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FOR NON-LINEAR PROBLEMS

Summary

The quasistatical approach to the non-linear shakedown problems for elastic-plastic materials with hardening is presented.. A given new statical shakedown theorem is a generalization of Gross-Weege's one. It contains known formulations of the literature as special cases.

Zusammenfassung

Es wird die quasistatische Näherung des nichtlinearen Einspielproblems für elastisch-plastisches Material mit Verfestigung untersucht. Ein hergeleitetes statisches Shakedown Theorem stellt eine Verallgemeinerung des Theorems von Groß-Weege dar. Es enthält darüber hinaus viele aus der Literatur bekannte Formulierungen als Spezialfälle.

Contents

Introduction

In many engineering situations, machines and structures are often subjected to loads and temperatures that act mostly repetitive and vary within given limits. The essence of the shakedown analysis is concerned with the limiting behaviour of a structure under such type of loading. In this study, a convex loading domain is considered and any load within this domain is assumed to be applied an infinite number of times during the lifetime of the structure.

The significance of such analysis is obvious. It allows to release our calculations from necessity regarding a real, as a rule, *a priori* an unknown loading history of the structure or its elements.

The theory of shakedown is based, in principle, on the assumption of geometrical linearity, so that the influence of progressive changes of the shape of the structure during the deformation process cannot be taken into account. In many practical applications, however, especially for mechanical structures, unserviceability may occur only beyond the range of the applicability of the geometrically linear theories. Examples are shell-like structures, where the conventional theories already fail when the displacements are of the order of the thickness of the wall structures. Moreover, the classical shakedown approach will not be adequate at finite deformations because the shakedown limits of loads must be path-dependent in this case. Nevertheless, in some special cases, exists a few partial results concerning such approach. The first was MAIER [5:1973] who has extended the classical approach taking into account the so-called 'second order' effects and using piecewise linear yield conditions. The influence of geometrical effects on the stability of the deformation process for particular structures was investigated by KÖNIG [2:1980]. NGUYEN and GARY [7:1983] studied the possibility of destabilization of the shakedown process due to successive plastic deformation. WEICHERT [8,9:1986; 11:1988] and GROSS-WEGGE [12:1989] investigated the geometrically non-linear shakedown problem for some restricted classes of non-linearities and gave an extension of MELAN's theorem for shell-like bodies undergoing moderate rotation at small strains. The influence of material hardening of the shakedown behaviour has been studied by several authors, among others, by MANDEL [20:1976], WEICHERT and GROSS-WEGGE [10:1988].

In the paper an extension of the Gross-Weege's shakedown theorem to more general non-linear problems is proposed. The requirement of an additive decomposition of the strain tensor into a purely elastic part and a purely plastic part restricts the proposed theory to moderate deformations. According to the classical shakedown theory the proposed theorem presumes the existence of a convex yield surface and the validity of the normality rule for the plastic strain rates. Considerations for elastic-plastic materials with hardening are given. Moreover, the given shakedown theorem for the generalized standard material is reformulated.

Notations

$\mathbb{R}$ - a field of real numbers;

$\mathcal{B}$ - an elastic-plastic body (continuum of class C^p, $p \geq 1$);

${}^iC$ - an initial configuration of $\mathcal{B}$;

${}^tC$ - an actual configuration of $\mathcal{B}$;

${}^RC$ - a reference configuration of $\mathcal{B}$;

${}^oC$ - a fictitious configuration of $\mathcal{B}$;

$\mathcal{T}^s$ - an Euclidean tensor field of the valence s;

$\mathcal{S}^s := \mathcal{T}^s|_{T=T^T}$, $s \geq 2$;

$\mathcal{A}^s := \mathcal{T}^s|_{T=-T^T}$, $s \geq 2$;

$\mathcal{T} := \mathcal{T}^s|_{\det T \neq 0}$;

$\mathcal{T}^+ := \mathcal{T}|_{\det T > 0}$;

V - a volume element of the body $\mathcal{B}$;

$S = \partial\mathcal{B}$ - a surface element of the body $\mathcal{B}$;

$\Omega = V \times [\tau_1, \tau_2] \subset \mathbb{R}^3 \times \mathbb{R}_+$;

$\mathcal{V}_\beta$ - a class of all load paths on V;

For objects $S \in \mathcal{T}$ by trS and detS we denote their trace and determinant, respectively. $\partial\mathcal{D}$ means a boundary of a subset $\mathcal{D}$ of a vector space. When $\partial\mathcal{B}$ is a boundary of a body $\mathcal{B}$, we assume $\partial\mathcal{B}$ to be also an orientable and measurable surface.

In the work bold-faced letters denote tensors of valence 1, at least. We adopt here, as far as possible, an absolute tensor notation and, if it is necessary, we use the index tensor notation. If necessary the summation convention is used.

1. Formulation of the problem

The purpose of this chapter is to point out the elementary aspects of the problem. We will present mathematical process necessary to describe the shakedown of the structure.

1.1. Kinematic-static equations of the problem

In the formulation of continuum mechanics the configuration of the body $\mathcal{B}$ is described by the continuous mathematical model whose geometric points are identified with places of material particles of the given body $\mathcal{B}$. By the body $\mathcal{B}$ we shall understand, further on, an elastic-plastic body.

Let the body $\mathcal{B}$ occupies the region $\Omega = V\times[\tau_1,\tau_2]$, where V is a simply connected subset in $\mathbb{R}^3$ and $[\tau_1,\tau_2]$ is the time-interval in $\mathbb{R}_+$.

Let ${}^1C$ and ${}^tC$ denote two different configurations of the body $\mathcal{B}$, $\mathbf{X}$ and $\mathbf{x}$ - places in ${}^1C$ and ${}^tC$ of the same particle X of the $\mathcal{B}$.

Let the mapping χ: ${}^1C \longrightarrow {}^tC$, called a deformation function, be a continuous deformation process defined by

$$\text{(1.1)}\qquad \mathbf{x} = \chi(\mathbf{X},\tau), \quad \mathbf{X} \in V, \quad \tau \in [\tau_1,\tau_2].$$

We consider only deformation processes χ in which displacements $\mathbf{u} = \chi(\mathbf{X},\tau) - \mathbf{X}$, and the deformation gradient $\mathbf{F} = \mathrm{Grad}\chi := \nabla\chi \in \mathcal{T}^+$ are continuous functions in $V\times[\tau_1,\tau_2]$.

The knowledge of the deformation gradient tensor $\mathbf{F}$ at the point X implies the knowledge of a deformation of the first order in a certain neighbourhood of this point.

A division of sufficiently smooth surface $S = \partial\mathcal{B}$ of the body $\mathcal{B}$ on two disjoint parts S_u and S_t where static and kinematic boundary conditions are given, respectively, it can be various for different components of external loads given. We consider volume forces $\mathbf{b}^\#$ in V, surface tractions $\mathbf{t}^\#$ on S_t and surface displacements $\mathbf{u}^\#$ on S_u. It is assumed that suitable components $t_i^\#$ and $u_i^\#$ are complementary. The body force $\mathbf{b}^\#$ is prescribed in the volume V in such a way that the entire body is in equilibrium.

Under this assumptions the boundary value problem refered to the initial undeformed configuration is represented by:

(1) Kinematic equations

(i) the compatibility conditions for displacement vector field

$$\text{(1.2)}\quad \begin{aligned} \mathbf{F} &= \mathbf{G} + \nabla\mathbf{u} \quad \text{in V,} \\ \mathbf{E} &= \frac{1}{2}\,(\mathbf{F}^T\mathbf{g}\mathbf{F} - \mathbf{G}) \quad \text{in V,} \end{aligned}$$

where **G** and **g** are the metric tensors of the undeformed and deformed configurations, respectively.

(ii) the kinematic boundary conditions

$$\text{(1.3)}\quad \mathbf{u} = \mathbf{u}^{\#} \quad \text{on } S_u .$$

(2) Static equations

(i) the equations of internal equilibrium

$$\text{(1.4)}\quad \mathrm{Div}(\mathbf{T}) = -\mathbf{b}^{\#} \quad \text{in V,}$$

(ii) the boundary conditions for surface tractions

$$\text{(1.5)}\quad \mathbf{T}\mathbf{n} = \mathbf{t}^{\#} \quad \text{on } S_t ,$$

$$\mathbf{T} = \mathbf{F}\mathbf{S}.$$

Here **T** is the first Piola-Kirchhoff stress tensor, **S** is the second Piola-Kirchhoff stress tensor, $\mathbf{t}^{\#}$ is the given traction vector field and **n** represents the unit normal vector in ${}^{1}C$.

The kinematic equations, the static ones and the constitutive relations (Section 1.2.) should be satisfied for every $\tau > 0$ together with

(4) initial conditions

$$\text{(1.5a)}\quad \mathbf{S} = \mathbf{0} \quad \text{and} \quad \mathbf{E} = \mathbf{0} \quad \text{at} \quad \tau = 0 \quad \text{in V.}$$

1.2. Constitutive relations for the elastic-plastic material with hardening

The proper formulation of elastic-plastic constitutive laws at finite strains is of great importance in engineering problems for large-scale computation. One of the methods, of which this is adopted, is based on the multiplicative decomposition as the basic kinematic assumption (LEE [28:1969], LUBARDA and LEE [34:1981], SIMO [31:1988]). In practice it is assumed

(1.6) $$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$$

to be satisfied in any neighbourhood of a body. From the definition $\mathbf{F}^p$ designates the plastic part of the deformation process and the elastic one $\mathbf{F}^e$ is obtained by unloading all infinitesimal neighbourhoods of the body, In general, $\mathbf{F}^e$ and $\mathbf{F}^p$ are incompatible point functions.

Starting from (1.6) the following Lagrangian strain measures can be defined

(1.7)
$$\mathbf{E} = \frac{1}{2}\,(\mathbf{F}^T\mathbf{g}\mathbf{F} - \mathbf{G})$$
$$\mathbf{E}^p = \frac{1}{2}\,[(\mathbf{F}^p)^T\mathfrak{G}\mathbf{F}^p - \mathbf{G}],$$

where $\mathbf{G}$, $\mathbf{g}$, and $\mathfrak{G}$ are the metric tensors of the undeformed, deformed and intermediate configurations. As a consequence of the definitions (1.7) the elastic part of the total strain tensor $\mathbf{E}^e$ admits the following definition, i.e.,

(1.8) $$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p.$$

The definitions (1.7) and (1.8) are suitable for general classes of deformation processes. In spite of it, the statement (1.8) is of no practical value for the present shakedown analysis because of the coupling between $\mathbf{E}$, $\mathbf{F}^p$ and $\mathbf{E}^e$. In this instance, we need a representation of $\mathbf{E}^e$ in a form appropriate for a shakedown analysis. Here we will follow CASEY [18:1985] who considered simplified expressions valid for restricted classes of geometrical non-linearities.

In our exposition we restrict ourselves to quasi-statical deformation process. At present, attention is focused on the pure mechanical theory of elastic-plastic materials. From the requirements of objectivity of the intermediate configuration and of the spatial covariance (cf. GREEN and NAGHDI [27:1965], SIMO and ORTIZ [23:1985], SIMO [31:1988]) it follows that the free energy potential of the elastic-plastic body should have the form

(1.9) $$\hat{W} = \hat{W}(\mathbf{E}, \mathbf{E}^p, \mathbf{Q}),$$

where $\mathbf{Q}$ designates a set of internal plastic variables, that characterizes the plastic response. The nature of these variables depends on the particular plastic model under consideration.

Now we confine ourselves to the theory of the hyperelastic elastic stress-strain relations and a postulated yield condition. Without loss of generality we assume an uncoupled free energy in the form (cf. SIMO [31:1988])

$$(1.10) \qquad \hat{W}(\mathbf{E},\mathbf{E}^{p},\mathbf{Q}) = W(\mathbf{E},\mathbf{E}^{p}) - \Theta(\mathbf{Q}).$$

Based on the hyperelastic response we get the following constitutive equation for the second Piola-Kirchhoff stress tensor

$$(1.11) \qquad \mathbf{S} = \rho_o \frac{\partial W(\mathbf{E},\mathbf{E}^{p})}{\partial \mathbf{E}},$$

where W denotes the free energy potential of the elastic-plastic body and ρ_o is the initial mass density.

To describe the plastic part of the material law first we postulate the form of a yield condition in the strain space (SIMO [31:1988])

$$(1.12) \qquad \hat{f}(\mathbf{E},\mathbf{E}^{p},\mathbf{Q}) \le 0$$

In the stress space the above condition has the form (cf. also SHRIVASTAVA, MROZ and DUBEY [21:1973])

$$(1.13) \qquad \hat{f}(\mathbf{E},\mathbf{E}^{p},\mathbf{Q}) = f(\rho_o \frac{\partial \hat{W}(\mathbf{E},\mathbf{E}^{p},\mathbf{Q})}{\partial \mathbf{E}},\mathbf{E},\mathbf{Q}) \le 0.$$

In practice the last form of the yield condition (for instance, the Huber-Mises yield condition) is reduced to

$$(1.14) \qquad f(\mathbf{S},k) \le 0,$$

in which k denotes a time-independent scalar. In the general case k can be dependent on the temperature. It is assumed f to be a regular (continuously differentiable) function of its variables.

The evolution of the internal plastic variable vector $\mathbf{Q}$ has the following form (SIMO [31:1988])

$$(1.15) \qquad \dot{\mathbf{Q}} = \dot{\gamma}\mathbf{H}(\mathbf{E},\mathbf{E}^{p},\mathbf{Q}).$$

Here, $\mathbf{H} \in \mathcal{T}^2$ is the generalized plastic hardening moduli, and $\dot{\gamma}$ denotes the plastic parameter given by (1.32).

The associative flow rule compatible with the kinematic decomposition (1.6) along with appropriate loading/unloading criterion can be derived as Kuhn-Tucker optimality conditions ensuing from the principle of maximum plastic dissipation (SIMO [31:1988]). At the thermodynamic state characterized by the variables $\{\mathbf{E},\mathbf{E}^{p},\mathbf{Q}\}$ the plastic dissipation is defined as

$$(1.16)\qquad \mathbb{D}^p(\mathbf{E},\mathbf{E}^p,\mathbf{Q};\dot{\mathbf{E}}^p,\dot{\mathbf{Q}}) = -\frac{\partial\hat{W}}{\partial\mathbf{E}^p}\cdot\dot{\mathbf{E}}^p - \frac{\partial\hat{W}}{\partial\mathbf{Q}}\cdot\dot{\mathbf{Q}}.$$

Taking into account a free energy in the form (1.10) the principle of maximum dissipation can be formulated as follows:

there exists a closed convex set $P = \{\mathbf{E} : \hat{f}(\mathbf{E},\mathbf{E}^p,\mathbf{Q}) \le 0\}$ such that

$$(1.17)\qquad \mathbb{D}^p(\mathbf{E},\mathbf{E}^p,\mathbf{Q};\dot{\mathbf{E}}^p,\dot{\mathbf{Q}}) \ge \mathbb{D}^p(\tilde{\mathbf{E}},\mathbf{E}^p,\mathbf{Q};\dot{\mathbf{E}}^p,\dot{\mathbf{Q}})$$

or, equivalently, using (1.10)

$$(1.18)\qquad -\frac{\partial W(\mathbf{E},\mathbf{E}^p)}{\partial\mathbf{E}^p}\cdot\dot{\mathbf{E}}^p \ge -\frac{\partial W(\tilde{\mathbf{E}},\mathbf{E}^p)}{\partial\mathbf{E}^p}\cdot\dot{\mathbf{E}}^p$$

for any $\tilde{\mathbf{E}} \in P$.

Now, maximum plastic dissipation implies that the actual strain tensor $\mathbf{E}$ is the argument of the maximum principle:

$$(1.19)\qquad \mathbf{E} = \arg\left\{\min_{\tilde{\mathbf{E}}\in P}\left[-\frac{\partial W(\tilde{\mathbf{E}},\mathbf{E}^p)}{\partial\mathbf{E}^p}\cdot\dot{\mathbf{E}}^p\right]\right\}.$$

As a result we obtain the following flow rule:

$$(1.20)\qquad \mathbb{M}\cdot\dot{\mathbf{E}}^p = -\dot{\gamma}\cdot\frac{\partial\hat{f}(\mathbf{E},\mathbf{E}^p,\mathbf{Q})}{\partial\mathbf{E}},$$

$$(1.21)\qquad \begin{aligned} &\dot{\gamma} \ge 0, \quad \hat{f}(\mathbf{E},\mathbf{E}^p,\mathbf{Q}) \le 0 && \text{(loading)},\\ &\dot{\gamma}\hat{f}(\mathbf{E},\mathbf{E}^p,\mathbf{Q}) = 0 && \text{(unloading)},\end{aligned}$$

where

$$(1.22)\qquad \mathbb{M} = \rho_o\,\frac{\partial^2 W(\mathbf{E},\mathbf{E}^p)}{\partial\mathbf{E}\partial\mathbf{E}^p}, \qquad \mathbb{M} \in \mathcal{T}^4.$$

The relation (1.20) defines the normality rule of the elasto-plasticity at finite strains in the strain space. However, the statical shakedown theory requires the normality rule in the stress space formulation. Let us now define it. To do this, let us first rewrite the equation (1.20) in the form

$$(1.23)\qquad \dot{\mathbf{E}}^p = \lambda\,\mathbb{M}^{-1}\cdot\mathbf{N}, \qquad \lambda \ge 0,$$

where $\mathbf{N}$ designates a tensor proportional to the normal to the loading surface.

Then, using (1.10), (1.13) and the chain rule, we have (LUBLINER [41:1984])

$$
(1.24)\qquad \begin{aligned} \mathbf{N} &= \alpha\, \frac{\partial f\left[\rho_o \frac{\partial W(\mathbf{E},\mathbf{E}^P)}{\partial \mathbf{E}}\Big|_{\mathbf{E}=\tilde{\mathbf{E}}}, \mathbf{E}, \mathbf{Q}\right]}{\partial \tilde{\mathbf{E}}}\Bigg|_{\mathbf{E}=\tilde{\mathbf{E}}} = \\ &= \alpha\, \mathbb{N}\cdot \frac{\partial f(\mathbf{S},\mathbf{E},\mathbf{Q})}{\partial \mathbf{S}}\Bigg|_{\mathbf{S}=\rho_o \frac{\partial W(\mathbf{E},\mathbf{E}^P)}{\partial \mathbf{E}}} \end{aligned}
$$

for some $\alpha \geq 0$, where

$$
(1.25)\qquad \mathbb{N} = \rho_o\, \frac{\partial^2 W(\mathbf{E},\mathbf{E}^P)}{\partial \mathbf{E}\partial \mathbf{E}}, \qquad \mathbb{N} \in \mathcal{T}^4;
$$

and the strain tensor $\tilde{\mathbf{E}}$ is defined by the loading function

$$
f\left[\rho_o \frac{\partial W(\tilde{\mathbf{E}},\mathbf{E}^P)}{\partial \tilde{\mathbf{E}}}, \mathbf{E}, \mathbf{Q}\right] = 0
$$

in the stress space.

Introducing $(1.24)_2$ to (1.23) we get the following form of the normality rule in the stress space:

$$
(1.26)\qquad \dot{\mathbf{E}}^P = \lambda\, \mathbb{M}^{-1}\mathbb{N}\cdot \frac{\partial f(\mathbf{S},\mathbf{E},\mathbf{Q})}{\partial \mathbf{S}}, \qquad \lambda \geq 0.
$$

The above normality rule reduces to the classical one

$$
(1.27)\qquad \dot{\mathbf{E}}^P = \lambda\, \frac{\partial f(\mathbf{S},\mathbf{E},\mathbf{Q})}{\partial \mathbf{S}},
$$

which is well known in the classical plasticity theory, if $\mathbb{M}^{-1}\mathbb{N} \equiv \mathbb{I} \in \mathcal{T}^4$. Likewise, this result is obtained also if the free energy potential W in (1.10) is assumed to have the form

$$
(1.28)\qquad W(\mathbf{E},\mathbf{E}^P) = W_e(\mathbf{E} - \mathbf{E}^P),
$$

as in GREEN and NAGHDI's theory [27:1965]. In this case, as well as within the framework of infinitesimal theory, the maximum dissipation energy implies the following inequality

$$
(1.29)\qquad (\mathbf{S} - \tilde{\mathbf{S}})\cdot\dot{\mathbf{E}}^P \geq 0,
$$

where

$$\tilde{\mathbf{S}} = \left.\frac{\partial W_e(\mathbf{E}-\mathbf{E}^p)}{\partial \mathbf{E}^p}\right|_{\mathbf{E}=\tilde{\mathbf{E}}}.$$

The deficiencies of (1.28) are discussed, among others, by SIMO [31:1988].

In turn, (1.29) and (1.27) lead to the following inequality (LUBLINER [41:1984])

$$(1.30)\qquad (\mathbf{S} - \tilde{\mathbf{S}})\cdot\frac{\partial f(\mathbf{S},\mathbf{E},\mathbf{Q})}{\partial \mathbf{S}} \geq 0$$

with $f(\tilde{\mathbf{S}},\mathbf{E},\mathbf{Q}) \leq 0$. In other words, $\partial f/\partial\mathbf{S}$ is the strain state associated with the stress state $\mathbf{S}$ and $\tilde{\mathbf{S}}$ is an arbitrary stress state such that $f(\tilde{\mathbf{S}},\mathbf{E},\mathbf{Q}) \leq 0$.

We now turn to define the plastic parameter $\dot{\gamma}$ in (1.15). It is determined from the consistency condition

$$(1.31)\qquad \frac{d}{d\tau}\,\hat{f}(\mathbf{E},\mathbf{E}^p,\mathbf{Q}) = 0$$

which is required for plastic loading to be $\dot{\gamma} > 0$. In order to define $\dot{\gamma}$ first, we differentiate with respect to time the constitutive equation (1.11) and use the flow rule (1.20), next, obtained result we substitute to (1.31). In effect we get the following expression for the plastic parameter (SIMO [31:1988])

$$(1.32)\qquad \dot{\gamma} = \frac{\dfrac{\partial\hat{f}}{\partial\mathbf{E}}\cdot\dot{\mathbf{E}}}{\dfrac{\partial\hat{f}}{\partial\mathbf{E}^p}\cdot\mathbb{M}^{-1}\cdot\dfrac{\partial\hat{f}}{\partial\mathbf{E}} - \dfrac{\partial\hat{f}}{\partial\mathbf{Q}}\cdot\mathbf{H}}.$$

On the basis of the above results the rate form of the stress-strain relation for elastic-plastic materials we can write as (SIMO [31:1988])

$$(1.33)\qquad \dot{\mathbf{S}} = \left[\rho_o\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}} - \frac{\dfrac{\partial\hat{f}}{\partial\mathbf{E}}\otimes\dfrac{\partial\hat{f}}{\partial\mathbf{E}}}{\dfrac{\partial\hat{f}}{\partial\mathbf{E}^p}\cdot\mathbb{M}^{-1}\cdot\dfrac{\partial\hat{f}}{\partial\mathbf{E}} - \dfrac{\partial\hat{f}}{\partial\mathbf{Q}}\cdot\mathbf{H}}\right]\cdot\dot{\mathbf{E}},$$

or in the compact form

$$(1.34)\qquad \dot{\mathbf{S}} = \mathbb{L}\cdot\dot{\mathbf{E}},$$

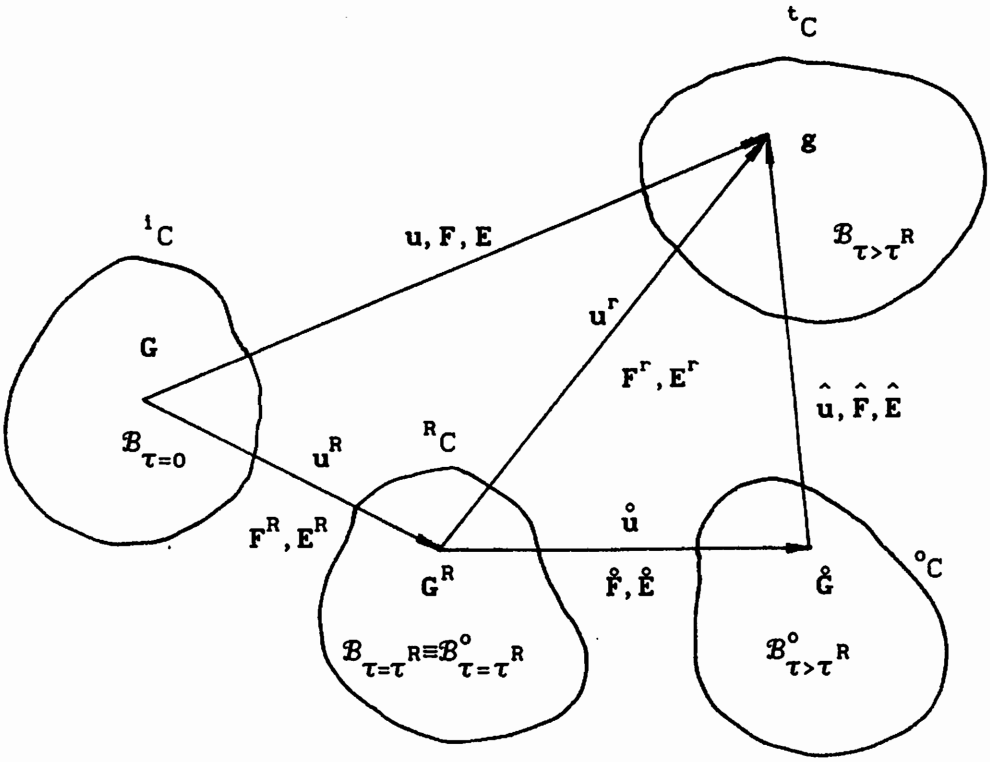
where

$$(1.35) \qquad \mathbb{L} = \left[\rho_o \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} - \frac{\frac{\partial \hat{f}}{\partial \mathbf{E}} \otimes \frac{\partial \hat{f}}{\partial \mathbf{E}}}{\frac{\partial \hat{f}}{\partial \mathbf{E}^p} \cdot \mathbb{M}^{-1} \cdot \frac{\partial \hat{f}}{\partial \mathbf{E}} - \frac{\partial \hat{f}}{\partial \mathbf{Q}} \cdot \mathbf{H}} \right].$$

The above form of the constitutive equation includes a wide class of material models with time-independent properties, namely, general anisotropic elastic-plastic materials, materials with a general shape of the plasticity surface, general work-hardening and work-softening materials, etc. (HILL [13:1959], HILL and RICE [19:1973]).

1.3. Residual stress distribution in the body $\mathcal{B}$

Our kinematical idea is focused on a specific conception of an evolution of the body $\mathcal{B}$ (Fig.1) (WEICHERT [8:1988], GROSS-WEEGE [12:1989]). Observe that such conception of the evolution of the body $\mathcal{B}$ enables to define a residual stress distribution in the body $\mathcal{B}$ as required by the shakedown theorem.



$\chi : {}^iC \longrightarrow {}^tC$

$\chi^R : {}^iC \longrightarrow {}^RC$

$\chi^r : {}^RC \longrightarrow {}^oC$

$\mathring{\chi} : {}^RC \longrightarrow {}^oC$

$\hat{\chi} : {}^oC \longrightarrow {}^tC$

Fig. 1. Evolution of body $\mathcal{B}$ and comparison body $\mathcal{B}^o$.

Let V_β denotes r-dimensional space of load paths β_s which defines the range of possible variations of the loads acting upon the structure. The loading, generally in the classical approach, are defined in such a way that every load type is a constant loading system multiplied by an appropriate load factor, for example

$$
\begin{aligned}
&\mathbf{t}^{\#}(X,\tau) = \beta_s(\tau)\mathbf{t}^{s\#}(X), \\
(1.37) \qquad & \qquad\qquad X \in V, \quad \tau \in [\tau_1,\tau_2], \\
&\mathbf{b}^{\#}(X,\tau) = \beta_s(\tau)\mathbf{b}^{s\#}(X).
\end{aligned}
$$

In our case we assume that at a fixed time τ^R the body $\mathcal{B}$ has already undergone deformations with finite displacements with respect to the initial configuration ${}^iC$ at time $\tau = 0$ so that $\mathcal{B}$ is at time τ^R in the reference configuration ${}^RC$ in quasi static and stable equilibrium with the external agencies $\mathbf{a}^R = (\mathbf{b}^{\#R},\mathbf{t}^{\#R})$, consisting of the prescribed loads and surface displacements. For times $\tau > \tau^R$ the body is submitted to additional variable loads $\mathbf{a}^r = (\mathbf{b}^{\#r},\mathbf{t}^{\#r})$ such that

$$
(1.38) \qquad \mathbf{a}(X,\tau) = \mathbf{a}^R(X) + \mathbf{a}^r(X,\tau),
$$

and $\mathcal{B}$ occupies the actual configuration ${}^tC$. Since ${}^tC$ should also be an equilibrium configuration the following equations hold:

$$
\begin{aligned}
&\mathbf{F}^R = \mathbf{G} + \mathbf{H}^R, \\
(1.39) \qquad &\mathbf{F}^r = \mathbf{G}^R + \mathbf{H}^r, \\
&\mathbf{F} = \mathbf{F}^r\mathbf{F}^R = \mathbf{G} + \mathbf{H} \quad \text{in } V,
\end{aligned}
$$

where displacement gradients

$$
\begin{aligned}
&\mathbf{H} = \nabla\mathbf{u}, \\
(1.40) \qquad &\mathbf{H}^R = \nabla\mathbf{u}^R, \\
&\mathbf{H}^r = \nabla_R\mathbf{u}^r
\end{aligned}
$$

are defined on the initial configuration ${}^iC$ and the reference configuration ${}^RC$, respectively. The tensors $\mathbf{G}$ and $\mathbf{G}^R$ in (1.39) are the metric tensors (Fig. 2) in the initial configuration ${}^iC$ and the reference configuration ${}^RC$, respectively. It is useful, in the context of $\mathbb{R}^3$, to think of $\mathbf{G}$, $\mathbf{G}^R$, $\mathring{\mathbf{G}}$, and $\mathbf{g}$

(for tensors $\mathring{G}$, g, see below) by making no notational distinction between them. In practice, it makes the identification $G \equiv G^R \equiv \mathring{G} \equiv g$. For conceptual clarity it convenient here to maintain the notational distinction between these metric tensors.

The geometrical boundary conditions are:

$$u = u^{\#R} + u^{\#r} \quad \text{on } S_u. \tag{1.41}$$

For the Green strain tensors we have:

$$E = E^R + E^r \quad \text{in } V, \tag{1.42}$$

where

$$
\begin{aligned}
E &= \frac{1}{2}\,(F^T g F - G),\\
E^r &= \chi^{R*}(\mathcal{E}^r) = (F^R)^T \mathcal{E}^r F^R,\\
E^R &= \frac{1}{2}\,[(F^R)^T G^R F^R - G],\\
\mathcal{E}^r &= \frac{1}{2}\,[(F^r)^T g F^r - G^R].
\end{aligned}
\tag{1.43}
$$

Here, and further on, $E^r = \chi^{R*}(\mathcal{E}^r)$ denotes the pull back operation of the strain measure $\mathcal{E}^r$ through the motion χ^R (cf. MARSDEN and HUGHES [32:1983], and g is the metric tensor in an actual configuration ${}^tC$.

The equilibrium conditions in the configuration ${}^tC$, therefore, have a form:

$$
\begin{aligned}
\mathrm{Div}(FS) &= -b^{\#R} - b^{\#r} \quad \text{in } V,\\
FSn &= t^{\#R} + t^{\#r} \quad \text{on } S_t,
\end{aligned}
\tag{1.44}
$$

where F is given by $(1.39)_3$ and the second Piola-Kirchhoff stress tensor S is calculated from (1.34) (for the part of the stress in ${}^RC$ see (1.46)) for the strain tensor (1.42).

Let us go to another group of relations important in the shakedown analysis. Especially important for our purposes is the connection between the stress state in the reference configuration ${}^RC$ and the actual stress state $\hat{S}$ (and a stress distribution $\bar{S}$ postulated by the shakedown theorem as well). The stress state $\bar{S}$ is any time-independent stress distribution with the property that superposition of $\bar{S}$ with any possible elastic stress distribution in a structure is everywhere less than the fully plastic stress. First, let us define equilibrium conditions in ${}^{R}C$.

Having defined the strain tensor $\mathbf{E}^{R}$ $(1.43)_3$ we can define the stress tensor $\mathbf{S}^{R}$. To this end, let us consider the constituive equation (1.34). Since the strain tensor $\mathbf{E}^{R}$ is time-independent, the relation (1.34) reduces to

$$\mathbf{S} = \mathbb{L}\cdot\mathbf{E}. \tag{1.45}$$

REMARK 1.1. Notice that the equation (1.45), for simplicity, is identified with the form $\Delta\mathbf{S} = \mathbb{L}\cdot\Delta\mathbf{E}$, where $\Delta(\cdot)$ denotes the tensor increment.

Substituting $\mathbf{E}$ in (1.45) by $\mathbf{E}^{R}$ from $(1.43)_3$ we find

$$\mathbf{S}^{R} = \mathbb{L}\cdot\mathbf{E}^{R}. \tag{1.46}$$

By assumption, the body $\mathcal{B}$ in the reference configuration ${}^{R}C$ is in equilibrium under loads $\mathbf{a}^{R}$. Hence, the equilibrium conditions in ${}^{R}C$ take the form

$$\begin{aligned} &\mathrm{Div}(\mathbf{F}^{R}\mathbf{S}^{R}) = -\mathbf{b}^{\#R} \quad \text{in } V, \\ &\mathbf{F}^{R}\mathbf{S}^{R}\mathbf{n} = \mathbf{t}^{\#R} \quad \text{on } S_t. \end{aligned} \tag{1.47}$$

We proceed now to define the time-dependent residual stress distribution in the body $\mathcal{B}$ in the reference configuration ${}^{R}C$. To do this we impose some constraints on the deformation process χ^{r} in the following way.

First, let us restrict ourselves to a class of deformations $\mathfrak{P}(\mathring{\mathbf{H}},\mathbf{H}^{r})$, where $\mathfrak{P}$ is a permutation of $\{O(\varepsilon),O(\varepsilon^{1/2})\}$, and $O(\cdot)$ denotes order of magnitude of $\mathring{\mathbf{H}}$ and $\mathbf{H}^{r}$, respectively. Here, the displacement gradient $\mathbf{H}^{r}$ is defined by $(1.40)_3$ and the displacement gradient $\mathbf{H}^{\circ}$is equal (cf. Fig.1)

$$\mathring{\mathbf{H}} = \nabla_R\mathring{\mathbf{u}}. \tag{1.48}$$

Next, we will consider loading histories characterized by the motion of a fictitious comparison body $\mathcal{B}^{\circ}$, having at τ^{R} the same field quantities as $\mathcal{B}$ but reacting in contrast to $\mathcal{B}$ purely elastically to the additional external loads $\mathbf{a}^{r}$, superimposed on $\mathbf{a}^{R}$ for $\tau > \tau^{R}$. Moreover, it is assumed that the state of deformation and the state of stress in $\mathcal{B}$ are subjected to variations in time (Fig.1).

Then, in accordance with Fig.1, we have

$$\mathbf{F}^{r} = \hat{\mathbf{F}}\mathring{\mathbf{F}}, \tag{1.49}$$

where

(1.50) $\mathring{F} = G^R + \mathring{H}$

is, by assumption, the time-dependent purely elastic deformation gradient, and its displacement gradient (1.48) is defined on the reference configuration ${}^R C$; in turn, the $\hat{F}$ is also the time-dependent deformation gradient defined as follows.

Let

(1.51) $\hat{u} = u^r - \mathring{u}$ in V,

be the displacement field such that

(1.52) $\hat{u} = 0$ on S_u.

At a level of accuracy of the assumed class of deformations $\mathfrak{D}(\mathring{H}, H^r)$, it follows from (1.49) that

(1.53) $\hat{F} = G^R + \hat{H}$

where

(1.54) $\hat{H} = H^r - \mathring{H}$

denotes the desired displacement gradient.

In a standard way, using (1.54), we get the strain tensor

(1.55) $\hat{E} = \chi^{R*}(\hat{\mathcal{E}}) = (F^R)^T \hat{\mathcal{E}} F^R,$

where

(1.56) $\hat{\mathcal{E}} = \frac{1}{2}(\hat{H} + \hat{H}^T + \hat{H}^T\hat{H}).$

Proceeding analogically as in (1.76), for the assumed class of deformations, the strain tensor (1.55) can be split into

(1.57) $\hat{E} = \hat{E}^e + \hat{E}^p + O(\varepsilon^{3/2}),$

where $\hat{E}^e$ and $\hat{E}^p$ can be expressed by one of the formulae (1.77), (1.79) or (1.80).

To obtain the time-dependent stress tensor $\hat{S}$ we start from the constitutive relation in the rate form (1.34). Hence, taking the derivative of $\hat{E}$ with respect to time, we have

(1.58) $\dot{\hat{S}} = \mathbb{L} \cdot \dot{\hat{E}}.$

After integrating (1.58) with respect to time, we get

$$(1.59) \qquad \hat{S}(\tau) - \hat{S}(\tau^R) = \int_{\tau^R}^{\tau} \dot{\hat{S}}(t)dt$$

under condition

$$(1.60) \qquad \hat{S}(\tau^R) = S^R \quad \text{in } V,$$

where S^R is given by (1.46).

We see therefore that the stress tensor $\hat{S}$ represents the time-dependent residual second Piola-Kirchhoff stress distribution in the structure refered to the reference configuration ${}^R C$. It is noteworthy that the stress state $\hat{S}$ is calculated from kinematical requirements, oposite to that used in literature.

Since the time-dependent residual stress field $\hat{S}$ superimposed on the stress field S^R must be an equilibrium configuration the following equations are satisfied:

$$(1.61) \qquad \begin{aligned} &\mathrm{Div}[\hat{F}F^R(S^R + \hat{S})] = -b^{\#R} \quad \text{in } V, \\ &\hat{F}F^R(S^R + \hat{S})n = t^{\#R} \quad \text{on } S_t. \end{aligned}$$

According to (1.53), the equations (1.61) are equivalent (1.47) and

$$(1.62) \qquad \begin{aligned} &\mathrm{Div}(\hat{T}) = 0 \quad \text{in } V, \\ &\hat{T}n = 0 \quad \text{on } S_t, \end{aligned}$$

where

$$(1.63) \qquad \hat{T} = \hat{H}F^R S^R + \hat{F}F^R \hat{S}$$

is the time-dependent residual first Piola-Kirchhoff stress distribution in the body $\mathcal{B}$.

1.4. Stress distribution for purely elastic body $\mathcal{B}^{o}$

In this section all quantities describing a purely elastic behaviour of the body are indicated additionally by mark "∘".

For a given structure and a given domain of loads the necessary condition for shakedown is the existence of a steady residual stress field (called the shakedown stress field) such that

$$(1.64)\qquad f\left[\mathring{S}(X,\tau) + \bar{S}(X), \mathring{E}(X,\tau) + \bar{E}(X), Q(X,\tau)\right] \le 0$$

for all loads defined by the domain and all points of the body volume. Here $\mathring{S}$ denotes any elastic stress destribution which is obtainable under the given loading conditions.

Inequality (1.64) is sufficient for ideal case. However, for a real structure, some plastic flow may occur locally of a section without the entire section becoming plastic. Hence, in practice, the condition (1.64) is equivalent to (cf. KÖNIG [1:1987])

$$(1.65)\qquad \max_{X\in V}\ \max_{\beta_s \in \mathcal{V}_\beta}\ f\left[\mu[\mathring{S}(X) + \bar{S}(X)], \mathring{E}(X) + \bar{E}(X), Q(X)\right] \le 0,$$

where $\mu > 1$ denotes a safety factor.

Let us now turn to our task. Based on the multiplicative decomposition of a deformation gradient (1.6), we will define the elastic part of the strain tensor $\mathbf{E}$ for χ^R and χ^r, respectively.

From $(1.7)_2$ and (1.8), in relation to $(1.43)_3$ and $(1.43)_2$, respectively, follow (Fig.2) that

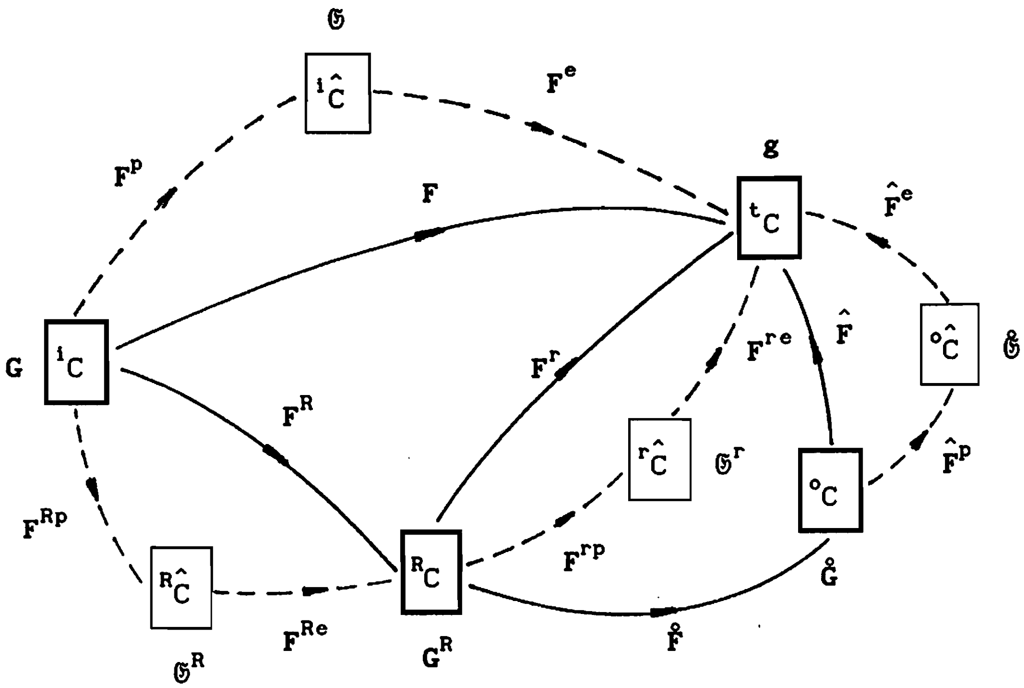
$$(1.66)\qquad \begin{aligned} \mathbf{E}^{Rp} &= \frac{1}{2}\,[(\mathbf{F}^{Rp})^T \mathfrak{G}^R \mathbf{F}^{Rp} - \mathbf{G}], \\ \mathbf{E}^{Re} &= \mathbf{E}^R - \mathbf{E}^{Rp} = \frac{1}{2}\,[(\mathbf{F}^R)^T \mathbf{G}^R \mathbf{F}^R - (\mathbf{F}^{Rp})^T \mathfrak{G}^R \mathbf{F}^{Rp}]. \end{aligned}$$

$$(1.67)\qquad \begin{aligned} \mathbf{E}^{rp} &= \chi^{R*}(\mathcal{E}^{rp}) = (\mathbf{F}^R)^T \mathcal{E}^{rp} \mathbf{F}^R, \\ \mathbf{E}^{re} &= \mathbf{E}^r - \mathbf{E}^{rp} = \frac{1}{2}\,(\mathbf{F}^R)^T[(\mathbf{F}^r)^T \mathbf{g} \mathbf{F}^r - (\mathbf{F}^{rp})^T \mathfrak{G}^r \mathbf{F}^{rp}]\mathbf{F}^R, \end{aligned}$$

where

$$(1.68)\qquad \mathcal{E}^{rp} = \frac{1}{2}\,[(\mathbf{F}^{rp})^T \mathfrak{G}^r \mathbf{F}^{rp} - \mathbf{G}^R].$$

In the above $\mathfrak{G}^R$ and $\mathfrak{G}^r$ denote the metric tensors in the intermediate configurations ${}^R\hat{C}$ and ${}^r\hat{C}$ for deformations χ^R and χ^r, respectively. The intermediate configuration is defined as the collection of all unloaded local neighbourhoods. This situation is graphically shown in Fig. 2. In general, the local unloading process is an abstract idea, since it does not take place in reality.



$\mathfrak{G}$
${}^{i}\hat{C}$
F^{e}
g
F^{p}
F
${}^{t}C$
$\hat{F}^{e}$
G
${}^{i}C$
F^{r}
F^{re}
$\hat{F}$
${}^{o}\hat{C}$
$\mathring{\mathfrak{G}}$
F^{R}
${}^{r}\hat{C}$
$\mathfrak{G}^{r}$
${}^{o}C$
$\hat{F}^{p}$
F^{Rp}
${}^{R}\hat{C}$
${}^{R}C$
F^{rp}
$\mathring{G}$
F^{Re}
G^{R}
$\mathring{F}$
$\mathfrak{G}^{R}$

Fig. 2.

For our immediate purposes we need a definition of the elastic part of strain tensor, say $\mathbf{E}^e$, not of the difference $\mathbf{E} - \mathbf{E}^p$. To utilize formulae (1.66) further, let us introduce simplifications to it. Here, we will go more deeply into the differential-geometric structure of these definitions. We shall now sketch how ideas demanded might be defined.

To this end we introduce a measure ε of smallness defined by

$$(1.69) \qquad \varepsilon = \max\left\{\sup_{^1\hat{C}} \|\mathbf{H}^e\|,\ \sup_{^1C} \|\mathbf{H}^p\|\right\},$$

where, f.e. $\|\cdot\| = \sqrt{\mathrm{tr}(\cdot)^2}$.

Let $\mathbf{H}$ be any tensor-valued function of $(\mathbf{H}^e, \mathbf{H}^p)$ defined in a neighbourhood of $(\mathbf{0}, \mathbf{0})$. If $\mathbf{H}$ satisfies the condition

$$(1.70) \qquad \|\mathbf{H}\| < K\varepsilon^n \quad \text{as } \varepsilon \longrightarrow 0,$$

where K is a nonnegative real constant and n is positive number, then we write

$$(1.71) \qquad \mathbf{H} = O(\varepsilon^n).$$

In the light of (1.69) and (1.71) let us assume each of $\mathbf{H}^e$, $\mathbf{H}^p$ to have order of magnitude $O(\varepsilon)$ or $O(\varepsilon^{1/2})$. The magnitude $O(\varepsilon^{1/2})$ corresponds to moderate deformations with respect to ε (CASEY [18:1985]).

To simplify the relations (1.66) we will use the following identities:

$$(1.72) \qquad \mathbf{E} = (\mathbf{H})_S + \frac{1}{2}\left[(\mathbf{H})_S^2 + (\mathbf{H})_S(\mathbf{H})_A - (\mathbf{H})_A(\mathbf{H})_S - (\mathbf{H})_A^2\right],$$

where

$$(1.73) \qquad (\mathbf{H})_S = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T), \qquad (\mathbf{H})_A = \frac{1}{2}(\mathbf{H} - \mathbf{H}^T);$$

and

$$(1.74) \qquad \mathbf{H} = \mathbf{H}^e + \mathbf{H}^p + \mathbf{H}^e\mathbf{H}^p$$

which follows from (1.6), $(1.39)_3$ and the definitions

$$(1.75) \qquad \mathbf{H}^e = \mathbf{F}^e - \mathfrak{G}, \qquad \mathbf{H}^p = \mathbf{F}^p - \mathbf{G}.$$

In order to express $\mathbf{E}^{Re}$ and $\mathbf{E}^{Rp}$ in (1.78) explicitly, it is possible, following CASEY [18:1985], to distinguish three physically possible cases of such decomposition. In particularly, the following types of deformations are of special interest:

(i) small plastic deformations, moderate elastic deformations:
Then, from (1.74) follows:

$$(1.76)\qquad \begin{aligned} &(H^{Re})_S = O(\varepsilon^{1/2}), \quad (H^{Re})_A = O(\varepsilon^{1/2}),\\ &(H^{Rp})_S = O(\varepsilon), \quad (H^{Rp})_A = O(\varepsilon). \end{aligned}$$

Moreover, with the aid of (1.76), we deduced from (1.72) that

$$(1.77)\qquad \begin{aligned} E^{Re} &= (H^{Re})_S + \frac{1}{2}\Big[(H^{Re})_S^2 + (H^{Re})_S(H^{Re})_A - (H^{Re})_A(H^{Re})_S - \\ &\quad - (H^{Re})_A^2\Big],\\ E^{Rp} &= (H^{Rp})_S + O(\varepsilon^{3/2}). \end{aligned}$$

Under these assumptions (1.66) reduces to

$$(1.78)\qquad E^R = E^{Re} + E^{Rp} + O(\varepsilon^{3/2}).$$

Expressions complementary to (1.77) may now be readily deduced by the same type of arguments. As a result, one can obtain for:

(ii) small elastic deformations, moderate plastic deformations:

$$(1.79)\qquad \begin{aligned} E^{Re} &= (H^{Re})_S + O(\varepsilon^{3/2}),\\ E^{Rp} &= (H^{Rp})_S + \frac{1}{2}\Big[(H^{Rp})_S^2 + (H^{Rp})_S(H^{Rp})_A - (H^{Rp})_A(H^{Rp})_S - \\ &\quad - (H^{Rp})_A^2\Big]; \end{aligned}$$

(iii) small strains, moderate rotations:

$$(1.80)\qquad \begin{aligned} E^{Re} &= (H^{Re})_S - \frac{1}{2}(H^{Re})_A^2 + O(\varepsilon^{3/2}),\\ E^{Rp} &= (H^{Rp})_S - \frac{1}{2}(H^{Rp})_A^2 + O(\varepsilon^{3/2}), \end{aligned}$$

where $(H)_S$ and $(H)_A$ are defined by (1.73).

To obtain the stress tensor of the elastic process we can use the equation (1.34) which reduces, when **E** and **S** are time-independent, to the form (cf. Remark 1.1)

(1.81) $\mathbf{S} = \mathbb{D}\cdot\mathbf{E},$

where

(1.82) $\mathbb{D} = \mathbb{L}_{|\mathbf{E}=\mathbf{E}^e} = \rho_o \dfrac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}}$

is the time independent and positive definite tensor of the elastic coefficients with the symmetries

(1.83) $\mathbb{D} = \mathbb{D}^{\overset{3-4}{T}} = \mathbb{D}^{\overset{1-2}{T}\ \overset{3-4}{T}} = \mathbb{D}^{\overset{1-3}{T}\ \overset{2-4}{T}}.$

Substituting $\mathbf{E}$ by $\mathbf{E}^{Re}$ from (1.78) in (1.81) we obtain the stress tensor

(1.84) $\mathring{\mathbf{S}}^R = \mathbb{D}\cdot\mathbf{E}^{Re}.$

The same analysis is also true for the strain tensor $\mathbf{E}^r$. Here, however, we can utilize the introduced assumption, namely, that the body $\mathcal{B}^o$ deformes as a perfectly elastic body $\mathcal{B}$. From this, using the definition (1.48), follows

(1.85) $\mathring{\mathbf{E}} = \mathring{\chi}^{R*}(\mathring{\mathcal{E}}) = (\mathring{\mathbf{F}}^R)^T\mathring{\mathcal{E}}\mathring{\mathbf{F}}^R,$

where

(1.86) $\mathring{\mathcal{E}} = \frac{1}{2}\,(\mathring{\mathbf{H}} + \mathring{\mathbf{H}}^T + \mathring{\mathbf{H}}^T\mathring{\mathbf{H}}).$

The motion $\mathring{\chi}^R$ in the pull back operation (1.85) denotes the completely elastic motion of the body $\mathcal{B}$ under the given loads a^R. According to (1.74), for the assumed class of deformations, the deformation gradient $\mathring{\mathbf{F}}^R$ can easily be defined. With these tools, in the spirit of (1.59), we conclude

(1.87) $\mathring{\mathbf{S}}(\tau) - \mathring{\mathbf{S}}(\tau^R) = \int_{\tau^R}^{\tau} \dot{\mathring{\mathbf{S}}}(t)dt.$

The natural condition under which the stress distribution $\mathring{\mathbf{S}}$ can be calculated is the following

(1.88) $\mathring{\mathbf{S}}(\tau^R) = \mathring{\mathbf{S}}^R$ in V,

where $\mathring{\mathbf{S}}^R$ is given by (1.84).

To summ up, the elastic stress distribution can be computed by any standard elastic technique for any combination of allowable loads.

1.5. Definition of shakedown

We assume the program of loading and the basic configuration of the body (structure) to be settled.

DEFINITION 1.2. It is said that the structure will shake down over any programme of loading if the total plastic energy dissipated W^P is bounded during the deformation process, i.e.,

$$(1.89) \qquad W^P = \int_V \int_0^\infty S \cdot \dot{E}^P dt dV < \infty,$$

where $S \in \mathcal{S}^2$ denotes the second Piola-Kirchhoff stress appearing in the elastic-plastic structure and $\dot{E}^P$ is the plastic part of strain tensor.

Shakedown of a structure represents a safe occurrence. This implies that, after some time, plastic strains cease to develop further and response to subsequent loading cycles is purely elastic.

2. Shakedown theorem for non-linear cases

The aim of the present section is to construct the non-linear shakedown theorem being sufficiently simple so as to be incorporated into effective methods of the shakedown structural analysis.

In order to define the problem, it will be assumed that the points or regions of applications of all loads are known and that loads may vary completely arbitrarily between prescribed limits. The problem is to determine the safety factor for which shakedown will occur. Since the loading is not specified, we cannot attack this problem directly. Instead, we must make use of the shakedown theorem. In essence, in the light of definition 1.2, a structure is safe if only a limited amount of plastic work can be done on the structure by any allowable application of the loads.

Keeping the notations of Section 1, the problem can be formulated as follows:

Let an elastic-plastic body $\mathcal{B}$ be in the reference configuration ${}^{R}C$ at the time $\tau = \tau^R$ equilibrium with time- independent external loads $\mathbf{a}^R$. Will the body $\mathcal{B}$ shake down under the action of additional variable loads $\mathbf{a}^r$?

Under assumptions explained in Section 1 the following theorem can be formulated.

SHAKEDOWN THEOREM 2.1. If there exists a time-independent state of residual stresses $\bar{S}$, such that the following relations hold:

(i) $\mathrm{Div}(\mathbf{F}^R\mathbf{S}^R) = -\mathbf{b}^{\#}$ in V,

$\mathbf{F}^R\mathbf{S}^R\mathbf{n} = \mathbf{t}^{\#}$ on S_t,

$\mathbf{F}^R = \mathbf{G} + \nabla\mathbf{u}^R$ in V, (2.1)

$\mathbf{u}^R = \mathbf{u}^{\#R}$ on S_u,

$\frac{\partial}{\partial\tau}(\mathbf{S}^R) = \frac{\partial}{\partial\tau}(\mathbf{u}^R) = \mathbf{0}$;

(ii) $\bar{\mathbf{T}} = \bar{\mathbf{H}}\mathbf{F}^R\mathbf{S}^R + \bar{\mathbf{F}}\mathbf{F}^R\bar{\mathbf{S}}$,

$\mathrm{Div}(\bar{\mathbf{T}}) = \mathbf{0}$ in V,

$\bar{\mathbf{T}}\mathbf{n} = \mathbf{0}$ on S_t,

$\bar{\mathbf{F}} = \mathbf{G}^R + \bar{\mathbf{H}}$ in V,

$\bar{\mathbf{H}} = \nabla_R\bar{\mathbf{u}}$ in V, (2.2)

$\bar{\mathbf{u}} = \mathbf{0}$ on S_u,

$\bar{\mathbf{E}} = \frac{1}{2}(\mathbf{F}^R)^T(\bar{\mathbf{H}} + \bar{\mathbf{H}}^T + \bar{\mathbf{H}}^T\bar{\mathbf{H}})\mathbf{F}^R$ in V,

$\frac{\partial}{\partial\tau}(\bar{\mathbf{S}}) = \frac{\partial}{\partial\tau}(\bar{\mathbf{u}}) = \mathbf{0}$;

(iii) $f\left[\mathring{\mathbf{S}}^R + \mathring{\mathbf{S}}(\tau) + \bar{\mathbf{S}}, \mathring{\mathbf{E}}^R + \mathring{\mathbf{E}}(\tau) + \bar{\mathbf{E}}, \mathbf{Q}\right] < 0$ (2.3)

for all $\tau > \tau^R$ in V, then the body $\mathcal{B}$ will shake down under given programme of loading.

The following points about the shakedown theorem may be noted:

1. The postulated shakedown stress field $\bar{\mathbf{S}}$ need not be the same as the actual residual stress field $\mathbf{S} - \mathbf{S}^*$ (here $\mathbf{S}$ means an actual elastic-plastic stress tensor, and $\mathbf{S}^*$ is a stress tensor for perfectly elastic body) which would actually exist in the structure after it had shaken down.
2. The presence of some initial self-equilbrated stresses has no influence on the shakedown.
3. The order in which loads are applied (during each χ^R and χ^r deformation

process separately) has no effect on whether a structure can shake down.

4. The elastic stress fields may include changes of stress induced by variations of temperature.

P r o o f: Let $\hat{\mathbf{S}}$ and $\hat{\mathbf{E}}$ be the actual residual stress and residual strain in the structure at any instant. Next, a shakedown stress distribution $\bar{\mathbf{S}}$ is defined as a residual stress state which satisfies (2.2) and (2.3) for all $\overset{\circ}{\mathbf{S}}{}^{R}$ and $\overset{\circ}{\mathbf{S}}$ which are obtainable under the given loading (formulae (1.84) and (1.87)).

We proof the shakedown theorem considering the quadratic form Π defined by

$$(2.4)\qquad \Pi(\tau) = \frac{1}{2}\int_V \left\{ \mathbb{D}^{-1}[\tilde{\mathbf{S}}]\cdot\tilde{\mathbf{S}} + (\tilde{\mathbf{F}}\mathbf{F}^R)^T\tilde{\mathbf{F}}\mathbf{F}^R\cdot\mathbf{S}^R \right\} dV,$$

where

$$\tilde{\mathbf{S}} = \hat{\mathbf{S}} - \bar{\mathbf{S}}$$

$$(2.5)\qquad \tilde{\mathbf{F}} = \mathbf{G}^R + \hat{\mathbf{H}} - \bar{\mathbf{H}}.$$

$$\bar{\mathbf{H}} = \nabla_R\bar{\mathbf{u}}, \qquad \hat{\mathbf{H}} = \nabla_R\hat{\mathbf{u}}.$$

We shall show that $\Pi(\cdot)$ is constant when there is no plastic flow and decreases when plastic flow occurs.

REMARK 2.2. The restriction (25) used in GROSS-WEGGE's paper [12:1989] is superfluous. It suffices to notice that

$$(2.6)\qquad \int_V \left\{ \mathbb{D}^{-1}[\tilde{\mathbf{S}}]\cdot\tilde{\mathbf{S}} + (\tilde{\mathbf{F}}\mathbf{F}^R)^T\tilde{\mathbf{F}}\mathbf{F}^R\cdot\mathbf{S}^R \right\} dV = \int_V \mathbf{T}\cdot\mathbf{F}\,dV$$

where

$$(2.7)\qquad \begin{aligned} \mathbf{F} &= \tilde{\mathbf{F}}\mathbf{F}^R, \\ \mathbf{T} &= \tilde{\mathbf{F}}\mathbf{F}^R(\mathbf{S}^R + \tilde{\mathbf{S}}). \end{aligned}$$

As is well known, the right hand side of (2.6) represents the strain energy of the present stress and the displacement fields.

By the definition $(2.2)_8$ $\bar{\mathbf{S}}$ does not change with time, hence the derivative of Π is

(2.8) $$\frac{d\Pi}{d\tau} = \int_V \mathbb{D}^{-1}[\hat{S} - \bar{S}]\cdot\frac{d}{d\tau}(\hat{S})dV + \int_V \tilde{F}F^R S^R\cdot\frac{d}{d\tau}(\hat{H})F^R dV.$$

Substituting the time-derivative of the equation (1.57) into equation (2.8) we obtain

(2.9) $$\frac{d\Pi}{d\tau} = \int_V (\hat{S} - \bar{S})\cdot\frac{d}{d\tau}(\hat{E} - \hat{E}^P)dV + \int_V \tilde{F}F^R S^R\cdot\frac{d}{d\tau}(\hat{H})F^R dV.$$

Using (1.63) and $(2.2)_1$ the equation (2.9) can be further transformed to

(2.10) $$\frac{d\Pi}{d\tau} = -\int_V (\hat{S} - \bar{S})\cdot\frac{d}{d\tau}(\hat{E}^P)dV + \int_V (\hat{T} - \bar{T})\cdot\frac{d}{d\tau}(\hat{H})F^R dV.$$

Using the Gauss' theorem the second term on the right hand side of the equation (2.10) can be written as follows:

(2.11) $$\int_V (\hat{T} - \bar{T})\cdot\frac{d}{d\tau}(\hat{H})F^R dV = \int_{S_t} (\hat{T} - \bar{T})n\cdot\frac{d}{d\tau}(\hat{u})F^R dS - \int_V [\mathrm{Div}(\hat{T}) - \mathrm{Div}(\bar{T})]\cdot\frac{d}{d\tau}(\hat{u})F^R dV.$$

In view of equations (1.62), $(2.2)_{2,3}$ and (1.52) it follows that the right hand side of (2.11) is equal to zero. Therefore, equation (2.10) reduces to

(2.12) $$\frac{d\Pi}{d\tau} = -\int_V (\hat{S} - \bar{S})\cdot\frac{d}{d\tau}(\hat{E}^P)dV.$$

Inequality (2.3) assures that

(2.13) $$\mathring{S}^R + \mathring{S} + \bar{S} = S^s,$$

being a "safe" state of stress fulfills the inequality (1.13). Inserting equations (2.13) into (2.12) we get

(2.14) $$\frac{d\Pi}{d\tau} = -\int_V (S - S^s)\cdot\frac{d}{d\tau}(\hat{E}^P)dV,$$

where S denotes the true state of total stresses and $d/d\tau(\hat{E}^P)$ is the true state of residual plastic strain rates. In view of inequality (1.30) $d\Pi/d\tau$ is always non-positive and moreover, $d\Pi/d\tau$ is equal to zero for $d/d\tau(\hat{E}^P) = \mathbf{0}$. Thus $d\Pi/d\tau < 0$ as long as $d/d\tau(\hat{E}^P) \neq \mathbf{0}$.

Since Π is non-negative one can conclude that

$$\Pi(0) \geq \Pi(\tau) \quad \text{for } \tau > 0$$

and

$$\frac{d\Pi}{d\tau} \longrightarrow 0, \quad \Pi(\tau) \longrightarrow \text{const} \quad \text{for } \tau \longrightarrow \infty.$$

It means that the residual stresses will no longer change with time, and the body will experience only elastic deformations as the loads are varied.

To prove the boundedness of the total energy dissipated we use the condition (2.12). Let $\mu > 1$ be the safety factor of the structure against failure due to non-shakedown. Then the stress state $\mu(\mathring{S}^R + \mathring{S} + \bar{S})$ is inside the elastic domain, i.e.,

$$(2.15) \qquad f[\mu(\mathring{S}^R + \mathring{S} + \bar{S}), \mathring{E}^R + \mathring{E} + \bar{E}, Q] \leq 0,$$

or with the help of equation (2.13)

$$(2.16) \qquad f(\mu S^s, \mathring{E}^R + \mathring{E} + \bar{E}, Q) \leq 0.$$

From the convexity of the yield surface and the validity of the normality rule follows

$$(2.17) \qquad (S - \mu S^s) \cdot \frac{d}{d\tau}(\hat{E}^P) \geq 0.$$

Transformation of equation (2.17) and integration over V gives

$$(2.18) \qquad S \cdot \frac{d}{d\tau}(\hat{E}^P) \leq \frac{\mu}{\mu-1} (S - S^s) \cdot \frac{d}{d\tau}(\hat{E}^P).$$

After integrating (2.18), first over the body volume, next with respect to time, we obtain

$$(2.19) \qquad \int_0^\tau \int_V S \cdot \frac{d}{d\tau}(\hat{E}^P) dV d\tau \leq \frac{\mu}{\mu-1} \Pi(0),$$

where we have used (2.12). In the presence of the definition (2.4), the above inequality implies the boundedness of the total energy dissipated. □

3. Comparison with Gross-Weege's paper

Our task is the following:

(1) Define a residual stress distribution on the equilibrium configuration ${}^{R}C$.

(2) Formulate a shakedown theorem for the elastic-plastic body with respect to the configuration ${}^{R}C$.

Assumptions restricted to that used in [12]:

(i) The body is in an equlibrium under both loads a^R and $a^R + a^r$.

(ii) It is assumed that a deformation process χ^R is a moderate and a deformation process χ^r is an infinitesimal one.

Before coming to technical details, it is worth recalling that the existing shakedown theories deal with a very simplified material model and its practical applicability is still unsatisfactory. The boundedness of the total energy dissipated, as required the shakedown definition, without specifying any definite bound for this energy is a considerable simplification. In spit of this all new viewpoint on the shakedown analysis are desirable. Now, let us turn to our task.

The assumption (i) implies that the equations (13) and (14) in [12] are exact. In the present work them correspond equations (1.39) ÷ (1.44).

REMARK 3.1. The relations $(15)_1$ in [12] does not result from kinematics of the problem. One should distinguish between a time-independent stress state S^R and a time dependent stress state S^r.

REMARK 3.2. All relations given by (16), (17) and (18) in [12] are superfluous.

If we wish to construct a residual stress distribution $\hat{S}$ in the class of deformation (ii) it is suffices to assume $(19)_1$. Analogical relations (1.51) have been used to obtain, first, the residual strain tensor (1.56), next, the residual stress distribution (1.59).

The assumed relation $(19)_1$ enables to define remaining equations exactly in the sense of (ii).

The definition $(20)_2$ should result from $(19)_1$ (cf. (1.55)). In turn, the defintion $(21)_3$ should come from the equilibrium conditions, as explained in (1.61) ÷ (1.63). The relations (1.55) and (1.63) oposite to $(29)_2$ and $(21)_2$ are exact in the assumed class of deformations. Observe that such relations do not require any simplification.

REMARK 3.3. The equations $(22)_2$ do not result from kinematics of the problem.

REMARK 3.4. The tensor S^R in the conditon (24) is qual to $\mathring{S}^R$. Moreover, the condition (25) is superfluous (cf. Remark 2.2).

Consistently with what we have said so far, the shakedown theorem reduces to the theorem 2.1. One can easily conclude that if

$$S^R \longrightarrow 0$$

and

$$F^R \longrightarrow G$$

then the shakedown theorem 2.1 is exactly the original Melan's theorem.

REMARK 3.5. In the equation (34) S^R should be substituted by $\mathring{S}^R$. By assumption S^R can be on the yield surface.

4. The generalized standard material model

The present section is devoted to the study of the shakedown problems for the generalized standard material. The used concept of internal parameters in the framework of the generalized standard material model assumes at each instant of the deformation process the actual state of hardening in an elastic-plastic material to be locally described by a finite number of process-dependent internal parametres. A practical form of this description was introduced by HALPHEN and NGUYEN [33:1975] and in the sequel developed and applied by several authors (e.g. MANDEL [20 :1976], RAFALSKI [29:1977], WEICHERT [11:1988], WEICHERT and GROSS-WEEGE [10:1988]). In this approach, generalized elastic and plastic strains, e^e and e^p, respectively, and generalized stresses s are introduced, defined by the sets

$$e = [E, \epsilon],$$

$$e^e = [E^e, \epsilon^e], \qquad e^p = [E^p, \epsilon^p], \tag{4.1}$$

$$s = [S, \sigma].$$

The quantities ϵ^e, ϵ^p and σ are the r-dimensional vectors of internal ("hidden") elastic and plastic parameters and "back-stresses", respectively. The dimension r depends upon the particular choice of hardening model.

Note that in the context of shakedown theory, kinematical hardening has first been discussed by MELAN [35:1938]. He used Prager's hardening rule and gave a criterion for shakedown under the assumption of unlimited hardening. This concept has been used by PONTER [39:1975], GOKHELD and CHERNIAVSKI [40:1980]. The used assumption of unlimited hardening implies that time-independent residual stress fields have not to fulfill any requirement of static admissibility. From physical point of view, it seems to be more realistic to consider an upper limit for hardening, as otherwise certain loading cases would lead to an unbounded loading capacity and only failure due to alternating plasticity can be detected (KÖNIG [1:1987]). This problem can be tackled by checking for limited ductility and imposing limitations on relevant parameters of plastic deformation (KÖNIG and SIEMASZKO [43:1988]). Such approach requires the computation of strains, say, by a step-by-step method. It is possible to use the method which includes the limitation of hardening by imposing limits on the internal parameters σ. Practically it can be interpreted as simple two-surface model for plastic behaviour (MROZ [36:1967), KRIEG [37:1975], where the limitation of internal parameters σ is equivalent to the assumption of a fixed loading surface (MROZ [36:1967], KRIEG [37:1975], WEICHERT and GROSS-WEEGE [10:1988]) and the calculation of strains can be avoided.

4.1. Plastic part of the material law

The yield condition given by (1.13) can be reduced to that in the (6+r)-dimensional space of generalized stresses **s** with the property, that all admissible states of observable stresses **S** and internal parameters σ are such that

(4.2) $$F(\mathbf{s},\mathbf{e}) \leq 0.$$

Then, just like for ideal plastic behaviour, F is time-independent. The motion of the yield surface in the space of observable stresses **S** is represented by a change of values of the internal parameters σ. It was proved by HALPHEN and NGUYEN [33:1975] that the properties of convexity and validity of normality rule are preserved in the space of generalized strains and stresses. Then, the normality rule can be given in the form (cf. (1.30))

(4.3) $$(\mathbf{s} - \tilde{\mathbf{s}})\cdot\dot{\mathbf{e}}^{p} \geq 0,$$

where $\tilde{\mathbf{s}}$ characterizes arbitrary admissible fields fulfilling inequality (4.2).

For instance, in the case of kinematical hardening following Prager's hardening rule (PRAGER [34:1959]) the evolution of the internal plastic parametres ϵ^p is linked to the plastic strain rates $\dot{E}^p$ by

$$(4.4) \qquad \dot{\epsilon}^p_a = -\dot{E}^p_{ij}$$

for $a = \frac{1}{2}(i+j)$ if $i = j$, and $a = i+j+1$ if $i \neq j$, $i,j = 1,2,3$.

The evolution if internal elastic parameters ϵ^e is in general given by

$$(4.5) \qquad \dot{\epsilon}^e_a = -\dot{\epsilon}^p_a, \qquad a = 1,2,\ldots,r$$

so that for initially virgin material we have

$$(4.6) \qquad \epsilon^e_a = -\epsilon^p_a \qquad a = 1,2,\ldots,r,$$

for all times (WEICHERT [38:1987], RAFALSKI [29:1977]).

4.2. The shakedown theorem

Using the material description given in Section 1 the shakedown theorem 2.1. given in Section 2 can be reformulated for the generalized standard material. From now on we assume all conditions given in Section 1 to be satisfied. We also use a fact that the internal parameteres to describe the state of hardening in the material vanish for the purely elastic reference problem.

Let the generalized strains and stresses in the configuration ${}^iC$ are given by

$$(4.7) \qquad \begin{aligned} &\mathbf{e}^R = [\mathbf{E}^R, \epsilon^R], \\ &\mathbf{e}^{Re} = [\mathbf{E}^{Re}, \epsilon^{Re}], \qquad \mathbf{e}^{Rp} = [\mathbf{E}^{Rp}, \epsilon^{Rp}], \\ &\mathbf{s}^R = [\mathbf{S}^R, \sigma^R]. \end{aligned}$$

Analogously, in the configuration ${}^RC$, the generalized residual stresses are equal to

$$(4.8) \qquad \begin{aligned} &\hat{\mathbf{e}} = [\hat{\mathbf{E}}, \hat{\epsilon}], \\ &\hat{\mathbf{e}}^e = [\hat{\mathbf{E}}^e, \hat{\epsilon}^e], \qquad \hat{\mathbf{e}}^p = [\hat{\mathbf{E}}^p, \hat{\epsilon}^p], \\ &\hat{\mathbf{s}} = [\hat{\mathbf{S}}, \hat{\sigma}]. \end{aligned}$$

We also assume that the solution of a reference problem is given, i.e. the solution of the purely elastic problem under originally given loads. All quantities refering to this reference problem will be indicated by upper index "∘".

SHAKEDOWN THEOREM 4.1. If there exists a time-independent state of generalized residual stresses $\bar{s} = [\bar{S}, \bar{\sigma}]$, such that the following relations hold:

(i) $$\mathrm{Div}\,(F^R S^R) = -b^{\#R} \quad \text{in } V,$$

$$F^R S^R n = -t^{\#R} \quad \text{on } S_t,$$

$$F^R = G + \nabla u^R \quad \text{in } V, \tag{4.9}$$

$$u^R = u^{\#R} \quad \text{on } S_u,$$

$$\frac{\partial}{\partial \tau}(S^R) = \frac{\partial}{\partial \tau}(u^R) = 0;$$

(ii) $$\bar{T} = \bar{H} F^R S^R + \bar{F} F^R \bar{S},$$

$$\mathrm{Div}\,(\bar{T}) = 0 \quad \text{in } V,$$

$$\bar{T} n = 0 \quad \text{on } S_t,$$

$$\bar{F} = G^R + \bar{H} \quad \text{in } V,$$

$$\bar{H} = \nabla_R \bar{u} \quad \text{in } V, \tag{4.10}$$

$$\bar{u} = 0 \quad \text{on } S_u,$$

$$\bar{E} = \frac{1}{2}(F^R)^T(\bar{H} + \bar{H}^T + \bar{H}^T\bar{H})F^R \quad \text{in } V,$$

$$\frac{\partial}{\partial \tau}(\bar{S}) = \frac{\partial}{\partial \tau}(\bar{u}) = 0;$$

(iii) $$\mathcal{F}[\overset{\circ}{s}{}^R + \overset{\circ}{s}(\tau) + \bar{s}, \overset{\circ}{e}{}^R + \overset{\circ}{e} + \bar{e}] < 0 \tag{4.11}$$

for all $\tau > \tau^R$ in V, then the body $\mathcal{B}$ will shake down under given programme of loading.

P r o o f: The proof is exactly the same as in the preceding theorem. The quadratic form Π in this case is defined by

$$\Pi(\tau) = \frac{1}{2}\int_V\left\{\mathbb{G}^{-1}[\tilde{\mathbf{s}}]\cdot\tilde{\mathbf{s}} + (\tilde{\mathbf{F}}\mathbf{F}^R)^T\tilde{\mathbf{F}}\mathbf{F}^R\cdot\mathbf{s}^R\right\}dV =$$

(4.12)
$$= \frac{1}{2}\int_V\left\{\mathbb{D}^{-1}[\tilde{\mathbf{S}}]\cdot\tilde{\mathbf{S}} + (\tilde{\mathbf{F}}\mathbf{F}^R)^T\tilde{\mathbf{F}}\mathbf{F}^R\cdot\mathbf{S}^R\right\}dV +$$

$$+ \frac{1}{2}\int_V\left\{\mathbb{Z}^{-1}[\tilde{\sigma}]\cdot\tilde{\sigma} + (\tilde{\mathbf{F}}\mathbf{F}^R)^T\tilde{\mathbf{F}}\mathbf{F}^R\cdot\sigma^R\right\}dV,$$

where

(4.13)
$$\tilde{\mathbf{s}} \equiv (\tilde{\mathbf{S}},\tilde{\sigma}) = \hat{\mathbf{s}} - \bar{\mathbf{s}},$$
$$\tilde{\mathbf{S}} = \hat{\mathbf{S}} - \bar{\mathbf{S}},$$
$$\tilde{\sigma} = \hat{\sigma} - \bar{\sigma},$$
$$\tilde{\mathbf{F}} = \mathbf{G}^R + \hat{\mathbf{H}} - \bar{\mathbf{H}},$$
$$\hat{\mathbf{H}} = \nabla_R\hat{\mathbf{u}}, \qquad \bar{\mathbf{H}} = \nabla_R\bar{\mathbf{u}}.$$

It is assumed in (4.12) that the generalized elastic strains $\mathbf{e}$ are related to $\mathbf{s}$ by

(4.14)
$$\mathbf{s} = \mathbb{G}\cdot\mathbf{e},$$

where

(4.15)
$$\mathbb{G} = (\mathbb{D},\mathbb{Z}).$$

Here $\mathbb{D}$ is the tensor of elastic moduli and the tensor $\mathbb{Z}$ represents a tensor of internal elastic moduli with the symmetries $\mathbb{Z} = \mathbb{Z}^T$. For ideal plastic material $\mathbb{Z}$ is equal to zero.

Next, proceeding as in the theorem 2.1. we obtain

(4.16)
$$\frac{d\Pi}{d\tau} = \int_V(\hat{\mathbf{S}} - \bar{\mathbf{S}})\cdot\frac{d}{d\tau}(\hat{\mathbf{E}}^P)dV + \int_V(\hat{\mathbf{T}} - \bar{\mathbf{T}})\cdot\frac{d}{d\tau}(\hat{\mathbf{H}})\mathbf{F}^R dV +$$
$$+ \int_V(\hat{\sigma} - \bar{\sigma})\cdot\frac{d}{d\tau}(\hat{\epsilon}^P)dV + \int_V(\hat{\sigma}_o - \bar{\sigma}_o)\cdot\frac{d}{d\tau}(\hat{\mathbf{H}})\mathbf{F}^R dV,$$

where

$$(4.17)\qquad \tilde{\sigma}_o \equiv \hat{\sigma}_o - \bar{\sigma}_o = \tilde{\mathbf{F}}\mathbf{F}^R\tilde{\sigma}.$$

In relation to the Gauss' theorem equation (4.16) reduces to

$$(4.18)\qquad \frac{d\Pi}{d\tau} = -\int_V (\hat{S} - \bar{S})\cdot\frac{d}{d\tau}(\hat{E}^P)dV - \int_V (\hat{\sigma} - \bar{\sigma})\cdot\frac{d}{d\tau}(\hat{\epsilon}^P)dV = = -\int_V (\hat{\mathbf{s}} - \bar{\mathbf{s}})\cdot\frac{d}{d\tau}(\hat{\mathbf{e}}^P)dV.$$

Inequality (4.11) assures that

$$(4.19)\qquad \overset{\circ}{\mathbf{s}}{}^R + \overset{\circ}{\mathbf{s}} + \bar{\mathbf{s}} = \mathbf{s}^s,$$

being a "safe" state of stress fulfills the inequality (4.2). Hence, using (4.19) we get

$$(4.20)\qquad \frac{d\Pi}{d\tau} = -\int_V (\mathbf{s} - \mathbf{s}^s)\cdot\frac{d}{d\tau}(\hat{\epsilon}^P)dV,$$

where $\mathbf{s}$ denotes the true state of total stresses and $d/d\tau(\hat{\epsilon}^P)$ is the true state of residual plastic strain rates. In view of inequality (4.3) $d\Pi/d\tau$ is always non-positive and moreover, $d\Pi/d\tau$ is equal to zero for $d/d\tau(\hat{\epsilon}^P) = \mathbf{0}$.

Since Π is non-negative one can conclude that

$$\Pi(0) \geq \Pi(\tau) \quad \text{for } \tau > 0$$

and

$$\frac{d\Pi}{d\tau} \longrightarrow 0, \quad \Pi(\tau) \longrightarrow \text{const} \quad \text{for } \tau \longrightarrow \infty.$$

Now, we prove the boundedness of the total energy dissipated. Let $\mu > 1$ be the safety factor of the structure against failure due to non-shakedown. Then the condition (4.2) holds

$$(4.21)\qquad F[\mu(\overset{\circ}{\mathbf{s}}{}^R + \overset{\circ}{\mathbf{s}} + \bar{\mathbf{s}}), \overset{\circ}{\mathbf{e}}{}^R + \overset{\circ}{\mathbf{e}} + \bar{\mathbf{e}}] = F(\mu\mathbf{s}^s, \overset{\circ}{\mathbf{e}}{}^R + \overset{\circ}{\mathbf{e}} + \bar{\mathbf{e}}) \leq 0.$$

for the stress state $\mu(\overset{\circ}{\mathbf{s}}{}^R + \overset{\circ}{\mathbf{s}} + \bar{\mathbf{s}})$. From the convexity of the yield surface and the validity of the normality rule follows

$$(4.22)\qquad (\mathbf{s} - \mu\mathbf{s}^s)\cdot\frac{d}{d\tau}(\hat{\epsilon}^P) \geq 0.$$

Transformation of equation (4.22) and integration over V gives

(4.23) $$\mathbf{s}\cdot\frac{d}{d\tau}(\hat{\epsilon}^P) \le \frac{\mu}{\mu-1}\,(\mathbf{s} - \mathbf{s}^s)\cdot\frac{d}{d\tau}(\hat{\epsilon}^P).$$

After integrating (4.23), first over the body volume, next with respect to time, we finally get

(4.24) $$\int_0^\tau\int_V \mathbf{s}\cdot\frac{d}{d\tau}(\hat{\epsilon}^P)dVd\tau \le \frac{\mu}{\mu-1}\,\Pi(0),$$

and this proves our theorem. □

Conclusions

Collecting our results we can formulate conditions under which they agreed with the conditions given by Gross-Weege (cf. GROSS-WEEGE [12:1989]). If the additional following assumptions hold:

(i) it is restricted to linear relations between variables;

(ii) it is restricted to elastic-perfectly plastic material;

(iii) all metric tensors in the specified configurations are identified, i.e. $\mathbf{G} \equiv \mathbf{G}^R \equiv \mathring{\mathbf{G}} \equiv \mathbf{g} \equiv \mathfrak{G} \equiv \mathfrak{G}^R \equiv \mathfrak{G}^r \equiv \mathring{\mathfrak{G}} \equiv \mathbf{1}$;

(iv) it is assumed the deformation process χ^r to be an infinitesimal one,

then considerations presented agree with the Gross-Weege's ones.

Moreover, the following special cases can be cited:

1) The shakedown theorem 2.1. reduces to the Gross-Weege's shakedown theorem (GROSS-WEEGE [12:1989]) if the above assumptions hold;

2) If (1) holds and moreover we additionally neglect all mixed terms, i.e. between quantities describing state $^R C$ and those ones describing the residual or the shakedown state, respectively, we get the extension of Melan's theorem (WEICHERT [8:1986]);

3) If in addition to the simiplifications in (2) we put $\mathbf{S}^R = \mathbf{0}$, $\mathbf{F}^R = \mathbf{G}$, we get the original geometrically linear Melan's theorem.

4) If (2) holds and it is used the concept of "generalized standard material" (RAFALSKI [29:1977]), we obtain the extended shakedown theorem (WEICHERT [30:1987, 11:1988]);

5) Under condition $\mathbf{S}^R = \mathbf{0}$, it is possible to use $\mathbf{F}^R$ to consider the influence of initial deformations or imperfections on the shakedown of the structure.

References

1. KÖNIG, J.A. [1987]. *Shakedown of Elastic-Plastic Structures*, PWN - Polish Scientific Publishers, Warsaw.

2. KÖNIG, J.A. [1980]. *On stability of the incremental collapse process*, Arch. Inz. Lad., Vol.**26**, Z.**1**, pp.219-229.

3. SAWCZUK, A. [1967]. *On incremental collapse of shells under cyclic loading*, IUTAM Symp. Theory of Thin Shells, Copenhagen 1967, Springer Verlag, Berlin 1969, pp.328-340.

4. KOITER, W.T. [1960]. *General theorems for elastic-plastic solids*, in: Progress in Solid Mechanics, J.N.Sneddon and R.Hill (eds), North-Holland, Amsterdam, Vol.**1**, pp.165-221.

5. MAIER, G. [1973]. *A shakedown matrix theory allowing for work-hardening and second-order geometric effects*, in: Foundations in Plasticity, A.Sawczuk (ed.), North-Holland, Amsterdam, pp.417-433.

6. MELAN, E. [1936]. *Theorie statisch unbestimmter Systeme aus ideal-plastischem Baustoff*, Sitzungsbericht der Akad. D. Wiss. (Wien), Abt.IIa, **145**, pp.195-218.

7. NGUYEN, Q.S. and GARY, G. [1983]. *Flambage par deformations plastiques accumulées sous charge cyclique additonelle*, J. Mécan. Théor. Appl., Vol.**2**, pp.351-373.

8. WEICHERT, D. [1986]. *On the influence of geometrical nonlinearities on the shakedown of elastic-plastic structures*, Int. J. Plasticity, Vol.**2**, pp.135-148.

9. WEICHERT, D. [1986]. *Elastic-plastic structures under variable loads at small strains and moderate rotations*, in: Finite Rotations in Structural Mechanics, W.Pietraszkiewicz (ed.), Lecture Notes in Engng, Springer, Berlin, **19**, pp.343-356.

10. WEICHERT, D. and GROSS-WEEGE, J. [1988]. *The numerical assessment of elastic-plastic sheets under variable mechanical and thermal loads using a simplified two-surface yield condition*, Int. J. Mech. Sci., Vol.**30**, pp.757-767.

11. WEICHERT, D. [1988]. *Shakedown of shell-like structures allowing for certain geometrical non-linearities*, Arch. Mech. (in press).

12. GROSS-WEGGE, J. [1989]. *A unified formulation of statical shakedown criteria for geometrically nonlinear problem* (to appear in Int. J. Plasticity).

13. HILL, R. [1959]. *Some basic principles in the mechanics of solids without a natural time*, J. Mech. Phys. Solids, Vol.**7**, pp.209-225.

14. KOITER, W.T. [1956]. *A new general theorem on shake-down of elastic-plastic structures*, Proc. Koninkl. Ned. Akad. Wet., **B59**, pp.23-34.

15. GOKHFELD, D.A. [1966]. *Some problems of shakedown of plates and shells*, in: Proc. of VII Union Conf. on Plates and Shells, Baku, Press 'Nauka', pp.284-291 (in Russian).

16. HILL, R. [1978]. *Aspects of invariance in solids mechanics*, Adv. Appl. Mech., Vol.**18**, pp.1-75.

17. CASEY, J. and NAGHDI, P.M. [1980]. *A remark on the use of the decomposition* $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ *in plasticity*, ASME J. Appl. Mech., Vol.**47**, p.672.

18. CASEY, J. [1985]. *Approximate kinematical relations in plasticity*, Int. J. Solids Struct., Vol.**21**, pp.671-681.

19. HILL, R. and RICE, J.R. [1973]. *Elastic potentials and the structure of inelastic constitutive laws*, SIAM J. Appl. Math., Vol.**25**, p.448.

20. MANDEL, J. [1976]. *Adaptation d'une structure plastique ecrouissable et approximations*, Mech. Res. Comm., Vol.**3**, pp.483-488.

21. SHRIVASTAVA, H.P., MROZ, Z. and DUBEY, R.N. [1973]. *Yield criterion and the hardening rule for a plastic body*, ZAMM, Vol.**53**, pp.625-633.

22. GREEN, A.E. and NAGHDI, P.M. [1971]. *Some remarks on elastic-plastic deformation at finite strain*, Int. J. Engng Sci., Vol. **9**, pp. 1219-1229.

23. SIMO, J.C. and ORTIZ, M. [1985]. *A unified approach to finite deformations elastoplastic analysis based on the use of hyperelastic constitutive equations*, Comput. Meths. Appl. Mech. Engng, Vol. **49**, pp. 221-245.

24. HILL, R. [1950]. *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford.

25. GAO, Y. and WIERZBICKI, T. [1989]. *Bounding theorem in finite plasticity with hardening effect*, Quart. Appl. Maths, Vol. **47**, pp. 395-403.

26. LEE, E.H. [1969]. *Elastic-plastic deformation at finite strains*, ASME J. Appl. Mech., Vol. **36**, pp. 1-6.

27. GREEN, A.E. and NAGHDI, P.M. [1965]. *A general theory of an elastic-plastic continuum*, Arch. Rat. Mech. Anal., Vol. **18**, pp. 251-281.

28. DRUCKER, D.C. [1951]. *A more fundamental approach to plastic stress-strain relations*, Proc. 1st U.S. Nat. Congr. Appl. Mech., ASME, Chicago, pp. 487-491.

29. RAFALSKI, P. [1977]. *Solution of the elastic-viscoplastic boundary value problem*, Int. J. Engng Sci., Vol. **15**, pp. 193-197.

30. WEICHERT, D. [1987]. *An improved shakedown theory for plates and shells allowing for a certain class of geometrical nonlinearities*, 24th Annual Technical Meeting Society of Engineering Science, Sept. 20-23, 1987, Univ. of Utah, Salt Lake City, Utah.

31. SIMO, J.C. [1988]. *A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation*, Comp. Meths Appl. Mech. Engng, Vol. **66**, pp. 199-219.

32. MARSDEN, J.E. and HUGHES, T.J.R. [1983]. *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs.

33. HALPHEN, B. and NGUYEN QUOC SON [1975]. *Sur les materiaux standards generalises*, J. de Mec., Vol. **14**, pp. 39-63.

34. PRAGER, W. [1959]. *An introduction to plasticity*, Reading, London, Addison, Wesley.

35. MELAN, E. [1938]. *Zur Plastizität des räumlichen Kontinuums*, Ing. Archive Vol. **9**, pp. 116-126.

36. MROZ, Z. [1967]. *On the description of anisotropic work-hardening*, J. Mech. Phys. Solids Vol. **15**, pp. 163-175.

37. KRIEG, R.D. [1975]. *A practical two surface plasticity theory*, J. Appl. Mech. (Trans. ASME), Vol. **47**, pp. 641-646.

38. WEICHERT, D. [1987]. *Zum Problem geometrischer Nichtlinearitäten in der Plastitätstheorie*, Mitt. Inst. für Mech., **53**, Ruhr-Universität Bochum.

39. PONTER, A.R.S. [1975]. *A general shakedown theorem for elastic/plastic bodies with work-hardening*, Proc. SMIRT-3, paper L5/2.

40. GOKHFELD, D.A. and CHERNIAVSKI, O.T. [1980]. *Limit analysis of structures at thermal cycling*, Noordhoff, Leyden.

41. LUBLINER, J. [1984]. *A maximum-dissipation principle in generalized plasticity*, ZAMM, Vol. **52**, pp. 225-237.

42. PHILLIPS, A. and WANG, G.J. [1975]. *An analytical study of an experimentally verified hardening rule*, J. Appl. Mech., (Trans. ASME), Vol. **42**, p. 375.

43. KÖNIG, J.A. and SIEMASZKO, A., *Strainhardening effects in shakedown processes*, Ing. Arch., Vol. **58**, pp. 58-66.

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