been obtained for groups of global symmetry stating that if the action integral is invariant under the n-parameter symmetry group then n linearly independent combinations of Lagrange expressions $L_M = \frac{\partial \mathcal{L}}{\partial U_M} - d_\mu \frac{\partial \mathcal{L}}{\partial U_{M,\mu}}$ can be written as divergences, NOETHER [1918]:

$$\delta I = \int dx^\mu \left(L_M \bar{\delta} U_M^a - d_\nu J^{\nu a} \right) \equiv 0 \quad , \qquad a = 1,2,\ldots,n \tag{27}$$

with the additional assumption that all field equations follow from the action integral $I(U_M(x^\mu)) = \int dx^\mu \mathcal{L}(x^\nu, U_M, U_{M,\nu})$ in a simple form. Here $(\cdot)_M$ denotes a common index depending on vector, spinor or tensor properties of U_M. The conserved quantity $J^{\mu a}$ in (27) is known as the symmetry current and from the mathematical point of view it is the vector with n values in the Lie algebra of a suitable group of transformation. For instance, the symmetry current connected with the Lorentz group is known as the energy-momentum tensor. Similar sources has the Eshelby tensor used in fracture mechanics and here interpreted as the current of $T(3)_o$ symmetry group (LE, STUMPF, WEICHERT [1989]). The conserved currents $J^{\mu a}$corresponding to invariance under global symmetry groups have now a remarkable property: they are not only linear combinations of L_M but also the divergence of an antisymmetric tensor $\partial_\mu M^{\nu\mu a}$ too. Such conservation current E. Noether called improper one in contrast to other proper conservation current.

Additionaly, if conservation laws $d_\mu J^{\mu a} = 0$ hold independent of a fulfillment of the equation of motion $L_M \neq 0$ we speak about strong conservation laws. However, if conservation laws take place when $L_M = 0$ we speak about weak conservation laws.

The second part of the fundamental work of Noether is concerned with the n-parameter local symmetry group. The second Noether theorem states that if the action integral is invariant under n-parameter local group of transformations then the Lagrangean expressions L_M and their derivatives commonly with symmetry currents generally fulfill n-identities:

$$\delta I = \int dx^\mu \Big[[L_M U^a_M - d_\mu (L_N U^{\mu a}_N)] \ \varepsilon_a - d_\mu (J^{\mu a} \ \varepsilon_a - L_N U^{\mu a}_N \ \varepsilon_a) \Big] = 0 \qquad (28)$$

where, in oder to express this theorem similar to (27) we have assumed that total variation of the function $U_M(x)$ is given by $\bar{\delta} U_M = U^a_M \varepsilon_a(x) + U^{\mu a}_M \partial_\mu \varepsilon_a(x)$. The above relations, usually refered to as Noether identities establish the dependence between Lagrangean expressions meaning that n of the Euler-Lagrangean equations are dependent (BERGMAN [1949], OLVER [1986]).

Note furthermore that in the framework of Lagrangean field theory, the action integral may be simmultenously related with some symmetry groups, global as well as local. It means that a field-theoretical model can possess a few conserved currents simmultaneously.

Let us point out, that in spite of their formal similarity, the Hamiltonian field theory does not explain mutual relations between invariance and conservation. Applications of the Hamiltonian approaches to continuum mechanics can be found in the papers of DZYALOSHINSKII and VOLOVIK [1980] and Simo et al. [1988]. Most hope is connected with the fact that the Hamiltonian formulation of mechanics is now incorporated into the differential-geometrical analysis on manifolds. Similar to the Lagrangean, we have now two equivalent Hamiltonian formulations of classical mechanics: in the physical language of canonical variables, Poisson and Dirac brackets and canonical transformation on one side and the geometrical formulation in terms of symplectic spaces, Hamiltonian vector fields, exterior calculus, etc, on the other.

Probably a first application of the Noether theorem had been given by BESSEL-HAGEN [1921] who derived ten integrals of the n-body problem and conservation of momentum and energy of the classical Maxwell electrodynamics. The conservation laws expressed in terms of energy-momentum tensor and angular momentum tensor were also discussed in the papers of WEYSSENHOFF and RAABE [1947] and BERGMAN [1949]. It should be noted also that an application of Noether's theorems some time requieres to determine the symmetry of nonlinear differential equations under which the equations remain invariant. For this reason, the investigation of conserved currents, especially for nonlinear equations such as Korteweg-deVries, Hopf, time-dependent Ginzburg-Landau, Langevin, Fokker-Plank equations, etc., remain still open, (OLVER [1986]).

Stimulated by the papers of Cosserat and own work GÜNTER [1958], [1962] has applied the first Noether theorem to the linear elastostatics. He extended the Noether theorem to covariant form and discussed conservation laws which correspond to the rotation and similarity invariances. Following Cosserat TOUPIN [1964] postulated that the action integral I is invariant under global seven-parameter group of Euclidian displacements and obtained seven strong conservation laws

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial t} = 0 \ , \qquad K_{ij} - K_{ji} = 0 \ , \qquad i,j=1,2,3$$

$$K_{ij} = x_i\frac{\partial \mathcal{L}}{\partial x_j} + d_{ai}\frac{\partial \mathcal{L}}{\partial d_{aj}} + \dot{x}_i\frac{\partial \mathcal{L}}{\partial x_j} + \dot{d}_{ai}\frac{\partial \mathcal{L}}{\partial d_{aj}} + x_{i,\alpha}\frac{\partial \mathcal{L}}{\partial x_{j,\alpha}} + d_{ai,\alpha}\frac{\partial \mathcal{L}}{\partial d_{aj,\alpha}} . \tag{29}$$

including the original Cosserat conservation laws (17). The above formulae establish the fundamental equivalence between conservation and invariance for generalized Cosserat continua.

The unification of Toupin's and Günter's approach one can find in the monograph of WOZNIAK [1968] where different forms of conservation laws are presented based on the Euclidian group of transformation for many sophisticated generalized continua.

At the present time most works, devoted to conserved quantities (momentum, energy, angular momentum, etc.), take into account the material as well as spatial symmetry transformations. Such a distinction is particularly important in modeling continuous distribiutions of defects in solid media, where conserved quantities are present accompanying spatial as well as material defects. It is worth noting that the problem of material conservation laws also become important in the context of fracture mechanics where, according to Eshelby, the material energy-momentum tensor describes the material force acting on a crack (see LE, STUMPF, WEICHERT [1989]).

New interesting results concerning conservation laws, which seem to have no counterparts in continuum physics, we can find in papers of KLUGE [1969], [1981]. It is shown that for the Cosserat continuum with a dense distribiution of dislocation and disclination the field equations can be written in terms of stress functions and appropriate densities of defects only. Therefore, similar to the field theory, the laws of conservarion of energy, momentum and angular momentum for such "dual" problem may also be derived. The balance equations introduced in natural way contain the Peach-Köhler and the Nabarro force for dislocations and analogous forces connected with disclinations (KLUGE [1981]).

From the point of view of applications other difficulties arise from the form of the conservation laws. Since the generators of symmetry cannot be put in a natural correspondence to classical objects, the physical interpretation of conserved quantities requires additional transformations (OLVER [1986]).

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