Mitteilungen aus dem Institut für Mechanik

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Mechanics of Irregular Shell Structures

Heft Nr. 95

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MITTEILUNGEN AUS DEM INSTITUT FÜR MECHANIK NR. 95

1994

Herausgeber:
Institut für Mechanik der Ruhr-Universität Bochum
Schriftenreihe
Universitätsstr. 150
44780 Bochum

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Introduction

Shells and shell structures. In a common sense, the term shell refers to specific shapes of artificial (man-made) structures or to shapes in nature. However, shapes which feature shells are so numerous that it is impossible to lay down any general rules for their rigorous characterization. Roughly speaking, shells are curved surface-like structures which, when acted upon by external loads and its own weight, can develop load-carrying abilities in shear in addition to extension (compression) and bending. When suitably designed, each factor can dominate the other two. The stone and brick Roman vaults (such as shown in Fig. 1) can well be taken as typical examples of the oldest, thick structures featuring shell roofs, which carry its own weight and imposed loads mainly by compressive membrane stresses.

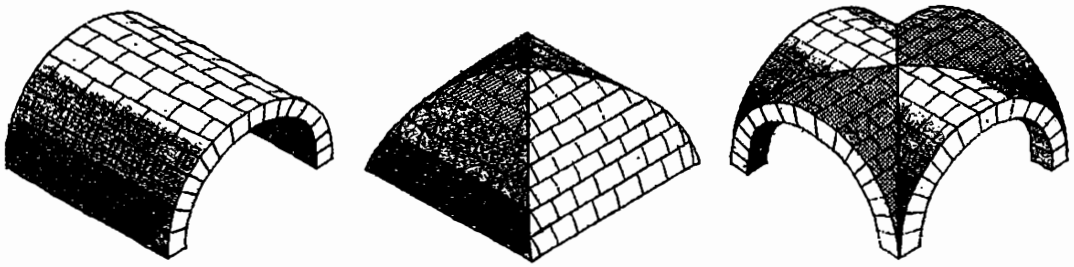

Fig. 1

Now-a-days man-made shells are utilized in almost every engineering field: aerospace and medical technology, automotive and shipbuilding industries, mechanical and civil engineering or chemical, nuclear reactor and electronic technologies. Aircraft fuselages, spacecraft and missile boosters, pipes and pressure vessels, liquid storage tanks, cooling towers and domes, as well as many off-shore installations may be cited as examples of shell structures.[1] The primary aim in designing artificial shells is to minimize the materials to be used for maximum strength (strength through form as opposed to strength through mass). Modern constructions of shells often combine strength and enclosing proportions through rings (circumferential) and stringers (longitudinal) stiffeners (Fig. 2

[1] An interesting account of shell examples and the development of shell technology can be found in SECHLER [1974].

shows a typical example) and new material technology, which makes it possible to construct even very thin, light weight shells that can span large distances and support large loads.

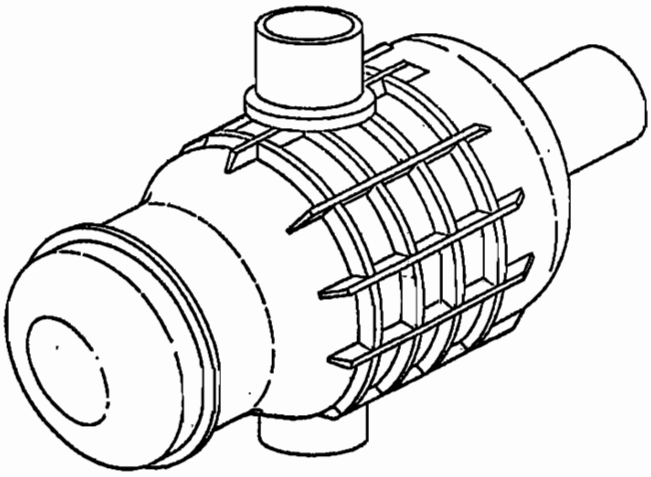

Fig. 2

There is also an abundance of shell-like shapes in nature. Shells of eggs, nuts, snails, turtles, skulls and hollow horns are just a few commonplace examples. All these shells express an unbelievable richness of surface forms as well as strength to bending. The various spiral geometries of seashells shown in Fig. 3 are typical examples.[2]

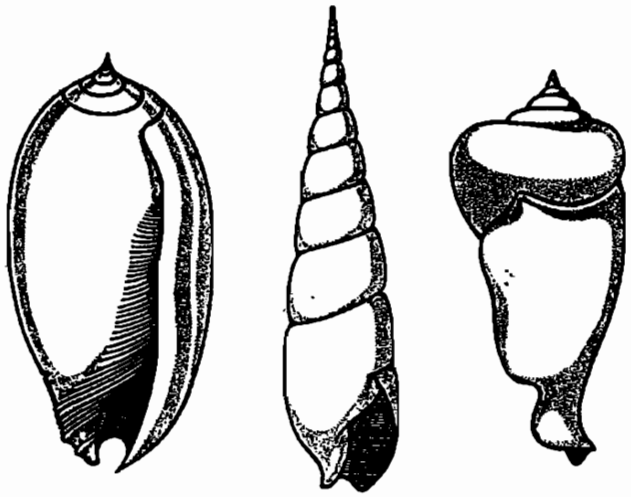

Fig. 3

[2] See ILLERT [1989,1990].

A special class of shells constitute the membranes. However, it must be distinguished between the membrane response of an idealized shell structure and the response of flexible membranes, which may be called soft shells. Flexible membranes cannot support compression or couples. They respond to external force action by pure tension by adjusting their form in a manner similar to a single cable system. Pneumatic structures are typical examples (Fig. 4). Historical use of membranes in engineering structures may be traced back to sails and tents. Kites, parachutes, balloons, and other flaying structures are of more recent use. The red blood cells or sea urchin eggs are just two examples of flexible membranes occurring in nature.

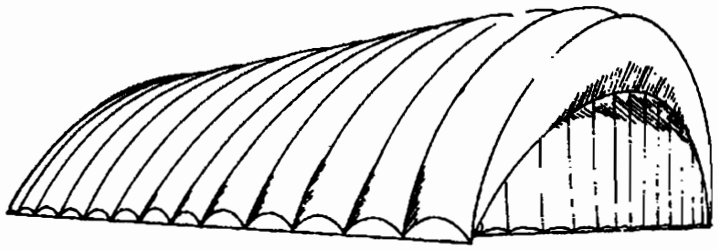

Fig. 4

The few examples presented above cannot be considered as a representative list of shell shapes. Our guideline in their selection was rather to give an imagination of shapes that must be taken into account in any attempt to develop a general shell theory.

From the point of view of shell theory, man-made and natural shells can be classified in many ways. The following factors are of particular importance:

- geometry, including characteristic dimensions and their relative proportions,
- construction and composition,
- mechanical properties of materials,
- supports and external loads.

With respect to the geometry, a shell is generally regarded as a material body having two basic identifying features: its reference surface and its thickness. The reference surface and thickness can thus be used to classify various shells. Thus, shells may be classified by the same criteria as mathematical surfaces:

- singly or doubly curved surfaces,
- closed (complete) or open (incomplete) surfaces,
- simply connected or multiply-connected surfaces.

Other important criteria are:

- rate of variation of the curvature,
- discontinuities in slopes and curvatures.

Singly curved surfaces, like cylinders and cones, are developable. They can be developed into a plane without stretching, shrinking, or tearing. Most shells occurring in nature are doubly curved, they are non-developable. Shells of double curvature are among the most efficient of known structural forms, they do not tend to flatten out under applied loads. This explains their superior performance. A further classification of shell surfaces is possible on the basis of the Gauss curvature. A subclassification is possible depending upon whether a shell surface is a translational, a ruled, or a revolution one. It is clear, however, that various classifications refer to parts of the real shell structures rather than to the whole structures, which are typically composed of some number of identical or different geometries (Fig. 5).

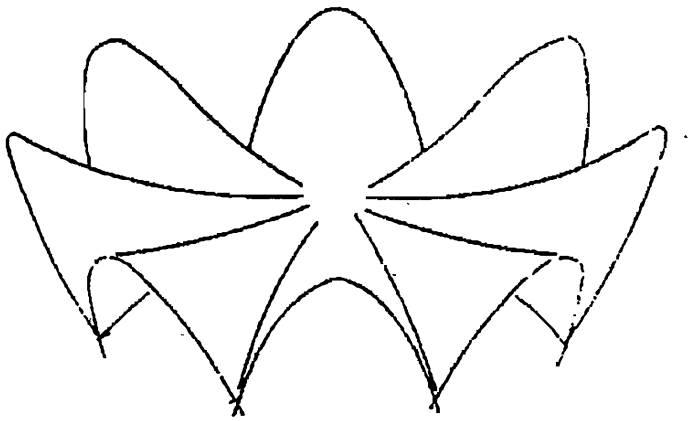

Fig. 5

With respect to the thickness shells can be classified as:

- thin or thick shells,
- shells of uniform, smoothly varying or abruptly changing thickness.

As a rule one assumes that the shell thickness must be small in comparison with the remaining two characteristic dimensions. A characteristic behavior of thin shells is the intricate interaction between the membrane action and the relatively small bending stiffness. However, thick shells are equally important structures in engineering practice.

To predict the behavior of shell structures as well as to be able to construct them, not only their geometry must be known, but also the type of construction and the mechanical properties of the materials they are constructed of. Thus, independently of their geometry, a few classes of shells can be distinguished:

- shells made of a single isotropic and homogenous material,
- anisotropic and reinforced shells,
- composite-material and layered shells.

Sandwich shells, which can be regarded as special cases of multilayered shells, typically consist of a soft core that resists very little to bending or stretching (such as a honeycomb core) and thin facing sheets with high bending and stretching stiffness. The behavior of anisotropic and layered shells differs substantially form the behavior of isotropic and homogenous shells when subjected to the same load and support conditions. A few of the many applications of composite-material shells are aircraft and helicopter fuselage sections, wing leading edges or tubular drive shafting.

Different classifications of shells can be based on the loads, constraints and supports. The boundaries of shells are often restrained elastically, which means that the edge support permit displacements and rotations depending upon the edge reaction. Free edges are the simplest case, they are not restricted at all. The so-called fixed or built-in edges are totally restricted against any deformation. In reality, shell edges are nearly always restrained elastically. For example, a dome with a built-in ring along the boundary has restricted deflections and rotations. This restriction is imposed by a ring, which itself is deformed under loading. The fixed edges are no more no less than a convenient approximation which simplifies the analysis.

Shells can be subjected to many types of external loading. The most obvious loads are the gravity action caused by the weight of the structure and snow. Lateral forces are exerted upon structures by wind and earthquakes, as well as earth and hydrostatic pressure. The loads may be permanent, such as dead loads, or temporary such as live loads. The duration of live loads is important especially for the analysis of the deflection. Generally, a few classes of forces can be distinguished:

- volume or surface forces,
- static (time independent) or dynamic forces,
- dead or configuration dependent forces.

In contrast to dead forces, live forces are variable; they change with time in magnitude and location.

Various specific features of shells as discussed above provide convenient means for classification of shell problems. However, it must be stressed that such classifications are based on idealizations, which in many cases cannot be uncritically accepted. For example, many shell shapes cannot be described as surfaces in the usual sense. As a typical example we can pick up the seashell shapes shown in Fig. 3 and the shell-like shape presented in Fig. 6, which are certainly not surfaces in the sense of differential geometry nor even topological surfaces. This is also true for many shell shapes of engineering importance.

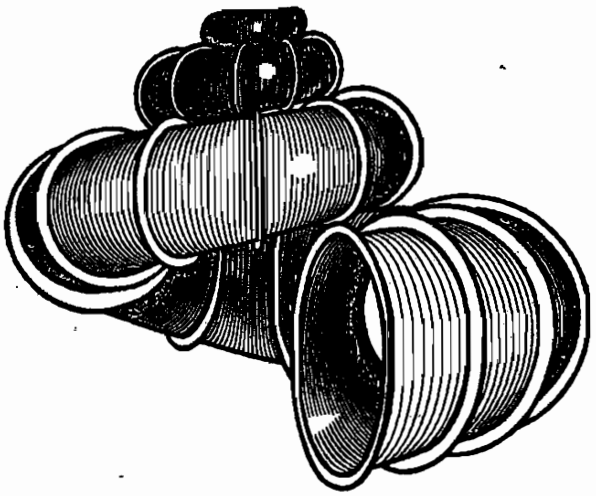

Fig. 6

In general, shells may have any of a great variety of geometric configurations and they may contain various kinds of discontinuities, such as holes, bosses, changes in thickness, etc. For many shell structures discontinuities in geometry, stiffness and loading may be dominant factors. For example, shell branching (three or more shell segments intersecting at a common juncture) is an important factor in the buckling analysis. Moreover, many engineering structures are not really shells in the common sense but rather they are composed of rod-like and plate/shell-like segments interconnected pointwise at joints and along junctions in a widely varying manner to form, in overall, fairly complex structures. Folded plate structures, multicell box girders, stiffened prismatic or non-prismatic shell structures, shell structures sustained by columns and stiffened with beams are typical examples. A rather simple example of such a structure is shown in Fig. 7.

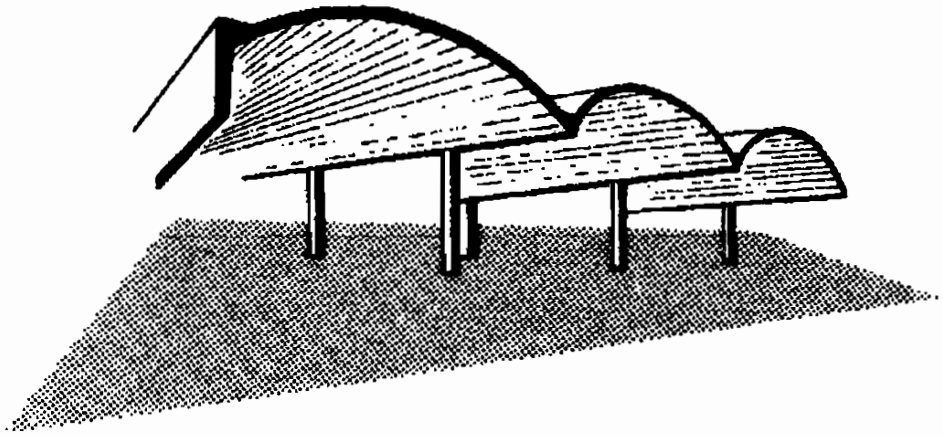

Fig. 7

Remarks on shell theory. There is no doubt that since the time, shell-like structures were built and constructed, some empirical rules were known concerning their strength. However, it was not until the beginning of the twenties century that the theory of shells started to influence the engineering practice. Historically, the interest in the formulation of shell theories was motivated by the desire to understand the basic parameters of vibrating drums, bells and plates. The earliest tentative efforts to formulate the problems in mathematical form were undertaken, when the theory of strings and beams was advanced but before the discovery of the general (linear) equations of elasticity.[3] But even after the general equations of elasticity had been formulated little advance has been made to treat shell problems. Only the special case of plates was formulated by Kirchhoff (1850) in a form, which even today is generally accepted. The pioneering work of Aron (1874) may well be considered as marking the origin of a general theory of thin elastic shells, and the classical paper of Love (1888) influenced for many decades the development of the shell theory.[4]

The possibility of formally extending the classical notion of a continuous body as a collection not only of points but also of directions associated with points, suggested by Duhem, was adopted by the brother Cosserat[5] to develop a general

[3] See TRUESDELL [1960] on a historical development of the theory of strings, beams, bars and rods.

[4] See LOVE [1927] for earlier developments of plate and shell theories and references to the original papers.

[5] COSSERAT [1907].

theory of elastic shells. While the work of the Cosserats went unnoticed for a half of the century, Love's pioneering works stimulated considerably during the twenty's century research efforts on the shell theory in several directions. One direction has been concerned with the nonlinear theory, another one with the derivation of theories for thin and thick shells from the three-dimensional elasticity without the Kirchhoff-Love hypothesis.[6]

Modern shell literature tends to make a clear distinction between the two possible methods to formulate a general shell theory: the direct approach and the derived approach.[7] Conceptually, these two approaches are very different as sketched in Fig. 8.

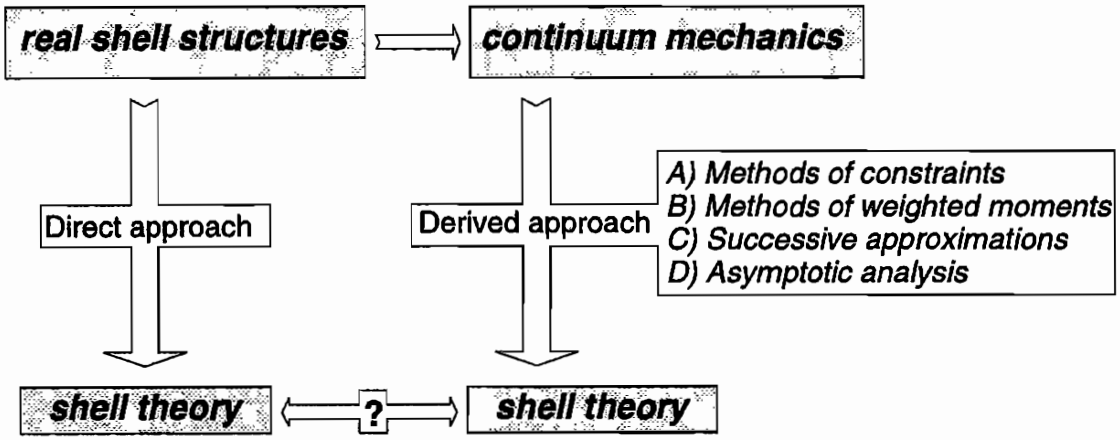

Fig. 8

Within the direct approach a shell is regarded from the outset as an intrinsically two-dimensional continuum governed by its own laws, which in principle can be independent of the laws underlying the classical continuum mechanics. As such the direct approach permits the formulation of a shell theory without recourse to the three-dimensional continuum mechanics. This approach has been invented in the pioneering works of the Cosserat brothers (cited above), who introduced the concept of a geometric surface, to each point of which a rigid triad of vectors is attached. Ericksen and Trusedell[8] generalized this concept to allow for three linearly independent vectors, and they called them directors. A surface with a

[6] Among the papers, where a detailed account of classical shell theories can be found, we cite here only NAGHDI [1963], KOITER [1966], KOITER AND SIMMONDS [1972], PIETRASZKIEWICZ [1979, 1989] and STUMPF [1986].
[7] See e.g. NAGHDI [1972,1980].
[8] ERICKSEN AND TRUSEDELL [1958].

single vector was developed by Green et al.[9] The two-dimensional continuum defined in such a manner is commonly called a Cosserat surface. When a particular model is adopted, the complete formulation of the shell theory proceeds parallelly to the methodology of continuum mechanics. Thus one needs to postulate suitable two-dimensional balance laws (physical principles) and suitable constitutive laws specifying the physical response to applied generalized forces. In this respect the shell equations are not regarded as approximations to the three-dimensional equations of classical continuum mechanics but rather as an independent theory to predict some of the main properties of the three-dimensional shell-like bodies. In this sense the shell theory formulated within the direct approach is exact. But it should be clear that exact means here exact by definition.

Within the derived approach a shell is regarded as a conventional material body, whose motion and deformation is governed by laws of classical continuum mechanics. The main objective of the shell theory is then to develop a systematic procedure which enables to reduce the problem for three spatial variables and time to one having the coordinates of a certain surface and time as the only independent variables. This goal can be reached in substantially varying manner, which essentially can be grouped into four methods (Fig. 8).

The traditional way of deriving shell equations from the three-dimensional theory, having its origin in Kirchhoff's plate theory and Love's shell theory, starts with a set of kinematic hypothesis with respect to the variation of the deformation across the thickness. This approach may be formalized and based on a rigorous mathematical ground to the extent which makes the derivation of the shell equation a routine task. The basic idea is quite simple. Assuming that a three-dimensional motion of the shell is subjected to appropriate kinematic constraints, the current position vector can be expressed as a given function of a finite set of functions having the parameters of the reference surface as the only independent variables (generalized displacements). The basic field equations and the corresponding boundary conditions can then be obtained from the three-dimensional principle of virtual work by a standard variational procedure. Also the two-dimensional constitutive laws can be derived, at least in principle, from the corresponding three-dimensional constitutive equations by making use of the adopted hypothesis. This methodology (method of constraints) has been exposed with all rigor by Antman.[10] In the latter paper, the authors draw attention to one fact not noticed in all earlier papers on shell theory: the kinematical constraints

[9] GREEN, NAGHDI AND WAINWRIGHT [1965].
[10] ANTMAN [1976]. See also ANTMAN AND MARLOW [1991].

imposed on the motion of the body induce associated reactive forces, and these forces must appear in the resulting shell equations. In this way they actually demonstrated that all shell theories formulated earlier by applying the method of hypothesis are defective.

Another usual way to obtain the two-dimensional shell equations is based on the integration of the three-dimensional equations of continuum mechanics through the shell thickness. Higher order equations can be obtained using weighted functions. This methodology can also be formalized as a method of weighted moments. The method of successive approximations originates from Cauchy and Poisson. In general, it is based on a series expansion of displacements and stresses with respect to the thickness coordinate. Power series, Legendre polynomials, and trigonometric functions have been employed. Within the method of asymptotic integration appropriate length scales are introduced in the three-dimensional equations of continuum mechanics for the displacements, strains and stresses together with parametric (asymptotic) expansions of these quantities. In this manner the three-dimensional equations are reduced to recursive sets of two-dimensional equations governing the interior and edge zone responses of the shell. The edge zone or boundary layer is produced by self-equilibrated (in the thickness direction) boundary stresses. We omit here the reference to the relevant literature, since these methods to derive basic shell equations will not be our concern in this work. We rather note that non of the various methods to formulate a general shell theory has a unique status as preferable over others. All of them suffer from one essential drawback: the resulting shell theory is depending in one or another way on the method of its formulation. Moreover, it is nearly the rule that the formulation of the shell governing equations is based on strong regularity assumptions, and it essentially relays on the use of the coordinate description.[11] Specifically, one typically assumes, implicitly rather than clearly stated, that:

- A shell is geometrically represented by a smooth regular surface (called a shell reference surface) in the sense of classical differential geometry.
- The reference surface admits a global, regular parametrization.
- A deformation of the reference surface is described by an injective, globally invertible and smoothly differentiable map.
- All static and kinematic variables are smoothly differentiable fields over the shell reference surface.
- Material properties vary smoothly over the shell.

[11] This is exemplified by the well-known treatise of NAGHDI [1972].

On the other side, in papers devoted to the problem of the existence of solutions of shell boundary value problems (usually linear) one considers a class of generalized functions. In such papers no special attention is paid to the fact that the derivation of the shell equations presented in other papers was based on much more restrictive regularity assumptions. Moreover, shell equations are usually also derived starting with a number of simplifying hypotheses about the displacement and stress distribution across the shell thickness without paying any attention to the question, if such hypotheses are really needed. This situation is in sharp contrast with standards set in modern continuum mechanics, where regularity assumptions, conceptual clarity and the mathematical rigor of a truly coordinate free formulation are the main issues.[12] This unsatisfactory situation in shell theory is well recognized and many aspects have been clarified, but there still remains much to do.[13]

Goal of the present work. In this work we shall be concerned with a general shell theory formulated within a rational and convincing approach, which has been set down by Libai and Simmonds[14] and subsequently worked out in our previous works.[15] The main virtues of this approach, which substantially differs from the methods discussed above, are:

- It draws a clear destinction between the general physical laws, which are independent of specific material properties and the specific construction of the shell, from the constitutive relations, which define particular classes of shells.
- The mechanical balance laws for shells are derived by direct specification of the laws of continuum mechanics for a shell-like body with no simplifying hypothesis and/or ad hoc postulate of whatsoever nature. In effect, various simplifying assumptions underlying the classical derivation of basic shell governing equations appear to be superfluous.
- The kinematics of the shell is the outcome of the formulation and not a basic assumption or a postulate of the theory as it is the case in other formulations of shell theory.[16]
- Displacements, rotations and strains are not restricted in any way as to their magnitude. There is also no thinness assumption.

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[12] See e.g. NOLL AND VIRGA [1992].
[13] See e.g. GURTIN AND MURDOCH [1984], ANTMAN [1976].
[14] LIBAI AND SIMMONDS [1983] and SIMMONDS [1984].
[15] MAKOWSKI AND STUMPF [1988,1990].
[16] Within a purely two-dimensional formulation this was first clearly demonstrated by REISSNER [1974].

- Independent kinematical variables of the shell theory consist of the displacement field of a shell reference surface and a proper orthogonal tensor specifying independent mean rotations of the shell cross sections. This feature is particularly important from the computational point of view.[17]
- The only approximate character of the theory may appear in the form of two-dimensional constitutive equations.

In effect, the shell theory formulated within this approach, is rich enough to account for extension (compression), flexure, transverse shear and an arbitrary through-the-thickness deformation. The underlying kinematic model of this theory coincides with a geometric surface (a shell reference surface), each point of which has extra degrees of freedom of the rigid body. Thus this approach shows that the shell theory rigorously derived from the classical continuum mechanics (Cauchy continuum) has the structure of the two-dimensional Cosserat continuum. In other words, an exact reduction of the problem from three to two dimensions results in a richer continuum than the Cauchy continuum, but not richer than the classical Cosserat continuum. By implication, it turns out that the so-called higher order shell theories (such as multi-director shell theories) need not be more accurate.

The primary aim of this work is to present a refinement and substantial generalization of the approach outlined above mainly by an essential relaxation of various regularity assumptions with a special emphasis on non-smooth and irregular shell structures. In this way we are able to formulate a general shell theory with an arbitrary geometry. The main steps of the adopted approach are illustrated in Fig. 9. Its main feature is the fact that we take the dynamics rather than the kinematics as the underlying concept. Thus starting with the three-dimensional integral balance laws of mechanics, we obtain the resultant two-dimensional balance laws for shells in a logic and straightforward manner with no assumption of whatever nature. Under fairly weak regularity assumptions the local equilibrium equations and jump conditions are then obtained as a direct implication of the (generalized) surface divergence theorem. The same approach leads also to the exact static boundary conditions. In turn, the associated integral identity makes it possible to obtain the two-dimensional kinematical model for the shell and the work-conjugate strain measures. The same line of thinking also leads to the associated resultant boundary conditions for generalized displacements and the corresponding kinematic jump conditions. Finally, through an analysis of the stress power the general structure of the two-dimensional constitutive equations is established.

[17] See CHROSCIELEWSKI, MAKOWSKI AND STUMPF [1992,1994].

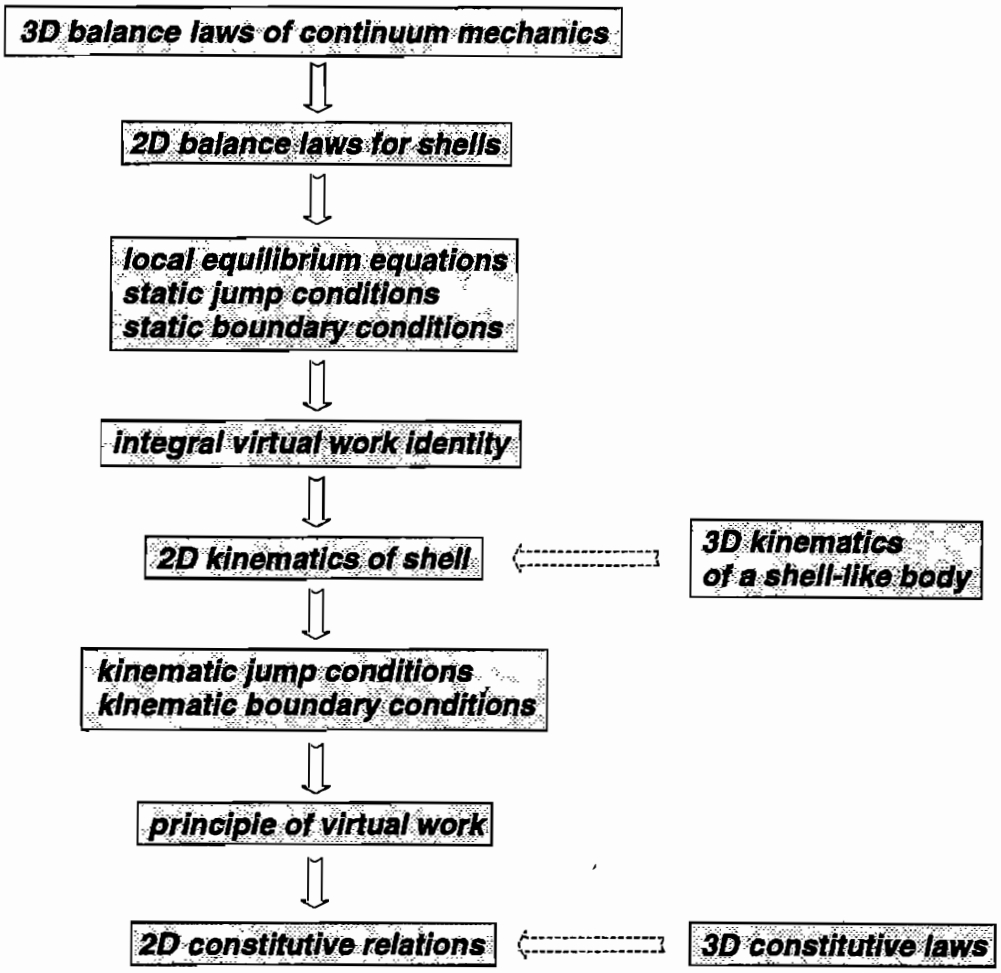

Fig. 9

One of our main concern in this work will be a theory of shells and shell structures of complex geometry (non-smooth and irregular shells). However, it must be stressed that in order to treat the problem of irregular shells with all possible rigor a proper setting would be the concept of rectifiable currents (measure-geometric surfaces) and not differential-geometric surfaces, the theory of Hausdorff measure and not the classical concept of area measure as well as the concept of functions of bounded variation and not classical differentiable functions. These are the concepts, which play an increasing role in modern continuum mechanics. However, this setting would take us into too many very subtle mathematical problems. Accordingly, we shall introduce a number of much stronger regularity assumptions: piecewise smooth surfaces and piecewise smoothly differentiable fields. Within these assumptions we are able to derive a complete set of shell

equations including side conditions (jump and boundary conditions) for shells which need not be smooth.

We now outline the contents of this work. In Chapt. I we provide a short summary of the key concepts of classical continuum mechanics. The presentation does not aim at a detailed exposition of the field. Our aim is rather to establish the notation and to emphasize those concepts, which are central for the subsequent formulation of the shell theory. We also make comments on various regularity assumptions.

Chapt. II contains the formulation of the resultant mechanical balance laws for smooth, regular and irregular shells as well as the derivation of the static field equations (equilibrium equations) and static side conditions (jump and boundary conditions). These results extend our earlier works through an explicit account for jump conditions across singular curves representing kinks, branches and multi-shell intersections.

In Chapt. III we formulate the general kinematics of the shell including suitable strain measures, strain rates, and kinematic jump and boundary conditions. It is also shown that there exist two entirely equivalent representations of the complete set of shell governing equations.

Finally, in Chapt. IV we discuss the general structure of shell constitutive relations. No attempt is made to derive the explicit form of the constitutive relations for special classes of shells. This problem is left for a separate study.

Notation and convention. Where feasible we shall adopt the notation and convention of modern continuum mechanics as well as coordinate free vector and tensor calculus.[18]

As a rule we use boldface lower case letters to denote vectors and vector-valued functions. Boldface capital letters will denote tensors and tensor-valued functions, and we write $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{u}\otimes\boldsymbol{v}$ for the usual inner product, cross product and tensor product of two vectors. However, $\boldsymbol{X}$ and $\boldsymbol{Y}$ will denote position vectors of material points in the undeformed configuration B of a body and of a shell reference surface M, respectively.

Throughout this paper we use standard summation convention that lower-case Greek indices have the range 1,2, lower-case Latin indices the range 1,2,3, and

[18] See e.g. NOLL AND VIRGA [1990], GURTIN AND MURDOCH [1975], MAN AND COHEN [1985].

that diagonally repeated indices are summed over their range. Also, if no confusion can arise, we make no distinction between functions and function values and, where convenient, we omit specific reference to the independent variables of a given function.

Index of notations

The following notations are consistently used throughout the work:

E	– three-dimensional Euclidean vector space, whose elements are called vectors,
$\mathcal{E}$	– three-dimensional Euclidean points space (physical space), whose translation space is E,
$B \subset \mathcal{E}$	– region in space $\mathcal{E}$ occupied by a three-dimensional material body in the undeformed configuration,
$\{\xi^A, A=1,2,3\}$	– curvilinear coordinates in B chosen in any convenient way,
$\boldsymbol{G}_A, \boldsymbol{G}^A$	– natural and reciprocal base vectors for the chosen curvilinear coordinates in B,
$M \subset B$	– undeformed shell reference surface,
$T_Y M$	– tangent space (the two-dimensional Euclidean subspace of $E = T_Y\mathcal{E}$) at each regular point $\boldsymbol{Y} \in M$ of the shell reference surface M,
$\{\xi^\Lambda, \Lambda=1,2\}$	– curvilinear coordinates on each smooth surface element of the shell reference surface M,
$\boldsymbol{A}_\Lambda, \boldsymbol{A}^\Lambda$	– natural and reciprocal base vectors for a surface coordinates on M.

Chapter I

Elements of classical continuum mechanics

1. Basic concepts

1.1 Preliminary remarks. In this chapter we provide a short summary of the key concepts of classical continuum mechanics. The presentation does not aim at a detailed exposition of the field, there is no need for that here.[1] The problems we shall consider in this chapter are classical. Our aim is rather to establish the notation and to emphasize those concepts, which are central for the subsequent formulation of the shell theory. We shall also make comments on various regularity assumptions, which too often are omitted in the literature.

Generally, continuum mechanics deals with the mechanical interactions between deforming bodies in motion. Motion and deformation of a body and the interactions between bodies must be formulated within the context of a space-time (event world). A space-time and a material body are thus the basic underlying concepts, on which the whole theory is founded. When these two concepts are mathematically characterized, the exposition of the complete theory proceeds in three main steps:[2]

1) *Kinematics* – description of motion and deformation of the body including the general theory of strains and strain rates.
2) *Dynamics* – statement of physical principles (laws of mechanics), which govern the motion and deformation of the body acted upon by forces including those which are specified at the boundary.
3) *Constitutive equations* – development of general constitutive equations, which describe specific mechanical properties of materials encountered in reality.

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[1] The famous treatise by TRUESDELL AND TOUPIN [1960] with its historical annotations still provides an excellent guide to continuum mechanics.
[2] Cf., e.g., TRUESDELL AND TOUPIN [1960], BOWEN [1989].

The study of the kinematics of the body is based on theorems of geometry and analysis. Basic theorems of kinematics are thus independent of the concept of forces and stresses and of physical principles governing the motion and deformation of the body. As such, they are applicable to all material bodies. Laws of mechanics are mathematical statements of physical principles. Like theorems of kinematics, they are applicable to all material bodies with arbitrary material behavior and undergoing arbitrary motion and deformation. Constitutive equations are mathematical statements and not physical principles. They represent the observed response of material bodies to external forces.

A clear distinction between general laws of mechanics and constitutive assumptions is one of the basic points of modern continuum mechanics. To keep this distinction will also be one of our main tasks in the subsequent formulation of the shell theory. It is also important to point out already here that the starting point of our considerations will be the dynamics rather then the kinematics, and finally we investigate the theory of constitutive equations. This appears to be crucial to the whole formulation for the shell theory.

1.2 Newtonian space-time. Classical theories of mechanics are based on the concept of Newtonian space-time $\mathcal{W}$, which may be mapped homeomorphically onto the product space $\mathcal{E} \times \mathcal{T}$. Here $\mathcal{E}$ denotes the three-dimensional Euclidean point space (physical space) referred to a fixed Cartesian coordinate system $\{x_k, k = 1,2,3\}$, and $\mathcal{T}$ is the one-dimensional Euclidean point space (time interval). In our subsequent considerations the translation space of $\mathcal{E}$, being the three-dimensional Euclidean vector space, will be denoted by E and $\{e_k, k = 1,2,3\}$ will denote the underlying orthonormal basis.

The map $\mathcal{W} \to \mathcal{E} \times \mathcal{T}$ is called the framing or the frame of reference or the observer.[3] Informally, frame of reference may be understood as a choice of a particular Cartesian coordinate system. Once a frame of reference is chosen, the motion is measured by specifying how coordinates of a particular object change as time goes on. We shall consider the motion and deformation of a body in a fixed frame of reference. Then places of the Newtonian space-time, which are stationary with respect to this reference frame, will be identified with places of the space $\mathcal{E}$.

1.3 Ordinary material continuum. Within continuum mechanics real material bodies are mathematically modeled in a number of different ways (e.g. continua

[3] WANG AND TRUESDELL [1973] and BOWEN [1989].

with microstructure, distributed material continuum, multicomponent continua). Classical theories are based on the concept of an ordinary material body, which is envisaged as a set consisting of point-like elements, called material particles, which at each time instant are continuously distributed over a region in the physical space. Such a region is then called an instantaneous current configuration of the body. A motion of the material body is thus perceived as a continuous sequence of its configurations. This picture is mathematically formalized by defining a body as a three-dimensional smooth manifold $\mathfrak{B}$ together with a class of imbeddings into the physical space (Fig. 1). This definition emphasizes that for conceptual clarity the body should not be confused with any of its spatial configurations.

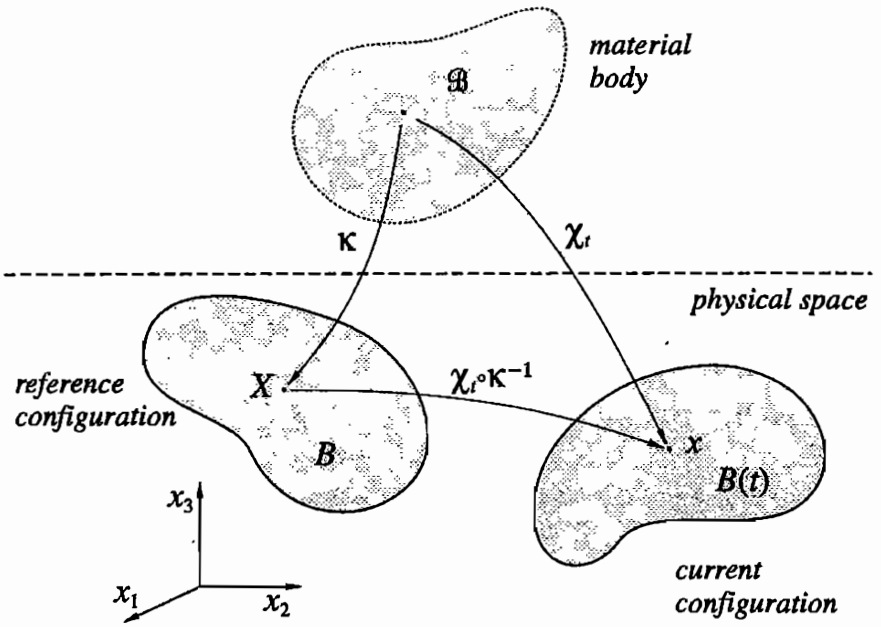

Fig. 1

It should be noted, however, that such concepts as sets of finite perimeter or reduced boundary, which are central in modern continuum mechanics, can hardly be defined on a smooth manifold (the concept of a manifold is strictly related to a differentiable structure and not to a measure geometric structure). It is this reason, why one usually identifies a body with a region $B \subset \mathscr{E}$ in the space the body occupies in a fixed reference configuration.[4] This configuration serves also to identify material particles. We shall follow this point of view. Relative to a fixed frame of reference, a typical point $X \in B$ is determined by its position vector X

[4] See e.g. NOLL AND VIRGA [1990].

having (X_K) as its Cartesian coordinates. In the sequel we shall not make a distinction between points (elements of the space $\mathscr{E}$) and their position vectors (elements of the space E). The (set topological) boundary of the region B will be denoted by ∂B (Fig. 2).

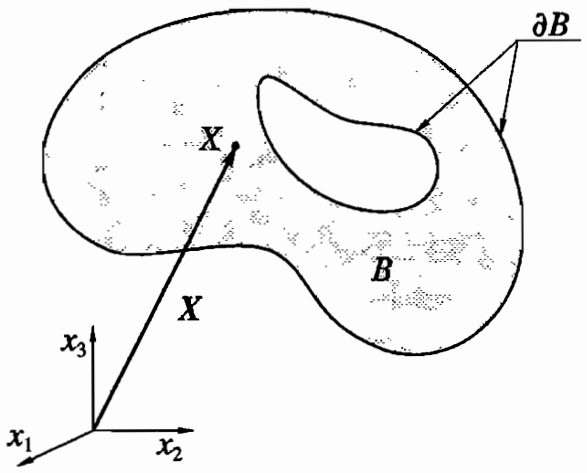

Fig. 2

Classical continuum mechanics is based on the idea that laws of mechanics are valid for every part of the body, called a subbody, regardless of its size. The concept of subbodies becomes a central one in the whole theory. In general, a subbody $P \subset B$ is defined as a standard domain in the sense of Whitney, or as a regular region in the sense of Kellogg.[5] However, such definitions appear to be too restrictive for the axiomatic setting of continuum mechanics. It has been shown that the suitable definition of subbodies should be based on the concept of sets of finite perimeter.[6] For the purpose of this work it will suffice to define a subbody as a domain (open connected set) with a smooth or piecewise smooth boundary. Clearly, this substantially simplifies the whole considerations.

1.4 Motion of the body. Relative to a fixed reference configuration B a motion of the body is described by a one–parameter family of maps

$$\chi_t : B \to \mathscr{E}, \qquad X \to x = \chi_t(X) = \chi(X,t), \tag{1.1}$$

where x denotes the place in the current configuration $\chi_t(B)$ occupied by the particle, whose initial place was X. The velocity, with which the body transverses

[5] GURTIN [1972].
[6] GURTIN, WILLIAMS AND ZIEMER [1986].

the space, is determined by the material velocity field $\dot{x}(X,t)=\dot{\chi}(X,t)$, where the superimposed dot denotes the material time derivative, i.e. the derivative with respect to t keeping X fixed.

In general, suitable regularity assumptions have to be introduced in order to ensure that the motion of the body represented by (1.1) be physically reasonable. However, the precise nature of such assumptions depends on the class of problems to be modeled. In general, we shall assume that at each time instant χ_t is smooth enough, one-to-one (injective) except possibly on the boundary, and orientation preserving. The injectivity assumption ensures that two distinct particles cannot simultaneously occupy the same place in space. The reason, why χ_t may loose its injectivity at the boundary, is the fact that a "self-contact" must be allowed. The term "smooth enough" is just a convenient way of saying that in a given definition, theorem, proof, etc. the smoothness of χ_t involved is such that all operations make sense. Clearly, the required degree of smoothness can vary from place to place depending on the intended applications and considered aspects of the theory. For most of our considerations in this chapter χ_t may be taken to be C^2 in the interior of B and C^1 on the closure of B. This means that χ_t is twice differentiable at each point $X \in intB$ and the second derivative is continuous on $int\, B$, while the first derivative has a continuous extension to the boundary ∂B. However, these regularity assumptions can substantially be relaxed. In fact, it is enough to assume that χ_t be a Lipschitz homeomorphism (this suffices for the classical gradient to exist almost everywhere). In our subsequent formulation of the shell theory it will be enough to assume that χ_t is piecewise smooth or even piecewise continuous, since we shall make no use of local field equations.

1.5 Change of frame of reference. The motion and deformation of the body cannot be specified physically in an absolute sense but only relative to a given frame of reference. A change of frame of reference is a transformation of space and time, which preserves distances in the physical space. In this sense it expresses the physical assumption that two observers should agree about distances. It should be noted that only the concept of change of observer, not the concept of observer itself, is regarded as having a mathematical meaning. If $x=\chi(X,t)$ and $x^*=\chi^*(X,t^*)$ describe the same deformation relative to different observers, then[7]

$$\chi^*(X,t^*)=o(t)+O(t)\chi(X,t), \qquad t^*=t+c, \tag{1.2}$$

[7] TRUESDELL AND NOLL [1965], p. 41.

where $\boldsymbol{o}$ is a vector, $\boldsymbol{O}$ is an orthogonal tensor, and c is a scalar constant. A change of observer induces transformations of quantities describing a material body.

1.6 Curvilinear coordinates. For analytical convenience the region B may be parametrized by curvilinear coordinates $\{\xi^A, A=1,2,3\}$ chosen in any convenient way. Then the position vector of a typical point relative to the fixed Cartesian coordinates in space may be expressed as a vector function of these curvilinear coordinates (Fig. 3):

$$\boldsymbol{X}(\xi^A) = X_K(\xi^A)\boldsymbol{e}_K. \tag{1.3}$$

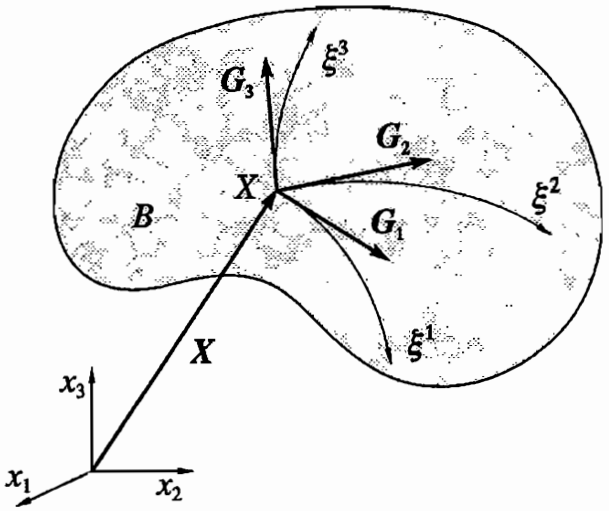

Fig. 3

Associated with such a coordinate system are the natural base vectors, the reciprocal base vectors, the components of the metric tensor and the permutation symbols, all being defined in a standard manner:

$$\begin{gathered} \boldsymbol{G}_A = \boldsymbol{X}_{,A}, \qquad \boldsymbol{G}^A \cdot \boldsymbol{G}_B = \delta^A_B, \\ G_{AB} = \boldsymbol{G}_A \cdot \boldsymbol{G}_B, \qquad G^{AB} = \boldsymbol{G}^A \cdot \boldsymbol{G}^B, \qquad G = det G_{AB} > 0, \\ \epsilon_{ABC} = (\boldsymbol{G}_A \times \boldsymbol{G}_B) \cdot \boldsymbol{G}_C, \qquad \epsilon^{ABC} = (\boldsymbol{G}^A \times \boldsymbol{G}^B) \cdot \boldsymbol{G}^C. \end{gathered} \tag{1.4}$$

Here the comma stands for partial derivative with respect to coordinates ξ^A. All other relevant relations follow now from (1.4) using the standard operation of rising and lowering of indices. In the same manner, the current configuration of the body may be parametrized by curvilinear coordinates $\{\xi^a, a=1,2,3\}$ chosen

entirely independent of the curvilinear coordinates ξ^A in the reference configuration B. Then the current position vector may be expressed in the form

$$\boldsymbol{x}(\xi^a) = x_k(\xi^a)\boldsymbol{e}_k \,. \tag{1.5}$$

The associated natural base vectors, the reciprocal base vectors, the components of the metric tensor and the permutation symbols are then defined by

$$\begin{gathered} \boldsymbol{g}_a = \boldsymbol{x}_{,a} \,, \qquad \boldsymbol{g}^a \bullet \boldsymbol{g}_b = \delta^a_b \,, \\ g_{ab} = \boldsymbol{g}_a \bullet \boldsymbol{g}_b \,, \qquad g^{ab} = \boldsymbol{g}^a \bullet \boldsymbol{g}^b \,, \qquad g = \det g_{ab} > 0 \,, \\ \epsilon_{abc} = (\boldsymbol{g}_a \times \boldsymbol{g}_b) \bullet \boldsymbol{g}_c \,, \qquad \epsilon^{abc} = (\boldsymbol{g}^a \times \boldsymbol{g}^b) \bullet \boldsymbol{g}^c \,. \end{gathered} \tag{1.6}$$

For any choice of curvilinear coordinates in the reference configuration and a possibly independent choice of curvilinear coordinates in the current configuration the motion $\boldsymbol{x} = \boldsymbol{\chi}_t(\boldsymbol{X})$ of the body is specified by

$$\boldsymbol{x}(\xi^a, t) = \boldsymbol{\chi}_t(\boldsymbol{X}(\xi^A)) \,, \qquad \xi^a = \xi^a(\xi^A, t) \,. \tag{1.7}$$

Whenever the map $\boldsymbol{\chi}_t$ is invertible for all time instants, the inverse motion is given by

$$\boldsymbol{X}(\xi^A, t) = \boldsymbol{\chi}_t^{-1}(\boldsymbol{x}(\xi^a)) \,, \qquad \xi^A = \xi^A(\xi^a, t) \,. \tag{1.8}$$

A special but convenient choice of curvilinear coordinates is such that for any time instant t, $\xi^a = \delta^a_A \xi^A$. Coordinates defined in this way are called convective ones (Fig. 4).

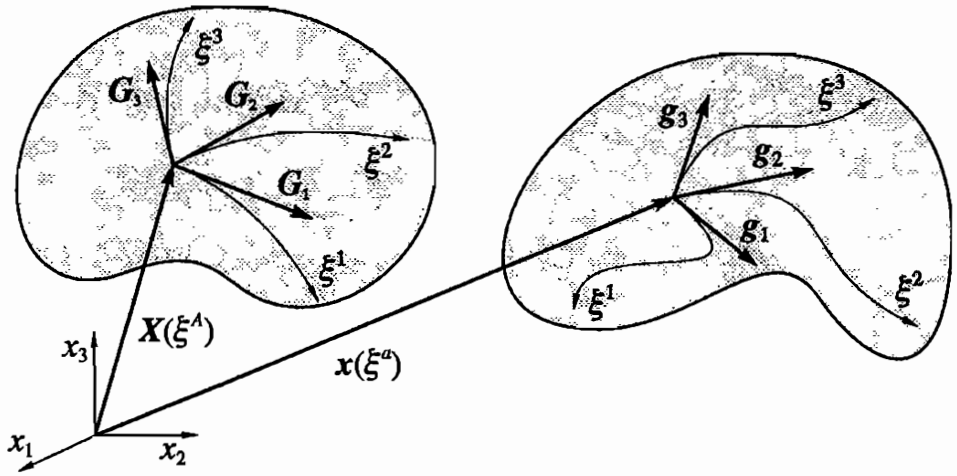

Fig. 4

2. Local deformation and strains

2.1 Deformation gradient. Generally, the moving body will change its orientation, shape and volume. This is loosely called a deformation. The deformation is thus a relative concept. Like geometry, deformation theory abounds in special theorems, concepts, and constructions. Thus, it should not be surprising that the main theorems of the deformation theory are, at the core, theorems of analysis.[8]

For a fixed time instant, the map $\chi_t : B \to \mathcal{E}$ is referred to as a global deformation of the body. On a large scale this map will be as a rule nonlinear(Fig. 5). Its differentiability makes it locally linear, the standard power expansion shows that the deformation gradient $\mathbf{F}(X,t) = \nabla\chi_t(X)$ approximates χ_t at X to within the first order. As such, it provides a complete description of a homogeneous local deformation, meaning both a rigid rotation and a pure deformation. The deformation gradient is thus considered as the primitive measure of local deformation. The requirement

$$\mathrm{J}(X,t) = \det \mathbf{F}(X,t) > 0 \tag{2.1}$$

assumed to hold almost everywhere (except possibly on subsets of measure zero) ensures that χ_t be locally invertible and orientation preserving.

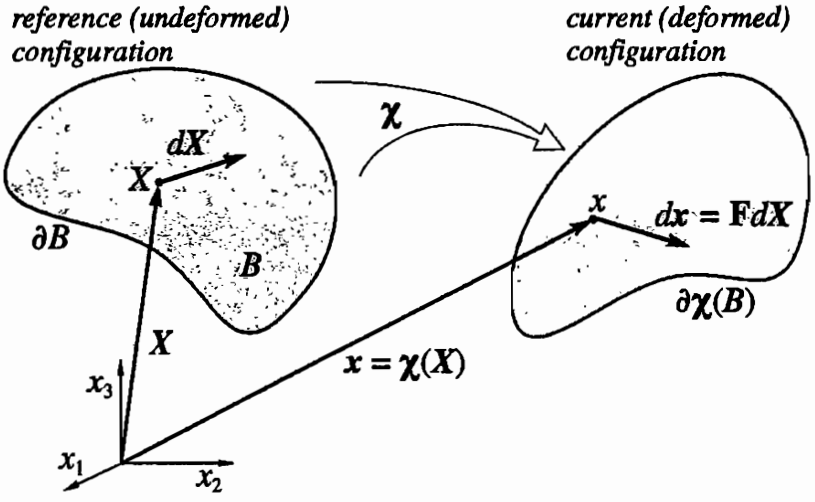

Fig. 5

[8] See e.g. Truesdell and Toupin [1960].

2.2 Decomposition of local deformation. As an implication of the assumption (2.1), the deformation gradient is nonsingular, and hence it admits the polar decomposition

$$\mathbf{F}(X,t)=\mathbf{R}(X,t)\mathbf{U}(X,t)=\mathbf{V}(X,t)\mathbf{R}(X,t), \tag{2.2}$$

which splits a local deformation into a local rigid rotation described by the proper orthogonal tensor **R**, called the rotation tensor, and a pure stretch described by the symmetric positive definite tensors **U** and **V**, called the right and left stretch tensors, respectively. The decomposition (2.2) shows that **F** is orthogonal only if $\mathbf{U}=\mathbf{1}$, and hence $\mathbf{V}=\mathbf{1}$. Moreover, in the operator norm, we have

$$\|\mathbf{F}\|_{\infty}=\|\mathbf{U}\|_{\infty}=\|\mathbf{V}\|_{\infty}. \tag{2.3}$$

Thus, the tensors **U** and **V** describe, how much **F** differs from the rotation tensor **R**. As such, they provide the basic measures of pure strain.

2.3 Measures of local deformation. Let us consider the differential line element dX, the oriented area element dA and the volume element dV at a point $X \in B$ Let dx, da and dv denote corresponding elements at the image point $x=\chi_t(X)$ in the current configuration of the body. Then the following three scalar quantities

$$\lambda(X,t)=\frac{\|dx\|}{\|dX\|}, \qquad \eta(X,t)=\frac{\|da\|}{\|dA\|}, \qquad \omega(X,t)=\frac{dv}{dV} \tag{2.4}$$

provide a complete description of the local, pure strain deformation. Taking into account the classical formulae

$$dx=\mathbf{F}dX, \qquad da=J(\mathbf{F}^{-1})^{T}dA, \qquad dv=JdV, \tag{2.5}$$

one finds

$$\lambda=\mathbf{t}\cdot\mathbf{C}\mathbf{t}, \qquad \eta=J\mathbf{n}\cdot\mathbf{C}^{-1}\mathbf{n}, \qquad \omega=J=\sqrt{\det\mathbf{C}}. \tag{2.6}$$

Here **t** is the unit vector along dX, **n** denotes the unit vector normal to the area element dA, and the symmetric tensor **C** is defined by

$$\mathbf{C}(X,t)=\mathbf{F}^{T}\mathbf{F}=\mathbf{U}^{2}. \tag{2.7}$$

The tensor **C**, called the right Cauchy-Green deformation tensor, plays a basic role in the description of local deformation of the body.

2.4 Strain measures. A general measure of strain may be defined by a smooth function $\mathbf{E}_{(g)} = \mathbb{G}(\mathbf{U})$ of the stretch tensor satisfying some conditions.[9] One usually assumes that: 1) it must vanish in the absence of change in length, 2) it should be a monotonously increasing function of stretch, and 3) it should coincide with the classical definition of infinitesimal strain. The most known examples of possible strain measures are

$$\mathbf{E}_{(n)} = \frac{1}{n!}(\mathbf{U}^n - \mathbf{1}), \quad n = 1,2,\ldots, \qquad \mathbf{H} = \mathbf{E}_{(0)} = \ln \mathbf{U}. \tag{2.8}$$

These tensors give rise to a family of tensorial strain measures, of which the Green-strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) \tag{2.9}$$

is the most familiar one.

While the polar decomposition (2.2) of the deformation gradient is commonly regarded as a reliable background for the analysis of finite deformation, the displacement field $\mathbf{u}$ defined by

$$x = \chi_t(X) = X + \mathbf{u}(X,t) \tag{2.10}$$

provides a basis for computations. From (2.10) we then have the classical relations

$$\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}, \qquad \mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\nabla\mathbf{u}\right). \tag{2.11}$$

2.5 Compatibility equations. When all kinematic quantities are calculated in terms of the function $\chi_t(X)$ or of the displacement field $\mathbf{u}(X,t)$, the continuity of deforming body is assured by the assumption that $\chi_t(X)$ be one-to-one and continuously differentiable. Alternatively, if the Green strain tensor $\mathbf{E}(X,t)$ is used as the fundamental kinematic quantity in the description of deformation of the body, then eqn $(2.11)_2$ constitutes a system of six partial differential equations for three unknown components of the displacement field $\mathbf{u}$. This over-determined system will not admit a solution unless the components of the strain tensor $\mathbf{E}$ satisfy suitable compatibility equations. The easiest way to infer the compatibility equations is to use some notions of geometry of Riemann spaces.[10]

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[9] See e.g. WANG AND TRUESDELL [1973].
[10] See e.g. FOSDICK [1966].

If strains in the body are assumed to be small in some sense, the question arises, what one can infer from such an assumption about the local rotation **R** and the deformation gradient **F**. The problem is by no means trivial, since for a given strain measure the deformation map χ_t is determined uniquely modulo rigid–body motion. By implication, **F** is determined uniquely up to a constant orthogonal tensor.[11]

2.6 Formulae in curvilinear coordinates. In terms of the curvilinear coordinates the motion of the body is described by (1.7), so that the deformation gradient and its inverse take the form

$$\begin{aligned} \mathbf{F} &= \boldsymbol{x}_{,A} \otimes \boldsymbol{G}^A = \xi^a_A \boldsymbol{g}_a \otimes \boldsymbol{G}^A , \qquad \xi^a_A \equiv \frac{\partial \xi^a}{\partial \xi^A} , \\ \mathbf{F}^{-1} &= \boldsymbol{X}_{,a} \otimes \boldsymbol{g}^a = \xi^A_a \boldsymbol{G}_A \otimes \boldsymbol{g}^a , \qquad \xi^A_a \equiv \frac{\partial \xi^A}{\partial \xi^a} , \end{aligned} \tag{2.12}$$

together with the obvious relations

$$\xi^a_A \xi^A_b = \delta^a_b , \qquad \xi^A_a \xi^a_B = \delta^A_B . \tag{2.13}$$

With the help of (2.12) various geometric relations can easily be calculated. For example, the two basic deformation tensors **C** and **E** take the form

$$\begin{aligned} \mathbf{C} &= C_{AB} \boldsymbol{G}^A \otimes \boldsymbol{G}^B , \qquad C_{AB} = \boldsymbol{x}_{,A} \cdot \boldsymbol{x}_{,B} = \xi^a_A \xi^b_B g_{ab} , \\ \mathbf{E} &= E_{AB} \boldsymbol{G}^A \otimes \boldsymbol{G}^B , \qquad E_{AB} = \frac{1}{2}(C_{AB} - G_{AB}) . \end{aligned} \tag{2.14}$$

Introducing the displacement field and its gradient

$$\mathbf{u} = u_A \boldsymbol{G}^A = u^A \boldsymbol{G}_A , \qquad \nabla \mathbf{u} = \mathbf{u}_{,B} \otimes \boldsymbol{G}^B = u_{A;B} \boldsymbol{G}^A \otimes \boldsymbol{G}^B , \tag{2.15}$$

the Green strain tensor is obtained in the form

$$\begin{aligned} E_{AB} &= \frac{1}{2}(\boldsymbol{G}_A \cdot \mathbf{u}_{,B} + \boldsymbol{G}_B \cdot \mathbf{u}_{,A} + \mathbf{u}_{,A} \cdot \mathbf{u}_{,B}) \\ &= \frac{1}{2}(u_{A;B} + u_{B;A} + u^C{}_{;A} u_{C;B}) , \end{aligned} \tag{2.16}$$

where $(.)_{;B}$ denotes the covariant derivative in the metric G_{AB}.

[11] KOHN [1982].

3. Principles of mechanics

3.1 Mechanical balance laws. Continuum mechanics is based on the fundamental assumption that on the large (macroscopic) scale the motion and deformation of a body are governed by a suitable set of global balance laws. Such laws entail interactions between bodies and rules responsible for their motion. All purely mechanical theories are based on three fundamental physical principles (recall that we identify the body with a region it occupies in the reference configuration):

I) *Balance law of mass* – The body B is equipped with a positive scalar quantity $\mathfrak{M}(B)$, called the mass of the body, which is independent of time, so that

$$\dot{\mathfrak{M}}(B)=0.$$

II) *Balance law of linear momentum* – The total force $\mathfrak{F}(B,t)$ acting on the body B equals the rate of change of its linear momentum $\mathfrak{L}(B,t)$,

$$\mathfrak{F}(B,t)=\dot{\mathfrak{L}}(B,t).$$

III) *Balance law of angular momentum* – The total torque $\mathfrak{T}(B,t)$ acting on the body B equals the rate of change of its angular momentum $\mathfrak{A}(B,t)$,

$$\mathfrak{T}(B,t)=\dot{\mathfrak{A}}(B,t).$$

These three physical principles are supposed to hold for all material bodies, whether solid or fluid, deformable or rigid, and whatever motion and deformation they undergo. Mathematically, all entities appearing in these laws are primitive concepts, and the laws themselves should be regarded as axioms of the theory. For the ordinary material continuum the three balance laws constitute the complete set of purely mechanical principles. For continua with microstructure additional balance laws can be needed.

Like motion and deformation, forces acting on the body cannot be specified in an absolute sense, but only relative to a given frame of reference. All frames in which the balance laws hold are called inertial frames. These frames may be in motion relative to each other, but their relative motion should be at constant translational velocity.

3.2 Non-polar body. In classical theories the mechanical balance laws are assumed to hold not only for the whole body but also for every subbody. Various theories, which are based on this postulate, are known as local theories. Within these theories the entities of the balance laws I)-III) are set-functions obeying the suitable additivity rules on separate subbodies. Their densities can be defined over the body itself (material description), over the current configuration of the body (spatial description), or over a fixed reference configuration of the body (referential description). In this work we restrict our considerations to the referential description and we identify the body with its reference configuration B. Thus the material and referential descriptions coincide.

The mass for every subbody $P \subset B$ is defined to be a time independent scalar $\mathfrak{M}(P)$, which is determined by a mass density $\rho(X)$ measured per unit volume of the reference configuration B:

$$\mathfrak{M}(P) = \int_P \rho(X)\, dV\,. \tag{3.1}$$

The linear momentum $\mathfrak{L}(P,t)$ and the angular momentum $\mathfrak{A}(P,t)$ are time dependent, vector-valued functions. Within classical continuum mechanics based on the concept of ordinary continuum, these functions are defined for every subbody and for each time instant by

$$\mathfrak{L}(P,t) = \int_P \rho \dot{x}(X,t)\, dV\,, \qquad \mathfrak{A}(P,t) = \int_P \rho x(X,t) \times \dot{x}(X,t)\, dV\,. \tag{3.2}$$

Let us note that from the point of view of generalized continua, definitions (3.2) should be regarded as a kind of constitutive assumptions.

The classical theory of continuum mechanics is concerned with three types of forces:

- Internal contact forces between separate parts of a body through a common smooth boundary.
- Body forces exerted on interior points of a body by its environment.
- External contact forces exerted on the boundary of a body by its environment.

In the material description, the body force $\mathbf{f}(X,t)$ is measured per unit volume of B, and the contact force $\mathbf{t}_N(X,t;\partial P)$ arising from the action of one part of the body on an adjacent part across a separating surface is measured per unit area of ∂P. These functions are assumed to be sufficiently smooth, and they must be

compatible with the laws of mechanics. The total force and the total torque acting on the subbody at time t are (Fig. 6)

$$\begin{aligned} \mathfrak{F}(P,t) &\equiv \int_P \mathbf{f}(X,t)\,dV + \int_{\partial P} \mathbf{t}_N(X,t,\partial P)\,dA\,, \\ \mathfrak{T}(P,t) &\equiv \int_P \boldsymbol{x}(X,t)\times\mathbf{f}(X,t)\,dV + \int_{\partial P} \boldsymbol{x}(X,t)\times\mathbf{t}_N(X,t,\partial P)\,dA\,. \end{aligned} \tag{3.3}$$

Properly, the torque $(3.3)_2$ should be called the torque with respect to the origin of the frame of reference.

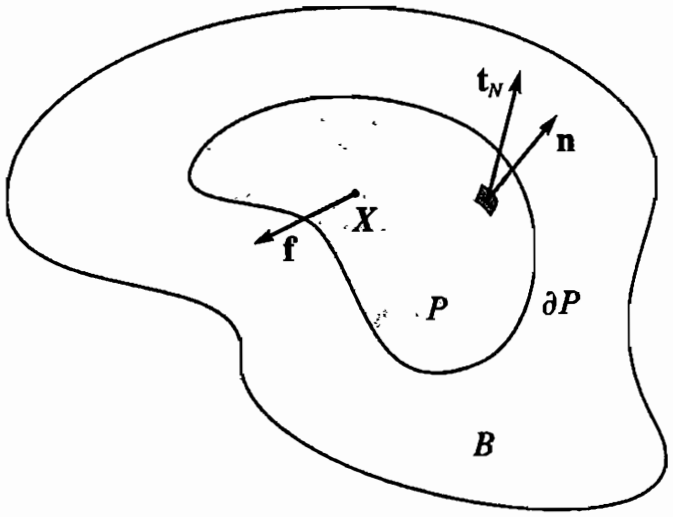

Fig. 6

3.3 Stress tensors. According to the classical Cauchy's hypothesis the contact force (stress vector) $\mathbf{t}_N(X,t;\partial P)$ depends upon the surface ∂P only through its positive unit normal vector $\mathbf{n}(X)$:

$$\mathbf{t}_N(X,t;\partial P) = \mathbf{t}_N(X,t;\mathbf{n}(X))\,. \tag{3.4}$$

The Cauchy's hypothesis is one of the most important and far reaching axioms of continuum mechanics. Under the additional assumption that $\mathbf{t}_N(X,.;.)$ is a continuous function with respect to X, it implies the existence of the first Piola-Kirchhoff stress tensor $\mathbf{T}(X)$ such that[12]

$$\mathbf{t}_N(X,\partial P) = \mathbf{T}(X)\mathbf{n}(X)\,. \tag{3.5}$$

[12] The existence of the stress tensor can be proved under weaker assumptions, cf. Gurtin and Martins [1976].

This is known as Cauchy's fundamental theorem. It enables one to replace the integral balance laws of linear and angular momentum by the local differential equations of motion.

It may be noted that under some regularity assumptions the Cauchy´s hypothesis follows from the balance of linear momentum. This is know as the Hamel-Noll theorem.[13] Moreover, applying a variational method Fosdick and Virga[14] proved Cauchy's hypothesis (3.4) can be weakened to allow the stress vector $\mathbf{t}_N(X,t;\partial P)$ acting on any oriented surface ∂P to depend upon both the unit normal vector $\mathbf{n}(X)$ and its surface gradient $\nabla_s\mathbf{n}(X)$.

From the point of view of generalized continua the Cauchy's hypothesis should be regarded as defining a specific property of the contact force. In this sense it expresses a kind of constitutive assumption. For example, in the theory of so-called higher–grad materials admitting hyperstresses the classical form (3.4) of this hypothesis cannot be uncritically accepted.[15]

3.4 Local equations of motion. For the integral balance laws I–III to make sense it is enough that all fields appearing in (3.2) and (3.3) be integrable over their domains. For example, if P is a regular region with a piecewise smooth boundary ∂P, and if the deformation map χ_t, the body force $\mathbf{f}$ and the stress tensor $\mathbf{T}$ are piecewise continuous and bounded, then the Riemann integrals exist. However, the integral balance laws make sense also under far weaker regularity assumptions.

However, in order to obtain the local field equations (equations of motion) implied by the integral balance laws much stronger regularity assumptions have to be made. Usually one assumes that the domain P, the motion χ_t and the stress tensor $\mathbf{T}$ be sufficiently regular for the classical divergence theorem to be applicable. Then the boundary term in the expression of the total force can be transformed into a volume integral

$$\int_{\partial P}\mathbf{T}\mathbf{n}\,dA=\int_P Div\,\mathbf{T}\,dV\,. \tag{3.6}$$

Taking further into account that

$$ad(\mathfrak{L}(P,\mathrm{t}))=-\int_P \boldsymbol{x}\wedge\mathbf{f}\,dV-\int_{\partial P}\boldsymbol{x}\wedge\mathbf{T}\mathbf{n}\,dA \tag{3.7}$$

[13] TRUESDELL [1977], p. 134.
[14] FOSDICK AND VIRGA [1989].
[15] See e.g. FORTE AND VIANELLO [1988].

and using a slightly more general form of the divergence theorem, we have[16]

$$\int_{\partial P} \boldsymbol{x} \wedge \mathbf{T}\boldsymbol{n}\, dA = \int_P (\mathbf{T}\mathbf{F}^T - \mathbf{F}\mathbf{T}^T + \boldsymbol{x} \wedge (Div\mathbf{T})\, dV\,. \tag{3.8}$$

Since the domain of integration is time independent, upon making use of (3.6) and (3.8) the balance laws of linear and angular momentum read:

$$\begin{aligned} \int_P (Div\,\mathbf{T} + \mathbf{f} - \rho\ddot{\boldsymbol{x}})\, dV &= \mathbf{0}\,, \\ \int_P \left(\mathbf{T}\mathbf{F}^T - \mathbf{F}\mathbf{T}^T + \boldsymbol{x} \wedge (Div\,\mathbf{T} + \mathbf{f} - \rho\ddot{\boldsymbol{x}})\right) dV &= \mathbf{0}\,. \end{aligned} \tag{3.9}$$

Under the assumption that the integrands in (3.9) are continuous functions, in view of the arbitrariness of P, the equations of motion implied by the integral balance laws are

$$Div\mathbf{T} + \mathbf{f} = \rho\ddot{\boldsymbol{x}}\,, \qquad \mathbf{T}\mathbf{F}^T = \mathbf{F}^T\mathbf{T}\,. \tag{3.10}$$

These two local field equations are known as Cauchy's first and second law of mechanics. In the spatial description, the second of these equations just asserts that the Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1}\mathbf{T}\mathbf{F}^T$ is symmetric. Since the symmetry of the Cauchy stress tensor is assured by the constitutive assumptions, the balance law of angular momentum leading to $(3.10)_2$ is not of central importance in classical continuum mechanics. However, this remark is no longer true for polar bodies and bodies with microstructure, i.e. for bodies with intrinsic spin, surface couples, and body couples. This is exactly the case for shells.

For any choice of the curvilinear coordinate systems in the reference and current configurations of the body (see Sect. 1.2.6), the first Piola-Kirchhoff stress tensor can be expressed in the form

$$\mathbf{T}(\boldsymbol{X},t) = \mathbf{t}^A(\boldsymbol{X},t) \otimes \boldsymbol{G}_A(\boldsymbol{X})\,, \tag{3.11}$$

where the so-called nominal stress vectors $\mathbf{t}^A$, $A = 1,2,3$, are defined by

$$\mathbf{t}^A = \mathbf{T}\boldsymbol{G}^A = T^{\alpha A}\boldsymbol{g}_\alpha\,. \tag{3.12}$$

With the help of (3.11) the equations of motion (3.10) can be written in the form

$$\mathbf{t}^A{}_{;A} + \mathbf{f} = \rho\ddot{\boldsymbol{x}}\,, \qquad \boldsymbol{x}_{,A} \times \mathbf{t}^A = \mathbf{0}\,, \tag{3.13}$$

[16] See e.g. TRUESDELL [1977].

where $(.)_{;A}$ denotes the covariant differentiation in the metric of the chosen curvilinear coordinate system $\{\xi^A\}$ in the region B.

3.5 Change of frame of reference. Like motion and deformation, the forces acting on the body can be specified not in an absolute sense but only relative to a given frame of reference.

In the spatial description, the body force, the internal contact force and the Cauchy (true) stress tensor are assumed to be frame-indifferent. These postulates imply that under the change of frame of reference the body force $\mathbf{f}$ and the first Piola-Kirchhoff stress tensor $\mathbf{T}$ obey the following transformation rules

$$\begin{aligned} \mathbf{f}^*(X,t^*) &= O(t)\mathbf{f}(X,t), \\ \mathbf{T}^*(X,t^*) &= O(t)\mathbf{T}(X,t). \end{aligned} \tag{3.14}$$

As a simple implication of $(3.14)_2$, (3.5) and (3.11) we have

$$\mathbf{t}_N^*(X,t^*) = O(t)\mathbf{t}_N(X,t) \tag{3.15}$$

and

$$(\mathbf{t}^A)^*(X,t^*) = O(t)\mathbf{t}^A(X,t), \qquad A=1,2,3. \tag{3.16}$$

It is interesting to note here that while the first Piola-Kirchhoff stress tensor is not objective (according to the standard definition), the internal contact force $\mathbf{t}_N$ and the nominal stress vectors $\mathbf{t}^A$ are objective.

3.6 Singular surfaces. A moving and deforming body may contain a surface, generally called a singular surface, which transverses the body with its own velocity and across which various fields may suffer jump discontinuity. Propagation of waves or formation of cracks are typical examples of such singular surfaces. In the presence of singular surfaces as well as other kinds of singularities the integral balance laws still remain valid, but the local theory requires a more subtle mathematical setting: sets of finite perimeter, functions of bounded variations and a good part of the measure theoretic theory.[17] However, for the purpose of this work it will be sufficient to consider the problem within a much more restrictive context: regular domains and piecewise continuous or smooth functions. Moreover, since our subsequent formulation of the shell theory will be

[17] See e.g. NOLL AND VIRGA [1990].

restricted primarily to the quasi-static case, we shall assume that possible singular surfaces are stationary.

If various fields appearing in the integral balance laws fail to satisfy the classical regularity assumptions, then from the integral balance laws, besides field equations (3.10), one obtains corresponding jump conditions. In the quasi-static case, let the deformation map χ and the stress tensor $\mathbf{T}$ be regular enough for the classical divergence theorem to hold for a finite partition of the region B. Then the jump conditions at any point $X \in S$ of a stationary singular surface $S \subset B$ take the form[18]

$$[\![\mathbf{T}]\!]\mathbf{n} = \mathbf{0}, \qquad [\![x \times \mathbf{T}\mathbf{n}]\!] = \mathbf{0}. \tag{3.17}$$

Here $\mathbf{n}$ denotes the unit normal vector to the singular surface S and the jump $[\![\Psi]\!]$ of any field Ψ is defined as the difference of the limits taken from both sides of S:

$$[\![\Psi]\!](X) = \Psi^{(+)}(X) - \Psi^{(-)}(X). \tag{3.18}$$

The jump conditions (3.17) hold across any smooth surface S provided the relevant limits taken from both sides of S exist.[19]

4. Weak form of the momentum balance laws

4.1 Classical boundary conditions. Generally, the boundary conditions at any point $X \in \partial B$ consist of a prescription of the position, or the traction, or some suitable combination of them. In classical form one assumes that there are two disjoint subsets of ∂B such that the traction boundary condition is specified on the part ∂B_f of the boundary in the form (analogue of the Neumann condition)

$$\mathbf{T}(X,t)\mathbf{n}(X) = \mathbf{t}^*(X,t), \qquad X \in \partial B_f. \tag{4.1}$$

On the complementary part $\partial B_d = \partial B \setminus \partial B_f$ the kinematic boundary condition is specified in the form (analogue of the Dirichlet condition)

$$\chi(X,t) = \chi^*(X,t), \qquad X \in \partial B_d. \tag{4.2}$$

---

[18] See e.g. STRIFORS [1990].

[19] See TRUESDELL AND TOUPIN [1960] for detailed account of the concept of singular surfaces and jump conditions in classical continuum mechanics.

Here $\mathbf{t}^*(X,t)$ and $\boldsymbol{\chi}^*(X,t)$ are given vector-valued functions of their arguments. Let us note that an asterisk used here has different meaning than in the previous subchapters.

4.2 Principle of virtual work. The classical way of deriving the weak form of the momentum balance laws is quite simple. Let $\mathbf{v}(X,t)$ be any vector-valued field (it may be called the virtual displacement or test function). Then, as a direct implication of Euler's first law $(3.10)_1$ and the dynamic boundary condition (4.1), we have

$$\int_B (Div\mathbf{T}+\mathbf{f}-\rho\ddot{\mathbf{x}})\cdot\mathbf{v}\,dV+\int_{\partial B_f}(\mathbf{t}^*-\mathbf{Tn})\cdot\mathbf{v}\,dA=\mathbf{0}. \tag{4.3}$$

Upon applying the divergence theorem to the first term in the first integral of (4.3) we obtain

$$\int_B \rho\ddot{\mathbf{x}}\cdot\mathbf{v}\,dV+\int_B \mathbf{T}\cdot\nabla\mathbf{v}\,dV=\int_B \mathbf{f}\cdot\mathbf{v}\,dV+\int_{\partial B_f}\mathbf{t}^*\cdot\mathbf{v}\,dA+\int_{\partial B_d}\mathbf{Tn}\cdot\mathbf{v}\,dA. \tag{4.4}$$

Assuming that $\mathbf{v}=\mathbf{0}$ on ∂B_d, in accordance with the kinematic boundary condition (4.2), the equation (4.4) reduces to

$$\int_B \rho\ddot{\mathbf{x}}\cdot\mathbf{v}\,dV+\int_B \mathbf{T}\cdot\nabla\mathbf{v}\,dV=\int_B \mathbf{f}\cdot\mathbf{v}\,dV+\int_{\partial B_f}\mathbf{t}^*\cdot\mathbf{v}\,dA \tag{4.5}$$

for all $\mathbf{v}$ satisfying the above condition. Eqn (4.5) expresses the principle of virtual work of forces. Reversing the process of derivation we can easily show that (4.5) implies the local equations of motion $(3.10)_1$ and the dynamic boundary condition (4.1). Applying the same arguments we easily see that from the second Euler law $(3.9)_2$ alone we have

$$\int_B \mathbf{F}^T\mathbf{T}\cdot\mathbf{W}\,dV=0, \tag{4.6}$$

for every skew-symmetric tensor field $\mathbf{W}$. This equation expresses the principle of virtual work of torque.

In the above considerations we have followed the conventional line of derivation. Antman and Osborn[20] have reconsidered the problem pointing out that this procedure is unsatisfactory for the following reason: both, the integral balance laws and the principle of virtual work, are regarded as valid under regularity assumptions far weaker than those used in the preceding arguments to show their

[20] ANTMAN AND OSBORN [1979].

equivalence. Using some techniques from modern analysis they next established the equivalence of the principles of virtual work and the integral balance laws without having introduced the local equations of motion in an intermediate step.

4.3 Stress power. Taking $\mathbf{v}$ to be the velocity field, the principle of virtual work of forces (4.5) takes the form

$$\int_B \rho \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, dV + \int_B \mathbf{T} \cdot \nabla \dot{\mathbf{x}}\, dV = \int_B \mathbf{f} \cdot \dot{\mathbf{x}}\, dV + \int_{\partial B_f} \mathbf{t}^* \cdot \dot{\mathbf{x}}\, dA. \tag{4.7}$$

This integral identity expresses the principle of virtual velocity. Taking further into account that $\dot{\mathbf{F}} = \nabla \dot{\mathbf{x}}$, we may rewrite (4.7) in the form

$$\dot{\mathfrak{K}}(B,t) + \int_B \Sigma(X,t)\, dV = \int_B \mathbf{f} \cdot \dot{\mathbf{x}}\, dV + \int_{\partial B_f} \mathbf{t}^* \cdot \dot{\mathbf{x}}\, dA, \tag{4.8}$$

where the kinetic energy of the body is defined by

$$\mathfrak{K}(B,t) = \frac{1}{2} \int_B \rho \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, dV \tag{4.9}$$

and

$$\Sigma(X,t) = \mathbf{T}(X,t) \cdot \dot{\mathbf{F}}(X,t) \tag{4.10}$$

is the stress power density (measured per unit volume of the reference configuration of the body).

4.3 Alternative stress measures. In the material description, stresses within the body are usually described by the first Piola-Kirchhoff stress tensor $\mathbf{T}(X,t)$. There are, however, many other stress tensors which are more convenient in the analysis of special problems. For example, recalling the definitions of the right Cauchy-Green deformation tensor and the Green strain tensor, from (4.10) we have

$$\Sigma = \mathbf{T} \cdot \dot{\mathbf{F}} = \frac{1}{2} \mathbf{S} \cdot \dot{\mathbf{C}} = \mathbf{S} \cdot \dot{\mathbf{E}}, \tag{4.11}$$

where $\mathbf{S}(X,t)$, known as the second Piola-Kirchhoff stress tensor, is defined by

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}. \tag{4.12}$$

In the same way we can introduce the work-conjugate stress tensor for any generalized strain measure.

5. Constitutive equations

5.1 Dynamic processes. Our considerations so far may be summarized as follows. A motion $\chi_t : B \to \mathcal{E}$ of the body assigns to every material particle, whose reference placement is $X \in B$, its spatial placement $x = \chi_t(X)$ in the region $\chi_t(B)$ instantaneously occupied by the body at time t. The body is endowed with the mass, the mass density in the reference configuration B being $\rho(X)$. The forces acting on the body are of two kinds: the body force $\mathbf{f}(X,t)$ and the contact force $\mathbf{t}_N(X,t;\partial P)$, the latter one being completely determined by the first Piola-Kirchhoff stress tensor $\mathbf{T}(X,t)$. The following four functions $(\chi_t, \mathbf{T}, \rho, \mathbf{f})$ are thus the basic variables of the theory.

The motion of the body is governed by three balance laws: the balance law of mass (satisfied identically in the material description), and two momentum balance laws represented in the local form by the field equations

$$Div\mathbf{T} + \mathbf{f} = \rho\ddot{x}, \qquad \mathbf{T}\mathbf{F}^T = \mathbf{F}\mathbf{T}^T. \tag{5.1}$$

Like the acceleration $\ddot{x}(X,t) = \ddot{\chi}_t(X)$, the deformation gradient $\mathbf{F}(X,t) = \nabla\chi_t(X)$ is not an independent variable, since it is defined in terms of the motion. In general, the body force $\mathbf{f}$ and the mass density ρ are assumed to be given as a part of the data. Then the motion χ_t and the stress tensor $\mathbf{T}$ should be determined from the field equations (5.1) and suitable side conditions (initial, jump and boundary conditions). However, the system of field equations and side conditions derived in this way is underdetermined, in general. This merely shows that principles of mechanics alone do not suffice to determine the motion of the body even if applied loads and side conditions are given. Constitutive equations, i.e. relationships between stresses and deformation, are needed to specify a diversity of materials.

Constitutive equations are mathematical statements and not physical principles. They represent the observed response of material bodies to external loadings. There are many approaches to the constitutive theory, but they all fall into two categories: Within the first category one seeks constitutive relations for a restricted class of materials in an unrestricted class of deformations. Within the second category one seeks constitutive relations for an unrestricted class of materials in a restricted class of deformations. These two approaches are not mutually distinct. The procedure in utilizing the basic laws of mechanics rests upon the following:

a) The field equation $(5.1)_1$ is assumed to hold for an arbitrary choice of the motion including, if required, an arbitrary choice of space and time derivatives.
b) The stress tensor is calculated from its respective constitutive equation.
c) The value of the body force can be found from the balance of linear momentum.
d) The equations resulting from the principle of moment of momentum and the side conditions are regarded as identities for every motion.

This procedure imposes some restrictions upon possible forms of the constitutive relations, which are needed to close the system of field equations and side conditions.

The ordered pair $(\chi_t, \mathbf{T})$ is called a dynamic process for the body, if the motion and the stress tensor are related in such a way that they satisfy the momentum balance laws, i.e. the local field equations (5.1). Two dynamic processes $(\chi_t, \mathbf{T})$ and $(\chi^*_{t^*}, \mathbf{T}^*)$ are said to be equivalent, if they are related by

$$\chi^*_{t^*}(X) = o(t) + O(t)\chi_t(X), \qquad \mathbf{T}^*(X, t^*) = O(t)\mathbf{T}(X, t). \tag{5.2}$$

A change of observer induces transformations of quantities describing a material body. This definition stipulates that the equivalent processes describe the same dynamic process recorded by two observers.

5.2 General principles of constitutive theory. The general theory of the constitutive relations rests upon its own postulates, which are believed to be reasonable physical assumptions for all kind of materials encountered in reality:[21]

1) *Principle of determinism.* The stress at the material particle X at the time t is determined by the motion χ_t of the body up to the time t. Said differently, the past and present motion determine the present stresses, future has no influence upon it.

2) *Principle of local action.* The present stress at a material particle X is entirely determined by the history of motion of an arbitrarily small neighborhood of that particle. The motion of the particles at a finite distance from X may be disregarded in calculating the stress at X.

[21] See Truesdell and Noll [1965] or Wang and Truesdell [1973].

3) *Principle of material frame-indifference.* If a constitutive relation is satisfied by the dynamic process, it is satisfied by every equivalent process.

According to the first of these principles, the general constitutive equations expressed in mathematical form reads:

$$\mathbf{T}(X,t) = \mathfrak{h}^{\infty}_{s=0}\left(\chi^t(X,s); X, t\right), \tag{5.3}$$

where $\mathfrak{h}^{\infty}_{s=0}$ denotes a constitutive functional (response functional), i.e. a function, whose argument is the history $\chi^t(X,s)$ of the motion of the body. According to the general theory the response functional must be compatible with the balance law of angular momentum.

Let us first recall that given a time dependent field $\mathfrak{f}(X,t)$ with values in a finite dimensional inner product vector space F, the history of $\mathfrak{f}(X,t)$ up to the present time instant t is defined by

$$\mathfrak{f}^t(X,s) = \mathfrak{f}(X,t-s), \qquad t \in [0,\infty). \tag{5.4}$$

The variable s is called the time-lapse from the past instant t–s to the present time instant t and $\mathfrak{f}^t(X,s)$ is the value of $\mathfrak{f}(X,t)$ at a time s units before the present time instant t. Note that $\mathfrak{f}^t(X,s)$ is defined only for $s \leq 0$, though $\mathfrak{f}(X,t)$ may be well defined for all t, and we have $\mathfrak{f}^t(X,0) = \mathfrak{f}(X,t)$.

According to the principle of determinism the stress at a particle X may depend on the complete history of the motion of the whole body B. Various special classes of materials can be defined by introducing additional assumptions, which delimit such dependence.

In particular, a theory of non-simple materials admits a dependence of the response functional on the history of the first and higher deformation gradients. A material, whose constitutive equations have the form

$$\mathbf{T}(X,t) = \mathfrak{h}^{\infty}_{s=0}\left(\mathbf{F}^t(X,s), \ldots, \nabla^{(n)}\mathbf{F}^t(X,s); X\right), \tag{5.5}$$

is called n–grad material.[22]

[22] TRUESDELL AND NOLL [1965].

5.3 Simple materials. In the theory of simple materials mechanical properties are assumed to be determined by functionals having the history of the first deformation gradient as the only arguments. In other words, its mechanical response is governed by the constitutive equations in the form

$$\mathbf{T}(X,t)=\mathfrak{h}_{s=0}^{\infty}\left(\mathbf{F}^{t}(X,s);X\right). \tag{5.6}$$

The statement (5.5) says that the mechanical response of the body to the deformation is sensitive only within an arbitrarily small neighborhood of a point X. In this sense it expresses the principle of local action, which is assumed to be valid in the theory of simple materials. The fact that the constitutive equation (5.6) may depend on the past values of $\mathbf{F}$ indicates that in general a simple material may have memory effects.

For a simple material the principle of frame indifference asserts that the response functional must satisfy the relation

$$\mathfrak{h}_{s=0}^{\infty}\left(O(s)\mathbf{F}^{t}(X,s);X\right)=O(0)\mathfrak{h}_{s=0}^{\infty}\left(\mathbf{F}^{t}(X,s);X\right) \tag{5.7}$$

for all histories of proper orthogonal tensors. This requirement may be viewed as a transformation law of the constitutive relation under a change of frame of reference. Making use of the polar decomposition theorem, the following Noll's representation theorem can be proved: A constitutive functional satisfies the principle of frame-indifference if and only if it can be represented by the restriction of it to positive symmetric histories

$$\mathfrak{h}_{s=0}^{\infty}\left(\mathbf{F}^{t}(X,s);X\right)=\mathbf{R}(t)\mathfrak{h}_{s=0}^{\infty}\left(\mathbf{U}^{t}(X,s);X\right). \tag{5.8}$$

This theorem asserts that the response functional is entirely independent of all past rotations and it depends on the present rotation in a limited way.

5.4 Elastic constitutive laws. An elastic material is one, which has no memory. Its mechanical response depends on the current state of the body only and the constitutive functionals reduce to functions:

$$\mathbf{T}(X,t)=\mathfrak{h}\left(\mathbf{F}(X,t),X\right). \tag{5.9}$$

Below, for simplicity of writing, the material particle X will generally be suppressed from the arguments of the response functions. The response function $\mathfrak{h}$ must be compatible with the balance of angular momentum, and it is delimited by the principle of frame-indifference, i.e.

$$\mathfrak{h}(\mathbf{F})\mathbf{F}^T = \mathbf{F}\mathfrak{h}(\mathbf{F})^T, \qquad O\mathfrak{h}(\mathbf{F}) = \mathfrak{h}(O\mathbf{F}), \tag{5.10}$$

and possible symmetries of the material. Making use of the polar decomposition of the deformation gradient, $\mathbf{F} = \mathbf{RU}$, the representation theorem implies that

$$\mathbf{T} = \mathbf{R}\mathfrak{h}(\mathbf{U}) = \mathbf{R}\hat{\mathfrak{h}}(\mathbf{C}), \qquad \mathbf{C} \doteq \mathbf{U}^2, \tag{5.11}$$

where $\mathbf{C}$ is the right Cauchy-Green deformation tensor. It can be proved that the constitutive equations (5.11) satisfy the principle of frame-indifference if and only if they can be reduced to the form

$$\mathbf{S} = \mathfrak{g}(\mathbf{C}) = \hat{\mathfrak{g}}(\mathbf{E}), \qquad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}). \tag{5.12}$$

Here $\mathbf{S}$ denotes the 2nd Piola-Kirchhoff stress tensor and $\mathbf{E}$ is the Green strain tensor.

5.5 Hyperelastic material. In the particular case of a hyperelastic material, whose mechanical properties are governed by a strain energy function $W = W(\mathbf{F}, \boldsymbol{X})$ per unit volume of the reference configuration B, we have

$$\mathfrak{h}(\mathbf{F}, \boldsymbol{X}) = \partial_{\mathbf{F}} W(\mathbf{F}, \boldsymbol{X}). \tag{5.13}$$

The frame-indifference requirement implies that $W = W(\mathbf{C}, \boldsymbol{X})$. This expresses our understanding that the strain energy W, like the stresses in the body, is caused by pure deformation and not by rigid rotations.

For an isotropic homogeneous hyperelastic material the strain energy density is a function of the principal invariants of the right Cauchy-Green deformation tensor:

$$I_1 = tr\mathbf{C}, \qquad I_2 = \frac{1}{2}\left((tr\mathbf{C})^2 - tr\mathbf{C}^2\right), \qquad I_3 = \det\mathbf{C} = J^2 . \tag{5.14}$$

Taking into account that

$$\partial_{\mathbf{C}} I_1 = \mathbf{1}, \qquad \partial_{\mathbf{C}} I_2 = I_1\mathbf{1} - \mathbf{C}, \qquad \partial_{\mathbf{C}} I_3 = I_3\mathbf{C}^{-1}, \tag{5.15}$$

the constitutive equations (5.11) read

$$\mathbf{T} = 2\left((W_1 + I_1W_2)\mathbf{F} - W_2\mathbf{FC} + I_3W_3(\mathbf{F}^{-1})^T\right), \tag{5.16}$$

where

$$W_k(I_1, I_2, I_3,) = \frac{\partial W(I_1, I_2, I_3,)}{\partial I_k}, \qquad k = 1,2,3. \tag{5.17}$$

Recalling that $\mathbf{T} = \mathbf{FS}$ we may rewrite the constitutive equation (5.16) in terms of the second Piola-Kirchhoff stress tensor:

$$\mathbf{S} = 2\left((W_1 + I_1 W_2)\mathbf{1} - W_2\mathbf{C} + I_3 W_3 \mathbf{C}^{-1}\right). \tag{5.18}$$

Chapter II

Resultant laws of mechanics for shells

1. Preliminary considerations

1.1 Basic concepts and definitions. From the point of view of continuum mechanics, a shell is a three-dimensional material body B, called a shell-like body, enjoying a specific shape. The motion and deformation of the shell-like body is thus governed by general principles of continuum mechanics (Chapt. I). The general aim of shell theory is then to reduce an otherwise three-dimensional problem of continuum mechanics to the one having coordinates of a certain surface as the only independent spatial variables. Within the shell theory the shell-like body is thus geometrically represented by a distinguished surface $M \subset B$, called a shell reference surface (or a shell carrying surface). The shell-like body and the shell reference surface are thus the basic underlying concepts of shell theory.

Let us make it clear from the very beginning that a shell (in the sense of shell theory) is not merely a surface but rather a two-dimensional continuum. In general, such a continuum is defined as a shell reference surface M endowed with certain kinematic and dynamic properties, which reflect the dominant features of the body it represents. Moreover, within the general theory we shall be concerned in this work, the shell reference surface M need not be the surface in the sense of classical differential geometry.[1]

The principal aim of shell theory is to describe, how a shell-like body will deform under applied forces. The rules responsible for the deformation are imbedded in the mechanical balance laws. The first step of the shell theory is to formulate appropriate resultant balance laws. Like in continuum mechanics, the resultant balance laws for shells can be formulated in the spatial description or in the

[1] Such surfaces exclude self-intersections, self-tangencies, etc. (see Appendix D).

referential (material) description. While in continuum mechanics the emphasize is placed on the former one, the latter one is more appropriate in the formulation of shell theory. Under suitable regularity assumptions both descriptions are completely equivalent, as it has to be, since laws of mechanics refer to the body and not to its particular configurations. In this work we shall be concerned with the derivation of the shell governing equations in the material description alone. Corresponding form of shell governing equations in the spatial description can be easily derived, if necessary, by appropriate transformation rules.

For the time being, we leave unspecified the precise meaning of the shell-like body. Using the notation of Chapt. I.3, we denote by $\mathfrak{F}(P,t)$ and $\mathfrak{T}(P,t)$ the total force and the total torque, respectively, acting on any shell-like subbody $P \subset B$ at the time instant t. Then for $\Pi = M \cap P$, being the corresponding part $\Pi \subset M$ of the shell reference surface, we define the through-the-thickness resultant total force $\mathfrak{f}(\Pi,t)$ and torque $\mathfrak{t}(\Pi,t)$ by

$$\mathfrak{f}(\Pi,t) \equiv \mathfrak{F}(P,t), \qquad \mathfrak{t}(\Pi,t) \equiv \mathfrak{T}(P,t). \tag{1.1}$$

Within a purely mechanical theory the basic laws governing the motion and deformation of the body are the balance laws of linear and angular momentum. In the quasi-static case they assert that the total force $\mathfrak{F}(P,t)$ and the total torque $\mathfrak{T}(P,t)$ vanish in every (dynamic) equilibrium configuration. In view of definitions (1.1), the resultant balance laws for the shell in the quasi-static case take the form

$$\mathfrak{f}(\Pi,t) = \mathbf{0}, \qquad \mathfrak{t}(\Pi,t) = \mathbf{0}. \tag{1.2}$$

The resultant dynamic equations for the shell can be obtained exactly in the same manner. Denoting by $\mathfrak{L}(P,t)$ and $\mathfrak{A}(P,t)$ the linear and angular momentum of the shell-like subbody P, the through-the-thickness resultant linear momentum $\mathfrak{l}(\Pi,t)$ and angular momentum $\mathfrak{a}(\Pi,t)$ are defined by

$$\mathfrak{l}(\Pi,t) \equiv \mathfrak{L}(P,t), \qquad \mathfrak{a}(\Pi,t) \equiv \mathfrak{A}(P,t). \tag{1.3}$$

Then the resultant balance laws of linear and angular momentum for the shell take the form

$$\mathfrak{f}(\Pi,t) = \dot{\mathfrak{l}}(\Pi,t), \qquad \mathfrak{t}(\Pi,t) = \dot{\mathfrak{a}}(\Pi,t). \tag{1.4}$$

If the motion of the shell is very slow, the inertial forces $\dot{\mathfrak{l}}(\Pi,t)$ and $\dot{\mathfrak{a}}(\Pi,t)$ can be neglected. Then (1.4) reduces to the quasi-static form (1.2). In this work we

confine ourselves to the quasi-static case. The fully dynamic shell theory should include also such phenomena as wave propagation, formation of cracks, etc. A detailed exposition of such problems will be our concern in a separate study. Restricting our considerations to the quasi-static case, all what remains to do is to deduce the explicit forms of $\mathbf{f}(\Pi,t)$ and $\mathbf{t}(\Pi,t)$ satisfying the conditions (1.1). We shall gain this aim using a technique, which is essentially customary in shell theory. However, when applied with great care, it yields the resultant laws enjoying a generality, which cannot be found in the literature.

1.2 General features of shells. Intuitively, the concept of a shell might seem to be obvious. However, a rigorous description of shell shapes is a quite subtle analytical problem. In general, a shell has three basic identifying features: its reference surface, its thickness, and its edges. As a rule, one assumes that a shell reference surface is a smooth differential-geometric surface admitting a global parametrization.[2] However, such an assumption excludes many shell structures of engineering importance. In fact, many shell shapes cannot be described as smooth or even piecewise smooth differential-geometric surface (see Introduction).

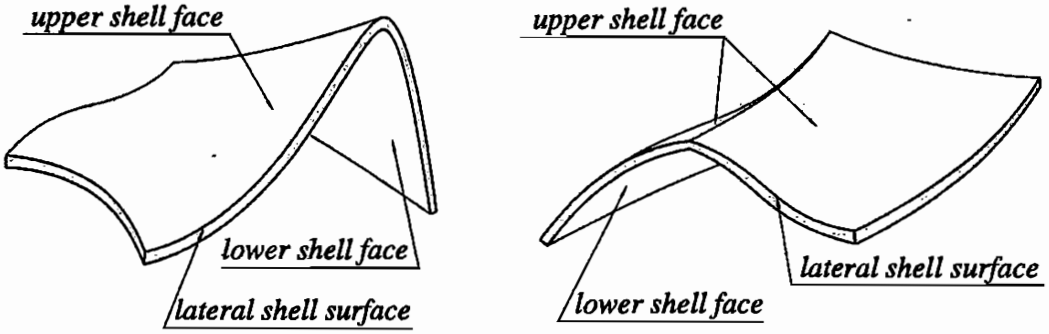

Fig. 1

The basic geometric features of shells come from the concept of a shell-like body, a three-dimensional body B, whose boundary ∂B is assumed to consist of three parts: an upper shell face M^+, a lower shell face M^- and a lateral surface (edge) $\partial B^\diamond$,

$$\partial B = M^+ \cup M^- \cup \partial B^\diamond , \tag{1.5}$$

having no internal points in common. Two examples of "nice" shell-like bodies are shown in Fig. 1. Mathematically, we shall assume B to be a regular region with

[2] See e.g. NAGHDI [1972] or ANTMAN [1976].

a piecewise smooth boundary. Thus, the shell faces $M^{\pm}$ and the lateral surface ∂B° will be assumed to be piecewise smooth surfaces. In general, neither $M^{\pm}$ nor ∂B° need to be even connected. However, we shall assume that the shell faces have no common points, i.e. $M^{+} \cap M^{-} = \varnothing$, since otherwise serious difficulties arise in the formulation of the resultant boundary conditions (as in the case shown in Fig. 2). We also note that in the case of closed shells the lateral surface ∂B° is an empty set (complete spherical shell is a typical example).

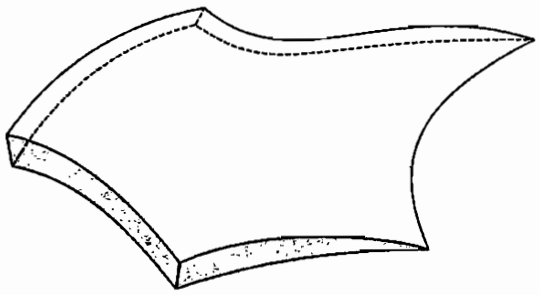

Fig. 2

Aside of the geometry, the definition of a shell-like body must also specify loads and boundary conditions. In general, loads acting on the body B consist of the external body force $\mathbf{f}(X)$ applied at each interior point $X \in B$ and the external surface force $\mathbf{t}^{*}(X)$ applied on the part $\partial B_f \subset \partial B$ of its boundary. On the complementary part $\partial B_d = \partial B \setminus \partial B_f$ of the boundary the deformation of the body is specified (Chapt. I.4). In the case of a shell-like body we assume that

$$\partial B_f = M^{+} \cup M^{-} \cup \partial B_f^{\circ}, \qquad \partial B^{\circ} = \partial B_f^{\circ} \cup \partial B_d^{\circ}. \tag{1.6}$$

The boundary of the shell reference surface is defined by $\partial M = M \cap \partial B^{\circ}$ so that

$$\partial M_f = M \cap \partial B_f^{\circ}, \qquad \partial M_d = M \cap \partial B_d^{\circ}, \qquad \partial M = \partial M_f \cup \partial M_d. \tag{1.7}$$

Thus the traction boundary conditions for the shell-like body take the form

$$\begin{aligned} &\mathbf{T}(X)\mathbf{n}^{\pm}(X) = \pm \mathbf{t}^{\pm}(X), && X \in M^{\pm}, \\ &\mathbf{T}(X)\mathbf{n}^{*}(X) = \mathbf{t}^{*}(X), && X \in \partial B_f^{\circ}. \end{aligned} \tag{1.8}$$

The minus sign in $(1.8)_1$ is conventional. The deformation of the body is described by the map $x = \chi(X)$, and the kinematic boundary condition are given by

$$x(X) = x^{*}(X), \qquad X \in \partial B_d^{\circ}. \tag{1.9}$$

Usually, the body force $\mathbf{f}(X)$ and the forces $\mathbf{t}^{\pm}(X)$ acting on the shell faces are supposed to be given as a part of data of the three-dimensional theory. But this assumption is not necessary for the subsequent formulation of the shell theory.

The above description provides a general characterization of shell-like bodies, which resemble general shell structures. However, from the point of view of shell theory this characterization is not precise enough. In fact, we shall see later on that several cases need to be considered separately:

I) Smooth shells – shells whose reference surface can be represented by a smooth regular surface.
II) Folded (non-smooth) shells – shells whose reference surface can be represented by a piecewise smooth, regular surface.
III) Branched shells – shell structures that consist of multiple shell intersections, i.e. three or more shell segments intersecting at a common juncture.
IV) Multi-shell structures – shell structures that consist of two or more distinct shells that are joined together in some technological manner along common boundaries.
V) Rod-shell structures – structures that consist of both shell-like and rod-like structural elements.

Smooth shells constitute a subclass of folded shells and they can be called collectively regular shells. Shells and shell structures belonging to the remaining three classes, called irregular shells, will be of our concern in Chapt. II.3.

1.3 Regular shell-like body. In general, we define the shell reference surface to be a geometric surface $M \subset B$ arbitrarily located within the region B. Relative to a fixed Cartesian system in space the position vector of a typical point $Y \in M$ will be denoted by Y. For simplicity, points of M and their position vectors will be denoted by Y leaving the context to make clear, which one is meant. Let us note that even in the case of non-smooth shell faces the shell reference surface M can still be defined to be smooth (Fig. 3), but such a choice of the reference surface needs not be the most appropriate one. Therefore, we shall take M to be an orientable, piecewise smooth and connected, but not necessarily simply connected, surface (Fig. 4). Thus, M is the union of disjoint smooth surface elements $M_{(k)}$ such that[3]

$$M = \bigcup_{k=1}^{K} M_{(k)}, \qquad int M_{(k)} \cap int M_{(l)} = \emptyset, \qquad k \neq l, \tag{1.10}$$

[3] See Appendix D.

where each $M_{(k)}$ is of class C^1 or higher and has a piecewise smooth boundary $\partial M_{(k)}$ consistently oriented with M. The unit normal vector at every interior point of smooth surface elements will be denoted by A_N, and we shall write $A_N^{(k)}$ for its limit value at the edges taken along paths in $M_{(k)}$.

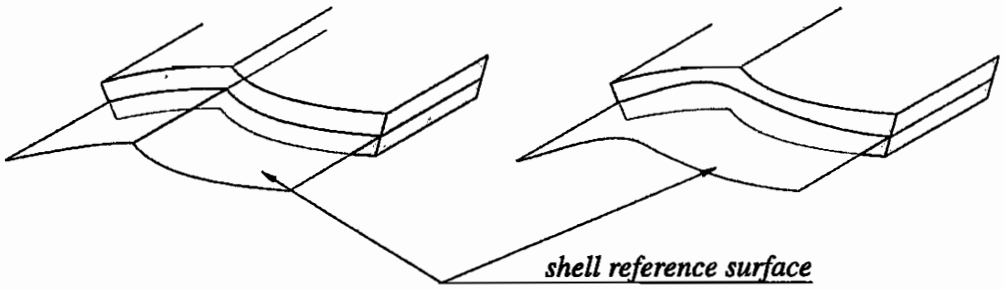

Fig. 3

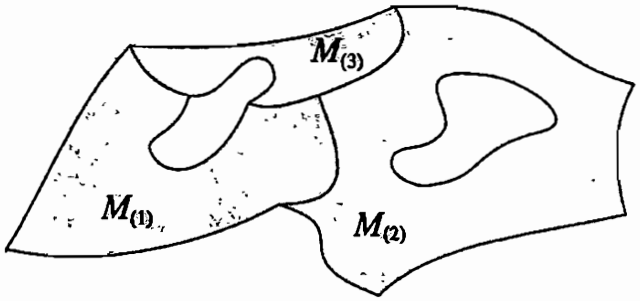

Fig. 4

Let us further assume that at each point of the shell reference surface M a unit vector $\boldsymbol{D}$ can be defined so that every point in the region B is uniquely determined by the position vector given in the form (Fig. 5)

$$\boldsymbol{X}(\boldsymbol{Y},\xi)=\boldsymbol{Y}+\xi\boldsymbol{D}(\boldsymbol{Y}), \qquad \xi\in[-h_0^-(\boldsymbol{Y}),+h_0^+(\boldsymbol{Y})], \tag{1.11}$$

where $h_0^{\pm}$ are given non-negative piecewise smooth functions on M such that

$$h_0(\boldsymbol{Y})=h_0^-(\boldsymbol{Y})+h_0^+(\boldsymbol{Y})>0. \tag{1.12}$$

The vector $\boldsymbol{D}$ is required to be not tangent to M, and it serves to define through-the-thickness fibres. Note that the functions $h_0^{\pm}$ define the location of the reference surface relative to the shell faces, whose position vectors are given by

$$\boldsymbol{X}^{\pm}(\boldsymbol{Y}) = \boldsymbol{Y} \pm h_0^{\pm}(\boldsymbol{Y})\boldsymbol{D}(\boldsymbol{Y}). \tag{1.13}$$

We shall refer to ξ as through-the-thickness or "normal" coordinate, and we call h_0 the initial shell thickness. It is clear that the thickness of the shell defined in this way depends upon the choice of the vector $\boldsymbol{D}$. We also note that the field $\boldsymbol{D}$ on M can be defined in a continuous manner even if M is not smooth. Moreover, without loss of generality we can assume that the lateral surface $\partial B^{\diamond}$ be a ruled surface, whose generators at each point $\boldsymbol{Y} \in \partial M$ are defined by (1.11).

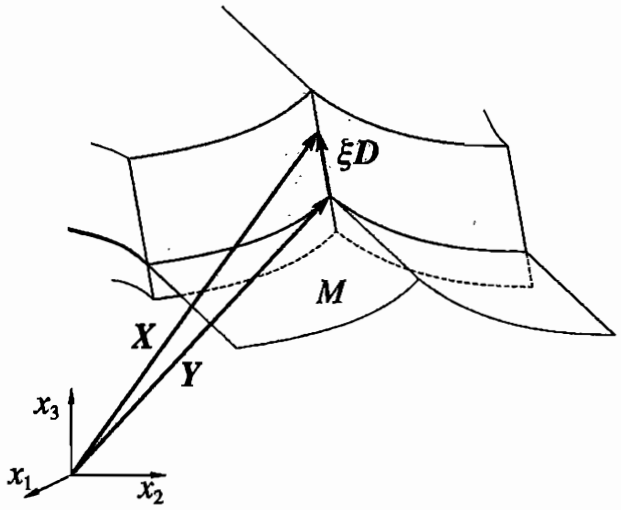

Fig. 5

When the above conditions are satisfied, we call B a regular shell-like body and, until Chapt. II.3, B will always have this meaning. Let us note that the shell faces $M^{\pm}$ of the regular shell-like body B must be connected (but not necessarily simply connected), while the lateral surface $\partial B^{\diamond}$ need not be even connected (Fig. 6). We also note, that as a special case, we can take $\boldsymbol{D}$ to coincide with the unit normal vector $\boldsymbol{A}_N$. However, this is possible only if M is smooth. Moreover, if the shell is composed of a single homogenous material, the reference surface can be taken to be the middle surface, that is the locus of all points equidistant to the shell faces. However, if the shell is layered or of other nonhomogeneous structure, in the case of relatively thick shells, in the analysis of contact problems, etc., it may be more convenient to take one of the shell faces as the reference surface. The shell reference surface can also be defined as a so-called neutral geometric surface (analogously to the neutral axis of a beam), whose position vector is determined by

$$Y = \int_{-}^{+} X(Y,\xi)\eta(Y,\xi)\,d\xi\,, \qquad \int_{-}^{+} \equiv \int_{h\bar{0}}^{h\dot{0}}\,, \tag{1.14}$$

where η is any given positive scalar function. For general considerations, it is then reasonable to leave the particular choice of the reference surface unspecified. We also note that even though the formulation of the shell theory is expected to be reasonable only for relatively thin bodies, no assumption of this kind is neither necessary nor invoked in this work.

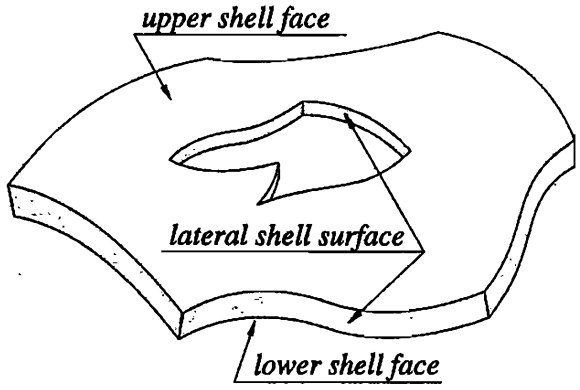

Fig. 6

1.4 Shell reference surface. At each regular point $Y \in M$ of a piecewise smooth surface M the tangent space $T_Y M$, i.e. the two-dimensional vector space, geometrically represented by the tangent plane (Fig. 7), is well defined. The ambient space $\mathcal{E}$, in which the surface M is immersed, induces in a natural way the direct decomposition of its translational space:[4]

$$E = T_Y\mathcal{E} = T_Y M \oplus T_Y M^{\perp}\,, \tag{1.15}$$

where $T_Y M^{\perp}$ denotes the orthogonal complement of the tangent space $T_Y M$. We shall then denote by

$$\boldsymbol{I}(Y): T_Y M \to E\,, \qquad \boldsymbol{P}(Y): E \to T_Y M\,, \tag{1.16}$$

the inclusion and projection operators, respectively. The tangent space $T_Y M$ with an inner product, being the restriction to $T_Y M$ of the inner product of E, becomes the two-dimensional Euclidean vector space. In turn, the inner product makes the

[4] Many concepts and definitions we shall be using below are treated in more detail in GURTIN AND MURDOCH [1975] and MURDOCH AND COHEN [1979], see also Appendix D.

surface M into a Riemannian manifold with the metric $A_Y = A(Y)$ defined at each point of M by

$$A_Y(u,v) = u \cdot v, \qquad \forall\, u,v \in T_Y M\,. \tag{1.17}$$

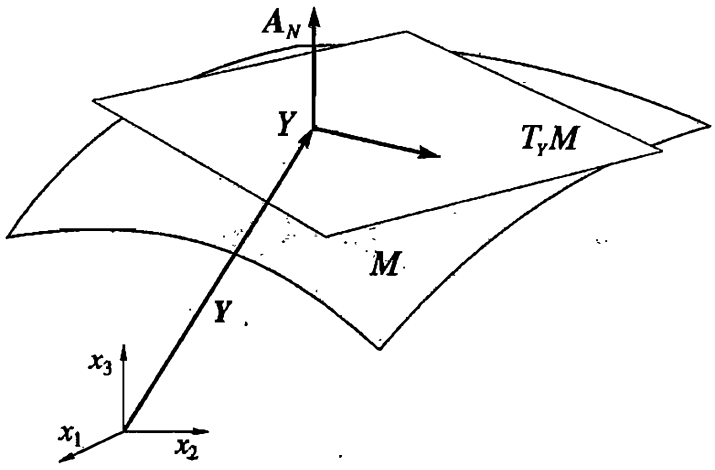

Fig. 7

The orientation of M is determined by the Gauss map, which assigns to every point $Y \in M$ a unit normal vector $A_N(Y)$. If M is of class C^2 or higher, the Gauss map is differentiable and its tangential gradient is the curvature tensor

$$B(Y) = P(Y)\nabla_s A_N(Y). \tag{1.18}$$

The principal invariants of B are the mean curvature H and the Gaussian curvature K, which are given by

$$H(Y) = \frac{1}{2} trB(Y), \qquad K(Y) = detB(Y). \tag{1.19}$$

Let us note that in the case of a piecewise smooth shell reference surface the definitions (1.15)-(1.19) make sense only at interior points of every smooth element $M_{(k)}$.

While all subsequent considerations will be carried out in a coordinate-free form, parallelly we shall also present their component form. To this end we introduce suitable coordinate systems. This also serves to fix notation.[5]

Locally, but not globally, the shell reference surface M (more precisely, any smooth surface element) may be parametrized by surface coordinates $(\xi^\Lambda, \Lambda = 1,2)$ chosen in any convenient way. Then its position vector $\boldsymbol{Y}$ can be expressed as a given function of ξ^Λ in the form (Fig. 8)

$$\boldsymbol{Y}(\xi^\Lambda) = Y_K(\xi^\Lambda)\boldsymbol{e}_K\,. \tag{1.20}$$

Assuming that (1.20) be a differentiable function of surface coordinates, the natural base vectors and the reciprocal base vectors at each regular point of M are defined in the usual way

$$\boldsymbol{A}_\Lambda(Y) = \boldsymbol{Y}_{,\Lambda}(Y)\,, \qquad \boldsymbol{A}^\Lambda(Y) \cdot \boldsymbol{A}_\Gamma(Y) = \delta^\Lambda_\Gamma\,. \tag{1.21}$$

It should be noted that the position vector $\boldsymbol{Y}$ of M is an E-valued function, and so are the partial derivatives $\boldsymbol{Y}_{,\Lambda}$ at every point of M. On the other hand, the natural base vectors are elements of the tangent space $T_Y M$. Therefore, it is not fully correct to write $\boldsymbol{A}_\Lambda = \boldsymbol{Y}_{,\Lambda}$. The correct definition is $\boldsymbol{A}_\Lambda = \boldsymbol{P}\boldsymbol{Y}_{,\Lambda}$, where $\boldsymbol{P}$ denotes the projection operator.

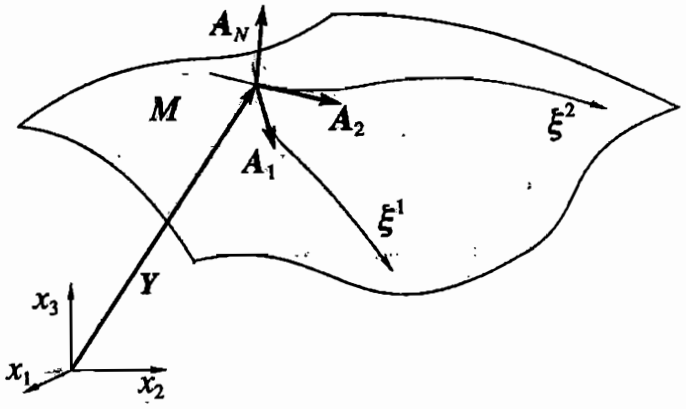

Fig. 8

For the chosen surface coordinates the metric tensor of M can be expressed in the familiar form

[5] Where feasible, we shall adopt the notation, which became standard in continuum mechanics and in part of the shell literature, e.g. TRUESDELL AND TOUPIN [1960], NAGHDI [1972].

$$A = A_{\Lambda\Gamma} A^{\Lambda} \otimes A^{\Gamma} = A^{\Lambda\Gamma} A_{\Lambda} \otimes A_{\Gamma} , \tag{1.22}$$

with components being just the inner product of the natural base vectors, and with a positive determinant:

$$A_{\Lambda\Sigma} = A_{\Lambda} \bullet A_{\Sigma} , \qquad A^{\Lambda\Sigma} = A^{\Lambda} \bullet A^{\Sigma} , \qquad A = det A_{\Lambda\Sigma} > 0 . \tag{1.23}$$

Similarly, the curvature tensor (1.18) of M has the component form

$$B = B_{\Lambda\Gamma} A^{\Lambda} \otimes A^{\Gamma} , \qquad B_{\Lambda\Gamma} = Y_{,\Lambda\Gamma} \bullet A_N . \tag{1.24}$$

Then the mean and Gaussian curvatures of M are given by

$$H = \frac{1}{2} A^{\Lambda\Sigma} B_{\Lambda\Sigma} = \frac{1}{2} B^{\Lambda}_{\Lambda} , \qquad K = det\, B^{\Lambda}_{\Sigma} = \frac{1}{2} \epsilon^{\Lambda\Theta} \epsilon^{\Sigma\Omega} B_{\Lambda\Sigma} B_{\Theta\Omega} . \tag{1.25}$$

Here the surface permutation symbols at any point of M are defined in the standard way

$$\epsilon_{\Lambda\Gamma} = \sqrt{A}\, e_{\Lambda\Gamma} , \qquad \epsilon^{\Lambda\Gamma} = \frac{1}{\sqrt{A}} e_{\Lambda\Gamma} , \tag{1.26}$$

where $e_{12} = -e_{21} = 1$ and $e_{11} = e_{22} = 0$. Noting that $\epsilon_{\Lambda\Gamma} = (Y_{,\Lambda} \times Y_{,\Gamma}) \bullet A_N$ the unit normal vector to M can be expressed in the classical form

$$A_N = \frac{1}{2} \epsilon^{\Lambda\Gamma} Y_{,\Lambda} \times Y_{,\Gamma} . \tag{1.27}$$

Moreover, we have the standard identities

$$A^{\Lambda\Gamma} = \epsilon^{\Lambda\Phi} \epsilon^{\Gamma\Psi} A_{\Phi\Psi} , \qquad \epsilon^{\Lambda\Phi} \epsilon^{\Gamma\Psi} = A^{\Lambda\Gamma} A^{\Phi\Psi} - A^{\Lambda\Psi} A^{\Gamma\Phi} , \tag{1.28}$$

together with a number of identities, which follow from the standard operation of rising and lowering indices.

Any piecewise smooth curve on M, such as the boundary curve ∂M, can be specified for the chosen surface coordinates implicitly, $\sigma(\xi^{\Lambda}) = 0$, or in a parametric form, $\xi^{\Lambda} = \xi^{\Lambda}(S)$. Here we assume that the curve is given in the parametric form $Y(S) = Y(\xi^{\Lambda}(S))$, where S denotes the arc length parameter along the curve. Then at each regular point of it we can define the orthonormal triad $\{\tau, \nu, A_N\}$ by (Fig. 9)

$$\tau = PY' = \tau^{\Lambda} A_{\Lambda} , \qquad \nu = P(A_N \times Y') = \nu_{\Lambda} A^{\Lambda} , \tag{1.29}$$

where τ is the tangent vector and ν the outward normal vector, both lying in the tangent plane to the reference surface at the underlying point. A prime will invariably stand for the derivative with respect to the arc length parameter. From the chain rule we have the familiar relations

$$\tau^\Lambda = \frac{d\xi^\Lambda}{dS}, \qquad \nu_\Lambda = \epsilon_{\Lambda\Gamma}\tau^\Gamma \tag{1.30}$$

and

$$A_\Lambda = \tau_\Lambda \tau + \nu_\Lambda \nu. \tag{1.31}$$

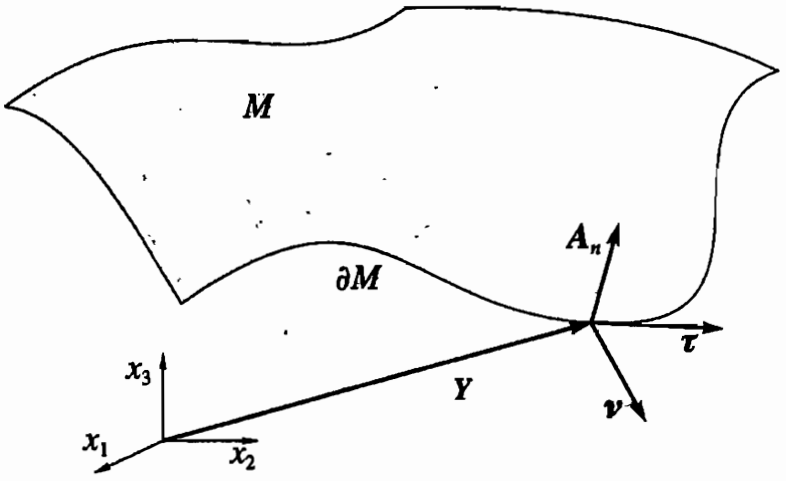

Fig. 9

1.5 Spatial coordinates. In principle, any choice of coordinates in the shell space B and on the reference surface M needs not be related at all. However, it is convenient and reasonable to assume that the local coordinates $(\xi^\Lambda) = (\xi^\Lambda, \xi)$ in B are chosen in such a way that the reference surface M is defined by the equation $\xi = 0$. Then the position vector (1.11) is given by

$$X(\xi^\Lambda, \xi) = Y(\xi^\Lambda) + \xi D(\xi^\Lambda), \qquad \xi \in [-h_0^-(\xi^\Lambda), +h_0^+(\xi^\Lambda)]. \tag{1.32}$$

From (1.32) all relevant geometric relations in the shell space can be derived in a standard way. In particular, the natural base vectors at any point $X \in B$ are obtained in the form (Fig. 10)

$$G_\Lambda = X_{,\Lambda} = A_\Lambda + \xi D_{,\Lambda}, \qquad G_3 = X_{,\xi} = D. \tag{1.33}$$

Here and in the sequel partial differentiation with respect to ξ will interchangeably be denoted by $(.)_{,3}$ or $(.)_{,\xi}$ as it will be convenient for typographical reasons. Taking further into account that

$$\sqrt{G} = (\boldsymbol{G}_1 \times \boldsymbol{G}_2) \cdot \boldsymbol{G}_3 = \frac{1}{2} \epsilon^{\Delta\Lambda} \sqrt{A} (\boldsymbol{G}_\Delta \times \boldsymbol{G}_\Lambda) \cdot \boldsymbol{D}, \tag{1.34}$$

we have

$$\mu \equiv \sqrt{\frac{G}{A}} = \boldsymbol{A}_N \cdot \boldsymbol{D} + \xi \epsilon^{\Lambda\Gamma} (\boldsymbol{Y}_{,\Lambda} \times \boldsymbol{D}_{,\Gamma}) \cdot \boldsymbol{D} + \frac{1}{2} \xi^2 \epsilon^{\Lambda\Gamma} (\boldsymbol{D}_{,\Lambda} \times \boldsymbol{D}_{,\Gamma}) \cdot \boldsymbol{D}. \tag{1.35}$$

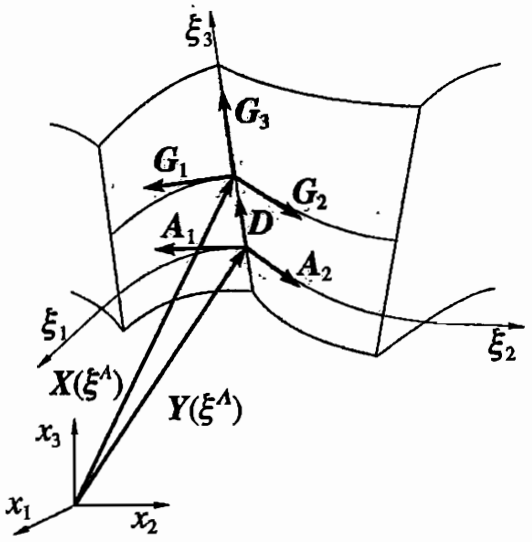

Fig. 10

Now the reciprocal base vectors can be expressed in the form

$$\boldsymbol{G}^\Lambda = \frac{1}{2} \mu^{-1} \epsilon^{\Gamma\Lambda} \boldsymbol{G}_\Gamma \times \boldsymbol{D}, \qquad \boldsymbol{G}^3 = \frac{1}{2} \mu^{-1} \epsilon^{\Lambda\Gamma} \boldsymbol{G}_\Lambda \times \boldsymbol{G}_\Gamma. \tag{1.36}$$

Since $\boldsymbol{D}$ is a field of unit vectors, $\boldsymbol{D} \cdot \boldsymbol{D}_{,\Lambda} = 0$, components of the metric tensor for the chosen coordinates in B are

$$\begin{aligned} G_{\Lambda\Gamma} &= A_{\Lambda\Gamma} + \xi (\boldsymbol{Y}_{,\Lambda} \cdot \boldsymbol{D}_{,\Gamma} + \boldsymbol{Y}_{,\Gamma} \cdot \boldsymbol{D}_{,\Lambda}) + \xi^2 \boldsymbol{D}_{,\Lambda} \cdot \boldsymbol{D}_{,\Gamma}, \\ G_{\Lambda 3} &= \boldsymbol{Y}_{,\Lambda} \cdot \boldsymbol{D}, \\ G_{33} &= 1. \end{aligned} \tag{1.37}$$

Reciprocal components of the metric tensor can be obtained by taking the inner product of the reciprocal base vectors given by (1.36).

As special case we can take the unit vector $\boldsymbol{D}$ to be the unit normal vector $\boldsymbol{A}_N$ to the reference surface. This choice defines the so-called normal coordinates in the shell space with the position vector given by

$$\boldsymbol{X}(\xi^\Lambda,\xi)=\boldsymbol{Y}(\xi^\Lambda)+\xi\boldsymbol{A}_N(\xi^\Lambda), \qquad \xi\in[-h_0^-(\xi^\Lambda),+h_0^+(\xi^\Lambda)]. \tag{1.38}$$

Then the natural base vectors can be expressed in the form

$$\boldsymbol{G}_\Lambda=\mu_\Lambda^\Gamma\boldsymbol{A}_\Gamma, \qquad \boldsymbol{G}^\Lambda=(\mu^{-1})_\Gamma^\Lambda\boldsymbol{A}^\Gamma, \qquad \boldsymbol{G}_3=\boldsymbol{G}^3=\boldsymbol{A}_N, \tag{1.39}$$

where the so-called shifters are given by

$$\begin{aligned}\mu_\Gamma^\Lambda&=\delta_\Gamma^\Lambda-\xi B_\Gamma^\Lambda,\\ (\mu^{-1})_\Lambda^\Gamma&=\frac{1}{\mu}[\delta_\Lambda^\Gamma+\xi(B_\Lambda^\Gamma-2H\delta_\Lambda^\Gamma)],\end{aligned} \tag{1.40}$$

and

$$\mu=\sqrt{\frac{G}{A}}=\det\mu_\Gamma^\Lambda=1-2\xi H+2\xi^2K. \tag{1.41}$$

Let us recall that H and K denote the mean and Gaussian curvatures of the reference surface M, respectively. The component of the metric tensor at any point of the shell space can now be obtained in the usual way.[6]

1.6 Volume and area elements. We shall consistently denote by $dV(X)$, $dA(Y)$ and $dS(Y)$ the differential volume element at any point of the region B, the differential area element of the reference surface M and the differential line element of any curve on M, respectively. In terms of the chosen coordinate system $(\xi^\Lambda)=(\xi^\Lambda,\xi)$ they are given by

$$dS=\sqrt{A_{\Lambda\Gamma}d\xi^\Lambda d\xi^\Gamma}, \qquad dA=\sqrt{A}d\xi^1d\xi^2, \qquad dV=\sqrt{G}d\xi^1d\xi^2d\xi. \tag{1.42}$$

By simple implication we have

$$dV(Y,\xi)=\mu(Y,\xi)d\xi dA, \qquad \mu(Y,\xi)=\sqrt{\frac{G(Y,\xi)}{A(Y)}}, \tag{1.43}$$

[6] NAGHDI [1972].

where the invariant μ is given by the formula (1.35).

In the subsequent considerations we shall also need the relations between oriented area elements of the shell faces and the shell lateral surface, and corresponding elements of the reference surface and its boundary.

The position vectors of the shell faces are given by

$$\boldsymbol{X}^{\pm}(\xi^{\Lambda}) = \boldsymbol{Y}(\xi^{\Lambda}) \pm \mathrm{h}_0^{\pm}(\xi^{\Lambda})\boldsymbol{D}(\xi^{\Lambda}), \tag{1.44}$$

and the position vector of the lateral surface can be expressed in the form

$$\boldsymbol{X}^{\diamond}(S,\xi) = \boldsymbol{Y}(S) + \xi \boldsymbol{D}(S), \qquad \xi \in [-h_0^{-}(S), +h_0^{+}(S)]. \tag{1.45}$$

The corresponding oriented area elements are defined by

$$\begin{aligned} d\boldsymbol{A}^{\pm} &= \mathbf{n}^{\pm} dA^{\pm} = \boldsymbol{X}^{\pm}{}_{,1} \times \boldsymbol{X}^{\pm}{}_{,2}\, d\xi^1 d\xi^2, \\ d\boldsymbol{A}^{\diamond} &= \mathbf{n}^{\diamond} dA^{\diamond} = (\boldsymbol{X}^{\diamond})' \times \boldsymbol{X}^{\diamond}{}_{,\xi}\, dS d\xi, \end{aligned} \tag{1.46}$$

and they can be expressed in the form

$$d\boldsymbol{A}^{\pm} = \frac{1}{2}\epsilon^{\Lambda\Gamma} \boldsymbol{X}^{\pm}{}_{,\Lambda} \times \boldsymbol{X}^{\pm}{}_{,\Gamma}\, dA, \qquad d\boldsymbol{A}^{\diamond} = (\boldsymbol{X}^{\diamond})' \times \boldsymbol{D} dS d\xi. \tag{1.47}$$

Keeping in mind that the position vectors of the shell faces depend on the surface coordinates also through the variable thickness, the partial derivatives of (1.44) are

$$\boldsymbol{X}^{\pm}{}_{,\Lambda} = \boldsymbol{A}_{\Lambda} \pm h_0^{\pm} \boldsymbol{D}_{,\Lambda} \pm h_0^{\pm}{}_{,\Lambda} \boldsymbol{D}. \tag{1.48}$$

Differentiating the position vector (1.45) of the lateral surface with respect to the arc length parameter we should take into account that $\xi^{\Lambda} = \xi^{\Lambda}(S)$. Then from the chain rule we obtain

$$(\boldsymbol{X}^{\diamond})' = \frac{d\xi^{\Lambda}}{dS} \boldsymbol{X}^{\diamond}{}_{,\Lambda} = \tau^{\Lambda} \boldsymbol{G}_{\Lambda} = -\epsilon^{\Gamma\Lambda} \boldsymbol{G}_{\Gamma} \nu_{\Lambda}. \tag{1.49}$$

Taking into account the definition of the natural base vectors in the shell space and on the reference surface we thus have

$$d\boldsymbol{A}^{\pm} = \pm (\boldsymbol{G}^3 \mp h_0^{\pm}{}_{,\Lambda} \boldsymbol{G}^{\Lambda})^{\pm} \mu^{\pm} dA, \qquad d\boldsymbol{A}^{\diamond} = \boldsymbol{G}^{\Lambda} \nu_{\Lambda} \mu d\xi dS. \tag{1.50}$$

By implication of (1.50), the unoriented (scalar) differential areae of the shell faces and the shell lateral surface are obtained in the form

$$dA^{\pm} = \alpha^{\pm} dA\,, \qquad dA^{\diamond} = \alpha^{\diamond} d\xi dS\,, \tag{1.51}$$

where

$$\begin{aligned} \alpha^{\pm} &= \pm\,\mu^{\pm}\sqrt{(G^{33} \mp 2h_0^{\pm}{}_{,\Lambda}\, G^{3\Lambda} + h_0^{\pm}{}_{,\Lambda}\, h_0^{\pm}{}_{,\Gamma}\, G^{\Lambda\Gamma})^{\pm}}\,, \\ \alpha^{\diamond} &= \mu\sqrt{G^{\Lambda\Gamma}\nu_{\Lambda}\nu_{\Gamma}}\,. \end{aligned} \tag{1.52}$$

2. Resultant laws - regular shell-like bodies

2.1 Regular shell-like subbody. The mechanical balance laws of continuum mechanics are postulated to hold not only for the whole body but also for every subbody. In order to derive corresponding resultant laws we shall consider a shell-like subbody $P \subset B$, which is regular in the sense of Sect. 1.3. This means that the position vector of every point in P can be expressed in the form (1.11), and that the boundary of P is the union of three disjoint parts, the faces $\Pi^{\pm} \subset M^{\pm}$ and the lateral surface $\partial P^{\diamond}$ so that

$$\partial P = \Pi^{+} \cup \Pi^{-} \cup \partial P^{\diamond}\,. \tag{2.1}$$

By definition, $\Pi \subset M$ is a part of the shell reference surface corresponding to the selected region P enclosed by a curve $\partial\Pi$, i.e.

$$\Pi = M \cap P\,, \qquad \partial\Pi = M \cap \partial P^{\diamond}\,. \tag{2.2}$$

Moreover, the lateral surface $\partial P^{\diamond}$ is a ruled surface described in the same way as the lateral surface of a regular shell-like body. A "nice" shell-like subbody is shown in Fig. 11.

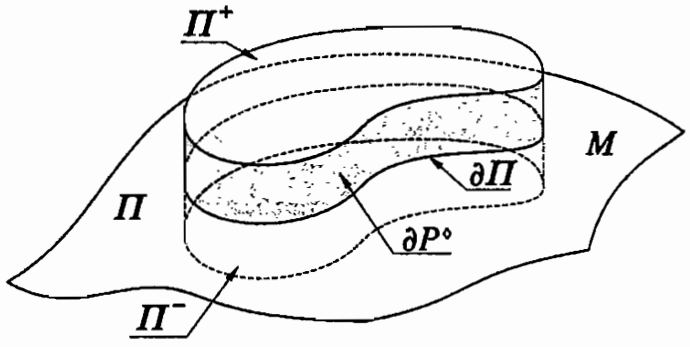

Fig. 11

The forces acting on P consist of the external body force $\mathbf{f}$ and the surface contact force $\mathbf{t}_N$. By virtue of Cauchy's theorem the contact force on the faces and on the lateral surface are given in terms of the first Piola-Kirchhoff stress tensor $\mathbf{T}$ by

$$\mathbf{Tn}^{\pm} = \pm \mathbf{t}^{\pm}, \qquad X \in \Pi^{\pm}, \qquad \mathbf{Tn}^{\diamond} = \mathbf{t}_N, \qquad X \in \partial P^{\diamond}. \tag{2.3}$$

In view of (2.1) and (2.3) the total force and the total torque acting on the subbody P can be written in the form

$$\begin{aligned} \mathfrak{F}(P) &= \int_P \mathbf{f}\, dV + \iint_{\Pi^+} \mathbf{t}_N^+ dA^+ - \iint_{\Pi^-} \mathbf{t}_N^- dA^- + \int_{\partial P^{\diamond}} \mathbf{Tn}^{\diamond} dA^{\diamond}, \\ \mathfrak{T}(P) &= \int_P x \times \mathbf{f}\, dV + \iint_{\Pi^+} x^+ \times \mathbf{t}_N^+ dA^+ - \iint_{\Pi^-} x^- \times \mathbf{t}_N^- dA^- + \int_{\partial P^{\diamond}} x \times \mathbf{Tn}^{\diamond} dA^{\diamond}, \end{aligned} \tag{2.4}$$

where $x = \chi(X) = \chi(Y, \xi)$ and

$$x^{\pm}(Y) = \chi\big(X(Y, \pm h_0^{\pm}(Y))\big). \tag{2.5}$$

In order to reduce (2.4) in an exact manner to a two-dimensional form appropriate for shell theory, we shall represent the three-dimensional deformation of the shell-like body in the form (Fig. 12)

$$x(Y, \xi) = \chi(X(Y, \xi)) = y(Y) + \zeta(Y, \xi). \tag{2.6}$$

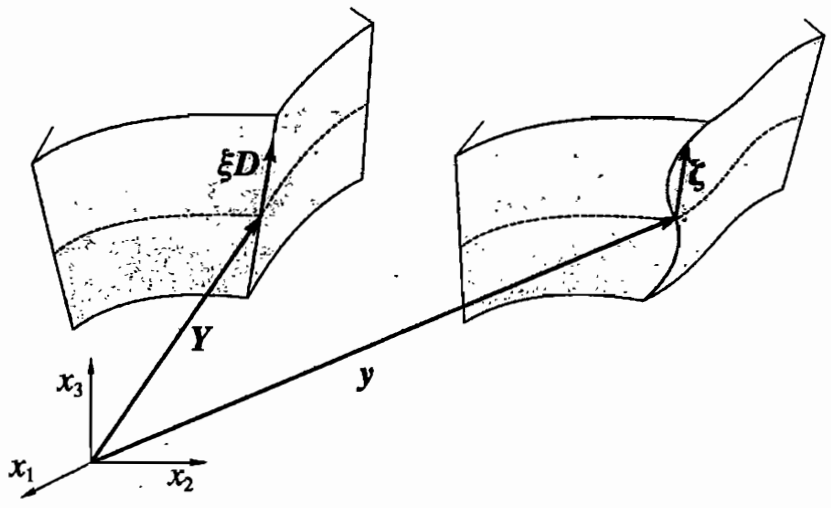

Fig. 12

Here $\chi: M \to \mathcal{E}$ with $y = \chi(Y)$ is the deformation map of the shell reference surface, and ζ is an unknown function which defines the location of the particle relative to the corresponding point on the deformed reference surface. It must be stressed that the representation (2.6) is purely formal and imposes no restrictions on the three-dimensional deformation of the shell-like body. This is reflected in the dependence of the function ζ on the through-the-thickness coordinate ξ. With the help of (2.6) the total force and the total torque (2.4) can easily be reduced to the two-dimensional form satisfying the requirements (1.1).

2.2 Resultant forces and couples. The contact forces $\mathbf{Tn}^\diamond$ acting on the lateral surface $\partial P^\diamond$ entail the internal stresses in the body and they can be reduced to the resultant force and couple along the corresponding part of the boundary curve $\partial \Pi$. To this end, we make use of (2.6) together with the geometric relation $(1.51)_2$ for the differential area element $dA^\diamond$, so that integrals in (2.4) over the lateral surface $\partial P^\diamond$ can be rewritten in the form

$$\begin{aligned} \int_{\partial P^\diamond} \mathbf{Tn}^\diamond \, dA^\diamond &= \int_{\partial \Pi} \Big(\int_-^+ \mathbf{Tn}^\diamond \alpha^\diamond d\xi \Big) dS \,. \\ \int_{\partial P^\diamond} \boldsymbol{x} \times \mathbf{Tn}^\diamond \, dA^\diamond &= \int_{\partial \Pi} \Big(\int_-^+ \boldsymbol{\zeta} \times \mathbf{Tn}^\diamond \alpha^\diamond d\xi + \boldsymbol{y} \times \Big(\int_-^+ \mathbf{Tn}^\diamond \alpha^\diamond d\xi \Big) \Big) dS \,. \end{aligned} \tag{2.7}$$

Now we define the resultant stress vector and the resultant stress couple vector by

$$\boldsymbol{n}_\nu(Y, \partial \Pi) = \int_-^+ \mathbf{Tn}^\diamond \alpha^\diamond d\xi \,, \qquad \boldsymbol{m}_\nu(Y, \partial \Pi) = \int_-^+ \boldsymbol{\zeta} \times \mathbf{Tn}^\diamond \alpha^\diamond d\xi \,, \tag{2.8}$$

so that

$$\begin{aligned} \int_{\partial P^\diamond} \mathbf{Tn}^\diamond \, dA^\diamond &= \int_{\partial \Pi} \boldsymbol{n}_\nu \, dS \,, \\ \int_{\partial P^\diamond} \boldsymbol{x} \times \mathbf{Tn}^\diamond \, dA^\diamond &= \int_{\partial \Pi} (\boldsymbol{m}_\nu + \boldsymbol{y} \times \boldsymbol{n}_\nu) \, dS \,. \end{aligned} \tag{2.9}$$

This shows that $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ entail the interaction along the boundary curve $\partial \Pi$ equipollent to the action of the stress vector $\mathbf{Tn}^\diamond$ upon the corresponding part of the lateral surface $\partial P^\diamond$ coinciding with $\partial \Pi$ on the reference surface.

In the same manner we can next reduce the remaining integrals in (2.4) to statically equivalent forces and couples defined over the shell reference surface. With the help of geometric relation (1.43), and making use of (2.6), the volume integrals in (2.4) can be written as

$$\begin{aligned}\int_P \mathbf{f}\, dV &= \iint_{\Pi}\Big(\int_-^+ \mathbf{f}\,\mu d\xi\Big)dA\,,\\ \int_P \boldsymbol{x}\times\mathbf{f}\, dV &= \iint_{\Pi}\Big(\int_-^+ \boldsymbol{\zeta}\times\mathbf{f}\,\mu d\xi + \mathbf{y}\times\Big(\int_-^+ \mathbf{f}\,\mu d\xi\Big)\Big)dA\,.\end{aligned} \tag{2.10}$$

Similarly, with the use of geometric relation $(1.51)_1$ the integrals in (2.4) over the shell faces can be written in the form

$$\begin{aligned}\iint_{\Pi^+}\mathbf{t}_N^+\, dA^+ - \iint_{\Pi^+}\mathbf{t}_N^-\, dA^- &= \iint_{\Pi}\big(\alpha^+\mathbf{t}_N^+ - \alpha^-\mathbf{t}_N^-\big)dA\,,\\ \iint_{\Pi^+}\boldsymbol{x}^+\times\mathbf{t}_N^+\, dA^+ - \iint_{\Pi^+}\boldsymbol{x}^-\times\mathbf{t}_N^-\, dA^- &= \iint_{\Pi}\big(\alpha^+\boldsymbol{\zeta}^+\times\mathbf{t}_N^+ - \alpha^-\boldsymbol{\zeta}^-\times\mathbf{t}_N^-\big)dA\\ &\quad + \iint_{\Pi}\big(\mathbf{y}\times\big(\alpha^+\mathbf{t}_N^+ - \alpha^-\mathbf{t}_N^-\big)\big)dA\,,\end{aligned} \tag{2.11}$$

where, analogously to (2.5), we write

$$\boldsymbol{\zeta}^{\pm}(Y) = \boldsymbol{\zeta}\big(X(Y, \pm h_0^{\pm}(Y))\big). \tag{2.12}$$

The body force $\mathbf{f}$ and the surface forces $\mathbf{t}_N^{\pm}$ acting on the shell faces are usually given as a part of data of the three-dimensional theory. Accordingly, they can be combined together in the definition of the resultant surface force and couple vectors:

$$\begin{aligned}\boldsymbol{p}(Y) &= \int_-^+ \mathbf{f}\,\mu d\xi + \alpha^+\mathbf{t}_N^+ - \alpha^-\mathbf{t}_N^-\,,\\ \boldsymbol{l}(Y) &= \int_-^+ \boldsymbol{\zeta}\times\mathbf{f}\,\mu d\xi + \alpha^+\boldsymbol{\zeta}^+\times\mathbf{t}_N^+ - \alpha^-\boldsymbol{\zeta}^-\times\mathbf{t}_N^-\,.\end{aligned} \tag{2.13}$$

This shows that $\boldsymbol{p}$ and $\boldsymbol{l}$ are statically equivalent to the external body force and the external surface forces acting on the shell faces.

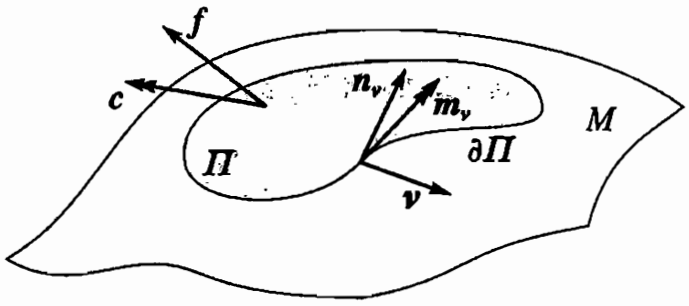

Fig. 13

When definitions (2.13) and (2.8) are substituted into (2.4), the resultant total force and the resultant total torque acting on the part $\Pi \subset M$ of the shell reference surface corresponding to the subbody $P \subset B$, are obtained in the form (Fig. 13)

$$\begin{aligned} \mathbf{f}(\Pi) &= \iint_{\Pi} \boldsymbol{p}\, dA + \int_{\partial\Pi} \boldsymbol{n}_{\nu}\, dS\,, \\ \mathbf{t}(\Pi) &= \iint_{\Pi} (\boldsymbol{l} + \boldsymbol{y} \times \boldsymbol{p})\, dA + \int_{\partial\Pi} (\boldsymbol{m}_{\nu} + \boldsymbol{y} \times \boldsymbol{n}_{\nu})\, dS\,. \end{aligned} \tag{2.14}$$

This derivation shows that the resultant force and torque (2.14) satisfy the requirement (1.1), and the resultant laws of mechanics (1.2) ensure the overall equilibrium of any shell-like subbody of finite thickness bounded by the shell faces and the lateral surface.

2.3 Resultant stress and couple stress tensors. It can be shown that at each regular point of the reference surface M the resultant stress vector $\boldsymbol{n}_{\nu}(\boldsymbol{Y};\partial\Pi)$ and the resultant stress couple vector $\boldsymbol{m}_{\nu}(\boldsymbol{Y};\partial\Pi)$ depend upon the curve $\partial\Pi$ only through the unit normal vector $\boldsymbol{\nu}(\boldsymbol{Y}) \in T_{\boldsymbol{Y}} M$. Moreover, there exist the resultant stress tensor $\boldsymbol{N}(\boldsymbol{Y}) \in E \otimes T_{\boldsymbol{Y}} M$ and the resultant stress couple tensor $\boldsymbol{M}(\boldsymbol{Y}) \in E \otimes T_{\boldsymbol{Y}} M$ such that

$$\boldsymbol{n}_{\nu}(\boldsymbol{Y};\partial\Pi) = \boldsymbol{N}(\boldsymbol{Y})\boldsymbol{\nu}(\boldsymbol{Y})\,, \qquad \boldsymbol{m}_{\nu}(\boldsymbol{Y};\partial\Pi) = \boldsymbol{M}(\boldsymbol{Y})\boldsymbol{\nu}(\boldsymbol{Y})\,. \tag{2.15}$$

This theorem, being the analogue of the Cauchy theorem in continuum mechanics, can be proved within purely two-dimensional considerations, or it can be obtained as implication of definitions (2.8). The latter way provides a clear three-dimensional interpretation of the resultant stress and stress couple tensors and is presented below.

For any choice of coordinates $(\xi^{\Lambda}) = (\xi^{\Lambda}, \xi)$ in the reference configuration of the shell-like body, the first Piola-Kirchhoff stress tensor $\mathbf{T}$ can be expressed in the form

$$\mathbf{T} = \mathbf{t}^{\Lambda} \otimes \boldsymbol{G}_{\Lambda} + \mathbf{t}^{3} \otimes \boldsymbol{G}_{3}\,. \tag{2.16}$$

Taking into account the geometric relations $(1.50)_2$ for the oriented area element of the lateral surface we have

$$\mathbf{T}\mathbf{n}^{\circ} dA^{\circ} = \mathbf{T}\boldsymbol{G}^{\Lambda} \nu_{\Lambda} \mu\, d\xi\, dS = \mathbf{t}^{\Lambda} \nu_{\Lambda} \mu\, d\xi\, dS\,. \tag{2.17}$$

Substituting (2.17) into (2.8) the resultant stress and stress couple vectors are obtained in the form

$$\boldsymbol{n}_\nu = \left(\int_-^+ \mathbf{t}^\Lambda \mu d\xi\right)\nu_\Lambda , \qquad \boldsymbol{m}_\nu = \left(\int_-^+ \boldsymbol{\zeta}\times\mathbf{t}^\Lambda \mu d\xi\right)\nu_\Lambda , \tag{2.18}$$

or

$$\boldsymbol{n}_\nu = \boldsymbol{n}^\Lambda \nu_\Lambda , \qquad \boldsymbol{m}_\nu = \boldsymbol{m}^\Lambda \nu_\Lambda , \tag{2.19}$$

where the resultant forces and couples along the coordinate curves on the shell reference surface are defined by

$$\boldsymbol{n}^\Lambda = \int_-^+ \mathbf{t}^\Lambda \mu d\xi , \qquad \boldsymbol{m}^\Lambda = \int_-^+ \boldsymbol{\zeta}\times\mathbf{t}^\Lambda \mu d\xi . \tag{2.20}$$

In this way we have shown that the resultant stress and stress couple tensors take the form

$$\boldsymbol{N} = \boldsymbol{n}^\Lambda \otimes \boldsymbol{A}_\Lambda , \qquad \boldsymbol{M} = \boldsymbol{m}^\Lambda \otimes \boldsymbol{A}_\Lambda . \tag{2.21}$$

This also indicates that $\boldsymbol{N}$ and $\boldsymbol{M}$ are surface tensors of first Piola-Kirchhoff type.

2.4 Change of frame of reference. Since our considerations are restricted to quasi-static deformations, we have not indicated the dependence of all variables on time. However, it is clear from the above considerations that all definitions and results remain valid for the dynamic case as well. Accordingly, whenever dependence on time of various variables is of importance, like in the analysis of frame-indifference properties given below, we shall add the time argument explicitly.

Under the change of frame of reference the three-dimensional motion of the body obeys the following transformation rule (Chapt. I.1)

$$\boldsymbol{x}^*(\boldsymbol{Y},t^*) = \boldsymbol{o}(t) + \boldsymbol{O}(t)\boldsymbol{x}(\boldsymbol{Y},t) , \tag{2.22}$$

where for each time instant t, $\boldsymbol{o}(t)\in E$ is a spatial vector and $\boldsymbol{O}(t)\in \mathrm{O}(3)$ is an orthogonal tensor. From (2.22) and the formal representation (2.6) of the motion of the shell-like body we easily find the following transformation rules

$$\boldsymbol{y}^*(\boldsymbol{Y},t^*) = \boldsymbol{o}(t) + \boldsymbol{O}(t)\boldsymbol{y}(\boldsymbol{Y},t), \qquad \boldsymbol{\zeta}^*(\boldsymbol{Y},\xi,t) = \boldsymbol{O}(t)\boldsymbol{\zeta}(\boldsymbol{Y},\xi,t). \tag{2.23}$$

Moreover, in continuum mechanics the Cauchy stress tensor is postulated to be frame-indifferent. In terms of the first Piola-Kirchhoff stress tensor this postulate takes the form

$$\mathbf{T}^*(X,t^*)=\boldsymbol{O}(t)\mathbf{T}(X,t) \quad \Rightarrow \quad \mathbf{t}_N^*(X,t^*)=\boldsymbol{O}(t)\mathbf{t}_N(X,t)\,. \tag{2.24}$$

When the transformation rules (2.24) and $(2.23)_2$ are substituted into definitions (2.8), we obtain

$$\begin{aligned} \boldsymbol{n}_\nu^*(\boldsymbol{Y}) &= \int_-^+ \boldsymbol{O}\mathbf{T}\mathbf{n}^*\mu^* d\xi = \boldsymbol{O}\Big(\int_-^+ \mathbf{T}\mathbf{n}^*\mu^* d\xi\Big)\,, \\ \boldsymbol{m}_\nu^*(\boldsymbol{Y}) &= \int_-^+ \boldsymbol{O}\boldsymbol{\zeta}\times\boldsymbol{O}\mathbf{T}\mathbf{n}^*\mu^* d\xi = \boldsymbol{O}\Big(\int_-^+ \boldsymbol{\zeta}\times\mathbf{T}\mathbf{n}^*\mu^* d\xi\Big)\,. \end{aligned} \tag{2.25}$$

This shows that under the change of frame of reference the resultant stress and stress couple vectors transform according to the rules

$$\boldsymbol{n}_\nu^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{n}_\nu(\boldsymbol{Y},t), \qquad \boldsymbol{m}_\nu^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{m}_\nu(\boldsymbol{Y},t)\,. \tag{2.26}$$

Moreover, since the unit normal vector $\boldsymbol{\nu}$ remains unchanged under the change of frame of reference, from (2.26) and (2.15) we have

$$\boldsymbol{N}^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{N}(\boldsymbol{Y},t), \qquad \boldsymbol{M}^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{M}(\boldsymbol{Y},t)\,. \tag{2.27}$$

Indeed, a direct calculation yields

$$\boldsymbol{n}_\nu^* = \boldsymbol{N}^*\boldsymbol{\nu}=\boldsymbol{O}\boldsymbol{n}_\nu=\boldsymbol{O}(\boldsymbol{N}\boldsymbol{\nu})=(\boldsymbol{O}\boldsymbol{N})\boldsymbol{\nu}, \tag{2.28}$$

and in the same manner for the stress couple vector. Thus (2.27) follows. This shows that the resultant stress and couple stress tensors obey the same transformation rules as the first Piola-Kirchhoff stress tensor, from which they are derived.

Exactly in the same manner we can show that the resultant surface force and couple vectors undergo the following transformations under the change of frame of reference:

$$\boldsymbol{p}^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{p}(\boldsymbol{Y},t), \qquad \boldsymbol{l}^*(\boldsymbol{Y},t^*)=\boldsymbol{O}(t)\boldsymbol{l}(\boldsymbol{Y},t)\,. \tag{2.29}$$

It has to be noted here that transformation rules for the shell variables derived above are direct implications of basic postulates of continuum mechanics, and not of the postulates of shell theory, as this is the case within the direct formulation of shell theory.

2.5 General remarks and comments. The easy with which the resultant balance laws for shells have been derived from the underlying principles of continuum mechanics is perhaps the best recommendation for the applied methodology. In this sense it may be regarded as the mathematical device designated to illuminate the intrinsic structure of the exact shell theory. The preceding details make it also clear that the presented derivation is rigorous in every aspect. There is no single assumption about the three-dimensional motion of the shell-like body, there are no restrictions on the mechanical properties of a material the shell-like body is made of, there is even no thickness assumption. We also note that the derived principles are valid for any choice of the reference surface. As such they are valid in the most general theory of shells, linear or nonlinear, elastic or unelastic, thin or thick. In this respect the presented derivation makes a clear distinction between the general principles valid for all shells and undergoing whatsoever deformation, and specific assumptions, which might be needed in the analysis of specific problems.

There is another important fact in our approach. The applied methodology relies on the most natural definitions of the resultant mechanical quantities involving solely integration through-the-thickness. As such regularity assumptions are far weaker than this is required in any other method of a shell theory formulation. Specifically, for the definitions (2.8) and (2.13) to make sense, we only need that the involved three-dimensional fields are integrable through the shell thickness, and hence they need not to be even continuous. A few special cases are worth to be considered in more details.

Shell irregularities such as folds and kinks are due to non-smoothness of the shell faces. From the above derivation it becomes clear, that irregularities of this kind are included in our formulation with no extra effort or assumptions. For example, the resultant total force and torque (2.14) remain valid for every regular shell-like subbody, like that shown in Fig. 14, and not necessarily for one having smooth faces and a smooth lateral surface.

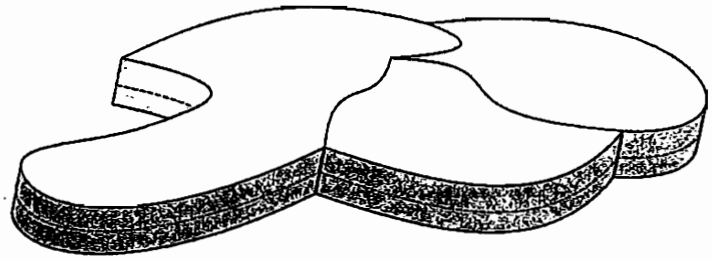

Fig. 14

An important class of shells are layered shells with jumps of various mechanical fields across inter-layers. The three-dimensional deformation of such shells need not be smooth across the thickness (Fig. 15). But such assumptions are not required in our formulation, and hence layered shells are also included here.

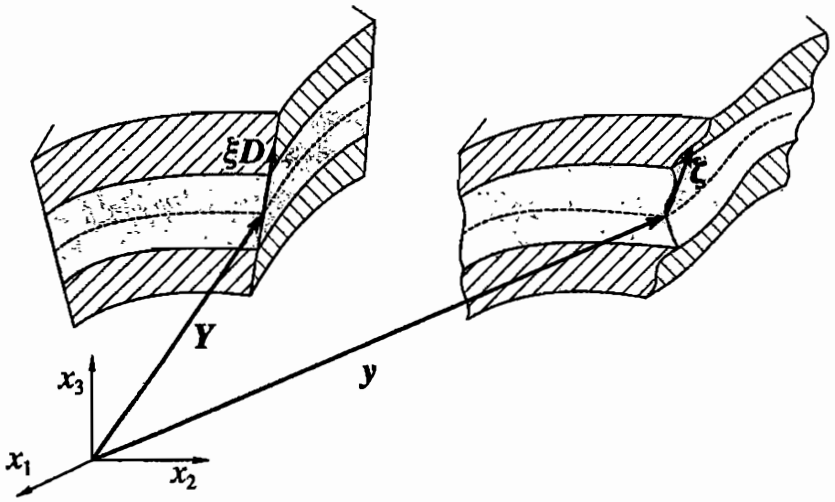

Fig. 15

As they stay, the resultant total force and torque (2.14) require only that the resultant mechanical quantities and the deformation of the shell reference surface are integrable fields. Thus we do not need to assume that the deformation of the reference surface is smooth (differentiable) or even continuous. A non-smooth deformation arises, e.g. for so-called non-elliptic materials, fracture, etc. All these problems are covered by the above formulation.

2.8 Discontinuous shell thickness. Most of the shells in engineering practice are characterized by a discontinuous thickness. A rigorous derivation of the resultant total force and torque leads then to the resultant total force and torque in the form

$$\begin{aligned} \mathfrak{f}(\Pi) &\equiv \iint_{\Pi\setminus\mathfrak{L}} \boldsymbol{p}\, dA + \int_{\partial\Pi} \boldsymbol{n}_\nu\, dS + \int_{\Pi\cap\mathfrak{L}} \boldsymbol{p}_\ell\, dS\,, \\ \mathfrak{t}(\Pi) &\equiv \iint_{\Pi\setminus\mathfrak{L}} (\boldsymbol{l} + \boldsymbol{\chi}\times\boldsymbol{p})\, dA + \int_{\partial\Pi} (\boldsymbol{m}_\nu + \boldsymbol{\chi}\times\boldsymbol{n}_\nu)\, dS \\ &\quad + \int_{\Pi\cap\mathfrak{L}} (\boldsymbol{l}_\ell + \boldsymbol{y}_\ell\times\boldsymbol{p}_\ell)\, dS\,, \end{aligned} \tag{2.30}$$

rather than (2.14), where $\mathfrak{L}$ is a piecewise smooth curve on the reference surface M, along which the force and couple vectors are given. The shell reference surface itself need not be smooth along this curve (Fig. 16).

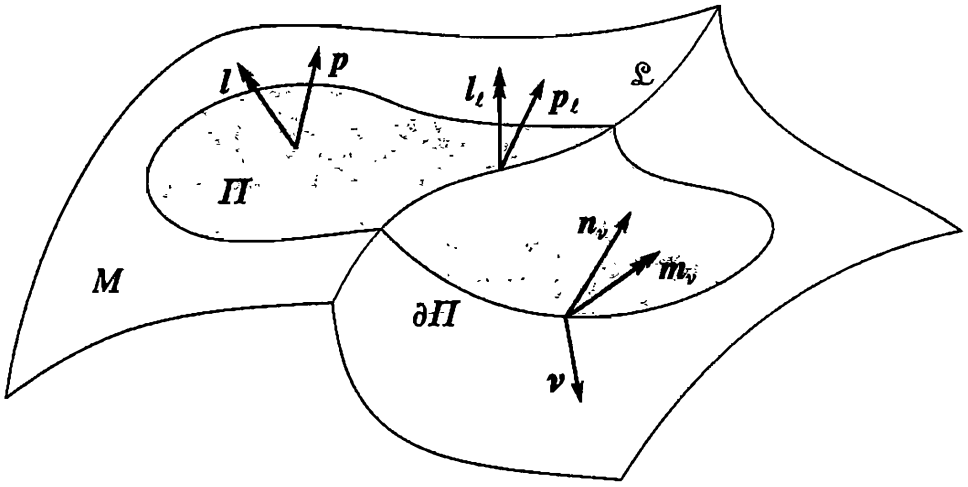

Fig. 16

3. Irregular shell structures

3.1 Preliminaries. The concept of regular shell-like bodies introduced in Sect. 1.3 and the resultant total force and torque derived in the previous chapter for such bodies cover a wide class of shell problems, wider than ever has been subject of a rigorous analysis before. Nevertheless, there are still many problems of engineering importance, which are not included within the presented setting.

Generally, engineering structures are typically composed of rod-like and shell-like elements interconnected pointwise at joints and along junctions in a widely varying manner to form, in overall, fairly complex structures. The possibilities are numerous, but, as we have already pointed out in Sect. 1.2, only three groups need to be considered separately.

The group III includes all structures, which resemble shells understood in a broader sense, but which are not regular in the sense of our definition. A typical example is shown in Fig. 17. Generally, structures of this group contain shell branching, i.e. three or more shell segments intersect at a common juncture. In such cases, the difficulties in a rigorous derivation of the resultant total force and torque lie in the fact that at the intersection it is not possible to define in a unique way the shell reference surface and the shell thickness. In this sense they are not a regular shell-like body, but rather a union of two or more regular shell-like bodies. Of course, we can simplify the problem ignoring the transition zone, what is the common practice in an engineering approach to the problem. But this need not be

our concern here, since we are seeking for a more reliable modelling of engineering structures.

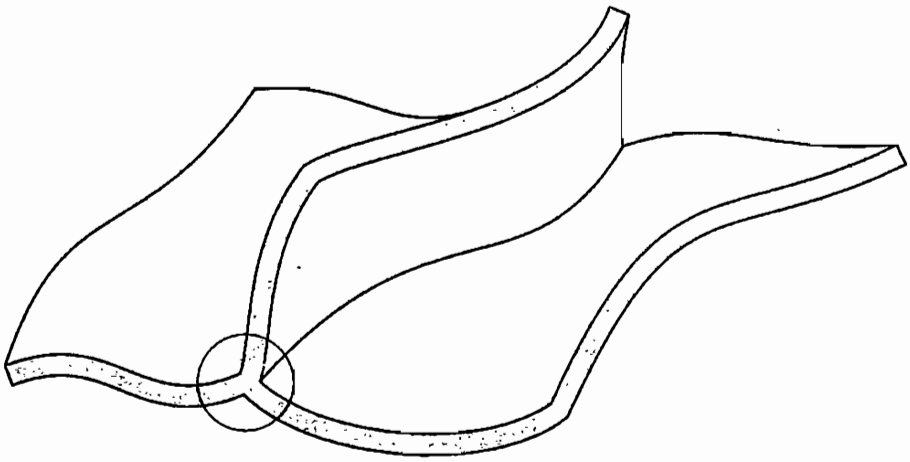

Fig. 17

An example shown in Fig. 18, while similar to the previous one, in essence is quite different. It shows not a single shell-like body but rather two shell-like bodies, which are interconnected along common boundaries in some technological manner. The kinematical and mechanical properties of the interconnection cannot be derived from the laws of continuum mechanics alone, but they must be supplemented by some a priori given technological data. Structures of this kind belong to the group IV).

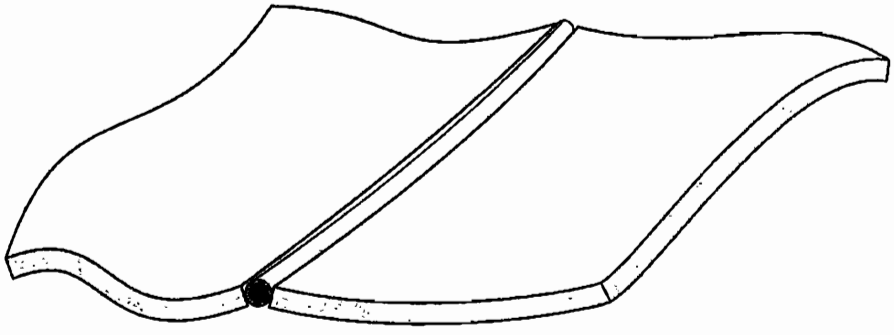

Fig. 18

Shell structures sustained by columns (Fig. 19) belong to the group V. They are easier to handle by methods of a theory of structures, except for a small region of the rod-to-shell transition.

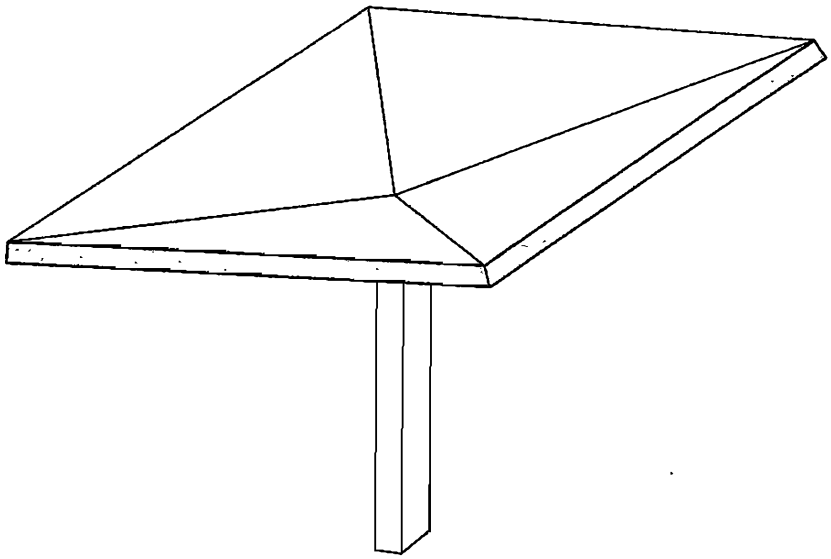

Fig. 19

It should be noted here that a rigorous description of the three-dimensional stress and strain fields in the aforementioned cases fall out of the realm of shell and rod theories in the same sense as non of them can provide a complete description of the boundary layer even for smooth shells and rods. But this fact in no way invalidates shell and rod theory based solutions at some distance of the irregularities. Furthermore, any theory of shells is distinguished from the exact three-dimensional theory of continuum mechanics insofar as one spatial variable is suppressed in the mathematical description. Such a theory, therefore, deal essentially merely with resultants of the local stresses. It is then obvious that only resultants can be correctly determined by shell theory. This is also true for rod theory. Moreover, rod theory can also be used for modelling of multiple shell intersections and technological interconnections of distinct shells. This is the viewpoint which we adopt in this chapter.

3.2 Rod equations. There is a strict analogue in the derivation of mechanical balance laws for shells and rods, as there is an analogue between the concept of a shell-like body and a rod-like body. This analogue is affected by interchanging the role of the shell faces $M^{\pm}$ and the lateral surface ∂B°. Within a rod theory a rod-like body is geometrically represented by a piecewise smooth curve $\mathfrak{L}$ (rod axis being a one-dimensional analogue of the shell reference surface M). Moreover, in the case of a regular rod-like body R, such as shown in Fig. 20, the rod cross sections $\Pi(S)$ play the role of the lateral shell surface ∂P°, and the lateral rod

surface $\partial R^{\diamond}$ correspond to the shell faces $M^{\pm}$. Here we assume that the rod axis $\mathfrak{L}$ is given in a parametric form, $\boldsymbol{Y} = \boldsymbol{Y}(S)$, where S denotes the arc length parameter.

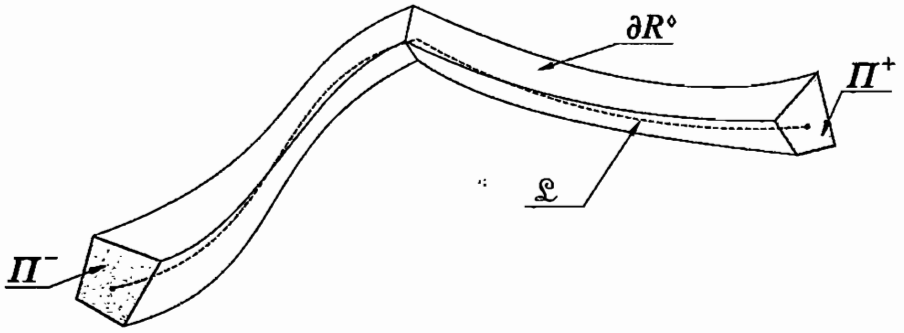

Fig. 20

Referring to Fig. 20 and applying exactly the same method, which we have used to derive the resultant balance laws for a regular shell-like body, we can write down at once the resultant total force and torque acting an any regular rod-like body:[7]

$$\begin{aligned} \mathbf{f}(R) &= \int_{\mathfrak{L}} \boldsymbol{f}\, dS + [\boldsymbol{n}]_{S_1}^{S_2}, \\ \mathbf{t}(R) &= \int_{\mathfrak{L}} (\boldsymbol{f} + \boldsymbol{y} \times \boldsymbol{c})\, dS + [\boldsymbol{m} + \boldsymbol{y} \times \boldsymbol{n}]_{S_1}^{S_2}. \end{aligned} \tag{3.1}$$

Here the resultant line force and couple vectors, measured per unit length of $\mathfrak{L}$ are defined by

$$\begin{aligned} \int_{\mathfrak{L}} \boldsymbol{f}\, dS &= \int_R \mathbf{f}\, dV + \iint_{\partial R^{\diamond}} \mathbf{t}_N\, dA, \\ \int_{\mathfrak{L}} (\boldsymbol{c} + \boldsymbol{y} \times \boldsymbol{f})\, dS &= \int_R \boldsymbol{x} \times \mathbf{f}\, dV + \iint_{\partial R^{\diamond}} \boldsymbol{x} \times \mathbf{t}_N\, dA. \end{aligned} \tag{3.2}$$

The resultant force and couple at the rod cross section are

$$\begin{aligned} \boldsymbol{n}(S) &= \iint_{\Pi(S)} \mathbf{t}_N\, dA, \\ (\boldsymbol{m} + \boldsymbol{y} \times \boldsymbol{n})(S) &= \iint_{\Pi(S)} \boldsymbol{x} \times \mathbf{t}_N\, dA. \end{aligned} \tag{3.3}$$

Integration by parts applied to (3.1) yields the total force and torque in the form

$$\begin{aligned} \mathbf{f}(R) &= \int_{\mathfrak{L}} \boldsymbol{p}_{\ell}\, dS, & \boldsymbol{p}_{\ell}(S) &\equiv \boldsymbol{n}' + \boldsymbol{f}, \\ \mathbf{t}(R) &= \int_{\mathfrak{L}} (\boldsymbol{l}_{\ell} + \boldsymbol{y}_{\ell} \times \boldsymbol{p}_{\ell})\, dS, & \boldsymbol{l}_{\ell}(S) &\equiv \boldsymbol{m}' + \boldsymbol{y}_{\ell}' \times \boldsymbol{n} + \boldsymbol{c}, \end{aligned} \tag{3.4}$$

[7] Aside of regularity assumptions, details of the derivation are given in SMOLENSKI [1994].

where a prime stands for the derivative with respect to the arc length parameter along the rod axis.

3.3 Structured continua. As we have pointed out in Sect. 3.1, structures of all three groups III–V are neither regular shell-like bodies nor regular rod-like bodies. However, their important feature is that they can be regarded as unions of such bodies. Therefore, multiple-shell intersections and technological interconnections of regular shell-like bodies can be modelled fairly correctly or even completely rigorously by a union of some number of shells, whose common boundaries are spatial curves having their own mechanical properties. This point of view can be also justified as follows.

Let us consider a material body (precisely, a region in the space occupied by a body) B, which can be written as a union of some number of separate regular shell-like bodies $B^{(A)}$, $A=1,2,\ldots,N$, and rod-like bodies $R^{(a)}$, $a=1,2,\ldots,n$, i.e.

$$B=\left(\bigcup_{A=1}^{N} B^{(A)}\right)\cup\left(\bigcup_{a=1}^{n} R^{(a)}\right). \tag{3.5}$$

Typically, each $R^{(a)}$ will represent a region of multiple intersection or a technological interconnection of regular shell-like bodies $B^{(A)}$.

Within an axiomatic formulation of the basic laws of continuum mechanics, the forces and torques are assumed to be vector-valued, additive measures on the set of pairwise disjoint subbodies. Without going into all details of this concept we only note that this implies that the total force and total torque acting on the body defined by (3.5) are the sum of forces and torques acting on each subbody:

$$\begin{aligned}\mathfrak{F}(B)&=\sum_{A=1}^{N}\mathfrak{F}(B^{(A)})+\sum_{a=1}^{n}\mathfrak{F}(R^{(a)}),\\ \mathfrak{T}(B)&=\sum_{A=1}^{N}\mathfrak{T}(B^{(A)})+\sum_{a=1}^{n}\mathfrak{T}(R^{(a)}).\end{aligned} \tag{3.6}$$

By virtue of the results of Chapt. 2 and of Sect. 3.2, we can model the body B as a structured continuum

$$M=\left(\bigcup_{A=1}^{N} M^{(A)}\right)\cup\left(\bigcup_{a=1}^{n} \mathfrak{L}^{(a)}\right), \tag{3.7}$$

which is the superposition of two-dimensional continua $M^{(A)}$ (shells in the sense of the previous chapter) and one-dimensional continua $\mathfrak{L}^{(a)}$ (having the properties of rods in the sense of Sect. 3.2). The resultant total force and torque are thus the appropriate sum of forces and torques given by (2.14) and (3.4).

A structured continuum M defined in this way may be called an irregular shell or the shell reference surface of an irregular shell-like body. Geometrically, M represents a "multi-surface" and, according to (3.7), it is defined as union of pairwise disjoint piecewise smooth, oriented and connected (but not necessarily simply connected) surfaces $M^{(A)}$. Two or more surfaces $M^{(A)}$ may have in common a piecewise smooth spatial curve $\mathfrak{L}^{(a)}$ being a boundary for all $M^{(A)}$. We shall denote by $\mathfrak{L}$ the union of all such curves $\mathfrak{L}^{(a)}$. Let us state it clearly that M is not a surface in the sense of classical differential geometry but rather a union of surfaces.

Admitting that $\mathfrak{L}$ is a one-dimensional continuum with its own mechanical properties and using the method of Chapt. II.2 and of Sect. 3.2 we can write down at once the resultant total force and the resultant total torque acting on any part of the irregular shell:

$$\begin{aligned} \mathbf{f}(\Pi) &= \iint_{\Pi\setminus\mathfrak{L}} \boldsymbol{p}\, dA + \int_{\partial\Pi} \boldsymbol{n}_\nu\, dS + \int_{\Pi\cap\mathfrak{L}} \boldsymbol{p}_\ell\, dS\,, \\ \mathbf{t}(\Pi) &= \iint_{\Pi\setminus\mathfrak{L}} (\boldsymbol{l} + \boldsymbol{y}\times\boldsymbol{p})\, dA + \int_{\partial\Pi} (\boldsymbol{m}_\nu + \boldsymbol{y}\times\boldsymbol{n}_\nu)\, dS + \int_{\Pi\cap\mathfrak{L}} (\boldsymbol{l}_\ell + \boldsymbol{y}_\ell\times\boldsymbol{p}_\ell)\, dS\,. \end{aligned} \tag{3.8}$$

Here $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ as well as $\boldsymbol{p}$ and $\boldsymbol{l}$ are defined exactly in the same way as for the regular shell-like body. The physical meaning of the line force $\boldsymbol{p}_\ell$ and couple $\boldsymbol{l}_\ell$ may vary substantially depending on the intended application of the theory.

In order to account for possible rod-to-shell interactions we can still enrich our model by admitting concentrated forces and couples acting at distinct points of the shell reference surface including the curves of intersections. In such cases the total force and couple (3.8) should be replaced by

$$\begin{aligned} \mathbf{f}(\Pi) &= \iint_{\Pi\setminus\mathfrak{L}} \boldsymbol{p}\, dA + \int_{\partial\Pi} \boldsymbol{n}_\nu\, dS \\ &\quad + \int_{\Pi\cap\mathfrak{L}} \boldsymbol{p}_\ell\, dS + \sum_{a=1}^{n} \boldsymbol{f}_a\,, \\ \mathbf{t}(\Pi) &= \iint_{\Pi\setminus\mathfrak{L}} (\boldsymbol{l} + \boldsymbol{y}\times\boldsymbol{p})\, dA + \int_{\partial\Pi} (\boldsymbol{m}_\nu + \boldsymbol{y}\times\boldsymbol{n}_\nu)\, dS \\ &\quad + \int_{\Pi\cap\mathfrak{L}} (\boldsymbol{l}_\ell + \boldsymbol{y}_\ell\times\boldsymbol{p}_\ell)\, dS + \sum_{a=1}^{n} (\boldsymbol{c}_a + \boldsymbol{y}_a\times\boldsymbol{f}_a)\,, \end{aligned} \tag{3.9}$$

where $\boldsymbol{f}_a$ and $\boldsymbol{c}_a$ are concentrated forces and couples acting at distinct points $\boldsymbol{Y}_a \in M$. Their physical meaning must be specified for each problem separately.

4. Equilibrium equations and static jump conditions

4.1 Regularity assumptions. In the most general case the shell is represented by the reference surface M, which is defined as a union of piecewise smooth surfaces $M^{(A)}$, $A=1,2,\ldots,N$. We shall further assume that each $M^{(A)}$ is partitioned into disjoint smooth surface elments $M^{(A)}_{(k)}$, $k=1,2,\ldots,N_A$, in such a way, that within every element $M^{(A)}_{(k)}$ all fields are continuous and smoothly differentiable as many times as needed. We can then renumber the elements $M^{(A)}_{(k)}$ so that the reference surface M will be a union of smooth surface elements $M_{(k)}$, $k=1,2,\ldots,n$, where $n=\sum_{A=1}^{N} N_A$. Two or more smooth surface elements $M_{(k)}$ can have in common a piecewise smooth curve. We shall then denote by $\mathfrak{L}$ the union of all such curves. In general, $\mathfrak{L}$ will represent kinks, multiple-shell intersections, technological interconnections or a curve on possible smooth parts of the reference surface, across which some fields may suffer jump discontinuities. With this convention, we shall refer to $\mathfrak{L}$ as a singular curve.

The orientation of each element $M_{(k)}$ is specified by the unit normal vector $\boldsymbol{A}_N$ being defined at every point $\boldsymbol{Y} \in M_{(k)}$. The boundary of $M_{(k)}$ is a closed piecewise smooth curve $\partial M_{(k)}$ with the outward normal vector denoted by $\boldsymbol{\nu}^{(k)}$. If the element $M_{(k)}$ has a part of $\mathfrak{L}$ as its boundary, then $\boldsymbol{\nu}^{(k)}$ is defined by

$$\boldsymbol{\nu}^{(k)} = \pm\, \boldsymbol{\tau}_\ell \times \boldsymbol{A}_N^{(k)}\,, \tag{4.1}$$

where $\boldsymbol{\tau}_\ell$ denotes the unit vector tangent to $\mathfrak{L}$. The sign in (4.1) must be chosen in such a way that the boundary $\partial M_{(k)}$ is consistently oriented with $M_{(k)}$. The boundary of the whole shell reference surface M will be denoted by ∂M and the outward unit normal vector at each regular point of ∂M will be denoted by $\boldsymbol{\nu}$. The same notation and convention will be applied to any part of M.

Omitting concentrated forces and couples, which within the local theory must be considered separately, and making use of the Cauchy's theorem (2.15) the resultant laws for shells (3.8) take the form

$$\begin{aligned} \iint_{\Pi\setminus\mathfrak{L}} \boldsymbol{p}\, dA + \int_{\partial\Pi} \boldsymbol{N}\boldsymbol{\nu}\, dS + \int_{\Pi\cap\mathfrak{L}} \boldsymbol{p}_\ell\, dS = \boldsymbol{0}\,, \\ \iint_{\Pi\setminus\mathfrak{L}} (\boldsymbol{l} + \boldsymbol{y}\times\boldsymbol{p})\, dA + \int_{\partial\Pi} (\boldsymbol{M}\boldsymbol{\nu} + \boldsymbol{y}\times\boldsymbol{N}\boldsymbol{\nu})\, dS + \int_{\Pi\cap\mathfrak{L}} (\boldsymbol{l}_\ell + \boldsymbol{y}_\ell\times\boldsymbol{p}_\ell)\, dS = \boldsymbol{0}\,. \end{aligned} \tag{4.2}$$

These two laws are assumed to hold for every part $\Pi \subset M$ of the shell carrying surface M. As they stay, the only underlying regularity assumptions are those, which ensure that the integrals in (4.2) make sense. However, in order to derive the corresponding local laws stronger regularity assumptions must be imposed on various fields in (4.2) as well as on their domains.

An essential initial step in order to derive the corresponding local laws is that of choosing a class of deformations for the shell reference surface. The usual assumption, as a rule implicit, that a deformation is injective (globally invertible), differentiable with differentiable inverse would exclude many of the problems we have considered in the previous chapters.

Generally, the deformation of the shell reference surface is described by a map $\chi : M \to \mathscr{E}$, which assigns to every surface particle Y its spatial place y it occupies in the current configuration,

$$y = \chi(Y) = Y + u(Y), \tag{4.3}$$

where u denotes the associated displacement field. We shall assume that (4.3) is a continuous function over each regular surface element $M_{(k)}$ and differentiable of class C^n, $n \geq 1$, in the interior $int\, M_{(k)}$ of each smooth surface element. Under this assumption the gradient

$$F(Y) = \nabla_s \chi(Y) \in E \otimes T_Y M, \tag{4.4}$$

exists at every point $Y \in int\, M_{(k)}$. We shall not assume a priori that the deformation $\chi : M \to \mathscr{E}$ is continuous across the singular curve $\mathfrak{L}$ or some parts thereof. Accordingly, we regard (4.3) as being defined for all $Y \in M \setminus \mathfrak{L}$, i.e. $\chi : M \setminus \mathfrak{L} \to \mathscr{E}$, and it has a finite limit at every point $Y \in \mathfrak{L}$,

$$y^{(k)} = \chi^{(k)}(Y) = \lim_{Z \to Y} \chi(Z) = Y + \lim_{Z \to Y} u(Z), \qquad Z \in \Pi_{(k)}, \tag{4.5}$$

whenever $\mathfrak{L}$ is a part of the boundary $\partial M_{(k)}$.

We shall admit that the singular curve may follow its own deformation

$$y_\ell = \chi_\ell(Y) = Y + u_\ell(Y), \tag{4.6}$$

where $\chi_\ell : \mathfrak{L} \to \mathcal{E}$ is a continuous map defined only along the singular curve. If the deformation (4.3) is continuous over the whole reference surface M, then (4.6) should be regarded as being the restriction of χ to the curve $\mathfrak{L}$, i.e. $\chi_\ell = \chi_{|\mathfrak{L}}$.

We shall also assume that the resultant stress tensor N and the resultant couple stress tensor M are of class C^n, $n \geq 1$, in the interior $int\, M_{(k)}$ of each smooth surface element and that they have finite limits

$$N^{(k)}(Y) = \lim_{Z \to Y} N(Z), \qquad M^{(k)}(Y) = \lim_{Z \to Y} N(Z), \qquad Z \in \Pi_{(k)}, \tag{4.7}$$

at every point $Y \in \mathfrak{L}$ taken along paths in $M_{(k)}$.

4.2 Derivation of field equations. Under the assumptions stated in the previous section, the boundary integrals in (4.2) containing N and M can be transformed into surface integrals using the generalized surface divergence theorem[8]

$$\begin{aligned} \int_{\partial\Pi} N\nu \, dS &= \iint_{\Pi \setminus \mathfrak{L}} Div_s N \, dA - \int_{\Pi \cap \mathfrak{L}} [\![n_\nu]\!] \, dS, \\ \int_{\partial\Pi} M\nu \, dS &= \iint_{\Pi \setminus \mathfrak{L}} Div_s M \, dA - \int_{\Pi \cap \mathfrak{L}} [\![m_\nu]\!] \, dS, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \int_{\partial\Pi} \chi \times N\nu \, dS = \iint_{\Pi \setminus \mathfrak{L}} \big(y \times Div_s N + ad^{-1}(NF^T - FN^T) \big) dA \\ - \int_{\Pi \cap \mathfrak{L}} [\![y \times n_\nu]\!] \, dS. \end{aligned} \tag{4.9}$$

Here the jumps are defined by

$$\begin{aligned} [\![n_\nu]\!] &= \sum_{k=1}^{n_p} n_\nu^{(k)} = \sum_{k=1}^{n_p} N^{(k)} \nu^{(k)}, \\ [\![m_\nu]\!] &= \sum_{k=1}^{n_p} m_\nu^{(k)} = \sum_{k=1}^{n_p} M^{(k)} \nu^{(k)}, \end{aligned} \tag{4.10}$$

and

$$[\![y \times n_\nu]\!] = \sum_{k=1}^{n_p} y^{(k)} \times n_\nu^{(k)} = \sum_{k=1}^{n_p} y^{(k)} \times N^{(k)} \nu^{(k)}. \tag{4.11}$$

Here it should be noted that $F(Y) \in E \otimes T_Y M$ and $N(Y) \in E \otimes T_Y M$ so that both NF^T and FN^T are well defined, they are elements of the tensor space $E \otimes E$ and their difference $NF^T - FN^T$ is a skew-symmetric tensor, whose axial vector is

[8] The details of the proof are given in the Appendix E.

$$ad^{-1}(NF^T - FN^T)\,. \tag{4.12}$$

Substituting now (4.5) and (4.6) into (4.2) and rearranging the terms the integral balance laws are obtained in the form

$$\begin{aligned} \iint_{\Pi\setminus\mathfrak{L}} (Div_s N + p)\,dA + \int_{\Pi\cap\mathfrak{L}} (p_\ell - [\![n_\nu]\!])\,dS = \mathbf{0}\,, \\ \iint_{\Pi\setminus\mathfrak{L}} \big(Div_s M + ad^{-1}(NF^T - FN^T) + l + y\times(Div_s N + f)\big)\,dA \\ + \int_{\Pi\cap\mathfrak{L}} (l_\ell - [\![m_\nu]\!] + y_\ell \times p_\ell - [\![y\times n_\nu]\!])\,dS = \mathbf{0}\,. \end{aligned} \tag{4.13}$$

4.3 Local equilibrium equations. Since the two integral laws (4.13) must hold simultaneously for each part $\Pi \subset M$ of the shell reference surface, and since the assumptions stated above ensure that the integrands are continuous functions on M except possibly at the singular curve $\mathfrak{L}$, the local (dynamic) equations of equilibrium at every regular point $Y \in M \setminus \mathfrak{L}$ are

$$\begin{aligned} Div_s N + p = \mathbf{0}\,, \\ Div_s M + ad^{-1}(NF^T - FN^T) + l = \mathbf{0}\,. \end{aligned} \tag{4.14}$$

In order to express the equilibrium equations (4.14) in a more familiar form, let us assume that M be given locally in the parametric form (see Sect. 1.4). Then the deformation (4.3) may be expressed in the form

$$y(\xi^\Lambda) = \chi(Y(\xi^\Lambda))\,. \tag{4.15}$$

The deformation gradient F at every regular point of M is given by

$$F(Y) = \nabla_s \chi(Y) = y_{,\Lambda}(Y) \otimes A^\Lambda(Y)\,, \tag{4.16}$$

where a comma denotes the partial derivative with respect to the corresponding surface coordinate. It is worthwhile to note here that for the deformation gradient to exist at a point Y it is not enough that partial derivatives $y_{,\Lambda}(Y)$ exist, but they must also be continuous at that point.

Recalling further that $N = n^\Lambda \otimes A_\Lambda$, the tensor defined by (4.12) may be obtained in the form

$$\begin{aligned} NF^T - FN^T &= (n^\Lambda \otimes A_\Lambda)(A^\Sigma \otimes y_{,\Sigma}) - (y_{,\Sigma} \otimes A^\Sigma)(A_\Lambda \otimes n^\Lambda) \\ &= n^\Lambda \otimes y_{,\Lambda} - y_{,\Lambda} \otimes n^\Lambda = n^\Lambda \wedge y_{,\Lambda}\,, \end{aligned} \tag{4.17}$$

and the associated axial vector is given by

$$ad^{-1}(NF^T - FN^T) = ad^{-1}(n^\Lambda \wedge y_{,\Lambda}) = -n^\Lambda \times y_{,\Lambda}\,. \tag{4.18}$$

Moreover, the surface divergence of N can be expressed as

$$Div_s N = n^\Lambda{}_{|\Lambda}\,, \tag{4.19}$$

and the same form for the stress couple tensor M, where the vertical stroke denotes the covariant derivative in the metric of the surface M. With the help of (4.17) and (4.18), the equilibrium equations (4.14) take the form

$$\begin{aligned} n^\Lambda{}_{|\Lambda} + p &= 0\,, \\ m^\Lambda{}_{|\Lambda} + y_{,\Lambda} \times n^\Lambda + l &= 0\,. \end{aligned} \tag{4.20}$$

With the definition of the covariant derivative

$$n^\Lambda{}_{|\Lambda} = n^\Lambda{}_{,\Lambda} - \Gamma^\Gamma_{\Gamma\Lambda} n^\Lambda \tag{4.21}$$

and

$$\Gamma^\Gamma_{\Gamma\Lambda} = \frac{1}{\sqrt{A}} \frac{\partial \sqrt{A}}{\partial \xi^\Lambda}\,, \qquad A = \det A_{\Gamma\Lambda}\,, \tag{4.22}$$

the equilibrium equations (4.20) can be rewritten in a form, which does not contain covariant derivatives:

$$\begin{aligned} \left(\sqrt{A}\, n^\Lambda\right)_{,\Lambda} + \sqrt{A} p &= 0\,, \\ \left(\sqrt{A} m^\Lambda\right)_{,\Lambda} + \sqrt{A}\left(y_{,\Lambda} \times n^\Lambda + l\right) &= 0\,. \end{aligned} \tag{4.23}$$

The formal similarity of the equations (4.20) and (4.23) to those known in the literature should not conceal the richness of their contents and the manner they have been obtained here. But we postpone any discussion of this up to the next chapter.

4.4 Jump conditions. In the integral laws (4.13) the integral along the curve $\mathfrak{L}$ must vanish separately. This leads to jump conditions at the singular curve to be

$$\begin{aligned} p_\ell - [\![n_\nu]\!] &= 0\,, \\ l_\ell - [\![m_\nu]\!] + y_\ell \times p_\ell - [\![y \times n_\nu]\!] &= 0\,. \end{aligned} \tag{4.24}$$

With the use of the jump condition $(4.24)_1$, the second term in the jump condition $(4.24)_2$ may be rewritten in the form

$$y_\ell \times p_\ell = y_\ell \times [\![n_\nu]\!] = [\![y_\ell \times n_\nu]\!]. \tag{4.25}$$

The jump conditions (4.24) can also be expressed in an equivalent form

$$\begin{aligned} p_\ell - [\![n_\nu]\!] &= 0, \\ l_\ell - [\![m_\nu]\!] + [\![(y_\ell - y) \times n_\nu]\!] &= 0. \end{aligned} \tag{4.26}$$

Taking further into account that

$$y_\ell - y = \chi_\ell(Y) - \chi^{(k)}(Y) = u_\ell(Y) - u^{(k)}(Y), \qquad Y \in \mathfrak{L}, \tag{4.27}$$

the second of the jump conditions becomes

$$l_\ell - [\![m_\nu]\!] + [\![(u_\ell - u) \times n_\nu]\!] = 0. \tag{4.28}$$

Various special cases can now be examined in detail under suitable regularity assumptions. But we postpone such an analysis until subsequent chapter.

5. Static boundary conditions

5.1 Integral form of 3D boundary conditions. Within the three-dimensional theory the shell-like body is subjected to the external body force $\mathbf{f}(X)$ acting at each interior point $X \in B$ and external surface forces given on a part of its boundary. The surface forces acting on the shell faces are included in the definition of the resultant surface force and couple vectors. Denoting by $\mathbf{t}^*(X)$ the surface force acting on a part ∂B_f° of the lateral surface ∂B°, the traction boundary conditions take the form:

$$\mathbf{T}(X)\mathbf{n}^\circ(X) = \mathbf{t}^*(X), \qquad X \in \partial B_f^\circ. \tag{5.1}$$

In order to obtain the resultant traction boundary conditions for the shell we consider the following integral form of (5.1),

$$\iint_{\partial B_f^\circ} (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\, dA^\circ = \mathbf{0}, \qquad \iint_{\partial B_f^\circ} x \times (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\, dA^\circ = \mathbf{0}, \tag{5.2}$$

where the deformation of the shell-like body is represented in the form

$$x(Y,\xi) = \chi(X(Y,\xi)) = y(Y) + \zeta(Y,\xi). \tag{5.3}$$

Applying to (5.2) the same technique, which has been used to derive the resultant balance laws, we can formulate the resultant dynamic boundary conditions for the shell.

5.2 Resultant traction force and couple vectors. We denote by $\partial M_f = M \cap \partial B_f^\circ$ the part of the boundary of the shell reference surface, which corresponds to the lateral surface on which the external surface loads are prescribed. Using (5.3) and the geometric relation $\mathbf{n}^\circ dA^\circ = \mathbf{n}^\circ \alpha^\circ d\xi dS$ for the oriented area element of the lateral surface we have

$$\begin{aligned} \iint_{\partial B_f^\circ} (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\, dA^\circ &= \int_{\partial M_f} \left(\int_-^+ (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\alpha^\circ d\xi \right) dS\,, \\ \iint_{\partial B_f^\circ} \boldsymbol{x} \times (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\, dA^\circ &= \int_{\partial M_f} \left(\boldsymbol{y} \times \int_-^+ (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\alpha^\circ d\xi + \int_-^+ \boldsymbol{\zeta} \times (\mathbf{T}\mathbf{n}^\circ - \mathbf{t}^*)\alpha^\circ d\xi \right) dS\,. \end{aligned} \tag{5.4}$$

By virtue of the definitions of the internal resultant stress and stress couple vectors we have (see Chapt. II.2)

$$\int_-^+ \mathbf{T}\mathbf{n}^\circ \alpha^\circ d\xi = \boldsymbol{N}\boldsymbol{\nu}\,, \qquad \int_-^+ \boldsymbol{\zeta} \times \mathbf{T}\mathbf{n}^\circ \alpha^\circ d\xi = \boldsymbol{M}\boldsymbol{\nu}\,. \tag{5.5}$$

By the same arguments the resultant boundary force and couple vectors, which are statically equivalent to the prescribed external load, are defined by

$$\boldsymbol{n}^*(\boldsymbol{Y}) = \int_-^+ \mathbf{t}^* \alpha^\circ d\xi\,, \qquad \boldsymbol{m}^*(\boldsymbol{Y}) = \int_-^+ \boldsymbol{\zeta} \times \mathbf{t}^* \alpha^\circ d\xi\,. \tag{5.6}$$

5.3 Resultant boundary conditions. In view of (5.5) and (5.6), the integral form (5.2) of the traction boundary conditions reads

$$\begin{aligned} \int_{\partial M_f} (\boldsymbol{N}\boldsymbol{\nu} - \boldsymbol{n}^*)\, dS &= \mathbf{0}\,, \\ \int_{\partial M_f} \left(\boldsymbol{M}\boldsymbol{\nu} - \boldsymbol{m}^* + \boldsymbol{y} \times (\boldsymbol{N}\boldsymbol{\nu} - \boldsymbol{n}^*) \right) dS &= \mathbf{0}\,. \end{aligned} \tag{5.7}$$

By implication, the resultant dynamic boundary conditions at each point $\boldsymbol{Y} \in \partial M_f$ take the form

$$\boldsymbol{N}(\boldsymbol{Y})\boldsymbol{\nu}(\boldsymbol{Y}) = \boldsymbol{n}^*(\boldsymbol{Y})\,, \qquad \boldsymbol{M}(\boldsymbol{Y})\boldsymbol{\nu}(\boldsymbol{Y}) = \boldsymbol{m}^*(\boldsymbol{Y})\,. \tag{5.8}$$

Chapter III

Kinetics of the shell

1. Virtual work identity

1.1 Summary of dynamic equations. Considerations of the previous chapter have led us in a clear and logic way to the complete set of dynamic shell equations implied by basic laws of continuum mechanics. In the local form this set of equations consists of:

the (dynamic) equilibrium equations at every regular point $\boldsymbol{Y} \in M \setminus \mathfrak{L}$ of the shell reference surface;

$$Div_s \boldsymbol{N} + \boldsymbol{p} = \boldsymbol{0}, \qquad Div_s \boldsymbol{M} + ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T) + \boldsymbol{l} = \boldsymbol{0}, \tag{1.1}$$

the (dynamic) jump conditions at every point $\boldsymbol{Y} \in \mathfrak{L}$ of the singular curve $\mathfrak{L}$,

$$\boldsymbol{p}_\ell - [\![\boldsymbol{n}_\nu]\!] = \boldsymbol{0}, \qquad \boldsymbol{l}_\ell - [\![\boldsymbol{m}_\nu]\!] + [\![(\boldsymbol{y}_\ell - \boldsymbol{y}) \times \boldsymbol{n}_\nu]\!] = \boldsymbol{0}, \tag{1.2}$$

the (dynamic) boundary conditions at every point $\boldsymbol{Y} \in \partial M_f$ of that part of the boundary of the reference surface, along which the external resultant force and couple vectors are prescribed,

$$\boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{n}_\nu^*, \qquad \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{m}_\nu^*. \tag{1.3}$$

The equations (1.1)-(1.3) appear to be all, what can be extracted from the underlying laws of three-dimensional theory by a most direct and natural approach involving no assumptions or ad hoc postulates whatsoever. While exact in this sense, this set of equations is obviously not complete, and it provides solely a firm foundation on which the theory of shells can be built. There is still an obvious lack of suitable kinematic variables, kinematic boundary and jump conditions, and appropriate constitutive relations.

In all foregoing considerations the only obvious kinematic concept has been the one of deformation of the shell reference surface. However, the kinematics of the shell is not merely the kinematics of its reference surface. The presented exact reduction of the problem from three to two dimensions enriches each particle of the reference surface with extra degrees of freedom. This is evident from the structure of the dynamic equations (1.1)-(1.3). Recalling the definitions of the resultant stress vectors $\boldsymbol{n}^{\Lambda}$ and the resultant stress couple vectors $\boldsymbol{m}^{\Lambda}$, it becomes obvious that both have all three components with respect to any basis (typically, two tangential components and one normal component). Thus, in scalar form the equilibrium equations (1.1) constitute a system of six independent equations involving six resultant forces and six resultant couples. Also, the dynamic jump conditions (1.2) and boundary conditions (1.3), when expressed in component form, constitute a systems of six scalar equations. For the shell reference surface to support these dynamic equations, extra kinematic variables are needed besides its deformation $y = \chi(Y)$. Referring to χ as the translational or macroscopic deformation, extra kinematic variables which the shell reference surface should be equipped with may be called internal or microscopic variables.

The crucial point in the theory is the derivation of a sufficiently general concept for the kinematics. In effect, the way this concept will be introduced will delimit the resulting structure of the shell theory. The usual treatment is based either on ad hoc postulated kinematics of the shell (direct approach) or it involves the representation of the three-dimensional deformation of the shell-like body as a given a priori function of some two-dimensional independent kinematical variables. In effect, the kinematics of the shell is no more no less but the basic postulate of the theory and the results to be obtained are embedded in the assumptions laying at the starting point.

Another way in approaching the problem is to take as a reasonable anticipation that the shell kinematics, like the dynamic shell equations, should be an outcome of the analysis, and not the basic postulate of the theory. This is exactly the point of view, which we take here. To this end, we first derive some integral identities, which are natural implications of the dynamic shell equations (1.1)-(1.3). Although our considerations are restricted to the quasi-static case, we shall consider not a single equilibrium state of the shell but a one-parameter family of such states. Consequently, we shall consider a one-parameter family $y = \chi_t(Y) = \chi(Y, t)$ of deformations of the reference surface. We may regard $t \in \mathcal{T}$ as time or just as a real parameter. The underlying maps $\chi_t : M \to \mathcal{E}$ can then be referred to as a real or virtual motion of the shell. For simplicity of presentation we

shall speak about the motion of the shell, and we define the time derivative as partial derivative with respect to t keeping Y fixed. Moreover, like in dynamic considerations, we admit that the motion of the shell reference surface needs not be even continuous across the singular curve $\mathfrak{L}$, which may follow its own motion $y_\ell = \chi_\ell(Y,t)$. If the motion is continuous over the entire reference surface, then χ_ℓ is defined to be the restriction of χ to the singular curve $\mathfrak{L}$.

1.2 Virtual work expressions. Let $v(Y,t)$ and $w(Y,t)$ be any two vector fields defined over the shell reference surface, except possibly at the singular curve. Thus, the fields v and w need not be defined along $\mathfrak{L}$. Let $v_\ell(Y,t)$ and $w_\ell(Y,t)$ be two other vector fields defined along the singular curve $\mathfrak{L}$. In the special case we may regard (v_ℓ, w_ℓ) as the restrictions of (v,w) to $\mathfrak{L}$.

Taking the dynamic shell equations (1.1)-(1.3) as the starting point, we shall consider the following expressions:

$$\big(Div_s N + p\big)\cdot v + \big(Div_s M + ad^{-1}(NF^T - FN^T) + l\big)\cdot w\,, \tag{1.4}$$

$$(n_\nu^* - Nv)\cdot v + (m_\nu^* - Mv)\cdot w\,, \tag{1.5}$$

$$\big(p_\ell - [\![n_\nu]\!]\big)\cdot v_\ell + \big(l_\ell - [\![m_\nu]\!] + [\![(y_\ell - y)\times n_\nu]\!]\big)\cdot w_\ell\,, \tag{1.6}$$

which are well defined at every point of $M\setminus\mathfrak{L}$, ∂M_f and $\mathfrak{L}$, respectively. Taking the integrals of (1.4)-(1.6) over corresponding domains and adding the results we obtain

$$\begin{aligned}\mathfrak{w} \equiv &-\iint_{M\setminus\mathfrak{L}}\big((Div_s N + p)\cdot v + (Div_s M + ad^{-1}(NF^T - FN^T) + l)\cdot w\big)\,dA\\ &+\int_{\mathfrak{L}}\big(\big(p_\ell - [\![n_\nu]\!]\big)\cdot v_\ell + \big(l_\ell - [\![m_\nu]\!] + [\![(y_\ell - y)\times n_\nu]\!]\big)\big)\cdot w_\ell\big)\,dS\\ &+\int_{\partial M_f}\big((n_\nu^* - Nv)\cdot v + (m_\nu^* - Mv)\cdot w\big)\,dS\,.\end{aligned} \tag{1.7}$$

The minus sign preceding the first integral in (1.7) is convention. The physical meaning of this expression is actually self-evident, but at this stage of analysis this needs not be our concern and all subsequent considerations can be regarded as purely formal. Let us only note that the quantity defined by (1.7) is in effect a set function having the entire shell reference surface and the fields $\mathbb{V} = (v,w,v_r,w_r)$ as its arguments. To emphasize this fact we may write $\mathfrak{w} = \mathfrak{w}(M;\mathbb{V})$. It may also be noted that we can equally define $\mathfrak{w}(\Pi;\mathbb{V})$ for any part $\Pi \subset M$ of the reference surface.

In order to obtain a more convenient form of $\mathfrak{w} = \mathfrak{w}(M; \mathbb{V})$ we assume that the fields v and w be of class C^n, $n \geq 1$, within the interior of each smooth part $M_{(k)}$ of the reference surface, and they have finite limits $v^{(k)}(Y)$ and $w^{(k)}(Y)$ at every point $Y \in \mathfrak{L}$ taken along paths in $M_{(k)}$, i.e.

$$v^{(k)}(Y) = \lim_{Z \to Y} v(Z), \qquad w^{(k)}(Y) = \lim_{Z \to Y} w(Z), \qquad Z \in M_{(k)} . \tag{1.8}$$

Moreover, we assume that the deformation of the reference surface and the resultant stress tensors satisfy the regularity conditions of Sect. II.4.1. Under these assumptions the generalized surface divergence theorem together with the following differential identity

$$(Div_s N) \cdot v = Div_s(N^T v) - N \cdot \nabla_s v \tag{1.9}$$

yields

$$\begin{aligned} \iint_{M \setminus \mathfrak{L}} (Div_s N) \cdot v \, dA = &- \iint_{M \setminus \mathfrak{L}} N \cdot \nabla_s v \, dA \\ &+ \int_{\mathfrak{L}} [\![n_\nu \cdot v]\!] \, dS + \int_{\partial M} N\nu \cdot v \, dS . \end{aligned} \tag{1.10}$$

In exactly the same manner

$$\begin{aligned} \iint_{M \setminus \mathfrak{L}} (Div_s M) \cdot w \, dA = &- \iint_{M \setminus \mathfrak{L}} M \cdot \nabla_s w \, dA \\ &+ \int_{\mathfrak{L}} [\![m_\nu \cdot w]\!] \, dS + \int_{\partial M} M\nu \cdot w \, dS . \end{aligned} \tag{1.11}$$

Here the jumps at every point $Y \in \mathfrak{L}$ of the singular curve are defined by

$$\begin{aligned} [\![n_\nu \cdot v]\!] &= \sum_{k=1}^{n} n_\nu^{(k)} \cdot v^{(k)} = \sum_{k=1}^{n} N^{(k)} \nu^{(k)} \cdot v^{(k)} , \\ [\![m_\nu \cdot w]\!] &= \sum_{k=1}^{n} m_\nu^{(k)} \cdot w^{(k)} = \sum_{k=1}^{n} M^{(k)} \nu^{(k)} \cdot w^{(k)} . \end{aligned} \tag{1.12}$$

The boundary of the shell reference surface can be written as union of two mutually complementary parts ∂M_f and ∂M_d, $\partial M = \partial M_f \cup \partial M_d$. Then integrals along ∂M can be expressed as sum of two parts:

$$\begin{aligned} \int_{\partial M} N\nu \cdot v \, dS &= \int_{\partial M_f} N\nu \cdot v \, dS + \int_{\partial M_d} N\nu \cdot v \, dS , \\ \int_{\partial M} M\nu \cdot w \, dS &= \int_{\partial M_f} M\nu \cdot w \, dS + \int_{\partial M_d} M\nu \cdot w \, dS . \end{aligned} \tag{1.13}$$

With the use of (1.10), (1.11) and (1.13) the expression (1.7) can be rewritten as

$$\mathfrak{w}(M; \mathbb{V}) = \mathfrak{w}_{int}(M; \mathbb{V}) + \mathfrak{w}_\ell(M; \mathbb{V}) - \mathfrak{w}_{ext}(M; \mathbb{V}) , \tag{1.14}$$

where

$$\mathfrak{w}_{int}(M;\mathbb{V}) = \iint_{M\setminus\mathfrak{L}} w(Y)\,dA, \tag{1.15}$$

$$w \equiv \boldsymbol{N}\bullet\nabla_s \boldsymbol{v} + ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\bullet\boldsymbol{w} + \boldsymbol{M}\bullet\nabla_s \boldsymbol{w} \tag{1.16}$$

may be called the internal virtual work (or the stress power),

$$\mathfrak{w}_\ell(\mathfrak{L};\mathbb{V}) = \int_{\mathfrak{L}} w_\ell(Y)\,dS\,, \tag{1.17}$$

$$\begin{aligned} w_\ell = {} & \boldsymbol{p}_\ell\bullet\boldsymbol{v}_\ell + \boldsymbol{l}_\ell\bullet\boldsymbol{w}_\ell + [\![\boldsymbol{n}_\nu\bullet\boldsymbol{v}]\!] - [\![\boldsymbol{n}_\nu]\!]\bullet\boldsymbol{v}_\ell \\ & + [\![\boldsymbol{m}_\nu\bullet\boldsymbol{w}]\!] - [\![\boldsymbol{m}_\nu]\!]\bullet\boldsymbol{w}_\ell + [\![(\boldsymbol{y}-\boldsymbol{y}_\ell)\times\boldsymbol{n}_\nu]\!]\bullet\boldsymbol{w}_\ell \end{aligned} \tag{1.18}$$

the virtual work (or the mechanical power) of all forces and couples acting along the singular curve $\mathfrak{L}$, and

$$\begin{aligned} \mathfrak{w}_{ext}(M;\mathbb{V}) = {} & \iint_{M\setminus\mathfrak{L}} (\boldsymbol{p}\bullet\boldsymbol{v} + \boldsymbol{l}\bullet\boldsymbol{w})\,dA + \int_{\partial M_f} (\boldsymbol{n}^*\bullet\boldsymbol{v} + \boldsymbol{m}^*\bullet\boldsymbol{w})\,dS \\ & + \int_{\partial M_d} (\boldsymbol{N}\boldsymbol{v}\bullet\boldsymbol{v} + \boldsymbol{M}\boldsymbol{v}\bullet\boldsymbol{w})\,dS \end{aligned} \tag{1.19}$$

the virtual work of external forces and couples acting on the shell reference surface and along its boundary, respectively. While these names are fully justified by the involved quantities, for the time being, they can equally be regarded as just convenient names. Let us only note that the last integral in (1.19) includes unknown resultant force and couple vectors being the reactions to possible constraints, which can be imposed on the deformation of the shell along the part ∂M_d of the boundary ∂M.

1.3 Stress power density. Let us denote by $\boldsymbol{W} : M\setminus\mathfrak{L} \to E\wedge E$ the field of skew-symmetric tensors associated with the vector field $\boldsymbol{w}$, i.e. $\boldsymbol{W} = ad\boldsymbol{w}$ at every regular point of the reference surface. Then

$$ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\bullet\boldsymbol{w} = \frac{1}{2}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\bullet\boldsymbol{W}\,. \tag{1.20}$$

From the classical property of the inner product of tensors we further have

$$\begin{aligned} \boldsymbol{N}\boldsymbol{F}^T\bullet\boldsymbol{W} &= \boldsymbol{N}\bullet\boldsymbol{W}\boldsymbol{F}\,, \\ \boldsymbol{F}\boldsymbol{N}^T\bullet\boldsymbol{W} &= \boldsymbol{N}^T\bullet\boldsymbol{F}^T\boldsymbol{W} = \boldsymbol{N}^T\bullet(\boldsymbol{W}^T\boldsymbol{F})^T \\ &= -\boldsymbol{N}^T\bullet(\boldsymbol{W}\boldsymbol{F})^T = -\boldsymbol{N}\bullet\boldsymbol{W}\boldsymbol{F}\,, \end{aligned} \tag{1.21}$$

and (1.20) is obtained in the form

$$ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)\bullet\boldsymbol{w}=\boldsymbol{N}\bullet\boldsymbol{W}\boldsymbol{F}\,. \tag{1.22}$$

Substituting (1.22) into (1.16) the stress power density (or the internal virtual work density) takes the form

$$w=\boldsymbol{N}\bullet(\nabla_s\boldsymbol{v}-\boldsymbol{W}\boldsymbol{F})+\boldsymbol{M}\bullet\nabla_s\boldsymbol{w}\,. \tag{1.23}$$

Since (1.23) is defined only at regular points of the reference surface, it can be rewritten in a more familiar form using surface coordinates. In local coordinates the surface gradients of the fields $\boldsymbol{v}$ and $\boldsymbol{w}$ are given by

$$\nabla_s\boldsymbol{v}=\boldsymbol{v}_{,\Lambda}\otimes\boldsymbol{A}^\Lambda\,,\qquad \nabla_s\boldsymbol{w}=\boldsymbol{w}_{,\Lambda}\otimes\boldsymbol{A}^\Lambda\,. \tag{1.24}$$

Consequently, we have

$$\begin{aligned}\boldsymbol{N}\bullet\nabla_s\boldsymbol{v}&=(\boldsymbol{n}^\Lambda\otimes\boldsymbol{A}_\Lambda)\bullet(\boldsymbol{v}_{,\Sigma}\otimes\boldsymbol{A}^\Sigma)=\boldsymbol{n}^\Lambda\bullet\boldsymbol{v}_{,\Lambda}\,,\\ \boldsymbol{M}\bullet\nabla_s\boldsymbol{w}&=(\boldsymbol{m}^\Lambda\otimes\boldsymbol{A}_\Lambda)\bullet(\boldsymbol{w}_{,\Sigma}\otimes\boldsymbol{A}^\Sigma)=\boldsymbol{m}^\Lambda\bullet\boldsymbol{w}_{,\Lambda}\,.\end{aligned} \tag{1.25}$$

Moreover, keeping in mind that $\boldsymbol{W}$ is a skew-symmetric tensor, whose axial vector is $\boldsymbol{w}$, we have

$$\boldsymbol{N}\bullet\boldsymbol{W}\boldsymbol{F}=(\boldsymbol{n}^\Lambda\otimes\boldsymbol{A}_\Lambda)\bullet(\boldsymbol{W}(\boldsymbol{y}_{,\Gamma}\otimes\boldsymbol{A}^\Gamma))=\boldsymbol{n}^\Lambda\bullet(\boldsymbol{y}_{,\Lambda}\times\boldsymbol{w})\,. \tag{1.26}$$

Substituting (1.24) and (1.26) into (1.23) the internal virtual work density is obtained in the form

$$w=\boldsymbol{n}^\Lambda\bullet(\boldsymbol{v}_{,\Lambda}+\boldsymbol{y}_{,\Lambda}\times\boldsymbol{w})+\boldsymbol{m}^\Lambda\bullet\boldsymbol{w}_{,\Lambda}\,. \tag{1.27}$$

Using the obvious relations

$$\begin{aligned}[\![\boldsymbol{n}_\nu\bullet\boldsymbol{v}]\!]-[\![\boldsymbol{n}_\nu]\!]\bullet\boldsymbol{v}_\ell&=[\![\boldsymbol{n}_\nu\bullet(\boldsymbol{v}-\boldsymbol{v}_\ell)]\!]\,,\\ [\![\boldsymbol{m}_\nu\bullet\boldsymbol{w}]\!]-[\![\boldsymbol{m}_\nu]\!]\bullet\boldsymbol{w}_\ell&=[\![\boldsymbol{m}_\nu\bullet(\boldsymbol{w}-\boldsymbol{w}_\ell)]\!]\end{aligned} \tag{1.28}$$

and the standard vector identity

$$[\![(\boldsymbol{y}-\boldsymbol{y}_\ell)\times\boldsymbol{n}_\nu]\!]\bullet\boldsymbol{w}_\ell=[\![((\boldsymbol{y}-\boldsymbol{y}_\ell)\times\boldsymbol{n}_\nu)\bullet\boldsymbol{w}_\ell]\!]=[\![\boldsymbol{n}_\nu\bullet((\boldsymbol{y}-\boldsymbol{y}_\ell)\times\boldsymbol{w}_\ell)]\!]\,, \tag{1.29}$$

the virtual work density (1.18) along the singular curve is obtained as

$$w_\ell=[\![\boldsymbol{n}_\nu\bullet(\boldsymbol{v}_\ell-\boldsymbol{v}+(\boldsymbol{y}_\ell-\boldsymbol{y})\times\boldsymbol{w}_\ell)]\!]+[\![\boldsymbol{m}_\nu\bullet(\boldsymbol{w}_\ell-\boldsymbol{w})]\!]-\boldsymbol{p}_\ell\bullet\boldsymbol{v}_\ell-\boldsymbol{l}_\ell\bullet\boldsymbol{w}_\ell\,. \tag{1.30}$$

1.4 Statement of the problem. The foregoing considerations show that if the equilibrium equations (1.1), the static jump conditions (1.2) and the static boundary conditions (1.3) hold, then

$$\mathfrak{w}(M;\mathbb{V}) = \mathfrak{w}_{int}(M;\mathbb{V}) + \mathfrak{w}_{\ell}(M;\mathbb{V}) - \mathfrak{w}_{ext}(M;\mathbb{V}) = 0\,, \tag{1.31}$$

for every set $\mathbb{V} = (\boldsymbol{v}, w, \boldsymbol{v}_{\ell}, w_{\ell})$ of vector fields satisfying the regularity assumptions stated in Sect. 1.2. Conversely, if (1.31) holds for every set $\mathbb{V} = (\boldsymbol{v}, w, \boldsymbol{v}_{\ell}, w_{\ell})$ of vector fields satisfying the same regularity assumptions, then the equilibrium equations (1.1), the static jump conditions (1.2) and the static boundary conditions (1.3) hold.

With the theorem (1.31) as the starting point we can now state the problem of the kinematics of the shell theory in a clear manner. By the way of preparation let us first make clear the physical sense of the above theorem.

The equilibrium equations (1.1), the jump conditions (1.2) and the boundary conditions (1.3) represent in the local form the balance laws of forces and torques acting at each regular point of the shell reference surface, at each point of the singular curve, and at every point of the boundary, respectively. Accordingly, $\mathfrak{w}_{int}(M;\mathbb{V})$, $\mathfrak{w}_{\ell}(M;\mathbb{V})$ and $\mathfrak{w}_{ext}(M;\mathbb{V})$, respectively, represent the associated virtual work densities. In effect, the theorem (1.31) expresses the principle of virtual work with $\mathbb{V} = (\boldsymbol{v}, w, \boldsymbol{v}_{\ell}, w_{\ell})$ being the test functions, more often called the virtual displacements. However, it must be stressed that the term "displacement" has to be understood here in a generalized sense. The local dynamic equations (1.1)-(1.3) are implications of the overall resultant balance laws expressed entirely in terms of through-the-thickness resultant quantities. Accordingly, the theorem (1.31), which has been derived from those equations, expresses the overall resultant balance laws in a weak form.

As to the theorem (1.31), we note that the stress tensor $\boldsymbol{N}(Y,t)$ and the stress couple tensor $\boldsymbol{M}(Y,t)$ represent through-the-thickness resultant stresses within the shell-like body. Accordingly, the expression (1.23) is just the overall virtual work performed by internal stresses with

$$\nabla_s \boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}\,, \qquad \nabla_s w\,, \tag{1.32}$$

as the work-conjugate, overall virtual strains. In general, these virtual strains need not be related to the virtual displacements $\mathbb{V} = (\boldsymbol{v}, w, \boldsymbol{v}_{\ell}, w_{\ell})$. The theorem (1.31)

shows that they are related by the formulae (1.32) in every dynamic equilibrium state of the shell.

The problem to be solved can now be stated in the following way: find the real motion of the shell and the real strain measures, whose virtual counterparts are $\mathbb{V} = (v, w, v_\ell, w_\ell)$ and (1.32), respectively.

2. Overall motion of the shell

2.1 Translational and rotational motion of the shell. If we take the motion $y = \chi(Y,t)$ of the shell reference surface as the primary kinematic variable, then the inspection of the virtual work expression (1.15)- (1.19) shows that the field $v(Y,t)$ entering those expressions should be regarded as the associated virtual motion or, what means the same, the associated velocity field. In other words, the translational velocity, with which the shell transverses the space, is defined as time derivative of the motion $y = \chi(Y,t)$, i.e.

$$v(Y,t) = \dot{\chi}(Y,t). \tag{2.1}$$

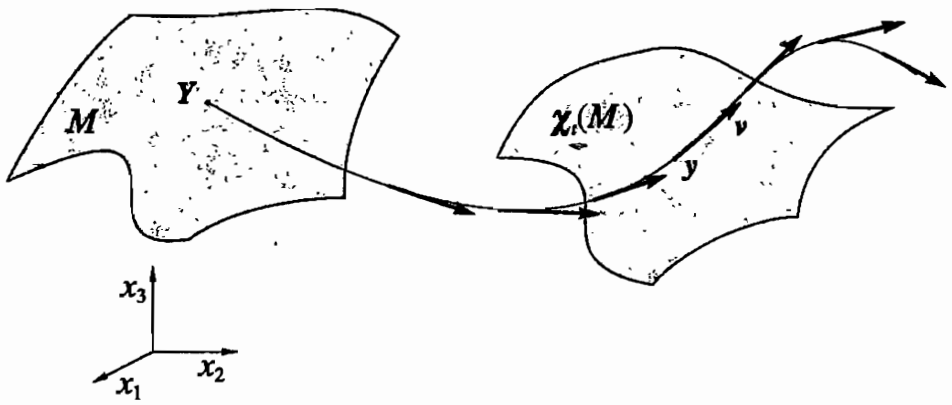

Fig. 1

However, we can also set the problem in the inverse form. Taking a field $v(Y,t)$ as primary variable, we want to find a one-parameter family of maps $\chi_t : M \to \mathcal{E}$ such that (Fig. 1)

$$\frac{d}{dt}\chi(Y,t) = v(Y,t), \qquad \forall\, t \in \mathcal{T}. \tag{2.2}$$

From the theory of first order differential equations it follows that the solution of (2.2) exists whenever suitable regularity assumptions are satisfied. In the language of differential geometry, χ_t is the flow of the vector field v.

Now it becomes clear that extra degrees of freedom, with which the carrying surface is equipped, must have the field $w(Y,t)$ appearing in the virtual work expressions (1.4) as the virtual counterpart or, in an other interpretation, as the velocity field. Formally, this problem can be set in the following way. Given a vector field $w(Y,t)$, find a one parameter family of maps $Q_t : M \to E \otimes E$ by $Q = Q(Y,t)$, such that

$$\frac{d}{dt}Q(Y,t) = W(Y,t)Q(Y,t), \qquad W(Y,t) = adw(Y,t). \tag{2.3}$$

Here and in the sequel for simplicity of writing we use the same symbol Q for the map and its values leaving the context to make clear, which one is meant. Since $W(Y,t)$ is a skew tensor, the solution of the differential equation (2.3) is the rotation tensor. Thus, the extra microstructure of the carrying surface is determined by the field of rotation tensors, which equips every point of M with three rotational degrees of freedom, $Q_t : M \to SO(3)$, where $SO(3)$ denotes the rotation group.

2.2 Angular velocity fields. A velocity field associated with the rotational motion of the shell can be defined in the same way as the translational velocity field, i.e. as time derivative of a given motion $Q_t : M \to SO(3)$. However, since $SO(3)$ lacks a vector space structure, a greater care is needed in the calculation of the time derivative of the rotation tensor. The basic fact here is that $SO(3)$ is the three-dimensional Lie group. Hence, for any fixed point $Y \in M$ the rotational motion of Y is a smooth curve on the rotation group. Then the time derivative $\dot{Q}(t)$ at a fixed time instant t is an element of the tangent space to $SO(3)$ at the "point" $Q(t)$, that is an element of the vector space $T_Q SO(3)$. In this consideration Y is an arbitrary but fixed point of M and we simply write $\dot{Q}(t)$ instead of $\dot{Q}(Y,t)$. The tangent space $T_Q SO(3)$ is isomorphic in two ways with the tangent space at the identity (Fig. 2), which is the vector space of skew-symmetric tensors $T_1 SO(3) = E \wedge E$. The two isomorphisms of these vector spaces are determined by the left and right translation on the group $SO(3)$. Using these facts we easily find the time derivative of the rotational motion $Q_t : M \to SO(3)$ of the shell in terms of angular velocity fields.

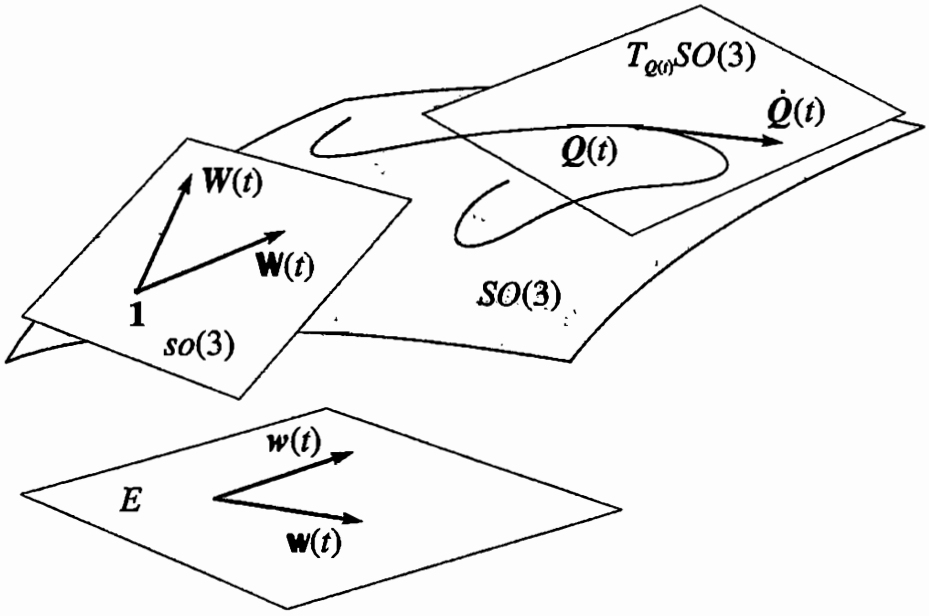

Fig. 2

The angular velocity fields can also be derived from elementary calculations in the following way. The rotation group $SO(3)$ can be considered as a "hypersurface" in the vector space $E \otimes E$ of tensors, i.e. the vector space of linear maps of E into itself. Then we can consider $Q_t : M \to SO(3)$ as a map having $E \otimes E$ as its range and satisfying the constraint

$$Q(Y,t)Q(Y,t)^T = Q(Y,t)^T Q(Y,t) = \mathbf{1}. \tag{2.4}$$

Within this setting the time derivative $\dot{Q}(t)$ has the meaning of derivative in the usual sense, and the differentiation of (2.4) yields

$$\dot{Q}(Y,t) = W(Y,t)Q(Y,t) = Q(Y,t)\mathbf{W}(Y,t), \tag{2.5}$$

where the two tensors

$$W(Y,t) = \dot{Q}Q^T, \qquad \mathbf{W}(Y,t) = Q^T\dot{Q}, \tag{2.6}$$

are necessarily skew-symmetric. We shall denote by

$$w(Y,t) = ad^{-1}W(Y,t), \qquad \mathbf{w}(Y,t) = ad^{-1}\mathbf{W}(Y,t), \tag{2.7}$$

the corresponding axial vectors. These two vectors, equivalently the skew tensors W and $\mathbf{W}$, represent the angular virtual motion or the angular velocity field. This

terminology is justified by the rigid body dynamics. Moreover, as simple implications we have the following relations

$$W = QWQ^T, \qquad w = Qw. \tag{2.8}$$

The first of these relations is a direct implication of (2.5), the second one follows from the property of the map *ad*,

$$ad^{-1}(QWQ^T) = Q(ad^{-1}W), \tag{2.9}$$

which is true for any two skew tensors and every rotation tensor.

For our subsequent considerations another point is to be noted. The fields (2.7) represent the same virtual or real angular velocity of the shell in two representations. This is the direct consequence of the fact that the tangent space $T_Q SO(3)$ is isomorphic with the tangent space at the identity of the group (this space is in turn isomorphic with the Lie algebra of the group). An important implication of this fact is that all shell equations have also two entirely equivalent representations. Moreover, various kinematic and dynamic variables of the shell theory in the two representations are related through the formulae similar to (2.8). In particular, it will become clear in the next chapters that it is convenient to introduce the translational velocity field defined by

$$v(Y,t) = Q(Y,t)v(Y,t), \tag{2.10}$$

in correspondence with the angular velocity fields.

2.3 Moving triads. To determine the rotational motion $Q_t : M \to SO(3)$ of the shell we have taken the field $w(Y,t)$ as the primary kinematic variable. This means that $Q(Y,t)$ determines the change of extra degrees of freedom with which the shell reference surface is equipped, and not the degrees of freedom themselves. Such degrees of freedom must be specified independently in every configuration of the shell reference surface. This can be done in many ways, but it is clear that it is enough to specify the extra degrees of freedom in the undeformed configuration M. The corresponding degrees of freedom in the current configuration $\chi_t(M)$ are related through the tensor Q to their counterpart in the reference configuration.

In the most general case the extra degrees of freedom of M can be specified by any invertible tensor field $T_0(Y)$. Then its spatial counterpart is the invertible tensor field $T(y,t)$ related to $T_0(Y)$ by

$$T(\chi(Y,t),t)=Q(Y,t)T_0(Y). \tag{2.11}$$

Let us note that neither $T_0(Y)$ nor $T(y,t)$ need to be rotation tensors, but solely invertible tensors. Since $T_0(Y)$ is a nonsingular tensor, it can be specified by assigning to each point $Y \in M$ the triad $\{D_K(Y)\}$ consisting of any three linearly independent vectors (Fig. 3). We can also introduce the reciprocal triad, which is defined pointwise in the usual way

$$D^K(Y)\cdot D_L(Y)=\delta^K_L. \tag{2.12}$$

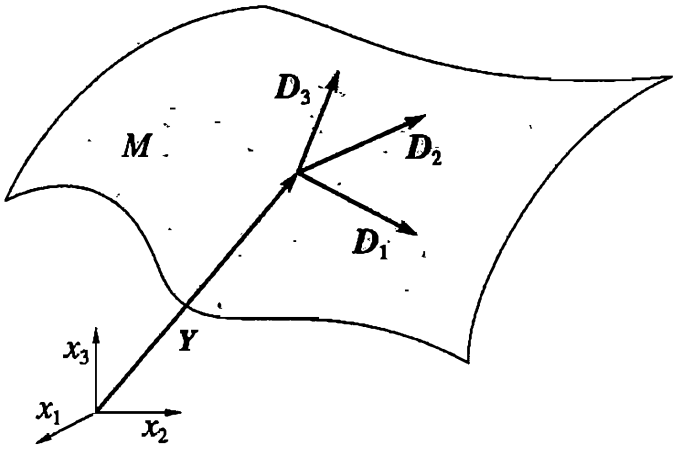

Fig. 3

The triad $\{D_K(Y)\}$ may be defined in any convenient way. For example, by the relation

$$D_K(Y)=T_0(Y)e_K, \tag{2.13}$$

where $\{e_K\}$ denotes a fixed orthonormal basis in the space. Then from (2.12) we have

$$D^K(Y)=T_0^{-1}(Y)e^K, \qquad e^K=e_K. \tag{2.14}$$

This definition is very convenient in the formulation of shell finite elements. Let us note here that the triad $\{D_K(Y)\}$ can be defined in a continuous manner even if the reference surface M itself is not smooth (Fig. 4). Moreover, without loss of generality we can always take

$$D(Y)\equiv D_3(Y)=D^3(Y). \tag{2.15}$$

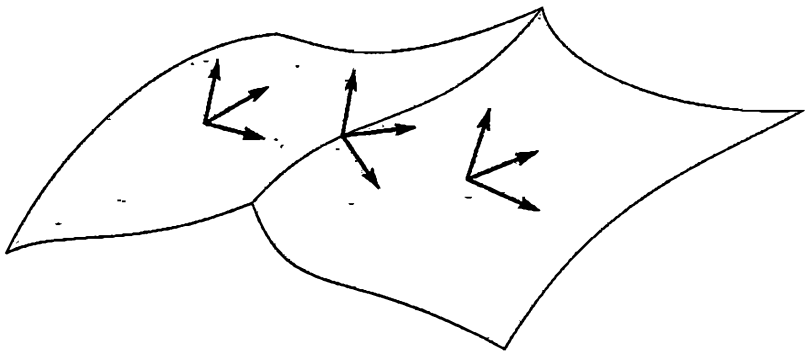

Fig. 4

As special case we can take the anholonomic triad $\{\boldsymbol{D}_K\}$ to coincide with the natural triad for the chosen surface coordinates:

$$\{\boldsymbol{D}_\Lambda, \boldsymbol{D}\} = \{\boldsymbol{A}_\Lambda, \boldsymbol{A}_N\}, \qquad \{\boldsymbol{D}^\Lambda, \boldsymbol{D}\} = \{\boldsymbol{A}^\Lambda, \boldsymbol{A}_N\}. \tag{2.16}$$

More generally, we can also define the anholonomic triad in terms of the natural triad, namely

$$\boldsymbol{D}_K = \boldsymbol{T}_0 \boldsymbol{A}_K, \qquad \boldsymbol{D}^K = (\boldsymbol{T}_0^{-1})^T \boldsymbol{A}^K. \tag{2.17}$$

However, the choice of the anholonomic triad in the form (2.16) or (2.17) is convenient only for smooth shells. Therefore, in general considerations it is preferable to leave unspecified the particular choice of the anholonomic triad.

For any choice of the anholonomic triad $\{\boldsymbol{D}_K\}$ over the undeformed shell reference surface M we can define at each point $y \in \chi(M)$ of the deformed shell reference surface the triad $\{\boldsymbol{d}_k(y,t)\}$ and the reciprocal triad

$$\boldsymbol{d}^k(y,t) \cdot \boldsymbol{d}_l(y,t) = \delta^k_l \tag{2.18}$$

such that

$$\boldsymbol{d}_k(Y,t) = \delta^K_k \boldsymbol{Q}(Y,t)\boldsymbol{D}_K(Y), \qquad \boldsymbol{d}^k(Y,t) = \delta^k_K \boldsymbol{Q}(Y,t)\boldsymbol{D}^K(Y). \tag{2.19}$$

Taking the time derivative of (2.19) and using the relations (2.5) we obtain

$$\dot{\boldsymbol{d}}_k = \delta^K_k \dot{\boldsymbol{Q}} \boldsymbol{D}_K = \boldsymbol{W}\boldsymbol{d}_k = \delta^K_k \boldsymbol{Q}(\boldsymbol{W}\boldsymbol{D}_K), \tag{2.20}$$

or in terms of the associated axial vectors

$$\dot{\boldsymbol{d}}_k = \boldsymbol{w} \times \boldsymbol{d}_k = \delta_k^K \boldsymbol{Q}(\mathbf{w} \times \boldsymbol{D}_K) . \tag{2.21}$$

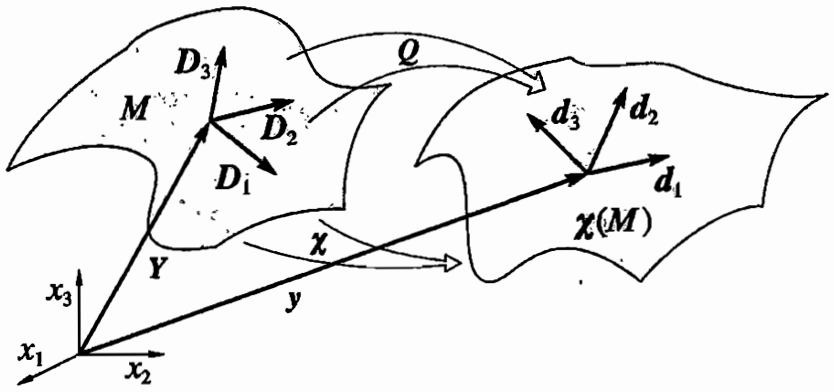

Fig. 5

The same formulae hold for the reciprocal triad. Now we can solve (2.21) for the angular velocity

$$\boldsymbol{w} = -\frac{1}{2}\dot{\boldsymbol{d}}^k \times \boldsymbol{d}_k = -\frac{1}{2}\dot{\boldsymbol{d}}_k \times \boldsymbol{d}^k . \tag{2.22}$$

This result can also be derived by direct calculations if we note that (2.19) implies the following expression for the rotation tensor

$$\boldsymbol{Q} = \delta_K^k \boldsymbol{d}_k \otimes \boldsymbol{D}^K = \delta_k^K \boldsymbol{d}^k \otimes \boldsymbol{D}_K . \tag{2.23}$$

Then the time derivative of (2.23) is given by

$$\dot{\boldsymbol{Q}} = \delta_K^k \dot{\boldsymbol{d}}_k \otimes \boldsymbol{D}^K = \delta_k^K \dot{\boldsymbol{d}}^k \otimes \boldsymbol{D}_K . \tag{2.24}$$

By implication,

$$\boldsymbol{W} = \delta_K^k \delta_L^l (\dot{\boldsymbol{d}}_k \otimes \boldsymbol{D}^K)(\boldsymbol{d}^l \otimes \boldsymbol{D}_L)^T = \delta_K^k \delta_L^l (\dot{\boldsymbol{d}}_k \otimes \boldsymbol{D}^K)(\boldsymbol{D}_L \otimes \boldsymbol{d}^l) . \tag{2.25}$$

Hence

$$\boldsymbol{W} = \dot{\boldsymbol{d}}_k \otimes \boldsymbol{d}^k = \dot{\boldsymbol{d}}^k \otimes \boldsymbol{d}_k . \tag{2.26}$$

Since $\boldsymbol{W}$ is a skew tensor, we have

$$\boldsymbol{W} = \dot{\boldsymbol{d}}_k \wedge \boldsymbol{d}^k = \dot{\boldsymbol{d}}^k \wedge \boldsymbol{d}_k , \tag{2.27}$$

whose axial vector is given by (2.22).

It is now an easy task to show that components of the velocity fields v and w taken with respect to the moving triad $\{d_k\}$ are equal to the components of the velocity fields $\mathbf{v}$ and $\mathbf{w}$ with respect to the referential triad $\{D_K\}$. Indeed, setting

$$v = v_k d^k = v^k d_k , \qquad w = w_k d^k = w^k d_k , \tag{2.28}$$

from transformation rules (2.10) and (2.8) we have

$$\mathbf{v} = v_K D^K = v^K D_K , \qquad \mathbf{w} = w_K D^K = w^K D_K , \tag{2.29}$$

where

$$v_k = \delta_k^K v_K , \qquad w_k = \delta_k^K w_K . \tag{2.30}$$

2.4 Virtual displacement and rotation. Within purely static considerations various kinematic fields can be interpreted in a different manner, but their sense remains unchanged. Let $y = \chi(Y)$ and $Q = Q(Y)$ be a given deformation of the shell. Then for any vector fields $v(Y)$ and $\mathbf{v}(Y)$ and any two skew tensor fields $W(Y)$ and $\mathbf{W}(Y)$, the virtual deformation of the shell is defined as a one-parameter family of deformations given in the form (suppressing the argument Y for simplicity of writing)

$$\begin{aligned} y(\lambda) &= y + \lambda v = y + \lambda Q\mathbf{v} , \\ Q(\lambda) &= exp(\lambda W)Q = Q exp(\lambda \mathbf{W}) , \end{aligned} \tag{2.31}$$

where λ denotes a real parameter. In this context $(v, \mathbf{v})$ and the axial vectors $(w, \mathbf{w})$ associated with the given skew tensors $(W \ \mathbf{W})$ can be referred to as virtual displacements and virtual rotations, respectively. Setting $\lambda = 0$ in (2.31) we obtain the given deformation of the shell. Then the virtual change of the given deformation is defined by

$$\delta y = \frac{d}{d\lambda} y(\lambda)|_{\lambda=0} , \qquad \delta Q = \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0} . \tag{2.32}$$

Now a simple calculation shows that

$$\delta y = v = Q\mathbf{v} , \qquad \delta Q = WQ = Q\mathbf{W} . \tag{2.33}$$

Thus, the resulting formulae are exactly the same as those derived in the previous sections.

2.5 Kinematics of irregular shells. So far we have considered the kinematics of the shell only for regular shell-like bodies. In order to model various irregularities, such as multiple-shell intersection, technological interconnections, etc., we have introduced the concept of irregular shells (Chapt. II.3). In such cases the above considerations should be regarded as being referred to points of $M \setminus \mathfrak{L}$ only, while the singular curve $\mathfrak{L}$ should be allowed to move and deform in its own way. This is reflected in the form of dynamic jump conditions (1.2) and in the corresponding virtual work expression (1.30), which contains the motion $y_\ell = \chi_\ell(Y,t)$ of the singular curve and the virtual fields $v_\ell(Y,t)$ and $w_\ell(Y,t)$ along this curve.

By the same arguments, which have led us to the real translational and rotational motions of every regular part of the shell, we can determine the real translational and rotational motion of the singular curve. They are related to the virtual fields by

$$\dot{\chi}_\ell(Y,t) = v_\ell(Y,t) = Q_\ell(Y,t)\mathbf{v}_\ell(Y,t) \tag{2.34}$$

and

$$\dot{Q}_\ell(Y,t) = W_\ell(Y,t)Q_\ell(Y,t) = Q_\ell(Y,t)\mathbf{W}_\ell(Y,t), \tag{2.35}$$

where $Q_\ell = Q_\ell(Y,t)$ is the field of rotation tensors. The angular velocity fields along the singular curve $\mathfrak{L}$ are then defined in the same manner as at regular points of the reference surface. They also obey the relations identical to those of (2.8).

Like for regular parts of the shell, the rotational degrees of freedom of the singular curve can be specified by the triad $\{D_K(Y)\}_\ell$ assigned at every point $Y \in \mathfrak{L}$ and the triad $\{d_k(y,t)\}_\ell$ at every point $y \in \chi_\ell(\mathfrak{L},t)$. These triads are related by

$$d_K(Y,t) = Q_\ell(Y,t)D_K(Y), \qquad d_K(Y,t) = \delta^k_K d_k(\chi_\ell(Y,t),t). \tag{2.36}$$

The corresponding reciprocal triads and various relations can now be derived in the same manner as in the foregoing considerations.

Thus, in the most general case the complete description of deformation and kinematics of the shell requires introduction of two sets of kinematic variables. The first set consists of the deformation function χ of the shell reference surface and of the rotation tensor Q representing the mean rotation of the shell cross

sections, both being defined at all points of the undeformed reference surface M except possibly at a singular curve $\mathfrak{L}$. The deformation of the curve $\mathfrak{L}$ itself is specified by the second set consisting of the deformation function χ_ℓ and the rotation tensor Q_ℓ. In the special case when the deformation of the shell is assumed to be smooth, χ_ℓ and Q_ℓ will denote the restrictions of χ and Q to the curve $\mathfrak{L}$.

3. Local deformation and strains

3.1 Surface deformation gradients. We have shown that the motion of the shell is completely determined by

$$\chi_t : M \to \mathcal{E}, \qquad Q_t : M \to SO(3), \tag{3.1}$$

Having the description of kinematics of the shell, the next step is to obtain a suitable description of strains and strain rates (equivalently, virtual strains). The general theory of strains for shells substantially differs from the theory of strains in classical continuum mechanics. The main difference is threefold:

1) The shell is a "curved continuum" imbedded in the Euclidean space (each configuration $\chi_t(M)$ of the shell reference surface is a geometric surface and not a region in the space $\mathcal{E}$). Hence, in the analysis of strains not only the changes of distances between surface particles but also changes of curvatures must be taken into account.
2) The shell is a two-dimensional continuum having an internal structure (each particle of the reference surface has extra degrees of freedom). Consequently, strains in the shell are of two kinds.
3) The internal structure of the shell is not a vectorial one (the rotation group $SO(3)$ is not a vector space but a Lie group).

All these facts must be taken into account in the analysis of shell strains.

For a fixed time instant t, the two maps (3.1) determine the global deformation of the shell relative to the reference configuration. Assuming that these maps be differentiable at all points of M, except possibly at points on the curve $\mathfrak{L}$, we denote by

$$\boldsymbol{F}(\boldsymbol{Y},t) = \nabla_s \chi(\boldsymbol{Y},t), \qquad \mathbb{F}(\boldsymbol{Y},t) = \nabla_s \boldsymbol{Q}(\boldsymbol{Y},t) \tag{3.2}$$

their surface gradients. It is clear that $\boldsymbol{F}(\boldsymbol{Y},t)\in E\otimes T_{\boldsymbol{Y}}M$, as we have already pointed out in many places. On the other hand, strictly speaking, $\nabla_s\boldsymbol{Q}(\boldsymbol{Y},t)$ is not defined, since the rotation group $SO(3)$ lacks a vector space structure. Leaving aside for a moment this aspect, we note that the deformation gradients (3.2) approximate the deformation maps (3.1) to within the first order. As such, they provide the basic measures of the local deformation of the shell. However, it must be stressed that the local deformation of the shell depends also on the rotation tensor $\boldsymbol{Q}(\boldsymbol{Y},t)$ and not only on its gradient $\nabla_s\boldsymbol{Q}(\boldsymbol{Y},t)$. Moreover, to determine the local deformation associated with the change of curvatures of the reference surface not only the deformation gradient $\boldsymbol{F}(\boldsymbol{Y},t)$ but also the second gradient of χ_t is needed. These are the main differences in comparison with classical continuum mechanics.

Using local surface coordinates (ξ^Λ) on the undeformed reference surface M, and possibly independent coordinates (ξ^λ) in the current configuration $\chi_t(M)$, the translational motion of the shell is given by

$$\boldsymbol{y}(\xi^\beta)=\chi(\boldsymbol{Y}(\xi^\Lambda)),\qquad \xi^\beta=\xi^\beta(\xi^\Lambda,t). \tag{3.3}$$

Hence, its deformation gradient reads

$$\boldsymbol{F}=\boldsymbol{y}_{,\Lambda}\otimes\boldsymbol{A}^\Lambda=\xi^\beta_\Lambda\boldsymbol{a}_{,\beta}\otimes\boldsymbol{A}^\Lambda,\qquad \xi^\beta_\Lambda\equiv\frac{\partial\xi^\beta}{\partial\xi^\Lambda}. \tag{3.4}$$

Assuming that the deformation of the reference surface be locally invertible, the inverse deformation is described by

$$\boldsymbol{Y}(\xi^\Lambda)=\chi^{-1}(\boldsymbol{y}(\xi^\beta,t),t),\qquad \xi^\Lambda=\xi^\Lambda(\xi^\beta,t). \tag{3.5}$$

Consequently, the inverse deformation gradient is given by

$$\boldsymbol{F}^{-1}=\boldsymbol{Y}_{,\beta}\otimes\boldsymbol{a}^\beta=\xi^\Lambda_\beta\boldsymbol{A}_\Lambda\otimes\boldsymbol{a}^\beta,\qquad \xi^\Lambda_\beta\equiv\frac{\partial\xi^\Lambda}{\partial\xi^\beta}. \tag{3.6}$$

Moreover, we have the obvious relations

$$\xi^\alpha_\Lambda\xi^\Lambda_\beta=\delta^\alpha_\beta,\qquad \xi^\Gamma_\alpha\xi^\alpha_\Lambda=\delta^\Gamma_\Lambda. \tag{3.7}$$

3.3 Surface gradient of the rotation tensor. In order to calculate the surface deformation gradient of the rotational motion of the shell we can proceed in two ways, which we have pointed out in Sect. 2.2. The first way requires to base our

calculations on the theory of Lie groups. The second one, which we take here, is based on the fact that the rotational motion of the shell can be regarded as the map $\boldsymbol{Q}_t : M \to SO(3) \subset E \otimes E$ having the tensor space $E \otimes E$ as its codomain and subjected to the constraint (2.4). Then the surface gradient operator is well defined and $\nabla_s \boldsymbol{Q}(\boldsymbol{Y},t) \in L(T_{\boldsymbol{Y}} M, E \otimes E)$.

In terms of surface coordinates the rotational motion of the shell is given in the form

$$\boldsymbol{Q}(\xi^\Lambda,t) = \boldsymbol{Q}(\boldsymbol{Y}(\xi^\Lambda),t), \tag{3.8}$$

so that the gradient can be written as

$$\nabla_s \boldsymbol{Q}(\boldsymbol{Y},t) = \boldsymbol{Q}_{,\Lambda}(\boldsymbol{Y},t) \otimes \boldsymbol{A}^\Lambda(\boldsymbol{Y}). \tag{3.9}$$

Moreover, since $\boldsymbol{Q}$ is the rotation tensor, its partial derivative with respect to the surface coordinates can be expressed in the form

$$\boldsymbol{Q}_{,\Lambda} = \boldsymbol{K}_\Lambda \boldsymbol{Q} = \boldsymbol{Q}\mathbf{K}_\Lambda, \tag{3.10}$$

where $\boldsymbol{K}_\Lambda(\boldsymbol{Y},t)$ and $\mathbf{K}_\Lambda(\boldsymbol{Y},t)$ are necessarily skew-symmetric tensors whose axial vectors we shall denote by $\boldsymbol{k}_\Lambda(\boldsymbol{Y},t)$ and $\mathbf{k}_\Lambda(\boldsymbol{Y},t)$, respectively, i.e.

$$\boldsymbol{K}_\Lambda = \boldsymbol{Q}_{,\Lambda}\boldsymbol{Q}^T = ad\boldsymbol{k}_\Lambda, \qquad \mathbf{K}_\Lambda = \boldsymbol{Q}^T\boldsymbol{Q}_{,\Lambda} = ad\,\mathbf{k}_\Lambda. \tag{3.11}$$

We can then define the following third-order tensors

$$\mathbb{K}(\boldsymbol{Y},t) = \boldsymbol{K}_\Lambda \otimes \boldsymbol{A}^\Lambda, \qquad \mathbb{K}(\boldsymbol{Y},t) = \mathbf{K}_\Lambda \otimes \boldsymbol{A}^\Lambda. \tag{3.12}$$

It is seen now that $\mathbb{K}\boldsymbol{u}$ and $\mathbb{K}\boldsymbol{u}$ are skew tensors for every vector $\boldsymbol{u} \in T_{\boldsymbol{Y}} M$, i.e.

$$\mathbb{K}(\boldsymbol{Y},t), \mathbb{K}(\boldsymbol{Y},t) \in L(T_{\boldsymbol{Y}} M, E \wedge E). \tag{3.13}$$

In effect, the above definitions and the derived results are two-dimensional counterparts of the calculations which we have carried out to obtain the angular velocity fields. Likewise, they are direct implications of the fact that the tangent space to the rotation group is ismomorphic in two ways with the space of skew tensors. This becomes obvious if we regard $\nabla_s \boldsymbol{Q}(\boldsymbol{Y},t)$ as the tangent map of the rotational motion $(3.1)_2$. Then $\nabla_s \boldsymbol{Q}(\boldsymbol{Y},t) \in L(T_{\boldsymbol{Y}} M, T_{\boldsymbol{Q}(\boldsymbol{Y},t)} SO(3))$ and all facts mentioned in Sect. 2.2 apply.

3.4 Strain measures. As basic measures of the local deformation of the shell we have $(\boldsymbol{F},\boldsymbol{Q},\mathbb{F})$. The above considerations show that, equivalently, we can take the triples $(\boldsymbol{F},\boldsymbol{Q},\mathbb{K})$ or $(\boldsymbol{F},\boldsymbol{Q},\mathbb{K})$. From them we can derive various strain measures and there are many possibilities. Among various possible strain measures there are some which are naturally distinguished by the dynamic shell equations, namely, those which are work-conjugate to the resultant stress and stress couple tensors. In view of the second theorem of Sect. 1.3, the problem to be solved is: find the strain tensors whose appropriate rates (or virtual changes) are tensors given in terms of the virtual displacement and rotation by relation (1.32).

Having in mind that there are two equivalent sets of velocity fields we can expect that there are also two equivalent sets of strain measures. In fact, we shall show that work-conjugate to the resultant stress and stress couple tensors $\boldsymbol{N}(\boldsymbol{Y},t)$ and $\boldsymbol{M}(\boldsymbol{Y},t)$ are the strain tensors $\boldsymbol{E}(\boldsymbol{Y},t), \boldsymbol{K}(\boldsymbol{Y},t) \in E \otimes T_Y M$ defined by

$$\boldsymbol{E} = \boldsymbol{F} - \boldsymbol{Q}\boldsymbol{I}\,, \qquad \boldsymbol{K}\boldsymbol{u} = ad^{-1}(\mathbb{K}\boldsymbol{u})\,, \quad \forall\, \boldsymbol{u} \in T_Y M\,. \tag{3.14}$$

Let us recall that $\boldsymbol{I}(\boldsymbol{Y}) \in E \otimes T_Y M$ denotes the inclusion map, and the rotation tensor $\boldsymbol{Q}$ is regarded here as an element of the space $E \otimes E$. The inclusion operator can be expressed in the form $\boldsymbol{I} = \boldsymbol{Y}_{,\Lambda} \otimes \boldsymbol{A}^{\Lambda}$, and we easily find that the strain tensors (3.14) can be expressed in the form

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{e}_{\Lambda} \otimes \boldsymbol{A}^{\Lambda}\,, \qquad \boldsymbol{e}_{\Lambda} = \boldsymbol{y}_{,\Lambda} - \boldsymbol{Q}\boldsymbol{Y}_{,\Lambda}\,, \\ \boldsymbol{K} &= \boldsymbol{k}_{\Lambda} \otimes \boldsymbol{A}^{\Lambda}\,, \qquad \boldsymbol{k}_{\Lambda} = ad^{-1}(\boldsymbol{Q}_{,\Lambda}\boldsymbol{Q}^{T})\,. \end{aligned} \tag{3.15}$$

Equivalent strain tensors $\mathbf{E}(\boldsymbol{Y},t), \mathbf{K}(\boldsymbol{Y},t) \in E \otimes T_Y M$ can be defined by

$$\mathbf{E} = \boldsymbol{Q}^{T}\boldsymbol{F} - \boldsymbol{I}\,, \qquad \mathbf{K}\boldsymbol{u} = ad^{-1}(\mathbb{K}\boldsymbol{u})\,, \quad \forall\, \boldsymbol{u} \in T_Y M\,. \tag{3.16}$$

In terms of surface coordinates these tensors are given by

$$\begin{aligned} \mathbf{E} &= \mathbf{e}_{\Lambda} \otimes \boldsymbol{A}^{\Lambda}\,, \qquad \mathbf{e}_{\Lambda} = \boldsymbol{Q}^{T}\boldsymbol{y}_{,\Lambda} - \boldsymbol{Y}_{,\Lambda}\,, \\ \mathbf{K} &= \mathbf{k}_{\Lambda} \otimes \boldsymbol{A}^{\Lambda}\,, \qquad \mathbf{k}_{\Lambda} = ad^{-1}(\boldsymbol{Q}^{T}\boldsymbol{Q}_{,\Lambda})\,. \end{aligned} \tag{3.17}$$

Let us note that $\mathbb{K}\boldsymbol{u}$ and $\mathbb{K}\boldsymbol{u}$ are skew tensors for every vector $\boldsymbol{u} \in T_Y M$, so we can compute their axial vectors. Thus, the two tensor fields $\boldsymbol{K}$ and $\mathbf{K}$ defined by $(3.14)_2$ and $(3.16)_2$ are well defined. These two definitions are equivalent to

$$(\boldsymbol{K}\boldsymbol{u}) \times \boldsymbol{w} = (\mathbb{K}\boldsymbol{u})\boldsymbol{w}\,, \qquad (\mathbf{K}\boldsymbol{u}) \times \boldsymbol{w} = (\mathbb{K}\boldsymbol{u})\boldsymbol{w}\,, \qquad \forall\, \boldsymbol{u} \in T_Y M\,,\ \boldsymbol{w} \in E\,. \tag{3.18}$$

From their definitions and the properties of the *ad* map we easily see that the two sets of strain measures are related by

$$\boldsymbol{E}(Y,t)=\boldsymbol{Q}(Y,t)\mathbf{E}(Y,t)\,,\qquad \boldsymbol{K}(Y,t)=\boldsymbol{Q}(Y,t)\mathbf{K}(Y,t) \tag{3.19}$$

and

$$\boldsymbol{e}_\Lambda=\boldsymbol{Q}\mathbf{e}_\Lambda\,,\qquad \boldsymbol{k}_\Lambda=\boldsymbol{Q}\mathbf{k}_\Lambda\,,\qquad \boldsymbol{K}_\Lambda=\boldsymbol{Q}\mathbf{K}_\Lambda\boldsymbol{Q}^T\,. \tag{3.20}$$

Components of the strain measures (3.15) are defined by

$$\boldsymbol{e}_\Lambda=E_{\alpha\Lambda}\boldsymbol{d}^\alpha+E_\Lambda\boldsymbol{d}\,,\qquad \boldsymbol{k}_\Lambda=\boldsymbol{d}\times(K_{\alpha\Lambda}\boldsymbol{d}^\alpha)+K_\Lambda\boldsymbol{d}\,. \tag{3.21}$$

Then, from (3.20) we obatin

$$\mathbf{e}_\Lambda=E_{\Gamma\Lambda}\boldsymbol{D}^\Gamma+E_\Lambda\boldsymbol{D}\,,\qquad \mathbf{k}_\Lambda=\boldsymbol{D}\times(K_{\Gamma\Lambda}\boldsymbol{D}^\Gamma)+K_\Lambda\boldsymbol{D}\,, \tag{3.22}$$

where

$$E_{\alpha\Lambda}=\delta^\Gamma_\alpha E_{\Gamma\Lambda}\,,\qquad K_{\alpha\Lambda}=\delta^\Gamma_\alpha K_{\Gamma\Lambda}\,. \tag{3.23}$$

Thus, the strains in the two representations have the same components but referred to the initial and rotated triads, respectively.

3.6 Strain rates. Differentiation of the stretch tensor $(3.14)_1$ with respect to time yields

$$\dot{\boldsymbol{E}}=\dot{\boldsymbol{F}}-\dot{\boldsymbol{Q}}\boldsymbol{I}=\dot{\boldsymbol{e}}_\Lambda\otimes\boldsymbol{A}^\Lambda\,. \tag{3.24}$$

In view of (3.2) and (2.5) we have

$$\dot{\boldsymbol{F}}=\frac{d}{dt}\nabla_s\boldsymbol{y}=\nabla_s\dot{\boldsymbol{y}}=\nabla_s\boldsymbol{v}=\boldsymbol{v}_{,\Lambda}\otimes\boldsymbol{A}^\Lambda\,,\qquad \dot{\boldsymbol{Q}}=\boldsymbol{W}\boldsymbol{Q}\,. \tag{3.25}$$

Substituting (3.25) into (3.24) yields

$$\dot{\boldsymbol{E}}=\dot{\boldsymbol{F}}-\boldsymbol{W}\boldsymbol{F}+\boldsymbol{W}\boldsymbol{E}\,. \tag{3.26}$$

Comparing (3.26) with $(1.32)_1$ we now see that the rate of the stretch tensor $\boldsymbol{E}$ is given by

$$\delta_t\boldsymbol{E}=\dot{\boldsymbol{E}}-\boldsymbol{W}\boldsymbol{E}=\dot{\boldsymbol{F}}-\boldsymbol{W}\boldsymbol{F}=\nabla_s\boldsymbol{v}-\boldsymbol{W}\boldsymbol{F}\,. \tag{3.27}$$

Moreover, in view of $(3.15)_1$ we obtain

$$\delta_t \boldsymbol{E} = \delta_t \boldsymbol{e}_\Lambda \otimes \boldsymbol{A}^\Lambda , \qquad \delta_t \boldsymbol{e}_\Lambda = \dot{\boldsymbol{e}}_\Lambda - \boldsymbol{w} \times \boldsymbol{e}_\Lambda = \boldsymbol{v}_{,\Lambda} + \boldsymbol{y}_{,\Lambda} \times \boldsymbol{w} . \tag{3.28}$$

Somewhat lengthy calculation based on definitions $(3.14)_2$ shows also that the rate of the bending tensor is given by

$$\delta_t \boldsymbol{K} = \dot{\boldsymbol{K}} - \boldsymbol{W}\boldsymbol{K} = \nabla_s \boldsymbol{w} . \tag{3.29}$$

It is next interesting to observe that equivalent definitions of the strain rates are

$$\delta_t \boldsymbol{E} = \boldsymbol{Q}\left(\frac{d}{dt}(\boldsymbol{Q}^T \boldsymbol{E})\right), \qquad \delta_t \boldsymbol{K} = \boldsymbol{Q}\left(\frac{d}{dt}(\boldsymbol{Q}^T \boldsymbol{K})\right). \tag{3.30}$$

Moreover, directly from the relations (3.19) we obtain

$$\delta_t \boldsymbol{E} = \boldsymbol{Q}\dot{\mathbf{E}} , \qquad \delta_t \boldsymbol{K} = \boldsymbol{Q}\dot{\mathbf{K}} . \tag{3.31}$$

From the component representation (3.21) of strains we have

$$\delta_t \boldsymbol{e}_\Lambda = \dot{E}_{\alpha\Lambda}\boldsymbol{d}^\alpha + \dot{E}_\Lambda \boldsymbol{d} , \qquad \delta_t \boldsymbol{k}_\Lambda = \boldsymbol{d} \times (\dot{K}_{\alpha\Lambda}\boldsymbol{d}^\alpha) + \dot{K}_\Lambda \boldsymbol{d}^3 . \tag{3.32}$$

Moreover, since the initial triads are time independent we obtain

$$\dot{\mathbf{e}}_\Lambda = \dot{E}_{\Gamma\Lambda}\boldsymbol{D}^\Gamma + \dot{E}_\Lambda \boldsymbol{D} , \qquad \dot{\mathbf{k}}_\Lambda = \boldsymbol{D} \times (\dot{K}_{\Gamma\Lambda}\boldsymbol{D}^\Gamma) + \dot{K}_\Lambda \boldsymbol{D} . \tag{3.33}$$

4. Kinematic side conditions

4.1 Kinematically admissible deformation. In Chapt. II.5 we have shown that it is rather a simple matter to formulate the resultant static boundary conditions for the shell. Given a three-dimensional distribution of external force $\mathbf{t}^*(X)$ acting on a part ∂B_f° of the shell lateral surface ∂B° we have derived the statically equivalent force vector $\boldsymbol{n}^*(Y)$ and couple vector $\boldsymbol{m}^*(Y)$ acting along the corresponding part $\partial M_f = M \cap \partial B_f^\circ$ of the boundary of the shell reference surface. Then the resultant static boundary conditions follow naturally from the balance laws of linear and angular momentum applied to the three-dimensional boundary element. We have also seen that the static jump conditions flow naturally out from the integral balance laws (Chapt. II.4). It may also be noted that the boundary conditions can actually be regarded as a special case of the jump conditions.

A different approach is needed in order to derive resultant kinematic boundary conditions and kinematic jump conditions. In general, they cannot be obtained directly from corresponding three-dimensional boundary and jump conditions, but should rather follow from the two-dimensional integral (virtual work) identity derived in Chapt. III.1. Taking into account the kinematic relations (3.27) and (3.29) the virtual work identity (1.31) can be rewritten in the form

$$\iint_{M\setminus\mathfrak{L}} (\boldsymbol{N} \bullet \delta_t \boldsymbol{E} + \boldsymbol{M} \bullet \delta_t \boldsymbol{K})\, dA = \int_{\mathfrak{L}} w_\ell \, dS + \mathfrak{w}_{ext}(M; \mathbb{V}), \tag{4.1}$$

where the virtual work density (1.29) of all forces and couples acting along the singular curve $\mathfrak{L}$ is given by

$$w_\ell = \boldsymbol{p}_\ell \bullet \boldsymbol{v}_\ell + \boldsymbol{l}_\ell \bullet \boldsymbol{w}_\ell - [\![\boldsymbol{n}_\nu \bullet (\boldsymbol{v}_\ell - \boldsymbol{v} + (\boldsymbol{y}_\ell - \boldsymbol{y}) \times \boldsymbol{w}_\ell)]\!] - [\![\boldsymbol{m}_\nu \bullet (\boldsymbol{w}_\ell - \boldsymbol{w})]\!] \tag{4.2}$$

and the last term in (4.1)

$$\begin{aligned} \mathfrak{w}_{ext}(M; \mathbb{V}) = \iint_{M\setminus\mathfrak{L}} (\boldsymbol{p} \bullet \boldsymbol{v} + \boldsymbol{l} \bullet \boldsymbol{w})\, dA + \int_{\partial M_f} (\boldsymbol{n}^* \bullet \boldsymbol{v} + \boldsymbol{m}^* \bullet \boldsymbol{w})\, dS \\ + \int_{\partial M_d} (\boldsymbol{N}\boldsymbol{v} \bullet \boldsymbol{v} + \boldsymbol{M}\boldsymbol{v} \bullet \boldsymbol{w})\, dS \end{aligned} \tag{4.3}$$

expresses the virtual work of all external forces and couples acting on the shell. The physical meaning of the first line integral in (4.3) is clear: it represents the virtual work of the resultant force and couple vectors which are statically equivalent to the external force $\mathbf{t}^*(X)$ acting on the corresponding part of the lateral surface.

4.2 Kinematic boundary conditions. In classical form the three-dimensional kinematic boundary condition requires that the deformation $x = \chi(X)$ of the shell-like body be prescribed on the part $\partial B_d^\diamond = \partial B^\diamond \setminus \partial B_f^\diamond$ of the lateral surface:

$$\chi(X) = \chi^*(X), \qquad X \in \partial B_d^\diamond . \tag{4.4}$$

In other words, the part $\partial B_d^\diamond$ of the lateral surface is assumed to deform in the prescribed manner. From the theory of constraints it then follows that there must exist reactive forces which support this kind of deformation. Let us further note that the complementary part ∂M_d of the shell boundary curve is defined by $\partial M_d = \partial M \setminus \partial M_f = M \cap \partial B_d^\diamond$. It is seen then that the second line integral in the virtual expression (4.3) represents the virtual work of the resultant force and couple vectors which are statically equivalent to the reactive force acting on $\partial B_d^\diamond$ due to the constraint (4.4). Moreover, we have seen in Chapt. III.2 that the deformation $y = \chi(Y)$ of the reference surface and the independent rotational deformation $Q = Q(Y)$ of the shell have the vector fields v and w appearing in (4.3) as their virtual counterparts. We then conclude that the prescription of the deformation map χ and of the rotation tensor Q along the complementary part ∂M_d of the shell boundary

$$\chi(Y) = \chi^*(Y), \qquad Q(Y) = Q^*(Y), \qquad Y \in \partial M_d \tag{4.5}$$

constitute the kinematic boundary conditions for the shell which are energetically equivalent to the pointwise, three-dimensional kinematic boundary condition (4.4). In (4.5), $\chi^*(Y)$ and $Q^*(Y)$ are functions which determine the prescribed deformation of the reference surface and the prescribed independent rotations along the part ∂M_d of the boundary curve.

From the above considerations it becomes clear that if

$$\chi^*(X) = X, \qquad X \in \partial B_d^\diamond , \tag{4.6}$$

then we must set

$$\chi^*(Y) = Y, \qquad Q^*(Y) = \mathbf{1}, \qquad Y \in \partial M_d , \tag{4.7}$$

where $\mathbf{1}$ denotes the identity tensor. The conditions (4.7) correspond to the fully clamped boundary of the shell. However, this is one of a few obvious cases. In general, it is not an easy task to determine the functions $\chi^*(Y)$ and $Q^*(Y)$ from the given three-dimensional function $\chi^*(X)$, unless one adopts ad hoc some classical hypotheses about through-the-thickness deformation of the shell-like body. But

then the derived two-dimensional kinematic boundary conditions for the shell represent the assumed kinematical hypothesis rather than real constraints imposed at the boundary of the shell-like body.

4.4 Rigid junctions between shells. In the remaining part of this chapter we shall be concerned with the formulation of kinematic jump conditions. Like in the case of static jump conditions, a few cases need to be examined separately. However, the general idea underlying the formulation of the kinematic jump conditions is the same and parallels the arguments we have used in the discussion of the kinematic boundary conditions. Our starting point will be always the virtual work expression (4.2). In general, we must make a clear distinction between two classes of problems.

The deformation of the shell reference surface is said to be continuous, if $\boldsymbol{\chi}_\ell$ and $\boldsymbol{Q}_\ell$ denote the restrictions of $\boldsymbol{\chi}$ and $\boldsymbol{Q}$ to the singular curve $\mathfrak{L}$, in other words, if

$$\boldsymbol{\chi}^{(k)}(\boldsymbol{Y}) = \lim_{\boldsymbol{Z}\to\boldsymbol{Y}} \boldsymbol{\chi}(\boldsymbol{Z}) = \boldsymbol{\chi}_\ell(\boldsymbol{Y}), \qquad \boldsymbol{Z} \in M_{(k)} \tag{4.8}$$

and

$$\lim_{\boldsymbol{Z}\to\boldsymbol{Y}} \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{Q}_\ell(\boldsymbol{Y}), \qquad \boldsymbol{Z} \in M_{(k)} \tag{4.9}$$

at every point $\boldsymbol{Y} \in \mathfrak{L}$. Then virtual displacements and virtual rotations must satisfy the same conditions, i.e. the fields $\boldsymbol{v}$ and $\boldsymbol{w}$ are continuous over the entire shell reference surface M including the singular curve $\mathfrak{L}$. In this case the virtual work expression (4.2) reduces to the form

$$w_\ell = \boldsymbol{p}_\ell \cdot \boldsymbol{v}_\ell + \boldsymbol{l}_\ell \cdot \boldsymbol{w}_\ell . \tag{4.10}$$

This shows that under the above assumptions the static jump conditions are satisfied identically in the weak form.

5. Equivalent forms of shell equations

5.1 Work-conjugate dynamic variables. Considerations presented in the previous two subchapters show that there are two equivalent forms of kinematic shell variables. Accordingly, there must also exist two equivalent forms of dynamic shell variables and dynamic shell equations. The most natural way to define the dynamic variables in the second representation is to take as a starting point the virtual work identity (1.31).

Let us consider first the internal virtual work (stress power) density (1.23), which in view of (3.27) and (3.29) can be expressed in the form

$$w = \boldsymbol{N} \cdot \delta_t \boldsymbol{E} + \boldsymbol{M} \cdot \delta_t \boldsymbol{K} = \boldsymbol{n}^{\Lambda} \cdot \delta_t \boldsymbol{e}_{\Lambda} + \boldsymbol{m}^{\Lambda} \cdot \delta_t \boldsymbol{k}_{\Lambda} . \tag{5.1}$$

We have shown that the virtual changes $(\delta_t \boldsymbol{E}, \delta_t \boldsymbol{K})$ and $(\delta_t \boldsymbol{e}_{\Lambda}, \delta_t \boldsymbol{k}_{\Lambda})$ are related to the material time derivative of the strains $(\mathbf{E}, \mathbf{K})$ and $(\mathbf{e}_{\Lambda}, \mathbf{k}_{\Lambda})$ through (3.31). Thus, we can rewrite (5.1) in the form

$$\begin{aligned} w &= \boldsymbol{N} \cdot \boldsymbol{Q}\dot{\mathbf{E}} + \boldsymbol{M} \cdot \boldsymbol{Q}\dot{\mathbf{K}} = \boldsymbol{n}^{\Lambda} \cdot \boldsymbol{Q}\dot{\mathbf{e}}_{\Lambda} + \boldsymbol{m}^{\Lambda} \cdot \boldsymbol{Q}\dot{\mathbf{k}}_{\Lambda} \\ &= \boldsymbol{Q}^T \boldsymbol{N} \cdot \dot{\mathbf{E}} + \boldsymbol{Q}^T \boldsymbol{M} \cdot \dot{\mathbf{K}} = \boldsymbol{Q}^T \boldsymbol{n}^{\Lambda} \cdot \dot{\mathbf{e}}_{\Lambda} + \boldsymbol{Q}^T \boldsymbol{m}^{\Lambda} \cdot \dot{\mathbf{k}}_{\Lambda} . \end{aligned} \tag{5.2}$$

We then define the work-conjugate stress resultant tensor $\mathbf{N}(Y,t)$ and the stress resultant couple tensor $\mathbf{M}(Y,t)$ by

$$\begin{aligned} \mathbf{N}(Y,t) &= \boldsymbol{Q}^T \boldsymbol{N} = \mathbf{n}^{\Lambda} \otimes \boldsymbol{A}_{\Lambda} , \qquad & \mathbf{n}^{\Lambda}(Y,t) &= \boldsymbol{Q}^T \boldsymbol{n}^{\Lambda} , \\ \mathbf{M}(Y,t) &= \boldsymbol{Q}^T \boldsymbol{M} = \mathbf{m}^{\Lambda} \otimes \boldsymbol{A}_{\Lambda} , \qquad & \mathbf{m}^{\Lambda}(Y,t) &= \boldsymbol{Q}^T \boldsymbol{m}^{\Lambda} . \end{aligned} \tag{5.3}$$

Then the stress power density (5.1) in the equivalent representation takes the form

$$w = \mathbf{N} \cdot \dot{\mathbf{E}} + \mathbf{M} \cdot \dot{\mathbf{K}} = \mathbf{n}^{\Lambda} \cdot \dot{\mathbf{e}}_{\Lambda} + \mathbf{m}^{\Lambda} \cdot \dot{\mathbf{k}}_{\Lambda} . \tag{5.4}$$

Excatly in the same manner we can introduce the remaining work-conjugate dynamic variables. Specifically, the force power (virtual work) of the external forces and couples acting on the shell is given by

$$\begin{aligned} \mathfrak{W}_{ext} = &\iint_{M \setminus \mathfrak{L}} (\boldsymbol{p} \cdot \boldsymbol{v} + \boldsymbol{l} \cdot \boldsymbol{w}) dA + \int_{\partial M_f} (\boldsymbol{n}^* \cdot \boldsymbol{v} + \boldsymbol{m}^* \cdot \boldsymbol{w}) dS \\ &+ \int_{\partial M_d} (\boldsymbol{N}\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{M}\boldsymbol{v} \cdot \boldsymbol{w}) dS . \end{aligned} \tag{5.5}$$

Then using the relations between translational and rotational velocity fields in the two representations (see Chapt. III.2) we define the resultant surface force vector $\mathbf{p}(Y,t)$, the surface couple vector $\mathbf{l}(Y,t)$, the external boundary force vector $\mathbf{n}^*(Y,t)$ and the external boundary couple vector $\mathbf{m}^*(Y,t)$ such that

$$p \cdot v + l \cdot w = \mathbf{p} \cdot \mathbf{v} + \mathbf{l} \cdot \mathbf{w} \tag{5.6}$$

and

$$n^* \cdot v + m^* \cdot w = \mathbf{n}^* \cdot \mathbf{v} + \mathbf{m}^* \cdot \mathbf{w}. \tag{5.7}$$

Recalling the relations between the velocity fields in the two representations we have

$$\begin{aligned} \mathbf{p}(Y,t) &= Q^T p, & \mathbf{l}(Y,t) &= Q^T l, \\ \mathbf{n}^*(Y,t) &= Q^T n^*, & \mathbf{m}^*(Y,t) &= Q^T m^*. \end{aligned} \tag{5.8}$$

Now the external virtual work (5.5) is obtained in the form

$$\begin{aligned} \mathfrak{W}_{ext} = &\iint_{M \setminus \mathcal{L}} (\mathbf{p} \cdot \mathbf{v} + \mathbf{l} \cdot \mathbf{w})\, dA + \int_{\partial M_f} (\mathbf{n}^* \cdot \mathbf{v} + \mathbf{m}^* \cdot \mathbf{w})\, dS \\ &+ \int_{\partial M_d} (\mathbf{N}\nu \cdot \mathbf{v} + \mathbf{M}\nu \cdot \mathbf{w})\, dS. \end{aligned} \tag{5.9}$$

5.2 Equilibrium equations and boundary conditions. Using the relation $N = Q\mathbf{N}$ between the resultant stress tensor in the two representations we obtain

$$Div_s N = Div_s(Q\mathbf{N}) = Q Div_s \mathbf{N} + \nabla_s Q[\mathbf{N}] = Q\big(Div_s \mathbf{N} + Q^T(\nabla_s Q[\mathbf{N}])\big). \tag{5.10}$$

Recalling next the definition of the bending strain tensor $\mathbf{K}$, (5.10) can be transformed into

$$Div_s N = Q\big(Div_s \mathbf{N} + ad^{-1}(\mathbf{N}\mathbf{K}^T - \mathbf{K}\mathbf{N}^T)\big). \tag{5.11}$$

Exactly in the same manner we obtain

$$Div_s M = Q\big(Div_s \mathbf{M} + ad^{-1}(\mathbf{M}\mathbf{K}^T - \mathbf{K}\mathbf{M}^T)\big). \tag{5.12}$$

From the relation $F = Q(I + \mathbf{E})$ it follows that

$$NF^T - FN^T = Q\big(\mathbf{N}(I + \mathbf{E})^T - (I + \mathbf{E})\mathbf{N}^T\big). \tag{5.13}$$

Substituting (5.11)-(5.13) into the equilibrium equations (1.1) in spatial representation, the equivalent form of the equilibrium equations in the material representation is obtained as

$$\begin{aligned} Div_s\mathbf{N}+ad^{-1}(\mathbf{M}\mathbf{K}^T-\mathbf{K}\mathbf{M}^T)+\mathbf{p}=\mathbf{0}, \\ Div_s\mathbf{M}+ad^{-1}(\mathbf{M}\mathbf{K}^T-\mathbf{K}\mathbf{M}^T)+ad^{-1}\big(\mathbf{N}(I+\mathbf{E})^T-(I+\mathbf{E})\mathbf{N}^T\big)+\mathbf{l}=\mathbf{0}. \end{aligned} \tag{5.14}$$

In terms of local coordinates on the reference surface M, the resultant stress tensor $\mathbf{N}$ and the stress couple tensor $\mathbf{M}$ are given by (5.3). Then the equilibrium equations (5.14) can also be represented by

$$\begin{aligned} \mathbf{n}^A{}_{|A}+\mathbf{k}_A\times\mathbf{n}^A+\mathbf{p}=\mathbf{0}, \\ \mathbf{m}^A{}_{|A}+\mathbf{k}_A\times\mathbf{m}^A+(A_A+\mathbf{e}_A)\times\mathbf{n}^A+\mathbf{l}=\mathbf{0}. \end{aligned} \tag{5.15}$$

In view of (5.3) and (5.8) the dynamic boundary conditions (1.3) can equivalently be expressed in the form

$$\mathbf{N}\nu=\mathbf{n}^*, \qquad \mathbf{M}\nu=\mathbf{m}^*, \qquad Y\in\partial M_f. \tag{5.16}$$

5.5 Change of frame of reference. The tranformation rules for the dynamic variables under the change of frame of reference can now be obtained

$$\mathbf{N}^*(Y,t^*)=\mathbf{N}(Y,t), \qquad \mathbf{M}^*(Y,t^*)=\mathbf{M}(Y,t), \tag{5.17}$$

and

$$\mathbf{p}^*(Y,t^*)=\mathbf{p}(Y,t), \qquad \mathbf{l}^*(Y,t^*)=\mathbf{l}(Y,t). \tag{5.18}$$

By simple calculation we have

$$\mathbf{N}^*=Q^{*T}N^*=Q^TO^TON=Q^TN=\mathbf{N}. \tag{5.19}$$

5.6 Component representation of stresses. Using the triad $\{D_K\}=\{D_A,D_3\}$ to define the components we have

$$\mathbf{n}^A=N^{\Gamma A}D_\Gamma+Q^AD, \qquad \mathbf{m}^A=D\times(M^{\Gamma A}D_\Gamma)+M^AD. \tag{5.20}$$

Now having in mind that the spatial triads are defined as the rotated material triads, the use of the relations between spatial and material representations of strains and stresses yields at once

$$\boldsymbol{n}^{\Lambda} = N^{\alpha\Lambda}\boldsymbol{d}_{\alpha} + Q^{\Lambda}\boldsymbol{d}\,, \qquad \boldsymbol{m}^{\Lambda} = \boldsymbol{d}\times(M^{\alpha\Lambda}\boldsymbol{d}_{\alpha}) + M^{\Lambda}\boldsymbol{d}\,, \tag{5.21}$$

where

$$N^{\alpha\Lambda} = \delta^{\alpha}_{\Gamma} N^{\Gamma\Lambda}\,, \qquad M^{\alpha\Lambda} = \delta^{\alpha}_{\Gamma} M^{\Gamma\Lambda}\,. \tag{5.22}$$

Thus the internal virtual work density when written in the component form is given by

$$\begin{aligned} w &= N^{\Gamma\Lambda}\dot{E}_{\Gamma\Lambda} + Q^{\Lambda}\dot{E}_{\Lambda} + M^{\Gamma\Lambda}\dot{K}_{\Gamma\Lambda} + M^{\Lambda}\dot{K}_{\Lambda} \\ &= N^{\alpha\Lambda}\dot{E}_{\alpha\Lambda} + Q^{\Lambda}\dot{E}_{\Lambda} + M^{\alpha\Lambda}\dot{K}_{\alpha\Lambda} + M^{\Lambda}\dot{K}_{\Lambda}\,. \end{aligned} \tag{5.23}$$

This form is identical in the two representations.

In the same manner we set

$$\mathbf{p} = p^{\Lambda}\boldsymbol{D}_{\Lambda} + p^{3}\boldsymbol{D}\,, \qquad \mathbf{l} = \boldsymbol{D}\times(l^{\Lambda}\boldsymbol{D}_{\Lambda}) + l^{3}\boldsymbol{D}\,. \tag{5.24}$$

Then in the spatial representation we obtain

$$\boldsymbol{p} = p^{\alpha}\boldsymbol{d}_{\alpha} + p^{3}\boldsymbol{d}\,, \qquad \boldsymbol{l} = \boldsymbol{d}\times(l^{\alpha}\boldsymbol{d}_{\alpha}) + l^{3}\boldsymbol{d}\,. \tag{5.25}$$

5.6 Jump conditions. In order to obtain the dynamic jump conditions we first need to define the line force and couple vectors in the material representation. Using the same arguments as for the surface and boundary forces and couples we obtain

$$\mathbf{p}_{\ell}(Y,t) = \boldsymbol{Q}_{\ell}(Y,t)^{T}\,\boldsymbol{p}_{\ell}(Y,t)\,, \qquad \mathbf{l}_{\ell}(Y,t) = \boldsymbol{Q}_{\ell}(Y,t)^{T}\boldsymbol{l}_{\ell}(Y,t)\,. \tag{5.26}$$

Substituting now (5.26) into the jump conditions (1.2) and taking subsequently into account (5.3) we have

$$\boldsymbol{Q}_{\ell}\mathbf{p}_{\ell} - [\![\boldsymbol{Q}\mathbf{n}_{\nu}]\!] = \mathbf{0}\,, \qquad \boldsymbol{Q}_{\ell}\boldsymbol{l}_{\ell} - [\![\boldsymbol{Q}\mathbf{m}_{\nu}]\!] + [\![(\boldsymbol{y}_{\ell}-\boldsymbol{y})\times\boldsymbol{Q}\mathbf{n}_{\nu}]\!] = \mathbf{0}\,, \tag{5.27}$$

or

$$\mathbf{p}_{\ell} - [\![\boldsymbol{Q}_{\ell}^{T}\boldsymbol{Q}\mathbf{n}_{\nu}]\!] = \mathbf{0}\,, \qquad \mathbf{l}_{\ell} - [\![\boldsymbol{Q}_{\ell}^{T}\boldsymbol{Q}\mathbf{m}_{\nu}]\!] + [\![\boldsymbol{Q}_{\ell}^{T}(\boldsymbol{y}_{\ell}-\boldsymbol{y})\times\boldsymbol{Q}_{\ell}^{T}\boldsymbol{Q}\mathbf{n}_{\nu}]\!] = \mathbf{0}\,. \tag{5.28}$$

6. Parametrization of rotations

6.1 Rotational parameters. So far we have considered the kinematics of the shell regarding the field $\boldsymbol{Q}_t : M \to SO(3)$ of rotation tensors, besides the deformation map $\boldsymbol{\chi}_t : M \to \mathcal{E}$ of the shell reference surface, as the primary independent kinematic variable. In this way we have been able to expose the general structure of the shell theory without restoring to a particular parametrization of the rotation group $SO(3)$. A choice of a particular parametrization of $SO(3)$ is a central issue for analytical and numerical solutions of the shell boundary value problems.

The familiar Euler (or Brayant) angles, Cayley-Klein parameters or quaternions are just a few examples of possible parametrizations of the rotation group $SO(3)$, none of which have a clear advantage over others. Formally, the parametrization of rotations is defined as an embedding of the rotation group $SO(3)$ in the real space $\mathbb{R}^N$:

$$SO(3) \to \mathbb{R}^N, \qquad \boldsymbol{Q}_t = \boldsymbol{Q}_t(\vartheta_k). \tag{6.1}$$

Then the N-tuple (ϑ_k) of real numbers is called the rotational parameters, and N is called the dimension of the parametrization. In evaluating the usefulness of a particular parametrization of $SO(3)$ the following points have to be considered:

1) the number of parameters needed and possible singularities,
2) the complexity of the resulting equations,
3) the susceptibility to numerical errors in the computer implementation of shell equations,
4) difficulties in the formulation of side conditions and complexity of kinematic relations.

It is well known that it is impossible to have a global singular free-parametrization of the rotation group in terms of less than five parameters. In turn, only three parameters are independent, and any more than three-dimensional parametrization results in redundant parameters which must satisfy suitable constraints. This leads to an extended system of field equations with corresponding Lagrange multipliers as additional unknown fields of the problem. Thus, from the computational point of view such an approach is of less importance. Hence, we shall consider only three-dimensional parametrizations in terms of the so-called finite rotation vectors. Such vectors can be defined in many ways, but all definitions have the same

underlying concept. In effect, various definitions of finite rotation vectors can be obtained as special cases of the following construction.

6.2 Geometric picture of rotations. Geometrically, the rotation tensor is an Euclidean, orientation-preserving isometry $\boldsymbol{Q}: E \to E$ leaving one point fixed. By virtue of the Euler theorem, every rotation of E is completely characterized by an axis of rotation and an angle of rotation. Given a rotation tensor $\boldsymbol{Q}$, the axis of rotation is determined by the unit vector $\boldsymbol{e}$ being the eigenvector of $\boldsymbol{Q}$ associated with the real eigenvalue $+1$. Both the unit vector $\boldsymbol{e}$ and the rotation angle ψ of a given rotation tensor $\boldsymbol{Q}$ are determined by the relations

$$\boldsymbol{Q}\boldsymbol{e} = +1\boldsymbol{e}, \qquad cos\psi = \frac{1}{2}(tr\boldsymbol{Q} - 1). \tag{6.2}$$

In terms of $\boldsymbol{e}$ and ψ the rotation tensor $\boldsymbol{Q}$ is given in the form

$$\boldsymbol{Q} = exp(\psi \boldsymbol{E}), \qquad \boldsymbol{E} = a\boldsymbol{d}\boldsymbol{e}. \tag{6.3}$$

Exceptionally, here $\boldsymbol{E}$ denotes the skew-symmetric tensor associated with the unit vector $\boldsymbol{e}$, and it should not be identified with the strain tensor denoted by the same symbol elsewhere. From the definition of the exponential function one can easily show that

$$\boldsymbol{Q} = \mathbf{1} + sin\psi \boldsymbol{E} + (1 - cos\psi)\boldsymbol{E}^2. \tag{6.4}$$

The inverse relationship, expressing $\boldsymbol{e}$ and ψ in terms of $\boldsymbol{Q}$ can be found directly by solving the equation (6.4). The result takes the form

$$sin\psi \boldsymbol{E} = \boldsymbol{Q} - \boldsymbol{Q}^T, \qquad cos\psi = \frac{1}{2}(tr\boldsymbol{Q} - 1). \tag{6.5}$$

In terms of the unit vector $\boldsymbol{e}$ and the rotation angle ψ the skew tensor $\boldsymbol{E}$ can also be expressed in the form

$$\boldsymbol{E} = -\frac{\psi}{sin\psi}\left((1 + 2cos\psi)\mathbf{1} - (1 + cos\psi)\boldsymbol{Q} + \boldsymbol{Q}^2/2\right). \tag{6.6}$$

6.3 Generalized finite rotation vector. Let $\theta(\psi)$ be any monotonously increasing function of the rotation angle ψ such that $\theta(0) = 0$. Then the generalized finite rotation vector $\boldsymbol{\theta}$ and the associated skew tensor $\boldsymbol{\Theta}$ are defined by

$$\boldsymbol{\theta} = \theta(\psi)\boldsymbol{e}\,, \qquad \boldsymbol{\Theta} = ad\,\boldsymbol{\theta} = \theta(\psi)\boldsymbol{E}\,. \tag{6.7}$$

Substituting (6.7) into (6.3) we have

$$\boldsymbol{Q}(\boldsymbol{\theta}) = 1 + \frac{\sin\psi}{\theta(\psi)}\boldsymbol{\Theta} + \frac{1-\sin\psi}{\theta^2(\psi)}\boldsymbol{\Theta}^2\,, \tag{6.8}$$

The monotonicity of the function $\theta(\psi)$ ensures that there exists a unique inverse function $\theta(\psi)$, and hence the rotation tensor $\boldsymbol{Q}$ given by (6.8) can be regarded as a function of the finite rotation vector $\boldsymbol{\theta}$ alone. Thus, three components of $\boldsymbol{\theta}$ provide the local singular-free parametrization of the rotation group.

The time derivative of the rotation tensor (6.8) is given by

$$\dot{\boldsymbol{Q}} = (\partial_{\boldsymbol{\theta}}\boldsymbol{Q})\dot{\boldsymbol{\theta}}\,. \tag{6.9}$$

Then the angular velocity fields are given by

$$\boldsymbol{W} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T = ((\partial_{\boldsymbol{\theta}}\boldsymbol{Q})\dot{\boldsymbol{\theta}})\boldsymbol{Q}^T\,, \qquad \mathbf{W} = \boldsymbol{Q}^T\dot{\boldsymbol{Q}} = \boldsymbol{Q}^T((\partial_{\boldsymbol{\theta}}\boldsymbol{Q})\dot{\boldsymbol{\theta}})\,. \tag{6.10}$$

The associated axial vectors can be expressed in the form

$$\boldsymbol{w} = \boldsymbol{L}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\,, \qquad \mathbf{w} = \mathbf{L}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\,, \tag{6.11}$$

where both tensors $\boldsymbol{L}(\boldsymbol{\theta})$ and $\mathbf{L}(\boldsymbol{\theta})$ are given functions of the finite rotation vector $\boldsymbol{\theta}$. In the general case these tensors are quite complex and involve trigonometric functions of the rotation angle ψ. In such a form they are not very useful in the formulation of basic shell equations. Therefore, we shall not present here all the relations using the generalized finite rotation vector (6.7). Instead we consider only one special case.

Let us consider the finite rotation vector defined by

$$\boldsymbol{\theta} = tg\frac{\psi}{2}\boldsymbol{e}\,, \qquad \boldsymbol{\Theta} = ad\boldsymbol{\theta} = tg\frac{\psi}{2}\boldsymbol{E}\,. \tag{6.12}$$

Substituting (6.12) into (6.8) and making use of some standard trigonometric identities the rotation tensor is obtained in the form

$$\boldsymbol{Q} = \mathbf{1} + \vartheta(\boldsymbol{\Theta} + \boldsymbol{\Theta}^2), \qquad \vartheta \equiv 2(1+\|\boldsymbol{\theta}\|^2)^{-1}\,. \tag{6.13}$$

Now simple calculations lead to the following representation of the two angular velocity vectors given in terms of time derivative of the finite rotation vector (6.7)

$$w = \vartheta(\dot{\theta} - \theta \times \dot{\theta}), \qquad \mathbf{w} = \vartheta(\dot{\theta} + \theta \times \dot{\theta}). \tag{6.14}$$

It is seen that the relations (6.14) do not contain trigonometric functions. It can also be shown that the finite rotation vector defined by (6.7) is the only one, up to an arbitrary constant multiplier, which leads to the formulae (6.11) containing no trigonometric functions.

In a similar way, substituting (6.13) into (3.15) we may obtain expressions for the strain measures

$$\boldsymbol{E} = \nabla_s \boldsymbol{u} - \vartheta(\boldsymbol{\Theta} - \boldsymbol{\Theta}^2)\boldsymbol{I}, \qquad \boldsymbol{K} = \vartheta(1 + \tfrac{1}{2}\boldsymbol{\Theta})\nabla_s \boldsymbol{\theta}, \tag{6.15}$$

and

$$\begin{aligned} e_\Lambda &= u_{,\Lambda} - \vartheta\big(\theta \times A_\Lambda - \theta \times (\theta \times A_\Lambda)\big), \\ k_\Lambda &= \vartheta(1 + \tfrac{1}{2}\boldsymbol{\Theta})\theta_{,\Lambda} = \vartheta(\theta_{,\Lambda} + \theta \times \theta_{,\Lambda}). \end{aligned} \tag{6.16}$$

6.4 Virtual work expressions. Let us consider the external virtual work

$$\mathfrak{W}_{ext} = \iint_{M \setminus \Sigma} (\boldsymbol{p} \cdot \boldsymbol{v} + \boldsymbol{l} \cdot \boldsymbol{w}) dA + \int_{\partial M_f} (\boldsymbol{n}^* \cdot \boldsymbol{v} + \boldsymbol{m}^* \cdot \boldsymbol{w}) dS. \tag{6.18}$$

The corresponding form of the virtual work expressions in terms of the generalized finite rotation vector (6.8) can be directly obtained using the formulae

$$\mathfrak{W}_{ext} = \iint_{M \setminus \Sigma} (\boldsymbol{p} \cdot \boldsymbol{v} + \boldsymbol{L}^T \boldsymbol{l} \cdot \dot{\boldsymbol{\theta}}) dA + \int_{\partial M_f} (\boldsymbol{n}^* \cdot \boldsymbol{v} + \boldsymbol{L}^T \boldsymbol{m}^* \cdot \dot{\boldsymbol{\theta}}) dS. \tag{6.19}$$

In the special case using the finite rotation vector (6.11) the external virtual work takes the form

$$\begin{aligned} \mathfrak{W}_{ext} = &\iint_M \big(\boldsymbol{p} \cdot \boldsymbol{v} + \vartheta(\boldsymbol{l} + \theta \times \boldsymbol{l}) \cdot \dot{\boldsymbol{\theta}}\big) dA \\ &+ \int_{\partial M_f} \big(\boldsymbol{n}^* \cdot \boldsymbol{v} + \vartheta(\boldsymbol{m}^* + \theta \times \boldsymbol{m}^*) \cdot \dot{\boldsymbol{\theta}}\big) dS. \end{aligned} \tag{6.20}$$

This form provides the interpretation of the static quantities which are work-conjugate to the chosen parameters representing rotational degrees of freedom. It becomes clear, however, that their resulting form is more complicated than in the

case when the rotation tensor is taken as the primary unknown field. Moreover, it is valid only locally due to the underling singularities of any three-dimensional parametrization of rotations, and hence we shall not go into details.

Chapter IV

Constitutive relations and other topics

1. General theory of constitutive relations

1.1 Nature of constitutive relations. We have been concerned so now with the derivation of field equations and side conditions for shells from general principles of classical continuum mechanics. We have shown that the resultant dynamic shell equations (Chapt. II) as well as the overall kinematics of the shell (Chapt. III) can be formulated in a clear and exact way without appealing to simplifying assumptions or ad hoc postulates. In this sense the formulated theory applies to all shells: smooth or irregular, thin or thick, homogeneous or multi-layered, etc. There is also no single restriction on the shell deformation: small strain or finite strain deformation, restricted or arbitrarily large displacements and rotations, reversible or irreversible deformation, etc. On the other hand, it is easily seen that the derived set of field equations and side conditions (boundary and jump conditions) is under-determined, and it has to be supplemented by suitable constitutive relations. Constitutive relations are needed in order to take into account the mechanical properties of the shell and to close the system of shell field equations and side conditions. In this sense, the constitutive relations in shell theory play the same role as in continuum mechanics (Chapt. I.5). However, the general theory of constitutive relations for shells is not merely an analogue of the theory of three-dimensional constitutive relations.

Mechanical properties of the shell depend not only upon mechanical properties of a material the shell is made of. The geometry of the shell, the shell thickness and the construction of the shell influence the mechanical response of the shell to external loads as well. For example, the deformed shape of a homogenous plate

substantially differs from the deformed shape of a sandwich plate, even though the thickness, load and boundary conditions are the same[1] (see Fig. 1).

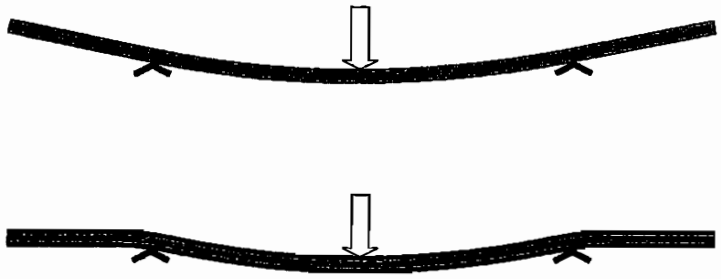

Fig. 1

The dependence of shell constitutive relation upon the geometry of a shell reference configuration can also be illustrated by the simple example shown in Fig. 2.

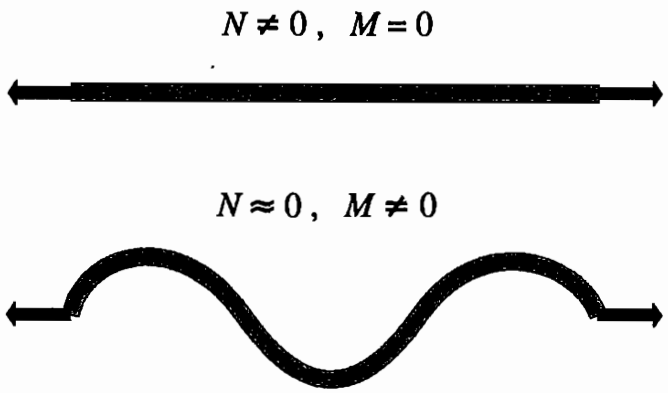

Fig. 2

A specific form of the constitutive relations for shells can be determined by at least two entirely different methods – the direct method and the reduction method.[2] The direct method requires to devise a suitable methodology of physical experiments for a piece of a shell, which would make it possible to establish directly two-dimensional stress-strain relations. The reduction method requires to develop a reliable mathematical method allowing to deduce constitutive relations for a shell from three-dimensional constitutive relations of continuum mechanics.

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[1] See experimental results presented by SAYIR AND KOLLER [1986].
[2] Cf. LIBAI AND SIMMONDS [1983], REISSNER [1974].

The usual way to determine constitutive relations for shells is based on three-dimensional constitutive relations for a material the shell is made of (the reduction method). Assuming some specific form of the deformation across the shell thickness, together with some smallness assumptions, one tries to derive two-dimensional constitutive relations for the resultant stresses and couples in terms of the associated strain measures. However, the shell constitutive relations formulated in this manner do not reflect the general structure of shell theory. They should rather be regarded as direct implications of the introduced assumptions.

In the following considerations we shall concentrate on the general structure of constitutive relations for shells. The formulation of particular types of constitutive relations is regarded as a problem requiring separate considerations for each class of shells. We take into account that a general form of constitutive relations is implied by the general theory of stresses and strains in the shell. In this sense the considerations given below provide the general framework, within which the specific forms of the constitutive relations can be sought by either of the two aforementioned methods.

1.2 Dynamical processes. Once dynamics and kinematics of the shell are well formulated, a general theory of constitutive equations can be developed in a way, which parallels the general theory of constitutive equations in continuum mechanics (Chapt. I.5). Let the reference configuration M of the shell reference surface be fixed, and let us identify a typical shell particle with the place $\boldsymbol{Y}$ it occupies in M. Analogous to the three-dimensional theory, a mechanical process of the shell is defined to be a collection of functions of $\boldsymbol{Y}$ and t. This collection consists of:

1) the generalized motion (deformation) pair

$$\mathcal{Y}(\boldsymbol{Y},t)=\big(y(\boldsymbol{Y},t),\boldsymbol{Q}(\boldsymbol{Y},t)\big), \tag{1.1}$$

2) the generalized internal stress defined as ordered pair involving the resultant stress and stress couple tensors

$$\mathcal{S}(\boldsymbol{Y},t)=\big(\boldsymbol{N}(\boldsymbol{Y},t),\boldsymbol{M}(\boldsymbol{Y},t)\big) \quad or \quad \mathcal{S}(\boldsymbol{Y},t)=\big(\mathbf{N}(\boldsymbol{Y},t),\mathbf{M}(\boldsymbol{Y},t)\big), \tag{1.2}$$

3) the generalized external force consisting of the surface force and couple vectors

$$\mathcal{P}(\boldsymbol{Y},t)=\big(\boldsymbol{p}(\boldsymbol{Y},t),\boldsymbol{l}(\boldsymbol{Y},t)\big) \quad or \quad \mathcal{P}(\boldsymbol{Y},t)=\big(\mathbf{p}(\boldsymbol{Y},t),\mathbf{l}(\boldsymbol{Y},t)\big), \tag{1.3}$$

4) the generalized boundary force given at all points $Y \in \partial M_f$ and consisting of the boundary force and couple vectors

$$s^*(Y,t)=\left(n^*(Y,t),m^*(Y,t)\right) \quad or \quad s^*(Y,t)=\left(\mathbf{n}^*(Y,t),\mathbf{m}^*(Y,t)\right), \tag{1.4}$$

5) the generalized deformation given at each point $Y \in \partial M_d$ of the complementary part of the shell boundary

$$y^*(Y,t)=(y^*(Y,t),Q^*(Y,t)). \tag{1.5}$$

We shall also denote by e the generalized strain defined as the ordered pair

$$e(Y,t)=\left(E(Y,t),K(Y,t)\right] \qquad or \qquad e(Y,t)=\left(\mathbf{E}(Y,t),\mathbf{K}(Y,t)\right]. \tag{1.6}$$

Let us note that e is not an independent variable of the shell theory, but it is defined in terms of the generalized deformation (1.1).

For regular but not necessarily smooth shells, (1.1)-(1.5) constitute the complete set of variables of the theory. For irregular shells the variables (1.1)-(1.3) and the generalized strain (1.6) are defined at every point of the shell reference surface M, except possibly at the singular curve $\mathcal{L}$, which can move and deform independently. In the latter case the shell variables (1.1)-(1.6) must be supplemented by:

6) the generalized deformation of the singular curve being defined as the ordered pair

$$y_\ell(Y,t)=\left(\chi_\ell(Y,t),Q_\ell(Y,t)\right); \tag{1.7}$$

7) the generalized line force at every point of the singular curve

$$p_\ell(Y,t)=\left(p_\ell(Y,t),l_\ell(Y,t)\right) \quad or \quad p_\ell(Y,t)=\left(\mathbf{p}_\ell(Y,t),\mathbf{l}_\ell(Y,t)\right). \tag{1.8}$$

All variables (1.1)-(1.8) are assumed to be consistent with the resultant laws of mechanics for shells, i.e. they must satisfy the equilibrium equations, jump conditions and boundary conditions, they must obey the corresponding transformation rules under the change of frame of reference and they must satisfy suitable regularity assumptions.

In general, the surface force (1.3), the boundary force (1.4) and the boundary deformation (1.5) are assumed to be given as a part of data. The role of the line

force (1.8) is less direct. This problem will not be our concern here. It is clear now that the most general constitutive relations for the shell have to be formulated as relations between the generalized stress (1.2) and the generalized motion (1.1). For irregular shells, additional constitutive relations can be needed in order to relate the line force (1.8) to the line deformation (1.7). A simple problem which requires such additional constitutive relations is shown in Fig. 3.

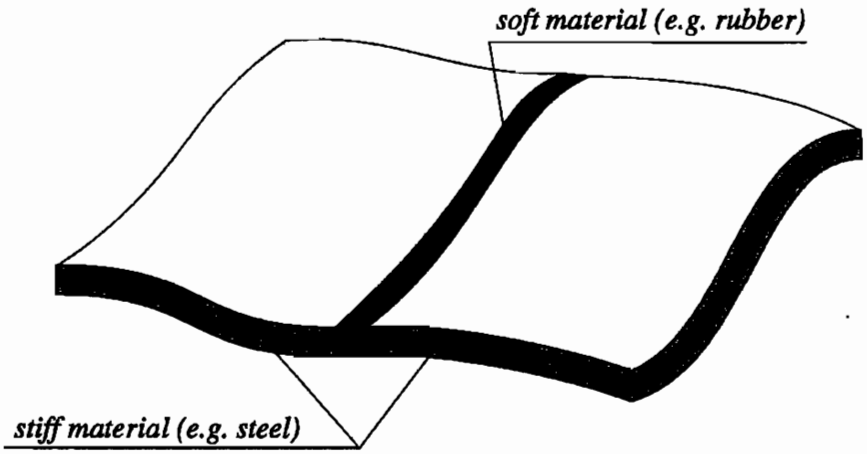

Fig. 3

1.3 General constitutive functional. In order to ensure that shell constitutive relations be physically meaningful, they must satisfy the same principles, which constitute the basis of the theory of constitutive relations in continuum mechanics (Chapt. I.5). According to the first of these principles, the principle of determinism, the generalized stress $\mathcal{S}$ at the particle Y at time t is determined by the history of motion $\mathcal{Y}$ up to time t:

$$\mathcal{S}(Y,t) = \mathcal{F}^{\infty}_{s=0}(\mathcal{Y}(Y,t-s);Y). \tag{1.9}$$

An additional constitutive relation, which might be needed for irregular shells, takes the form

$$\mathcal{P}_{\ell}(Y,t) = \mathcal{P}^{\infty}_{s=0}(\mathcal{Y}_{\ell}(Y,t-s);Y). \tag{1.10}$$

Here $\mathcal{F}^{\infty}_{s=0}$ and $\mathcal{P}^{\infty}_{s=0}$ denote constitutive functionals. The explicit dependence of these functionals on Y signifies the fact that the mechanical response of the shell depends not only upon material properties, but also upon the local geometry of the shell reference state through the curvature tensor $\boldsymbol{B}$ of the undeformed shell

reference surface M, the triad $\{D_K(Y)\}$ assigned at each point of M, and other parameters like a variable shell thickness $h_0(Y)$.

The constitutive relations (1.9) and (1.10) are further delimited by the principle of local action and the principle of material frame-indifference. According to the principle of local action it is possible to distinguish a particular class of shells, whose mechanical response is determined by the constitutive relation of the form

$$\mathcal{S}(Y,t) = \hat{\mathcal{S}}_{s=0}^{\infty}(\mathcal{E}^t(Y,s), \nabla_s \mathcal{E}^t(Y,s), \ldots, \nabla_s^{(n)} \mathcal{E}^t(Y,s); Y), \tag{1.11}$$

where the history of the generalized strain $\mathcal{E}$ is defined in the usual way,

$$\mathcal{E}^t(Y,s) = \mathcal{E}(Y,t-s), \tag{1.12}$$

and $\nabla_s^{(n)}\mathcal{E}$ denotes the *n-th* surface gradient of the strains. Following the terminology of continuum mechanics,[3] shells governed by the constitutive relation (1.11) can be called *n-grad* shells or, more general, non-simple shells.

1.4 Simple shells. A special and most important subclass of non-simple shells is the one, whose constitutive functional depends only on the history of strain measures and not on their gradients. In this case the constitutive equation (1.11) reduces to

$$\mathcal{S}(Y,t) = \hat{\mathcal{S}}_{s=0}^{\infty}(\mathcal{E}^t(Y,s); Y). \tag{1.13}$$

Using standard terminology of continuum mechanics, we shall refer to this class of shells as to simple shells. However, it must be stressed that the adjective "simple" refers here to the mechanical response of the shell and not to its geometry, which can be entirely arbitrary.

In the analysis of simple shells it will be more convenient to write the constitutive relation (1.13) in explicit form

$$\begin{aligned} N(Y,t) &= \hat{n}_{s=0}^{\infty}(E^t(Y,s), K^t(Y,s); Y), \\ M(Y,t) &= \hat{m}_{s=0}^{\infty}(E^t(Y,s), K^t(Y,s); Y), \end{aligned} \tag{1.14}$$

or

[3] See TRUESDELL AND NOLL [1965] or CHEVERTON AND BEATTY [1976].

$$\begin{aligned} \mathbf{N}(Y,t) &= \mathfrak{n}_{s=0}^{\infty}(\mathbf{E}^t(Y,s),\mathbf{K}^t(Y,s);Y), \\ \mathbf{M}(Y,t) &= \mathfrak{m}_{s=0}^{\infty}(\mathbf{E}^t(Y,s),\mathbf{K}^t(Y,s);Y). \end{aligned} \tag{1.15}$$

Recalling the relations between the resultant stress and stress couple tensors and the work-conjugate strain measures in the two representations (Chapt. III.4), it becomes obvious that (1.14) and (1.15) are two equivalent forms of the same constitutive relations, which can be obtained from each other through formal transformations.

The third basic principle of the general theory of constitutive equations – the principle of frame-indifference – asserts that the mechanical response of a shell has to be independent of the choice of the frame of reference. In order to examine consequences of this principle, let us recall that the stress resultant tensors (N,M) and the work-conjugate strain tensors (E,K) transform under the change of observer according to the following rule

$$\begin{aligned} E^*(t) &= O(t)E(t), \qquad K^*(t) = O(t)K(t), \\ N^*(t) &= O(t)N(t), \qquad M^*(t) = O(t)M(t), \end{aligned} \tag{1.16}$$

while the stress tensors $(\mathbf{N},\mathbf{M})$ and the strain tensors $(\mathbf{E},\mathbf{K})$ remain unchanged under the change of observer. For simplicity of writing, we have omitted Y from the argument of the corresponding stress and strain tensors. With the help of (1.16) we easily find that the principle of frame-indifference implies the following reduced forms of the constitutive relations (1.10):

$$\begin{aligned} \mathfrak{n}_{s=0}^{\infty}(E^t(s),K^t(s);Y) &= O(t)^T\,\mathfrak{n}_{s=0}^{\infty}(O(t)E^t(s),O(t)K^t(s);Y), \\ \mathfrak{m}_{s=0}^{\infty}(E^t(s),K^t(s);Y) &= O(t)^T\,\mathfrak{m}_{s=0}^{\infty}(O(t)E^t(s),O(t)K^t(s);Y). \end{aligned} \tag{1.17}$$

On the other hand, using the relations

$$\begin{aligned} E(t) &= Q(t)\mathbf{E}(t), \qquad K(t) = Q(t)\mathbf{K}(t), \\ N(t) &= Q(t)\mathbf{N}(t), \qquad M(t) = Q(t)\mathbf{M}(t), \end{aligned} \tag{1.18}$$

between the shell stress and strain measures in the two representations, and taking $O(t)$ to coincide with $Q(t)$, we find that the constitutive relations (1.15) remain unchanged under the change of frame of reference.

The three principles stated above are implied by corresponding principles underlying the general theory of constitutive equations in classical continuum mechanics. The constitutive equations for shells must obey two additional

requirements: they must be invariant under the change of orientation of the shell reference surface M, and they must be regular in the limit $H=0$ and $K=0$, where H and K denote the mean curvature and Gaussian curvature of M, respectively. The shell constitutive relations can be delimited further by possible material symmetries and suitable constitutive restrictions. However, neither the symmetry of constitutive relations nor the constitutive restrictions are easy to define for shells, and we shall not consider them here. We shall also not discuss here many other possible classes of shells, which can be defined by special constitutive relations. All such classes can be introduced in the same way as in continuum mechanics.

1.5 Consistency conditions. The above considerations provide the general framework for the theory of constitutive relations for shells. The constitutive relations, which can be formulated in this way, take the most general form which is consistent with the formal mathematical structure of the shell theory. In other words, nothing more general can be included into the two-dimensional constitutive relations than this what has been shown above. In this context an important question arises about possible interrelations between constitutive relations formulated within purely two-dimensional considerations as above, and constitutive relations for shells, which can be derived from constitutive laws of continuum mechanics.

Let us recall that the stress power density (per unit volume of the reference configuration) of the body B is given by (see Chapt. I.4)

$$\Sigma(X,t)=\mathbf{T}(X,t)\bullet\dot{\mathbf{F}}(X,t)=\frac{1}{2}\mathbf{S}(X,t)\bullet\dot{\mathbf{C}}(X,t)=\mathbf{S}(X,t)\bullet\dot{\mathbf{E}}(X,t), \tag{1.19}$$

where $\mathbf{T}$ and $\mathbf{S}$ denote the first and second Piola-Kirchhoff stress tensors, respectively, $\mathbf{F}$ is the deformation gradient and $\mathbf{C}$ denotes the right Cauchy-Green strain tensor. As usual, the superimposed dot stands for the material time derivative. We can then define a through-the-thickness resultant stress power (measured per unit area of the shell reference surface M) in a natural way:

$$\int_B \Sigma(X,t)\,dV=\iint_M \Sigma(Y,t)\,dA \quad\Rightarrow\quad \Sigma(Y,t)=\int_-^+ \Sigma(X,t)\mu\,d\xi\,. \tag{1.20}$$

On the other hand, it follows from considerations of Chapt. III.4 that the stress power density for the shell is given by

$$\begin{aligned}\sigma(Y;t)&=N(Y;t)\bullet\delta_t K(Y;t)+M(Y;t)\bullet\delta_t K(Y;t)\\&=\mathbf{N}(Y;t)\bullet\dot{\mathbf{E}}(Y;t)+\mathbf{M}(Y;t)\bullet\dot{\mathbf{K}}(Y;t)\,.\end{aligned} \tag{1.21}$$

Assuming that the material is simple, the constitutive law for the body can be expressed in either of the two forms (Chapt. I.5)

$$\mathbf{T}(X,t)=\mathfrak{h}^t_{s=0}(\mathbf{F}^t(X,t);X),\qquad \mathbf{S}(X,t)=\mathfrak{g}^t_{s=0}(\mathbf{C}^t(X,t);X). \tag{1.22}$$

Substituting (1.22) into the definition $(1.20)_2$ the resultant stress power for the shell made of a simple material is obtained in the form

$$\Sigma(Y,t)=\int_-^+ \mathfrak{h}^t_{s=0}(\mathbf{F}^t(s))\bullet\dot{\mathbf{F}}(t)\mu d\xi=2\int_-^+ \mathfrak{g}^t_{s=0}(\mathbf{C}^t(s))\bullet\dot{\mathbf{C}}(t)\mu d\xi\,. \tag{1.23}$$

On the other hand, substituting the general constitutive relation (1.9) into (1.21) we have

$$\sigma(Y;t)=\hat{\mathfrak{s}}^t_{s=0}(\mathcal{Y}(Y,s-t),Y)\bullet\delta_t\mathcal{E}(Y,t). \tag{1.24}$$

Now it becomes clear that the difference between the resultant stress power (1.23) and the stress power (1.24),

$$r(Y;t)=\Sigma(Y;t)-\sigma(Y;t), \tag{1.25}$$

provides a measure, how good the two-dimensional constitutive relations (1.9) approximate the constitutive relations, which can be derived from corresponding constitutive laws of continuum mechanics. In particular, if we could show that the difference (1.25) vanishes in some special cases, then the two-dimensional constitutive relations would be completely equivalent to those derived from corresponding three-dimensional ones. These remarks also indicate the way of deriving the constitutive laws for shells from the three-dimensional theory. A few other points should be noted here.

In the special case of simple shells, whose constitutive relations are given by (1.14) or (1.15), we have

$$\begin{aligned}\sigma(Y;t)&=\mathfrak{n}^\infty_{s=0}(\mathcal{E}^t(s))\bullet\delta_t\boldsymbol{E}(t)+\mathfrak{m}^\infty_{s=0}(\mathcal{E}^t(s))\bullet\delta_t\boldsymbol{K}(t)\\&=\mathfrak{n}^\infty_{s=0}(\mathcal{E}^t(s))\bullet\dot{\mathbf{E}}(t)+\mathfrak{m}^\infty_{s=0}(\mathcal{E}^t(s))\bullet\dot{\mathbf{E}}(t).\end{aligned} \tag{1.26}$$

It must be pointed out, however, that even if the shell-like body is made of a simple material, we cannot expect the resultant constitutive relations to have the form (1.13). In other words, the stress power (1.26) may not be equal to the resultant stress power (1.23), in general. The reduction of the three-dimensional problem of continuum mechanics to its two-dimensional form will, in general, give rise to some non-local effects.

It should be also clear that our approach to the shell constitutive relations is not restricted to shells made of a single material, but applies equally well to multi-layered shells. In such cases the definition (1.23) of the resultant stress power must be modified in order to take into account that materials of different layers are governed by their own constitutive laws.

2. Elastic shells

2.1 General constitutive relations. In agreement with the definition of an elastic material (Chapt. I.5), the shell is said to be elastic, if its mechanical response depends only on the actual strain state and not on the history of strains. Accordingly, for an elastic shell the constitutive equations can depend only on the strains and their gradients of any order. For simple shells they take the form

$$N(Y,t)=\mathcal{n}(E(Y,t),K(Y,t);Y), \qquad M(Y,t)=\mathcal{m}(E(Y,t),K(Y,t);Y). \tag{2.1}$$

Here $\mathcal{n}$ and $\mathcal{m}$ are response functions (not functionals). An equivalent form of the constitutive equations (2.1) is given by

$$\mathbf{N}(Y,t)=\mathfrak{n}(\mathbf{E}(Y,t),\mathbf{K}(Y,t);Y), \qquad \mathbf{M}(Y,t)=\mathfrak{m}(\mathbf{E}(Y,t),\mathbf{K}(Y,t);Y). \tag{2.2}$$

The constitutive equations (2.1) and (2.2) can also be expressed in terms of stress and strain vectors instead of stress and strain tensors:

$$\begin{aligned} n^{\Lambda}(Y,t)&=\mathcal{n}^{\Lambda}(e_{\Gamma}(Y,t),k_{\Gamma}(Y,t);Y),\\ m^{\Lambda}(Y,t)&=\mathcal{m}^{\Lambda}(e_{\Gamma}(Y,t),k_{\Gamma}(Y,t);Y), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \mathbf{n}^{\Lambda}(Y,t)&=\mathfrak{n}^{\Lambda}(\mathbf{e}_{\Gamma}(Y,t),\mathbf{k}_{\Gamma}(Y,t);Y),\\ \mathbf{m}^{\Lambda}(Y,t)&=\mathfrak{m}^{\Lambda}(\mathbf{e}_{\Gamma}(Y,t),\mathbf{k}_{\Gamma}(Y,t);Y), \end{aligned} \tag{2.4}$$

respectively.

In general, the constitutive relations (2.1)-(2.4) are delimited solely by the principle of frame-indifference. As a simple implication of (1.17) we have

$$\begin{aligned} \mathcal{n}^{\Lambda}(e_{\Gamma},k_{\Gamma};Y)&=O^{T}\mathcal{n}^{\Lambda}(Oe_{\Gamma},Ok_{\Gamma};Y),\\ \mathcal{m}^{\Lambda}(e_{\Gamma},k_{\Gamma};Y)&=O^{T}\mathcal{m}^{\Lambda}(Oe_{\Gamma},Ok_{\Gamma};Y), \end{aligned} \tag{2.5}$$

while the equations (2.4) remain unchanged. This principle is satisfied identically, if we express the constitutive laws for elastic shells in the component form. Moreover, since components of stress and strain vectors and tensors in the two representations are identical, the general constitutive equations for a simple elastic shell can be written in the following component form:

$$\begin{aligned} N^{\Gamma\Lambda} &= \mathfrak{n}^{\Gamma\Lambda}(E_{\Phi\Psi}, E_{\Phi}, K_{\Phi\Psi}, K_{\Phi}; Y), \\ Q^{\Lambda} &= \mathfrak{q}^{\Lambda}(E_{\Phi\Psi}, E_{\Phi}, K_{\Phi\Psi}, K_{\Phi}; Y), \\ M^{\Gamma\Lambda} &= \mathfrak{m}^{\Gamma\Lambda}(E_{\Phi\Psi}, E_{\Phi}, K_{\Phi\Psi}, K_{\Phi}; Y), \\ M^{\Lambda} &= \mathfrak{m}^{\Lambda}(E_{\Phi\Psi}, E_{\Phi}, K_{\Phi\Psi}, K_{\Phi}; Y). \end{aligned} \tag{2.6}$$

It should be noted that the drilling couples M^{Λ} do not appear in any known shell theory derived from elasticity theory by classical methods. However, the exact approach presented in this work shows that they have to be taken into account in the general shell theory.

2.2 Hyperelastic shells. Using the general consititutive equation (2.1) or (2.2) the stress power density (1.21) for a simple elastic shell takes the form

$$\sigma = \boldsymbol{n}(\boldsymbol{E}, \boldsymbol{K}) \cdot \delta_t \boldsymbol{E} + \boldsymbol{m}(\boldsymbol{E}, \boldsymbol{K}) \cdot \delta_t \boldsymbol{K} = \mathfrak{n}(\mathbf{E}, \mathbf{K}) \cdot \dot{\mathbf{E}} + \mathfrak{m}(\mathbf{E}, \mathbf{K}) \cdot \dot{\mathbf{K}}. \tag{2.7}$$

It can happen that (2.7) is an exact differential, i.e. there exists a function

$$\Phi(\boldsymbol{Y}, t) = \Phi(\boldsymbol{E}(\boldsymbol{Y}, t), \boldsymbol{K}(\boldsymbol{Y}, t); \boldsymbol{Y}) = \Phi(\mathbf{E}(\boldsymbol{Y}, t), \mathbf{K}(\boldsymbol{Y}, t); \boldsymbol{Y}) \tag{2.8}$$

such that its differential

$$\dot{\Phi} = (\partial_{\boldsymbol{E}} \Phi) \cdot \delta_t \boldsymbol{E} + (\partial_{\boldsymbol{K}} \Phi) \cdot \delta_t \boldsymbol{K} = (\partial_{\mathbf{E}} \Phi) \cdot \dot{\mathbf{E}} + (\partial_{\mathbf{K}} \Phi) \cdot \dot{\mathbf{K}} \tag{2.9}$$

is given by (2.7). In this case, the shell is called hyperelastic and the function (2.8) is called the strain energy function. It then follows that the constitutive relations for hyperelastic shells take the form

$$\boldsymbol{n}(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{Y}) = \partial_{\boldsymbol{E}} \Phi(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{Y}) \qquad \boldsymbol{m}(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{Y}) = \partial_{\boldsymbol{K}} \Phi(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{Y}), \tag{2.10}$$

or, equivalently,

$$\mathfrak{n}(\mathbf{E}, \mathbf{K}; \boldsymbol{Y}) = \partial_{\mathbf{E}} \Phi(\mathbf{E}, \mathbf{K}; \boldsymbol{Y}) \qquad \mathfrak{m}(\mathbf{E}, \mathbf{K}; \boldsymbol{Y}) = \partial_{\mathbf{K}} \Phi(\mathbf{E}, \mathbf{K}; \boldsymbol{Y}). \tag{2.11}$$

It is obvious that both forms of the constitutive relations are completely equivalent and they can be obtained from each other by simple transformation.

Taking further into account (1.23) we find that the resultant stress power for the shell made of hyperelastic material is given by

$$\Sigma(Y,t)=\frac{d}{dt}\int_{-}^{+}W(\mathbf{F})\mu d\xi=\frac{d}{dt}\int_{-}^{+}W(\mathbf{C})\mu d\xi\,. \tag{2.12}$$

However, it has to be noted that even if the shell-like body is made of hyperelastic material we cannot expect that the resultant two-dimensional constitutive relations will have the form (2.10) or (2.11).

2.3 Linearly elastic shells. An elastic shell is said to be linearly elastic, if the constitutive functions are linear with respect to the strains. The general constitutive relations take thus the form

$$\mathbf{N}=\mathbb{C}_1[\mathbf{E}]+\mathbb{C}_2[\mathbf{K}],\qquad \mathbf{M}=\mathbb{C}_3[\mathbf{E}]+\mathbb{C}_4[\mathbf{K}], \tag{2.13}$$

where

$$\mathbb{C}_k(Y)\in L(E\otimes T_Y M),\qquad k=1,2,3,4 \tag{2.14}$$

are elasticity tensors. If the elasticity tensors (2.14) satisfy the following symmetry conditions

$$\mathbb{C}_1=\mathbb{C}_1^T,\qquad \mathbb{C}_4=\mathbb{C}_4^T,\qquad \mathbb{C}_2=\mathbb{C}_3^T, \tag{2.15}$$

then there exists a strain energy function given in the form

$$\Phi=\mathbf{E}\bullet\mathbb{C}_1[\mathbf{E}]+2\mathbf{E}\bullet\mathbb{C}_2[\mathbf{K}]+\mathbf{K}\bullet\mathbb{C}_4[\mathbf{K}]. \tag{2.16}$$

If the elasticity tensors are invertible, their inverses

$$\mathbb{H}_k(Y)=\mathbb{C}_k(Y)^{-1},\qquad k=1,2,3,4 \tag{2.17}$$

are called the compliance tensors, and the constitutive relations can be solved for strains

$$\mathbf{E}=\mathbb{H}_1[\mathbf{N}]+\mathbb{H}_2[\mathbf{M}],\qquad \mathbf{K}=\mathbb{H}_3[\mathbf{N}]+\mathbb{H}_4[\mathbf{M}]. \tag{2.18}$$

The components of the elasticity tensors are defined by

$$\mathbb{C}_k=C_k^{K\Lambda L\Sigma}\boldsymbol{D}_K\otimes\boldsymbol{A}_\Lambda\otimes\boldsymbol{D}_L\otimes\boldsymbol{A}_\Sigma\,. \tag{2.19}$$

2.5 Thin shells. For thin shells undergoing small strain deformation the constitutive equations can be assumed in a simpler, uncoupled form

$$\begin{aligned} N^{\Lambda\Sigma} &= C^{\Phi\Lambda\Psi\Sigma}E_{\Phi\Psi}, \qquad Q^{\Lambda} = G^{\Lambda\Sigma}E_{\Sigma}, \\ M^{\Lambda\Sigma} &= D^{\Phi\Lambda\Psi\Sigma}K_{\Phi\Psi}, \qquad M^{\Lambda} = H^{\Lambda\Sigma}K_{\Sigma}, \end{aligned} \tag{2.20}$$

which for an isotropic material reduce to the form (here written in terms of physical components)

$$\begin{aligned} N^{\langle 11\rangle} &= C(E_{\langle 11\rangle} + \nu E_{\langle 22\rangle}), & N^{\langle 22\rangle} &= C(E_{\langle 22\rangle} + \nu E_{\langle 11\rangle}), \\ N^{\langle 12\rangle} &= C(1-\nu)E_{\langle 12\rangle}, & N^{\langle 22\rangle} &= C(1-\nu)E_{\langle 21\rangle}, \\ Q^{\langle 1\rangle} &= \frac{1}{2}\alpha_s C(1-\nu)E_{\langle 1\rangle}, & Q^{\langle 2\rangle} &= \frac{1}{2}\alpha_s C(1-\nu)E_{\langle 2\rangle}, \\ M^{\langle 11\rangle} &= D(K_{\langle 11\rangle} + \nu K_{\langle 22\rangle}), & M^{\langle 22\rangle} &= D(K_{\langle 22\rangle} + \nu K_{\langle 11\rangle}), \\ M^{\langle 12\rangle} &= D(1-\nu)K_{\langle 12\rangle}, & M^{\langle 21\rangle} &= D(1-\nu)K_{\langle 12\rangle}, \\ M^{\langle 1\rangle} &= \alpha_t D(1-\nu)K_{\langle 1\rangle}, & M^{\langle 2\rangle} &= \alpha_t D(1-\nu)K_{\langle 2\rangle}, \end{aligned} \tag{2.21}$$

where

$$C = \frac{Eh_0}{1-\nu^2}, \qquad D = \frac{Eh_0^3}{12(1-\nu^2)}. \tag{2.22}$$

Here E denotes the Young modulus, ν the Poisson ration, α_s and α_t stand for the shear and torsional coefficients, respectively. The constitutive equations (2.21) may be viewed as a simplest generalization of the classical ones. In particular, upon symmetrization and omitting the constitutive relations for drilling couples, they reduce to the form generally accepted in the Mindlin-Reissner type shell theory. Moreover, it can be shown that for smooth, relatively thin shells undergoing small strain deformation the contribution of strains $\kappa_{\langle 1\rangle}, \kappa_{\langle 2\rangle}$ to the two-dimensional strain energy function is of higher order small and can be neglected (taking $\alpha_t = 0$). However, from the computational point of view it is convenient to retain this small contribution and thus to preserve the complete structure of the general shell theory.

3. Approximations in shell theory

3.1 General and special theories. The formulation of the complete set of shell governing equations presented in this work clearly shows that no simplifying assumptions are needed to reduce the three-dimensional problems of continuum mechanics to the two-dimensional form appropriate for shell-like bodies. In this sense the derived shell theory can be called "exact" or "general theory". It is also important to note that while our approach has not been based on some a priori adopted assumptions, it also does not exclude any. The resulting shell theory enjoys thus a full generality, and it provides a convenient starting point for the examination of various special cases.

The terms "exact", "general" or "special" shell theories are used throughout the literature in various contexts and with different meaning assigned to them. For example[4], any shell theory formulated by the direct approach can be called "exact". In our case we shall use the term "exact" or "general" shell theory to mean that the shell governing equations are derived directly (static equilibrium equations and side conditions) or indirectly (kinematic relations and kinematic side conditions) from general principles of continuum mechanics without any restrictions imposed on the deformation of the shell-like body. Let us further note that our formulation shows in effect that the exact reduction of the three-dimensional problem of classical continuum mechanics (based on the concept of Cauchy's continuum) leads to the concept of two-dimensional continuum with internal microstructure but not richer than the classical two-dimensional Cossert continuum. In this context it is interesting to see, how the shell theory formulated in this work is related to various shell theories which use more kinematical variables, and which are generally supposed to give a more accurate description of the shell problem. The possible relationships between various shell theories is a somewhat controversial subject[5] and a detailed discussion of these relations would require a separate work, but a few points are worth to be noted here.

3.2 Successive approximations. While a more general case can be considered,[6] for our purpose it will suffice to assume here that the three-dimensional deformation $x = \chi(X)$ of the shell-like body is constrained in such a way that the current position vector x can be expressed in the form

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[4] Cf. NAGHDI [1972].
[5] Cf. GREEN AND NAGHDI [1974].
[6] See ANTMAN [1976].

$$x(\xi^\Lambda,\xi)=\chi\big(X(\xi^\Lambda,\xi)\big)=y(\xi^\Lambda)+\sum_{A=1}^{N} z^A(\xi^\Lambda,\xi)\boldsymbol{d}_A(\xi^\Lambda). \tag{3.1}$$

Here, like in the general shell theory, $y=\chi(Y)$ denotes the position vector of the deformed shell reference surface $\chi(M)$, the vectors

$$\boldsymbol{d}_A(\xi^\Lambda)=\boldsymbol{d}_A(\chi(\xi^\Lambda)), \qquad A=1,2,\ldots,N, \tag{3.2}$$

can be called generalized displacements (or directors), and $z^A=z^A(\xi^\Lambda,\xi)$ are given functions of all three spatial coordinates. For increasing values of N we can formulate higher-order theories, and with $N\to\infty$ we may view (3.1) as providing successive approximations to a three-dimensional deformation $x=\chi(X)$ of the shell-like body.

The functions $z^A=z^A(\xi^\Lambda,\xi)$ can be determined by any method known in the approximation theory. For example, assuming that $x(\xi^\Lambda,\xi)=\chi(X(\xi^\Lambda,\xi))$ is an analytical function with respect to the through-the-thickness coordinate ξ, standard power expansion yields

$$z^A=\xi^A, \qquad \boldsymbol{d}_A(\xi^\Lambda)=\frac{\partial^A x(\xi^\Lambda,\xi)}{\partial\xi^A}\Big|_{\xi=0}, \qquad A=1,2,\ldots. \tag{3.3}$$

Let us make clear that ξ^A denotes here the A-th power of through-the-thickness coordinate ξ and not the A-th spatial coordinate. The other possible way of determining the functions z^A and the generalized displacements $\boldsymbol{d}_A$ can be based on the interpolation theory. For example, piecewise linear interpolation can be applied to layered shells.[7]. But this problem need not be our concern here. Our aim is to show, how the shell theory based on the assumption (3.1) is related to the general theory presented in this work.

3.3 Static equations and boundary conditions. The complete set of shell governing equations consistent with the assumption (3.1) can now be derived using a standard variational procedure. This is the classical way of deriving shell governing equations, and it will suffice to outline here only the main points. For simplicity, we shall restrict our discussion to the case of a regular shell-like body in the sense of Chapt. II.1. In this case, the three-dimensional principle of virtual work (see Chapt. I.4) can be written in the form

[7] See PINSKY AND KIM [1986].

$$\int_B \mathbf{T}\cdot\nabla(\delta\boldsymbol{x})\,dV = \int_B \mathbf{f}\cdot\delta\boldsymbol{x}\,dV + \int_{M^+}\mathbf{t}_N^+\cdot\delta\boldsymbol{x}^+\,dA^+ - \int_{M^-}\mathbf{t}_N^-\cdot\delta\boldsymbol{x}^-\,dA^- + \int_{\partial B_f^\circ}\mathbf{t}_N^*\cdot\delta\boldsymbol{x}\,dA^\circ . \tag{3.4}$$

Here $\delta\boldsymbol{x}(\xi^\Lambda,\xi)$ denotes the three-dimensional virtual displacement field, which in consistency with (3.1) is given by

$$\delta\boldsymbol{x}(\xi^\Lambda,\xi) = \delta\boldsymbol{y}(\xi^\Lambda) + \sum_{A=1}^{N} z^A(\xi^\Lambda,\xi)\delta\boldsymbol{d}_A(\xi^\Lambda) . \tag{3.5}$$

The term on the left-hand side of (3.4) represents the internal virtual work, which for the regular shell-like body can be written in the form

$$\int_B \mathbf{T}\cdot\nabla(\delta\boldsymbol{x})\,dV = \iint_M w\,dA , \tag{3.6}$$

where the through-the-thickness resultant virtual work density is defined by

$$w = \int_-^+ \mathbf{T}\cdot\nabla(\delta\boldsymbol{x})\,\mu d\xi = \int_-^+ (\mathbf{t}^\Lambda\cdot\delta\boldsymbol{x}_{,\Lambda} + \mathbf{t}^3\cdot\delta\boldsymbol{x}_{,3})\,\mu d\xi . \tag{3.7}$$

By virtue of (3.5) we have

$$\delta\boldsymbol{x}_{,\Lambda} = \delta\boldsymbol{y}_{,\Lambda} + z^A{}_{,\Lambda}\,\delta\boldsymbol{d}_A + z^A\delta\boldsymbol{d}_{A,\Lambda} , \qquad \delta\boldsymbol{x}_{,3} = z^A{}_{,3}\,\delta\boldsymbol{d}_A . \tag{3.8}$$

Here and in the sequel summation with respect to A over the range $\{1,2,\ldots,N\}$ has to be applied. Substituting (3.8) into (3.7) we obtain

$$w = \boldsymbol{n}^\Lambda\cdot\delta\boldsymbol{v}_{,\Lambda} + \boldsymbol{m}^A\cdot\delta\boldsymbol{d}_A + \boldsymbol{m}^{A\Lambda}\cdot\delta\boldsymbol{d}_{A,\Lambda} , \tag{3.9}$$

where the resultant stress vectors and the resultant stress couple vectors are defined by

$$\begin{aligned} \boldsymbol{n}^\Lambda &= \int_-^+ \mathbf{t}^\Lambda\mu\,d\xi , \\ \boldsymbol{m}^A &= \int_-^+ (z^A{}_{,\Lambda}\,\mathbf{t}^\Lambda + z^A{}_{,3}\,\mathbf{t}^3)\mu\,d\xi , \\ \boldsymbol{m}^{A\Lambda} &= \int_-^+ z^A\mathbf{t}^\Lambda\mu\,d\xi . \end{aligned} \tag{3.10}$$

Exactly in the same manner, with the use of (3.5) and relevant geometric relations, we can reduce the right-hand side of the principle of virtual work (3.4) to its two-dimensional form. As a result, we obtain the following form of the two-dimensional principle of virtual work:

$$\iint_M w\,dA = \iint_M (\boldsymbol{p}\cdot\delta\boldsymbol{y} + \boldsymbol{l}^A\cdot\delta\boldsymbol{d}_A)\,dA + \int_{\partial M_f} (\boldsymbol{n}^*\cdot\delta\boldsymbol{y} + \boldsymbol{m}^{*A}\cdot\delta\boldsymbol{d}_A)\,dS . \tag{3.11}$$

Here the resultant surface and line force and couple vectors are defined by

$$\begin{aligned} \boldsymbol{p} &= \int_{-}^{+} \mathbf{f}\mu\, d\xi + \alpha^{+}\mathbf{t}_N^{+} - \alpha^{+}\mathbf{t}_N^{-}, \\ \boldsymbol{l}^A &= \int_{-}^{+} z^A \mathbf{f}\mu\, d\xi + \alpha^{+}(z^A)^{+}\mathbf{t}_N^{+} - \alpha^{-}(z^A)^{-}\mathbf{t}_N^{-}, \\ \boldsymbol{n}^* &= \int_{-}^{+} \mathbf{t}^*\alpha^*\, d\xi, \qquad \boldsymbol{m}^{*A} = \int_{-}^{+} z^A \mathbf{t}^*\alpha^*\, d\xi. \end{aligned} \tag{3.12}$$

Under usual regularity assumptions we can easily derive from (3.11) the following equilibrium equations

$$\boldsymbol{n}^{\Lambda}{}_{|\Lambda} + \boldsymbol{p} = \mathbf{0}, \qquad \boldsymbol{m}^{A\Lambda}{}_{|\Lambda} - \boldsymbol{m}^{A} + \boldsymbol{l}^{A} = \mathbf{0}, \qquad A = 1,2,\ldots,N. \tag{3.13}$$

together with the static boundary conditions

$$\boldsymbol{n}^{\Lambda}\nu_{\Lambda} = \boldsymbol{n}^*, \qquad \boldsymbol{m}^{A\Lambda}\nu_{\Lambda} = \boldsymbol{m}^{*A}. \tag{3.14}$$

An additional equation can be derived from the three-dimensional balance law of angular momentum, which in the local form reads

$$\boldsymbol{x}_{,K} \times \mathbf{t}^{K} = \boldsymbol{x}_{,\Lambda} \times \mathbf{t}^{\Lambda} + \boldsymbol{x}_{,3} \times \mathbf{t}^{3} = \mathbf{0}. \tag{3.15}$$

According to (3.1) we have

$$\boldsymbol{x}_{,\Lambda} = \boldsymbol{y}_{,\Lambda} + z^A \boldsymbol{d}_{A,\Lambda} + z^A{}_{,\Lambda}\boldsymbol{d}_A, \qquad \boldsymbol{x}_{,3} = z^A{}_{,3}\boldsymbol{d}_A. \tag{3.16}$$

Substituting (3.16) into (3.15), multiplying by the invariant μ, and integrating through the shell thickness we obtain

$$\boldsymbol{y}_{,\Lambda} \times \int_{-}^{+} \mathbf{t}^{\Lambda}\mu\, d\xi + \boldsymbol{d}_{A,\Lambda} \times \int_{-}^{+} z^A \mathbf{t}^{\Lambda}\mu\, d\xi + \boldsymbol{d}_A \times \int_{-}^{+} z^A{}_{,K}\, \mathbf{t}^{K}\mu\, d\xi = \mathbf{0}. \tag{3.17}$$

Taking into account the definitions (3.10), the equation (3.17) takes the form

$$\boldsymbol{y}_{,\Lambda} \times \boldsymbol{n}^{\Lambda} + \boldsymbol{d}_A \times \boldsymbol{m}^{A} + \boldsymbol{d}_{A|\Lambda} \times \boldsymbol{m}^{A\Lambda} = \mathbf{0}. \tag{3.18}$$

These are exactly the basic shell equations, which can be derived from the three-dimensional principle of virtual work using the assumption (3.1) or within the direct approach.

3.4 Kinematical and constitutive relations. Within the assumption (3.1) the kinematics of the shell is entirely determined by the deformation $\boldsymbol{y} = \boldsymbol{\chi}(\boldsymbol{Y})$ of the shell reference surface and the set of N generalized displacements (directors) $\boldsymbol{d}_A = \boldsymbol{d}_A(\boldsymbol{Y})$, $A = 1,2,\ldots,N$. Moreover, it is a simple matter to obtain the

corresponding two-dimensional constitutive relations from three-dimensional constitutive equations of continuum mechanics. For example, assuming that the material of the shell is hyperelastic we define the two-dimensional strain energy function by

$$\Phi(\boldsymbol{y}_{,\Lambda}, \boldsymbol{d}, \boldsymbol{d}_{,\Lambda}) = \int_{-}^{+} W(\boldsymbol{x}_{,\Lambda}, \boldsymbol{x}_{,3})\, \mu d\xi . \tag{3.19}$$

The two-dimensional constitutive equations follow at once in the form[8]

$$\boldsymbol{n}^{\Lambda} = \frac{\partial \Phi}{\partial \boldsymbol{y}_{,\Lambda}}, \qquad \boldsymbol{m}^{A} = \frac{\partial \Phi}{\partial \boldsymbol{d}^{A}}, \qquad \boldsymbol{m}^{A\Lambda} = \frac{\partial \Phi}{\partial \boldsymbol{d}^{A}{}_{,\Lambda}} . \tag{3.20}$$

3.5 Reduction of the general shell theory. Our aim now is to show how the governing equations of the general shell theory can be reduced to the form which is consistent with the kinematical hypothesis (3.1). We note first that the assumption (3.1) does not affect the definition of the resultant stress vectors $\boldsymbol{n}^{\Lambda}$ (aside a reactive force). Thus, in the theory consistent with this hypothesis, like within the general theory, the resultant stress vectors are defined by

$$\boldsymbol{n}^{\Lambda} = \int_{-}^{+} \mathbf{t}^{\Lambda} \mu\, d\xi . \tag{3.21}$$

Comparing next the kinematic assumption (3.1) with the formal representation $\boldsymbol{x}(\boldsymbol{Y}, \xi) = \boldsymbol{y}(\boldsymbol{Y}) + \boldsymbol{\zeta}(\boldsymbol{Y}, \xi)$ of the shell deformation, which we have used in our derivation of the resultant balance laws, one can see that in this case

$$\boldsymbol{\zeta}(\xi^{\Lambda}, \xi) = z^{A}(\xi^{\Lambda}, \xi)\boldsymbol{d}_{A}(\xi^{\Lambda}) . \tag{3.22}$$

Substituting (3.22) into the definition of the resultant stress couple vectors $\boldsymbol{m}^{\Lambda}$ (see Chapt. II.2) we obtain

$$\boldsymbol{m}^{\Lambda} = \int_{-}^{+} \boldsymbol{\zeta} \times \mathbf{t}^{\Lambda} \mu\, d\xi = \boldsymbol{d}_{A} \times \left(\int_{-}^{+} z^{A} \mathbf{t}^{\Lambda} \mu\, d\xi \right) \equiv \boldsymbol{d}_{A} \times \boldsymbol{m}^{A\Lambda} , \tag{3.23}$$

where the resultant stress couple vectors $\boldsymbol{m}^{A\Lambda}$ are defined by

$$\boldsymbol{m}^{A\Lambda} = \int_{-}^{+} z^{A} \mathbf{t}^{\Lambda} \mu\, d\xi . \tag{3.24}$$

[8] Constitutive equations of the type (3.20) have been discussed by NAGHDI [1972] within a purely direct approach.

In the same way the resultant surface and boundary couple vectors $\boldsymbol{l}$ and $\boldsymbol{m}^*$ can be obtained in the form

$$\boldsymbol{l}=\boldsymbol{d}_A\times\boldsymbol{l}^A\,,\qquad \boldsymbol{m}^*=\boldsymbol{d}_A\times\boldsymbol{m}^{*A}\,. \tag{3.25}$$

Using (3.23) and (3.25) the exact equilibrium equations of Chapt. II.4 can now be rewritten in the form

$$\begin{aligned}\boldsymbol{n}^{\Lambda}{}_{|\Lambda}+\boldsymbol{p}&=\boldsymbol{0}\,,\\ \boldsymbol{d}_A\times\boldsymbol{m}^{A\Lambda}{}_{|\Lambda}+\boldsymbol{d}_{A|\Lambda}\times\boldsymbol{m}^{A\Lambda}+\boldsymbol{y}_{,\Lambda}\times\boldsymbol{n}^{\Lambda}+\boldsymbol{d}_A\times\boldsymbol{l}^A&=\boldsymbol{0}\,.\end{aligned} \tag{3.26}$$

With the help of (3.18) the equilibrium equations (3.26) can be reduced to the form

$$\boldsymbol{n}^{\Lambda}{}_{|\Lambda}+\boldsymbol{p}=\boldsymbol{0}\,,\qquad \boldsymbol{d}_A\times(\boldsymbol{m}^{A\Lambda}{}_{|\Lambda}-\boldsymbol{m}^A+\boldsymbol{l}^A)=\boldsymbol{0}\,,\qquad A=1,2,\ldots,N\,. \tag{3.27}$$

In the same manner, the static boundary conditions of the exact shell theory take the form

$$\boldsymbol{n}^{\Lambda}\nu_{\Lambda}-\boldsymbol{n}^*=\boldsymbol{0}\,,\qquad \boldsymbol{d}_A\times(\boldsymbol{m}^{A\Lambda}\nu_{\Lambda}-\boldsymbol{m}^{*A})=\boldsymbol{0}\,. \tag{3.28}$$

Insisting the moment equilibrium equation $(3.27)_2$ and the moment boundary condition $(3.28)_2$ to hold for any choice of the vectors $\boldsymbol{d}_A$, the terms enclosed in the bracket must vanish for every $A=1,2,\ldots,N$. By this argument, the equilibrium equations and the static boundary conditions of the general shell theory reduce to the form (3.13) and (3.14), respectively.

4. General five parameter theory

4.1 Generalized Mindlin-Reissner hypothesis. In the context of the finite element analysis of shells much effort has been devoted to formulate a shell theory taking as starting point the classical Mindlin-Reissner hypothesis.[9] According to this hypothesis the current position vector is given by

$$\boldsymbol{x}(\xi^\Lambda,\xi)=\boldsymbol{\chi}\big(\boldsymbol{X}(\xi^\Lambda,\xi)\big)=\boldsymbol{y}(\xi^\Lambda)+\xi\boldsymbol{d}(\xi^\Lambda), \tag{4.1}$$

where $\boldsymbol{y}(\xi^\Lambda)=\boldsymbol{\chi}(\boldsymbol{Y}(\xi^\Lambda))$ and $\boldsymbol{d}(\xi^\Lambda)=\boldsymbol{d}(\boldsymbol{\chi}(\boldsymbol{Y}(\xi^\Lambda)))$ is a field of unit vectors defined over the deformed shell reference surface. It is clear that (4.1) is just a very special case of (3.1), and the basic shell equations follow from those derived in the previous subchapter by taking $N=1$. It is then interesting to see that the shell theory having the same mathematical structure can also be formulated under a much less restrictive assumption.

Instead of (4.1), let us assume that the three-dimensional deformation of the shell-like body is constrained in such a way that the current position vector is given in the form

$$\boldsymbol{x}(\xi^\Lambda,\xi)=\boldsymbol{\chi}\big(\boldsymbol{X}(\xi^\Lambda,\xi)\big)=\boldsymbol{y}(\xi^\Lambda)+\zeta(\xi^\Lambda,\xi)\boldsymbol{d}(\xi^\Lambda), \tag{4.2}$$

where $\boldsymbol{y}$ and $\boldsymbol{d}$ have the same meaning as in (4.1), and $\zeta=\zeta(\xi^\Lambda,\xi)$ is an arbitrary scalar function of all three spatial coordinates. It is obvious that (4.2) implies that through-the-thickness fibres remain straight during deformation. However, they can undergo an arbitrary extension/contraction defined by the function $\zeta(\xi^\Lambda,\xi)$, which has to be specified by some additional conditions. In fact, there are four possible ways in which we can treat the hypothesis (4.2).

The simplest way is to assume that $\zeta=\zeta(\xi^\Lambda,\xi)$ is a given function of all three coordinates, and it does not depend on the shell deformation. For example, taking $\zeta=\xi$ we obtain the classical Mindlin-Reissner hypothesis (4.1).

Another way is to assume that $\zeta=\zeta(\xi^\Lambda,\xi)$ is a given function but containing scalar variables which enrich the kinematical model of the shell. For example, we can take

[9] See e.g. SIMO [1993].

$$\zeta(\xi^\Lambda,\xi)=\xi\lambda(\xi^\Lambda), \tag{4.3}$$

where $\lambda(\xi^\Lambda)$ represents a homogenous through-the-thickness stretch of the shell fibres. The assumption (4.3) is equivalent to the assumption of linear variation of displacements across the shell thickness.[10] More generally, we can take

$$\zeta(\xi^\Lambda,\xi)=\xi\lambda(\xi^\Lambda)+\xi^2\kappa(\xi^\Lambda), \tag{4.4}$$

allowing a quadratic distribution of the through-the-thickness stretch with $\lambda(\xi^\Lambda)$ and $\kappa(\xi^\Lambda)$ to be regarded as additional independent variables of the theory.

Within an entirely different approach, we can assume that $\zeta=\zeta(\xi^\Lambda,\xi)$ is completely determined in terms of $\boldsymbol{y}$ and $\boldsymbol{d}$ and their surface derivatives. For example, it can be shown that this is the case, whenever the shell undergoes an isochoric deformation (this is the case for rubber-like materials, for example). The complete shell theory within this setting was formulated in our earlier papers.[11]

Within a still more general approach we can admit that $\zeta(\xi^\Lambda,\xi)$ describes an arbitrary but unspecified through-the-thickness deformation of the shell. This is much the same approach as we have applied in order to derive our general shell theory, where instead of (4.2) we have used a formal representation of the shell deformation in the form $\boldsymbol{x}(\xi^\Lambda,\xi)=\boldsymbol{y}(\xi^\Lambda)+\boldsymbol{\zeta}(\xi^\Lambda,\xi)$ of which (4.2) is a special case. This is the point of view we adopt here.

4.2 Equilibrium equations. Now, it is a rather simple task to reduce our general shell theory to the form, which is consistent with the kinematic hypothesis (4.2). When the hypothesis (4.2) is introduced into the definitions of the resultant stress couples, they take the form

$$\boldsymbol{m}^\Lambda=\boldsymbol{d}\times\left(\int_-^+\zeta\mathbf{t}^\Lambda\mu\,d\xi\right)=\boldsymbol{d}\times\hat{\boldsymbol{m}}^\Lambda, \qquad \hat{\boldsymbol{m}}^\Lambda=\int_-^+\zeta\mathbf{t}^\Lambda\mu\,d\xi. \tag{4.5}$$

In the same way the surface couple vector $\boldsymbol{l}$ is obtained to be

$$\boldsymbol{l}=\boldsymbol{d}\times\hat{\boldsymbol{l}}, \qquad \hat{\boldsymbol{l}}=\int_-^+\zeta\mathbf{f}\mu\,d\xi+\alpha^+\zeta^+\mathbf{t}_N^+-\alpha^-\zeta^-\mathbf{t}_N^-, \tag{4.6}$$

where we write $\zeta^\pm(\xi^\Lambda)=\zeta(\xi^\Lambda,\pm h_0^\pm(\xi^\Lambda))$. With the use of (4.5) and (4.6) the equilibrium equations of the general shell theory can be expressed in the form

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[10] See PIETRASZKIEWICZ [1977].

[11] MAKOWSKI AND STUMPF [1986], and in STUMPF AND MAKOWSKI [1986].

$$\hat{\boldsymbol{n}}^{\Lambda}{}_{|\Lambda} + \hat{\boldsymbol{p}} = \boldsymbol{0}, \qquad \hat{\boldsymbol{m}}^{\Lambda}{}_{|\Lambda} - \hat{\boldsymbol{m}} + \hat{\boldsymbol{l}} = \boldsymbol{0}, \tag{4.7}$$

together with an additional set of three scalar equations

$$\boldsymbol{y}_{,\Lambda} \times \hat{\boldsymbol{n}}^{\Lambda} + \boldsymbol{d} \times \hat{\boldsymbol{m}} + \boldsymbol{d}_{,\Lambda} \times \hat{\boldsymbol{m}}^{\Lambda} = \boldsymbol{0}, \tag{4.8}$$

where

$$\hat{\boldsymbol{m}}(\xi^{\Lambda}) = \int_{-}^{+} \zeta_{,J}\, \mathbf{t}^{J} \mu \, d\xi \ . \tag{4.9}$$

This set of equations can be derived applying the approach of Sect. 3.3. Eqns (4.8) play the role of constitutive restrictions, and they should be satisfied identically for any properly invariant constitutive equations.

As an obvious implication of (4.3) and (4.4) we have

$$\boldsymbol{d} \cdot \boldsymbol{m}^{\Lambda} = 0, \qquad \boldsymbol{d} \cdot \boldsymbol{l} = 0, \tag{4.10}$$

showing that the drilling couples are ruled out by the kinematical assumption (4.2) entirely independently of the particular form of the function ζ. It must be pointed out here that the absence of the drilling couples is not a property of a rigorously formulated shell theory, but becomes the obvious implication of the fairly restrictive kinematic hypothesis (4.2).

The external boundary couple vectors can also be obtained in this manner, but we shall not discuss here the complete theory.

4.3 Kinematical considerations. Next, the virtual work identity and kinematics of the shell can be constructed applying the approach of Chapt. III.2 and Chapt. III.3. The shell theory formulated in this way has the formal structure identical with the one derived under the classical Mindlin-Reissner hypothesis (4.1). However, it has to be noted that (4.2) is a far weaker assumption than the Mindlin-Reissner one, and that it admits a highly nonlinear through-the-thickness deformation. In any case, the only independent kinematical variables are the position vector $\boldsymbol{y}$ of the deformed shell reference surface and the field of unit vectors $\boldsymbol{d}$.

Since $\boldsymbol{d}$ is a field of unit vectors with only two independent components, the corresponding virtual field $\delta\boldsymbol{d}$ satisfies the constraint

$$\boldsymbol{d} \cdot \delta\boldsymbol{d} = 0. \tag{4.11}$$

Therefore, there is no drilling component here, what is a consistent implication of (4.2).

The above considerations illustrate how the general shell theory presented in the previous sections can be reduced to special shell theories consistent with some adopted kinematical hypotheses. It is trivial to say that the reverse construction is by no way unique: starting with the hypothesis (4.2) or with the more restrictive Mindlin-Reissner hypothesis (4.1) it is not possible to obtain the governing equations of the general shell theory. For the field of unit vectors $\boldsymbol{d}$ the condition $\boldsymbol{d} \cdot \delta \boldsymbol{d} = 0$ makes it possible to express $\delta \boldsymbol{d}$ in the form $\delta \boldsymbol{d} = \boldsymbol{\omega} \times \boldsymbol{d}$. Here $\boldsymbol{\omega}$ denotes the associated virtual rotation satisfying the condition $\boldsymbol{d} \cdot \boldsymbol{\omega} = 0$. The equation $\boldsymbol{d} \cdot \boldsymbol{\omega} = 0$ has the general solution

$$\boldsymbol{\omega} = \boldsymbol{d} \times \delta \boldsymbol{d} + \omega \boldsymbol{d} , \tag{4.11}$$

where ω is an undetermined scalar parameter (virtual drilling rotation). Then it is clearly seen that within the shell theory formulated under the hypothesis (4.2), and especially within the Mindlin-Reissner assumption (4.1), the (virtual) drilling rotation ω remains undetermined. This does not mean that we can set $\omega = 0$. Overlooking the obvious fact that this is a special choice for ω it can lead to the erroneous claim that the general shell theory incorporating drilling couples and drilling rotation can be derived form the Mindlin-Reissner hypothesis. Misconceptions can also arise, if one uses, on the one side, the general kinematics including the rotation tensor $\boldsymbol{Q}$ as independent variable, while, on the other side, the static considerations are based on the Mindlin-Reissner hypothesis. As it is apparent from the above considerations the Mindlin-Reissner hypothesis rules out the drilling couples and leaves the drilling rotation undetermined. Consequently, in this case the kinematics does not fit for the statics and artifices are needed to build the complete theory.

Appendices

Vectors, tensors and tensor fields

This appendix contains a collection of results on tensor algebra and analysis to the extent which makes the work essentially self-contained. Various symbols used here should not be identified with any quantity denoted by the same symbol in the main part of this work.

Appendix A

Operations on vectors and tensors

Euclidean vector spaces. In its most classical sense, vectors are represented by directed line segments with definite direction and length, and for which the parallelogram law of addition is valid. This geometric interpretation of vectors is advantageous, but we shall rather emphasize on the mathematical content of various vector operations. In the abstract approach, a real vector space is a set E, whose elements are called vectors but otherwise unspecified, together with two binary operations,

$$\begin{aligned} E\times E\to E, \qquad & (\boldsymbol{u},\boldsymbol{v})\to \boldsymbol{u}+\boldsymbol{v}, \\ \mathbb{R}\times E\to E, \qquad & (\lambda,\boldsymbol{v})\to \lambda\boldsymbol{u}, \end{aligned} \tag{A.1}$$

called addition and scalar multiplication, respectively, with the same properties as in the most classical case. The zero vector will be invariably denoted by $\mathbf{0}$.

In the geometric presentation, a vector is characterized by its magnitude (length) and direction, and there is the concept of an angle between two vectors. All three notions derive from the concept of inner product of vectors; the inner product of two vectors is a map

$$E\times E\to\mathbb{R}, \qquad (\boldsymbol{u},\boldsymbol{v})\to \boldsymbol{u}\cdot\boldsymbol{v}, \tag{A.2}$$

which assigns a real number to each pair of vectors with the properties of symmetry, bilinearity and positive definiteness. It is the inner product of vectors, which justifies the name of Euclidean vector space. The cosine of the angle between two vectors is defined by

$$\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}. \tag{A.3}$$

In view of Schwartz's inequality this definition is meaningful indeed. A unit vector is one for which $\|\boldsymbol{u}\|=1$. The two vectors are said to be mutually orthogonal if $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

All vector spaces, which are considered in this and the subsequent appendices, are assumed to be real (i.e. over the field of real numbers $\mathbb{R}$), finite dimensional, and equipped with an inner product. We denote vector spaces by E, F, E', etc., vectors by lower case bold letters, and we write $\boldsymbol{u}\cdot\boldsymbol{v}$ for the inner product of two vectors regardless of the vector space in question. The induced Euclidean norm is denoted by $\|\boldsymbol{u}\|$, i.e. $\|\boldsymbol{u}\|=\sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}>0$.

Basis. A basis for a vector space E is any linearly independent system of vectors of maximum order, i.e. equal to the dimension of the space. In general, any set $\{\boldsymbol{g}_i\}$ of n linearly independent vectors constitute a basis for the vector space E. These vectors need not be of unit length nor mutually orthogonal. Then there exists the unique reciprocal basis, denoted by $\{\boldsymbol{g}^i\}$, such that

$$\boldsymbol{g}^i\cdot\boldsymbol{g}_j=\delta^i_j \quad (\textit{Kronecker delta}). \tag{A.4}$$

Associated with these bases are the so-called components of the metric tensor which are defined by

$$g_{ij}=\boldsymbol{g}_i\cdot\boldsymbol{g}_j\,, \qquad g^{ij}=\boldsymbol{g}^i\cdot\boldsymbol{g}^j\,, \qquad g=\det g_{ij}>0\,. \tag{A.5}$$

From these definitions it follows that[1]

$$\boldsymbol{g}^i=g^{ij}\boldsymbol{g}_j\,, \qquad \boldsymbol{g}_i=g_{ij}\boldsymbol{g}^j\,, \qquad g_{ik}g^{kj}=\delta^j_i\,. \tag{A.6}$$

A distinct feature of the Euclidean vector space is the existence of an orthonormal basis, that is the one of n vectors each of unit length and any two vectors are mutually orthogonal, $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$. Given any basis the orthonormal basis can be constructed by a method known as Gram-Schmidt orthogonalization procedure.

[1] If in a term an index occurs twice, in which case it is called a summation index, then the term is to be summed with respect to this index over the range of its admissible values. When working with the three-dimensional vector space and unless stated otherwise, the following convention is adopted: Lower-case Latin indices i, j, etc. have range 1,2,3, lower-case Greek indices, etc. have range 1,2, and such diagonally repeated indices are summed over their range.

Components of vectors. Since the base vectors are linearly independent, every vector can be expressed as linear combination of the base vectors, i.e. it has the following component representations

$$\boldsymbol{u} = u^i \boldsymbol{g}_i = u_i \boldsymbol{g}^i . \tag{A.7}$$

The real numbers

$$u_i = \boldsymbol{u} \bullet \boldsymbol{g}_i = g_{ij} u^j , \qquad u^i = \boldsymbol{u} \bullet \boldsymbol{g}^i = g^{ij} u_j , \tag{A.8}$$

are often called the covariant and contravariant components of the vector, respectively. However, the names are somewhat arbitrary, since the underlying bases are mutually reciprocal and there is no way to choose one over the other. It should be obvious that a vector does not depend on any basis even though its components do.

A substantial part of the usefulness of the orthonormal bases is their self-reciprocities. Hence contravariant and covariant components of vectors coincide.

Lie bracket. Given a vector space E, a map

$$[.,.]: E \times E \to E , \qquad (\boldsymbol{u},\boldsymbol{v}) \to [\boldsymbol{u},\boldsymbol{v}] \tag{A.9}$$

with the following three properties

$$\begin{aligned} [\boldsymbol{u},\boldsymbol{v}] &= -[\boldsymbol{v},\boldsymbol{u}] , \\ [\lambda \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}] &= \lambda [\boldsymbol{u},\boldsymbol{w}] + [\boldsymbol{v},\boldsymbol{w}] , \\ [\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]] + [\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]] + [\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]] &= \boldsymbol{0} , \end{aligned} \tag{A.10}$$

is called Lie bracket on E. The first and second of the properties (A.10) say that the Lie bracket is skew-symmetric and bilinear. The third property is called the Jacobi identity.

The vector space E taken together with the Lie bracket is called the Lie algebra.

Cross product of vectors. In the case of the three dimensional Euclidean vector space, the cross product of vectors is denoted by $\boldsymbol{u} \times \boldsymbol{v}$. In geometric interpretation, the area spanned by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by the formula $ab\sin\theta$, where a and b are the length of the two vectors and θ is the angle between them. This can be taken as motivation to define a vector area with the help of the cross product of vectors. In this sense the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is another vector,

whose length is $absin\theta$. This definition is confined to the three dimensional space and even then it requires a conventional choice of the direction, into which the product vector points.

Formally, the cross product of vectors is a binary internal operation $(\boldsymbol{u},\boldsymbol{v})\to\boldsymbol{u}\times\boldsymbol{v}$, which is completely determined by the properties of skew-symmetry, bilinearity, and Jacobi identity

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{0}. \tag{A.11}$$

The vector space E taken together with the operation of cross product of vectors is the Lie algebra.

In the case of the three-dimensional Euclidean space with base vectors we can associate the so-called permutation symbols, which are defined by

$$\epsilon_{ijk}=(\boldsymbol{g}_i\bullet\boldsymbol{g}_j)\times\boldsymbol{g}_k\,,\qquad \epsilon^{ijk}=(\boldsymbol{g}^i\bullet\boldsymbol{g}^j)\times\boldsymbol{g}^k\,. \tag{A.12}$$

The properties of the vector product of vectors imply that the permutation symbols are totally skew-symmetric with the cyclic property. The permutation symbols defined by (A.12) are related through important identities

$$\begin{aligned}\boldsymbol{g}_i\times\boldsymbol{g}_j&=\epsilon_{ijk}\boldsymbol{g}^k\,,\qquad \boldsymbol{g}^i\times\boldsymbol{g}^j=\epsilon^{ijk}\boldsymbol{g}_k\,,\\ \boldsymbol{g}_k&=\frac{1}{2}\epsilon_{ijk}\boldsymbol{g}^i\times\boldsymbol{g}^j\,,\qquad \boldsymbol{g}^k=\frac{1}{2}\epsilon^{ijk}\boldsymbol{g}_i\times\boldsymbol{g}_j\,.\end{aligned} \tag{A.13}$$

Tensors and linear maps. A map $\boldsymbol{A}:E\to F$ of a vector space E into a vector space F is said to be linear, if the following two axioms hold:

$$\boldsymbol{A}(\lambda\boldsymbol{u})=\lambda\boldsymbol{A}(\boldsymbol{u})\,,\qquad \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{A}(\boldsymbol{u})+\boldsymbol{A}(\boldsymbol{v})\,,\qquad \forall\,\lambda\in\mathbb{R}\,,\ \boldsymbol{u},\boldsymbol{v}\in E\,. \tag{A.14}$$

To emphasize the linearity one writes $\boldsymbol{A}\boldsymbol{u}\equiv\boldsymbol{A}(\boldsymbol{u})$ for the image of a vector $\boldsymbol{u}$ under a map $\boldsymbol{A}$. The set $L(E,F)$ of all linear maps of E into F becomes a vector space with the scalar multiplication and addition defined by

$$(\lambda\boldsymbol{A})\boldsymbol{u}=\lambda\boldsymbol{A}\boldsymbol{u}\,,\qquad (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{u}=\boldsymbol{A}\boldsymbol{u}+\boldsymbol{B}\boldsymbol{u}\,,\qquad \forall\,\boldsymbol{u}\in E\,, \tag{A.15}$$

for any two linear maps $\boldsymbol{A},\boldsymbol{B}\in L(E,F)$ and every real number $\lambda\in\mathbb{R}$. The zero linear map $\boldsymbol{0}\in L(E,F)$ is one, which maps every vector of E into the zero vector of F, i.e. $\boldsymbol{0}\boldsymbol{u}=\boldsymbol{0}$ for all $\boldsymbol{u}\in E$. Moreover, $dimL(E,F)=(dimE)(dimF)$.

Following the standard practice in continuum mechanics, we use the term tensor as a synonym of a linear map. Thus, given any two vector spaces E and F, we write $F \otimes E = L(E,F)$ for the vector space of all linear maps of E into F.

The two vector spaces E and F are called isomorphic, if there exists a bijective (one-to-one and onto) linear map $\boldsymbol{L}: E \to F$. The most important fact about isomorphic vector spaces is that they are in a sense indistinguishable.

Dual vector space. The vector space $E^* = L(E,\mathbb{R})$ of all linear maps of E into the set of real numbers $\mathbb{R}$ is called the (algebraic) dual of a given vector space E. The elements of E^* are called co-vectors or 1-forms.

If E is an inner product vector space, then for every co-vector $\boldsymbol{\theta} \in E^*$ there exists exactly one vector $\boldsymbol{t} \in E$ such that $\boldsymbol{\theta}(\boldsymbol{u}) = \boldsymbol{t} \cdot \boldsymbol{u}$, for all $\boldsymbol{u} \in E$. In this way the dual space E^* becomes canonically isomorphic with the spaces E itself and the two vector spaces can be identified. This fact also justifies the identification of the two vector spaces $F \otimes E$ and $L(E,F)$, which are canonically isomorphic.

Composition of tensors. Given any three vector spaces E, F and F', the composition of two tensors $\boldsymbol{A} \in F' \otimes F$ and $\boldsymbol{B} \in F \otimes E$ is a tensor $\boldsymbol{AB} \in F' \otimes E$ defined by (whenever domains and codomains coincide)

$$(\boldsymbol{AB})\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{Bu}), \qquad \forall\, \boldsymbol{u} \in E. \tag{A.16}$$

The composition of linear maps is distributive,

$$\boldsymbol{C}(\boldsymbol{B}+\boldsymbol{A}) = \boldsymbol{CB} + \boldsymbol{CA}, \qquad \forall\, \boldsymbol{A},\boldsymbol{B} \in F \otimes E,\ \boldsymbol{C} \in F' \otimes F, \tag{A.17}$$

and associative

$$\boldsymbol{C}(\boldsymbol{BA}) = (\boldsymbol{CB})\boldsymbol{A} = \boldsymbol{CBA}, \qquad \forall\, \boldsymbol{A} \in F \otimes E, \boldsymbol{B} \in F' \otimes F,\ \boldsymbol{C} \in F'' \otimes F'. \tag{A.18}$$

Moreover, for every real number $\lambda \in \mathbb{R}$,

$$\boldsymbol{B}(\lambda \boldsymbol{A}) = (\lambda \boldsymbol{B})\boldsymbol{A} = \lambda(\boldsymbol{BA}), \qquad \forall\, \boldsymbol{A} \in F \otimes E,\ \boldsymbol{B} \in F' \otimes F. \tag{A.19}$$

Transpose of a tensor. The transpose of a given tensor $\boldsymbol{A} \in F \otimes E$ is the unique tensor $\boldsymbol{A}^T \in E \otimes F$ such that

$$\boldsymbol{Au} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{A}^T\boldsymbol{v}, \qquad \forall\, \boldsymbol{u} \in E,\ \boldsymbol{v} \in F. \tag{A.20}$$

In effect, the transposition is a linear map $F \otimes E \to E \otimes F$, i.e.

$$(\lambda A + B)^T = \lambda A^T + B^T, \qquad \forall\, A, B \in F \otimes E,\ \lambda \in \mathbb{R}. \tag{A.21}$$

Moreover, we have

$$(AB)^T = B^T A^T, \qquad \forall\, A \in F' \otimes F,\ B \in F \otimes E. \tag{A.22}$$

Algebra of tensors. The vector space $E \otimes E$ of linear maps of a vector space E into itself, taken together with the composition of tensors, constitutes the associative but not commutative, in general, algebra. The unit tensor (identity map) $\mathbf{1} \in E \otimes E$ is defined by $\mathbf{1}u = u$ for all $u \in E$.

The trace assigns to every tensor $A \in E \otimes E$ a real number, denoted by trA, i.e. it maps the vector space $E \otimes E$ into the set of real numbers with the following properties

$$\begin{aligned} &trA^T = trA, \qquad tr(\lambda A) = \lambda trA, \qquad \forall\, A \in E \otimes E, \lambda \in \mathbb{R}, \\ &tr(AB) = tr(BA), \qquad tr(A+B) = trA + trB, \qquad \forall\, A, B \in E \otimes E. \end{aligned} \tag{A.23}$$

The determinant assigns to every tensor $A \in E \otimes E$ a real number, denoted by $detA$, with the following properties

$$detA^T = detA, \qquad det(\lambda AB) = \lambda^n detA\, detB, \qquad \forall\, A, B \in E \otimes E, \lambda \in \mathbb{R}, \tag{A.24}$$

where n is the dimension of the vector space E.

In the case of the vector space, for which the cross product of vectors is defined, the determinant and trace of any tensor can be defined by

$$\begin{aligned} (detA)\big((u \times v) \cdot w\big) &= (Au \times Av) \cdot Aw, \\ (trA)\big((u \times v) \cdot w\big) &= (Au \times v) \cdot w + (u \times Av) \cdot w + (u \times v) \cdot Aw, \end{aligned} \tag{A.25}$$

for all vectors. It must be noted that the above definitions relay on the cross product of vectors and they are valid only for the three-dimensional Euclidean vector space.

General linear group. A tensor $A \in F \otimes E$ is called invertible or nonsingular, if there exists a tensor $A^{-1} \in E \otimes F$, called the inverse of A, such that

$$AA^{-1} = \mathbf{1}_E, \qquad A^{-1}A = \mathbf{1}_F, \tag{A.26}$$

where $\mathbf{1}_E \in A(E)$ and $\mathbf{1}_F \in A(F)$ are the identity maps. The inverse tensor is unique whenever it exists, and if it exists the linear equation $Au = v$ has the unique solution $u = A^{-1}v$.

For invertible tensors the following rules hold:

$$\begin{aligned} &(A^T)^{-1} = (A^{-1})^T, && \forall\, A \in F \otimes E, \\ &(\lambda A)^{-1} = \lambda^{-1} A^{-1}, && \forall\, A, \in F \otimes E, \lambda \in \mathbb{R}, \lambda \neq 0, \\ &(BA)^{-1} = A^{-1} B^{-1}, && \forall\, A \in F \otimes E, B \in F' \otimes F. \end{aligned} \tag{A.27}$$

The set of all invertible tensors is closed with respect to the composition of tensors, and hence it forms a group, denoted by $GL(E)$ and called general linear group. It is known that $A \in A(E)$ is invertible if and only if $\det A \neq 0$.

$$\det(A^{-1}) = (\det A)^{-1}. \tag{A.28}$$

Inner product of tensors. For any two tensors $A, B \in F \otimes E$, we have $AB^T \in E \otimes E$ and $A^T B \in F \otimes F$, so that their traces are well defined and they are equal. This makes it possible to define the inner product of A and B by

$$A \bullet B = tr(A^T B) = tr(AB^T). \tag{A.29}$$

It can be verified that (A.29) defines the inner product indeed. This makes the vector space $F \otimes E$ into the inner product vector space. Directly from the definition of the inner product of tensors we can derive the following formulae, which are valid for all tensors:

$$\begin{aligned} &A \bullet B = A^T \bullet B^T, \\ &AB \bullet C = A \bullet CB^T = B \bullet A^T C, \\ &A \bullet BC = AC^T \bullet B = B^T A \bullet C. \end{aligned} \tag{A.30}$$

The inner product (A.29) induces the Euclidean norm of a tensor $A \in F \otimes E$ defined by $\|A\| = \sqrt{A \bullet A}$. The Euclidean norm must be distinguished from the operator norm $\|A\|_\infty$. For example, for the identity tensor we have $\|\mathbf{1}\| = \sqrt{n}$ and $\|\mathbf{1}\|_\infty = 1$. Nevertheless both norms are equivalent (this is true for all finite dimensional vector spaces). For all tensors and vectors the following properties can be proved

$$\begin{aligned} \|A^T\| &= \|A\|, \\ \|Au\| &\le \|A\|\|u\|, \\ \|AB\| &\le \|A\|\|B\|, \\ |A \cdot B| &\le \|A\|\|B\|. \end{aligned} \tag{A.31}$$

Tensor product of vectors. The tensor product of two vectors $a \in F$ and $b \in E$ is defined to be the tensor (i.e. the linear map) $a \otimes b \in F \otimes E$ such that

$$(a \otimes b)u = (b \cdot u)a, \qquad \forall u \in E. \tag{A.32}$$

Tensors of the form $a \otimes b$ are called simple or decomposable. For simple tensors we then have the classical rules:

$$\begin{aligned} &(a \otimes b)(c \otimes d) = (b \cdot c)a \otimes d, \qquad \forall a \in F',\ b,c \in F.\ d \in E, \\ &(a \otimes b) \cdot (c \otimes d) = (a \cdot c)(b \cdot d), \qquad \forall a,c \in F,\ b,d \in E \end{aligned} \tag{A.33}$$

and

$$(a \otimes b)^T = b \otimes a, \qquad \forall a \in F,\ b \in E, \tag{A.34}$$

for any three vector spaces E, F and F'. By the linearity of the tensor product of vectors, these rules extend to all tensors.

Tensor basis. Not every tensor can be represented as the tensor product of two vectors. For example the unit tensor is not a simple one. On the other hand, the simple tensors span the tensor product space. In particular, if $\{g_i\}$ and $\{h_a\}$ are any bases for the vector spaces E and F, respectively, then the following sets of simple tensors

$$\{h_a \otimes g_j\}, \quad \{h^a \otimes g^j\}, \quad \{h_a \otimes g^j\}, \quad \{h^a \otimes g_j\}, \tag{A.35}$$

form the basis for the space $F \otimes E$. Thus every tensor $T \in F \otimes E$ has the following component representations

$$\begin{aligned} T &= t^j \otimes g_j = T^{aj} h_a \otimes g_j = T_a^{\cdot j} h^a \otimes g_j \\ &= t_i \otimes g^j = T_{aj} h^a \otimes g^j = T_j^{\cdot a} h_a \otimes g^j, \end{aligned} \tag{A.36}$$

where

$$\begin{aligned} t^j &= \boldsymbol{T}g^j = T^{aj}h_a = T^{\cdot j}_{a}h^a, \qquad t_j = \boldsymbol{T}g_j = T_{aj}h^a = T^{a}_{\cdot j}h_a, \\ T^{aj} &= h^a \cdot t^j = h^a \cdot \boldsymbol{T}g^j, \qquad T_{aj} = h_a \cdot t_j = h_a \cdot \boldsymbol{T}g_j. \end{aligned} \tag{A.37}$$

The various operations on tensors like the transposition, composition and trace follow from the definitions for general inner product vector spaces.

Special tensors. A tensor $\boldsymbol{S} \in E \otimes E$ is said to be symmetric if $\boldsymbol{S} = \boldsymbol{S}^T$ and a tensor $\boldsymbol{W} \in E \otimes E$ is said to be skew-symmetric (skew in short) if $\boldsymbol{W}^T = -\boldsymbol{W}$. The symmetric and skew-symmetric tensors constitute the subspaces $E \odot E$ and $E \wedge E$, respectively, of the vector space $E \otimes E$. A tensor $\boldsymbol{Q} \in F \otimes E$ is orthogonal if

$$\boldsymbol{Q}u \cdot \boldsymbol{Q}v = u \cdot v, \qquad \forall\, u, v \in E. \tag{A.38}$$

An orthogonal tensor $\boldsymbol{Q} \in E \otimes E$ is said to be proper orthogonal or a rotation tensor, if $\det \boldsymbol{Q} = 1$. The rotation tensors form the Lie group, called the special orthogonal group (the rotation group), denoted by $SO(E)$ or $SO(n)$, where $n = \dim E$.

Let vectors $u \in F$ and $v \in E$ be given. Then we have

$$\begin{aligned} \boldsymbol{A}u \otimes v &= \boldsymbol{A}(u \otimes v), \qquad \forall\, \boldsymbol{A} \in F \otimes F, \\ u \otimes \boldsymbol{B}v &= (u \otimes v)\boldsymbol{B}^T, \qquad \forall\, \boldsymbol{B} \in E \otimes E, \end{aligned} \tag{A.39}$$

and

$$\boldsymbol{A} \cdot (u \otimes v) = \boldsymbol{A}^T u \cdot v = u \cdot \boldsymbol{A}v, \qquad \forall\, \boldsymbol{A} \in F \otimes E. \tag{A.40}$$

Moreover, if two vectors u and v belong to the same space, then we have

$$det(\boldsymbol{1} + u \otimes v) = 1 + u \cdot v. \tag{A.41}$$

If in addition $u \cdot v \neq -1$, then the tensor $\boldsymbol{1} + u \otimes v$ is invertible and its inverse is given by

$$(\boldsymbol{1} + u \otimes v)^{-1} = \boldsymbol{1} - (1 + u \cdot v)^{-1} u \otimes v. \tag{A.42}$$

In the three-dimensional case, if $\boldsymbol{T}$ is a non-singular tensor, the following formula is valid

$$\boldsymbol{T}u \times \boldsymbol{T}v = (det\, \boldsymbol{T})(\boldsymbol{T}^{-1})^T (u \times v), \qquad \forall\, u, v \in E. \tag{A.43}$$

Appendix B

Skew-symmetric and rotation tensors

Lie algebra of skew-symmetric tensors. In the remaining part of the appendix (like throughout the main part of this work) E will always denote the three-dimensional Euclidean vector space, and we denote by $\boldsymbol{u}\times\boldsymbol{v}$ the usual cross product of two vectors.

Let us recall that a tensor $\boldsymbol{W}\in E\otimes E$ is said to be skew-symmetric (skew, in short) if

$$\boldsymbol{W}\boldsymbol{u}\cdot\boldsymbol{v}=-\boldsymbol{u}\cdot\boldsymbol{W}\boldsymbol{v}\,,\qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in E\,. \tag{B.1}$$

By direct implication we have

$$tr\boldsymbol{W}=0\,,\qquad \det\boldsymbol{W}=0\,,\qquad \|\boldsymbol{W}\|^2=-tr(\boldsymbol{W}^2)\,, \tag{B.2}$$

for any skew tensor. It then follows that a tensor $\boldsymbol{W}$ is skew, if and only if $\boldsymbol{W}^T=-\boldsymbol{W}$. Accordingly, we define the vector space of all skew tensors by

$$E\wedge E=\left\{\boldsymbol{W}\in E\otimes E\mid \boldsymbol{W}+\boldsymbol{W}^T=\boldsymbol{0}\right\}. \tag{B.3}$$

For any two skew tensors $\boldsymbol{V}$ and $\boldsymbol{W}$, the commutator $[\boldsymbol{V},\boldsymbol{W}]=\boldsymbol{V}\boldsymbol{W}-\boldsymbol{W}\boldsymbol{V}$ is necessarily a skew tensor. Accordingly, $E\wedge E$ is also an example of the Lie algebra with the commutator taken as the Lie bracket.

Let us also recall that the exterior product of any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is the skew tensor $\boldsymbol{v}\wedge\boldsymbol{w}$ defined by

$$\boldsymbol{v}\wedge\boldsymbol{w}=\boldsymbol{v}\otimes\boldsymbol{w}-\boldsymbol{w}\otimes\boldsymbol{v}\,. \tag{B.4}$$

Axial vector of a skew-symmetric tensor. The vector space E taken together with the cross product as the Lie bracket, $[\boldsymbol{u},\boldsymbol{v}]=\boldsymbol{u}\times\boldsymbol{v}$ for all vectors $\boldsymbol{u},\boldsymbol{v}\in E$, is the Lie algebra, which is isomorphic with the Lie algebra $E\wedge E$ of skew tensors. However, it must be stressed here that the dimension three is the basic fact in this case.

For every vector $w \in E$ there exists a unique skew tensor $W \in E \wedge E$, such that

$$Wu = w \times u, \qquad \forall u \in E. \tag{B.5}$$

Equivalently, the skew tensor W associated with a given vector w can be defined by

$$W = \frac{\partial (w \times u)}{\partial u}, \qquad \forall u \in E. \tag{B.6}$$

The vector w is then called the axial vector of the skew tensor W. The converse assertion is also true, i.e. for every skew tensor W there exists only one vector w such that (B.5), equivalently (B.6), holds. In fact, the map

$$ad : E \to E \wedge E, \qquad w \to W = adw, \tag{B.7}$$

which assigns to each vector the associated skew tensor is linear and it preserves the Lie bracket, i.e. for all skew tensors and the associated axial vectors we have

$$\begin{aligned} ad(\lambda v + w) &= \lambda V + W, \\ ad(v \times w) &= [V, W] = -v \wedge w. \end{aligned} \tag{B.8}$$

Moreover, the correspondence between vectors and skew tensors is invertible, i.e. the map (B.7) is invertible. These properties can easily be derived using standard vector identities. Thus the two vector spaces E and $E \wedge E$ are isomorphic not only as the vector spaces but also as the Lie algebras.

For any two vectors v and w and the associated skew tensors V and W we have

$$\begin{aligned} VW &= v \otimes w - (v \cdot w)\mathbf{1}, \\ Vu \times Wu' &= (v \otimes w)(u \times u'). \end{aligned} \tag{B.9}$$

Note that VW need not be a skew tensor even though V and W are. For any skew tensors we have

$$tr(VW) = -V \cdot W = -2(v \cdot w). \tag{B.10}$$

Moreover, for every nonsingular tensor T the following holds

$$ad(Tw) = (\det T)(T^{-1})^T W T^{-1}. \tag{B.11}$$

Basis. Given any mutually reciprocal bases $\{\boldsymbol{g}_i\}$ and $\{\boldsymbol{g}^i\}$ for the vector space E, each of the two families of skew tensors

$$\boldsymbol{G}_k = ad\,\boldsymbol{g}_k = -\epsilon_{ijk}\boldsymbol{g}^i \wedge \boldsymbol{g}^j\,, \qquad \boldsymbol{G}^k = ad\,\boldsymbol{g}^k = -\epsilon^{ijk}\boldsymbol{g}_i \wedge \boldsymbol{g}_j\,, \tag{B.12}$$

are linearly independent and they also form the basis for the space $E \wedge E$. Moreover, from the properties (A.13) and (B.8) we have

$$2\boldsymbol{G}^i \bullet \boldsymbol{G}_k = \delta^i_j\,, \qquad \boldsymbol{G}^k = \epsilon_{ijk}[\boldsymbol{G}^i, \boldsymbol{G}^j]\,, \qquad [\boldsymbol{G}_i, \boldsymbol{G}_j] = \epsilon_{ijk}\boldsymbol{G}^k\,, \tag{B.13}$$

together with the associated relations that can be obtained by rising and lowering the indices.

Given a vector and the associated skew tensor we have

$$\begin{aligned} &\boldsymbol{w} = w_k \boldsymbol{g}^k\,, \qquad \boldsymbol{W} = ad\,\boldsymbol{w} = W_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j\,, \\ &w_k = -\,\varepsilon_{ijk}W^{ij}\,, \qquad W_{ij} = -\,\varepsilon_{ijk}w^k\,, \end{aligned} \tag{B.14}$$

and in matrix notation

$$[W^{ij}] = \frac{1}{\sqrt{g}}\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \tag{B.15}$$

Rotation tensors. A tensor as a linear map of an inner product vector space E into itself is called orthogonal, if it preserves the inner product of vectors and hence the Euclidean norm of a vector:

$$\boldsymbol{Qu} \bullet \boldsymbol{Qv} = \boldsymbol{u} \bullet \boldsymbol{v}\,, \qquad \|\boldsymbol{Qu}\| = \|\boldsymbol{u}\|, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in E\,. \tag{B.16}$$

The second condition is obviously the implication of the first one. From the definition of the transposition of a tensor we have

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{1} \qquad \Leftrightarrow \qquad \boldsymbol{Q}^T = \boldsymbol{Q}^{-1}\,. \tag{B.17}$$

The set of all orthogonal tensors forms an orthogonal group $O(3)$ with the composition as the group operation.

A proper orthogonal tensor is a linear isometric map preserving the orientation of the vector space, i.e. a linear map satisfying the following two axioms:

$$\boldsymbol{Ru}\cdot\boldsymbol{Rv}=\boldsymbol{u}\cdot\boldsymbol{v}\,,\qquad \boldsymbol{Ru}\cdot(\boldsymbol{Rv}\times\boldsymbol{Rw})=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})\,,\qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in E\,. \quad \text{(B.18)}$$

The first condition implies that proper orthogonal tensors preserve the inner product and hence the norm of vectors. The second condition ensures that proper orthogonal tensors preserve the orientation.

It is clear that proper orthogonal tensors are non-singular and hence invertible. The inverse of a proper orthogonal tensor is again a proper orthogonal tensor and the unit tensor $\mathbf{1}$ is proper orthogonal in obvious way. Thus the set

$$SO(3)=\left\{\boldsymbol{Q}\in E\otimes E\,|\,\boldsymbol{QQ}^T=\boldsymbol{Q}^T\boldsymbol{Q}=\mathbf{1},\ det\boldsymbol{Q}=1\right\} \quad \text{(B.19)}$$

of all proper orthogonal tensors together with the compositon of tensors forms the group, called special orthogonal group. The group is non-commutative (non-Abelian). Proper orthogonal tensors are also called rotation tensors or simply rotations, and $SO(3)$ is sometimes called the rotation group. This group is a connected subgroup of the group $O(3)$ of orthogonal tensors.

The rotation group $SO(3)$ is a three-dimensional Lie group. Hence, for any smooth curve on the rotation group, the derivative $\dot{\boldsymbol{Q}}(t)$ is an element of the tangent space to $SO(3)$ at the "point" $\boldsymbol{Q}(t)$, that is an element of the vector space

$$T_{\boldsymbol{Q}}SO(3)=\left\{\,\boldsymbol{V}\in E\otimes E\;|\;\boldsymbol{VQ}^T+\boldsymbol{QV}^T=\mathbf{0}\right\}. \quad \text{(B.20)}$$

The tangent space (B.20) is isomorphic in two ways with the tangent space at the identity, which is the vector space of skew-symmetric tensors:

$$T_{\mathbf{1}}SO(3)=E\wedge E=\left\{\,\boldsymbol{W}\in E\otimes E\;|\;\boldsymbol{W}+\boldsymbol{W}^T=\mathbf{0}\right\}. \quad \text{(B.21)}$$

The two isomorphisms of the vector spaces (B.20) and (B.21) are determined by the left and right translation on the group defined by

$$\begin{aligned} &L_{\boldsymbol{Q}}:SO(3)\to SO(3)\,, && \boldsymbol{R}\to L_{\boldsymbol{Q}}(\boldsymbol{R})=\boldsymbol{QR}\,,\\ &R_{\boldsymbol{Q}}:SO(3)\to SO(3)\,, && \boldsymbol{R}\to R_{\boldsymbol{Q}}(\boldsymbol{R})=\boldsymbol{RQ}\,. \end{aligned} \quad \text{(B.22)}$$

These maps are differentiable and the tangent maps are

$$\begin{aligned} &T_{\boldsymbol{R}}L_{\boldsymbol{Q}}:T_{\boldsymbol{R}}SO(3)\to T_{L_{\boldsymbol{Q}}(\boldsymbol{R})}SO(3)\,, && \boldsymbol{V}\to T_{\boldsymbol{R}}L_{\boldsymbol{Q}}(\boldsymbol{V})=\boldsymbol{QV}\,,\\ &T_{\boldsymbol{R}}R_{\boldsymbol{Q}}:T_{\boldsymbol{R}}SO(3)\to T_{R_{\boldsymbol{Q}}(\boldsymbol{R})}SO(3)\,, && \boldsymbol{V}\to T_{\boldsymbol{R}}R_{\boldsymbol{Q}}(\boldsymbol{V})=\boldsymbol{VQ}\,. \end{aligned} \quad \text{(B.23)}$$

Using these well known facts we easily find that the derivative of a curve $\boldsymbol{Q}:\mathbb{R}\to SO(3)$ is given by

$$\dot{\boldsymbol{Q}}(t)=\boldsymbol{W}(t)\boldsymbol{Q}(t)=\boldsymbol{Q}(t)\mathbf{W}(t)\,, \tag{B.24}$$

where the two tensors

$$\boldsymbol{W}=\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\,, \qquad \mathbf{W}=\boldsymbol{Q}^T\dot{\boldsymbol{Q}}\,, \tag{B.25}$$

are necessarily skew-symmetric. We shall denote by

$$\boldsymbol{w}=ad^{-1}\boldsymbol{W}\,, \qquad \mathbf{w}=ad^{-1}\mathbf{W}\,, \tag{B.26}$$

their axial vectors. These two vectors, equivalently the skew tensors (B.25), represent the angular virtual motion or the angular velocity fields. This terminology is justified by the dynamics of the rigid body motion. As simple implications we have the following relations

$$\boldsymbol{W}=\boldsymbol{Q}\mathbf{W}\boldsymbol{Q}^T\,, \qquad \boldsymbol{w}=\boldsymbol{Q}\mathbf{w}\,. \tag{B.27}$$

The first of these relations is a direct implication of (B.25), the second one follows at from the property (B.11) of the map *ad*.

Appendix C

Mappings and fields

Euclidean point space. In this appendix we shall be concerned with fields on subsets of the n-dimensional Euclidean point space $\mathcal{E}^n$. By its very definition, $\mathcal{E}^n$ is the affine space modelled on the n-dimensional Euclidean vector space E^n. The elements of $\mathcal{E}^n$, called points or places, will be denoted by $\boldsymbol{x}$, $\boldsymbol{z}$, etc. The elements of E^n, called vectors, are denoted by $\boldsymbol{u}$, $\boldsymbol{v}$, etc. Mathematically, the spaces $\mathcal{E}^n$ and E^n, as well as the real space $\mathbb{R}^n$ with its natural vector space structure, are in a sense the same. They are isomorphic. However, $\mathcal{E}^n$ is not merely a fixed copy of E^n or $\mathbb{R}^n$.

Let $\boldsymbol{o} \in \mathcal{E}^n$ be a fixed point. Then by virtue of the definition of the affine space every other point $\boldsymbol{x} \in \mathcal{E}^n$ can be represented in the form $\boldsymbol{x} = \boldsymbol{o} + \boldsymbol{r}$, where the unique vector $\boldsymbol{r} \in E^n$ is called the position vector of $\boldsymbol{x}$ relative to a point $\boldsymbol{o}$. Given any two points $\boldsymbol{o}$ and $\boldsymbol{x}$ in the space $\mathcal{E}^n$, we can draw an arrow which begins at $\boldsymbol{o}$ and ends at $\boldsymbol{x}$. This arrow represents a vector $\boldsymbol{r}$ which is denoted by $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{o}$. Moreover, let further $\{\boldsymbol{e}_k\}$ be an orthonormal basis for E^n. Then we may write (here and in the sequeal, $k = 1,2,\ldots,n$, and the usual summation convention applies)

$$\boldsymbol{x} = \boldsymbol{o} + \boldsymbol{r} = \boldsymbol{o} + x_k \boldsymbol{e}_k . \tag{C.1}$$

The n-triple of real numbers $(x_k) \in \mathbb{R}^n$ are then called the Cartesian coordinates of the place $\boldsymbol{x}$. Since the representation (C.1) of a point $\boldsymbol{x} \in \mathcal{E}^n$ is unique, we have the sequence isomorphisms:

$$\mathcal{E}^n \xrightarrow{\text{fixed point}} E^n \xrightarrow{\text{fixed basis}} \mathbb{R}^n , \qquad \boldsymbol{x} \to \boldsymbol{r} \to (x_k) . \tag{C.2}$$

However, the first isomorphism depends on the choice of a point in $\mathcal{E}^n$. The vector space E^n has a distinguished element, the zero vector. In contrast, the space $\mathcal{E}^n$ has no naturally distinguished point. The second isomorphism depends on a choice of the basis for E^n. Hence they are not canonical. Accordingly, whenever a particular choice of a point in the Euclidean point space is important for some reasons, the identification of $\mathcal{E}^n$ and E^n cannot be made. On the other hand, once a

point in $\mathcal{E}^n$ has been fixed and we have chosen a basis for E^n, then such identification is legitimate.

Topology of the Euclidean space. The inner product on the translation space E together with the induced norm makes it possible to define the distance between points of the space $\mathcal{E}$. The distance $d(x,z)$ between two point x and z is defined to be the length of the vector that translates one point into another, $d(x,z)=\|z-x\|$. The distance function (metric) d makes $\mathcal{E}$ into a metric space:

$$space\ \mathcal{E} \xrightarrow{metric\ \rho} metric\ space\ (\mathcal{E},d).$$

In turn, the metric gives rise to a topology on $\mathcal{E}$ in a natural way and $\mathcal{E}$ becomes the topological space:

$$metric\ space\ (\mathcal{E},d) \xrightarrow{induced\ topolog\ \mathcal{T}} topological\ space\ (\mathcal{E},\mathcal{T}).$$

A topology on $\mathcal{E}$ consists of a family $\mathcal{T}$ of subsets of $\mathcal{E}$, called open sets, satisfying certain axioms. The notion of a topology gives sense to the intuitive ideas of nearness and continuity. Fundamental concepts of a topology are open and closed sets and the interior, closure and boundary of a set. Referring to the mathematical literature[1] for all details, here we only give a very short review of those concepts which are needed in the subsequent considerations.

Closed and open sets. An open ball with center at a point $x\in\mathcal{E}$ and radius $\delta>0$ is the set

$$B_\delta(x)=\{z\in\mathcal{E}\,|\,d(z,x)<\delta\}. \tag{C.3}$$

A set $A\subset\mathcal{E}$ is open, if for every point $x\in A$ there exists a ball $B_\delta(x)$ such that $B_\delta(x)\subset A$. Clearly, an open ball is an open set. A point $x\in A$ is said to be an interior point of a set $A\subset\mathcal{E}$, if there exists an open ball $B_\delta(x)$ such that $B_\delta(x)\subset A$. The collection of all interior points of a set $A\subset\mathcal{E}$ is called its interior and its is denoted by $intA$. It can be shown that the interior of a given set A is the largest open set contained in A. Moreover, it is easy to see that a set $A\subset\mathcal{E}$ is open if and only if $intA=A$. A set $A\subset\mathcal{E}$ is said to be closed if its complement $\mathcal{E}\setminus A$ in $\mathcal{E}$ is open. Given a set $A\subset\mathcal{E}$, we write $cl\Omega$ for its closure, which is a largest closed set containing A. Clearly there are subsets of $\mathcal{E}$ which are neither open nor closed (see Fig. 1).

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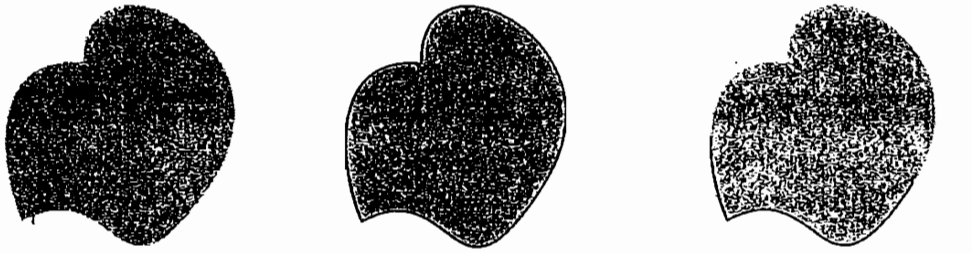
[1] R. ENGELKING [1989].

subsets of the two-dimensional Euclidean space (plane)



open *closed* *neither open nor closed*

Fig. 1

The collection of all open sets is called the topology induced by the metric $d(x,z)=\|z-x\|$. The topology turns out to be independent of the choice of the metric. This fact is a direct consequence of the equivalence of the norms on the finite dimensional vector spaces. The Euclidean topology has many important properties. In particular, it is Hausdorff and it has a countable basis.

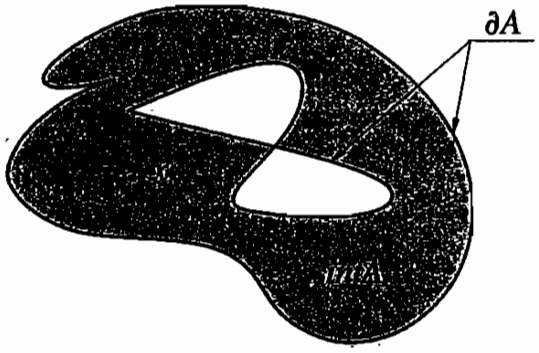


Fig. 2

A set-topological boundary (a set-topological boundary need to be distinguished from the measure theoretic boundary of a given set) ∂A of a given set $A \subset \mathcal{E}$ is defined by (Fig. 2)

$$\partial A = clA \cap cl(\mathcal{E} \setminus A). \tag{C.4}$$

Sequences. The notion of distance (metric) can be used to define limits, convergence, continuity, etc. in the customary manner. For example (Fig. 3), we say that a series $\{x_n,1,2,...\}$ of points converges to a point $x \in \mathcal{E}$ if

$\lim_{n\to\infty} d(x_n, x) = 0$. While the metric depends on the norm of the translation space E, the convergence is independent of the particular norm. A set is closed, if it contains the limits of all convergent sequences.

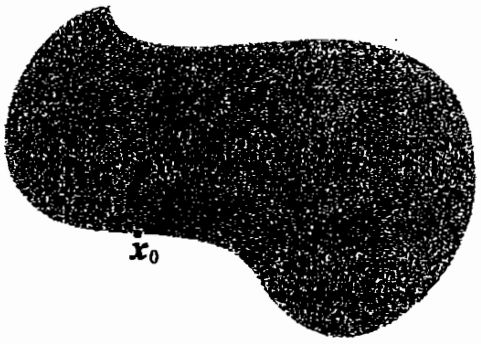


Fig. 3

Connected sets. A subset $\Omega \subset \mathcal{E}$ is connected if any two points in Ω can be connected by a curve in Ω. A subset $\Omega \subset \mathcal{E}$ is simply connected if any closed curve in Ω can be continuously deformed to a point without leaving Ω (see Fig. 4). A domain is an open connected (but not necessarily simply connected) set.

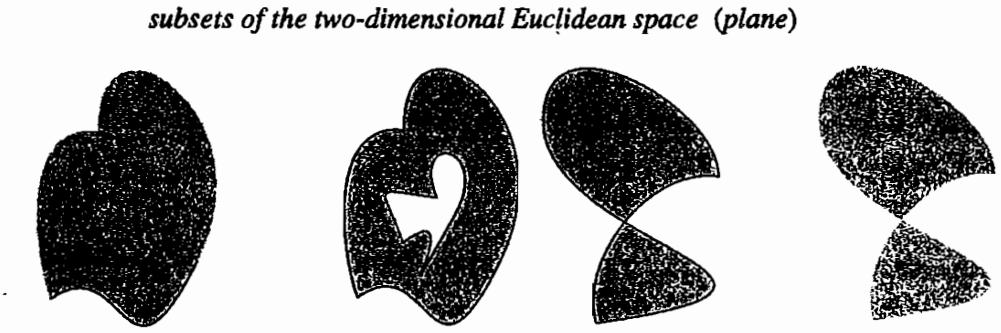


Fig. 4

We shall also need to consider bounded domains with the so-called Lipschitz boundary. The definition of this concept is rather complicated. Therefore, referring to the literature, we only note that to the class of domains with Lipschitz boundary

belong domains, whose boundary is smooth or piecewise smooth and has no singularities such as cuspidal edges, etc.

Mappings between Euclidean spaces. Let $\mathcal{E}$ and $\mathcal{F}$ be two Euclidean point spaces modelled on the Euclidean vector space E and F, respectively. Let $\Omega \subset \mathcal{E}$ be an open set. A mapping $\boldsymbol{\phi}: \Omega \to \mathcal{F}$ is a rule, which assigns to each point $\boldsymbol{x} \in \Omega$ an element $\boldsymbol{\phi}(\boldsymbol{x}) \in \mathcal{F}$.

Let us note that every Euclidean vector space is an Euclidean point space modelled on itself. Thus, we use the term field for a map $\boldsymbol{\phi}: \Omega \to F$ having a domain $\Omega \subset \mathcal{E}$ in the Euclidean point space and a range in a finite-dimensional inner product vectors space F. A scalar field $\phi: \Omega \to \mathbb{R}$ is an important special case with $F = \mathbb{R}$ taken to be the set of real numbers (which is the one-dimensional vector space). A vector field in the classical sense means a map $\boldsymbol{u}: \Omega \to E$, where E denotes the translation space of $\mathcal{E}$. More general, a tensor field of p-order is again a special case with $F = \otimes^p E$ taken to be the tensor product of p-copies of the translation space E. Still another possible choice of the space F is the direct sum $\oplus^p E$ of p-copies of the space E.

We shall be concerned with the differentiation of mappings and fields having as their domains open and connected, but not necessarily simply connected, subsets of the space $\mathcal{E}$. Accordingly, for the remaining part of this appendix, $\Omega \subset \mathcal{E}$ will always denote such subset.

Continuous mappings. A mapping $\boldsymbol{\phi}: \Omega \to \mathcal{F}$ is said to be continuous at a point $\boldsymbol{x} \in \Omega$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $\boldsymbol{z} \in \Omega$ the following condition holds

$$\|\boldsymbol{z} - \boldsymbol{x}\| < \delta \quad \Rightarrow \quad \|\boldsymbol{\phi}(\boldsymbol{z}) - \boldsymbol{\phi}(\boldsymbol{x})\| < \varepsilon . \tag{C.5}$$

Though the two norms in (C.5) may be different, being defined on different spaces E and F, we commonly use the same symbol, letting the context indicate which norm is meant. In the case of finite-dimensional vector spaces, with which we are concerned here, all norms are topological equivalent. Thus the continuity of a given field is norm-independent.

A mapping $\boldsymbol{\phi}: \Omega \to \mathcal{F}$ is said to be continuous on Ω, if it is continuous at each point $\boldsymbol{x} \in \Omega$. We shall denote by $C^0(\Omega, \mathcal{F})$ the set of all continuous fields having Ω as the common domain and taking their values in the space $\mathcal{F}$.

A mapping $\phi : \Omega \to \mathcal{F}$ is homeomorphic, if it is bijective (one-to-one and onto), continuous, and if its inverse is continuous.

Differentiable mappings. The concept of a differentiation can be developed at various levels of generality. The following definition will overcome the lack of many desirable properties of weaker differentiability requirements: A mapping $\phi : \mathcal{E} \to \mathcal{F}$ is said to be differentiable at a point $x \in \Omega$, if there exists a linear map (tensor) $\nabla\phi(x) \in F \otimes E$ such that

$$\phi(x+u) = \phi(x) + \nabla\phi(x)u + o(u), \tag{C.6}$$

for every vectors $u \in E$. If $\nabla\phi(x)$ exists, it is unique and it is called the gradient of a mapping ϕ at the point x.

The definition (C.6) allows most of the usual properties of derivatives in one-dimension to be carried out over the general case considered here. For example, whenever the gradient exists, it can be computed according to the rule

$$\nabla\phi(x)u = \frac{d}{d\lambda}\phi(x+\lambda u)_{|\lambda=0}. \tag{C.7}$$

Another way to compute the gradient of a differentiable field is the following one: let $x(t)$, $t \in I \subset \mathbb{R}$, be a smooth parametrized curve in the region Ω. Then

$$\dot{\phi}(x(t)) = \nabla\phi(x(t))\dot{x}(t). \tag{C.8}$$

Many important properties of a differentiable field can be derived as consequences of the following theorem: If a field $\phi : \Omega \to \mathcal{F}$ is differentiable at a point $x \in \Omega$, then it is continuous at x. More precisely, there exist constants $\delta > 0$ and $c \geq 0$ such that

$$\|u\| \leq \delta \quad \Rightarrow \quad \|\phi(x+u) - \phi(x)\| \leq c\|u\|. \tag{C.9}$$

The gradient of a given scalar field $\phi : \Omega \to \mathbb{R}$ is the unique vector field $\nabla\phi$, in which case the formula (C.8) takes the form

$$\dot{\phi}(x(t)) = \nabla\phi(x(t)) \cdot \dot{x}(t). \tag{C.10}$$

The gradient of the position vector is the unit tensor, $\nabla x = \mathbf{1}$.

Higher gradients. A field $\phi : \Omega \to \mathcal{F}$ is said to be differentiable on Ω, if the gradient $\nabla\phi(x)$ exists at each point of Ω. If ϕ is differentiable on Ω, then its gradient is the field

$$\nabla\phi : \Omega \to F \otimes E, \qquad x \to \nabla\phi(x). \tag{C.11}$$

We say that a field ϕ is of class C^1 on Ω, if it is differentiable at each point of Ω and the mapping $\nabla\phi(x) : \Omega \to F \otimes E$ is continuous on Ω.

Since $F \otimes E \cong L(E,F)$ is again a finite-dimensional inner product vector space, we may apply the definition of differentiation to (C.11) in order to obtain the second gradient of ϕ. Continuing in this manner, we denote by

$$\nabla^{(n)}\phi = \nabla(\nabla^{(n-1)}\phi), \qquad n = 2,3,..., \tag{C.12}$$

the n-th gradient (if it exists) of a given field $\phi : \Omega \to \mathcal{F}$. If the n-th gradient of a field $\phi : \Omega \to \mathcal{F}$ exists, it is a field of the type

$$\nabla^{(n)}\phi : \Omega \to F \otimes \underbrace{E \otimes ... \otimes E}_{n-copies} \cong L(\underbrace{E \otimes ... \otimes E}_{n-copies}, F). \tag{C.13}$$

A field $\phi : \Omega \to \mathcal{F}$ is said to be of class C^n, if it is differentiable at every point of Ω and its n-th gradient is continuous on Ω. We shall denote by $C^n(\Omega, F)$ the set of all fields of class C^n. The fields of class C^0 are just continuous fields on Ω.

The basic theorems in the differential calculus assert that for any two fields $\phi, \psi \in C^n(\Omega, \mathcal{F})$ of class $n \geq 0$ and every real number $\lambda \in \mathbb{R}$, their linear combination $\lambda\phi + \psi$ is a field of the same class and

$$\nabla^{(k)}(\lambda\phi + \psi) = \lambda\nabla^{(k)}\phi + \nabla^{(k)}\psi, \qquad k = 0,1,2,...n. \tag{C.14}$$

An important property of the higher gradients of differentiable fields is their symmetry: If a field $\phi : \Omega \to \mathcal{F}$ is twice differentiable at a point $x \in \Omega$, then

$$\nabla^{(2)}\phi(x)[u \wedge v] = 0, \qquad \forall\, u, v \in E. \tag{C.15}$$

Analogously, if a field $\phi : \Omega \to \mathcal{F}$ is n-times differentiable at a point $x \in \Omega$, then

$$\nabla^{(k)}\phi(x)[u_{\sigma(1)} \wedge ... \wedge u_{\sigma(k)}] = 0, \qquad \forall\, u_1, ... u_k \in E, \tag{C.16}$$

for every permutation $\sigma \in \Sigma_k$, where Σ_k denotes the permutation group.

Coordinate systems. The space $\mathscr{E}$ or any open subset of $\mathscr{E}$ may be parametrized by a curvilinear coordinate system of which the Cartesian coordinate system is just a special case. Formally, a coordinate system on an open set $\Omega \subset \mathscr{E}$ is a diffeomorphism of Ω onto an open set in the space $\mathbb{R}^n$:

$$\varphi : \Omega \to U = \varphi(\Omega) \subset \mathbb{R}^n , \qquad x \to (\xi^i) = \varphi(x). \tag{C.17}$$

The n-tuple of real numbers (ξ^i) is then called the coordinates of the point $x \in \Omega$. A particular coordinate system can be determined by specifying functions of Cartesian coordinates which have unique inverses. Then the position vector of any point in the domain of the given coordinates can be expressed as $x(\xi^i) = x_k(\xi^i)e_k$. Then at every point x one defines the natural base vectors and the reciprocal base vectors (Fig. 5):

$$g_i(x) = x_{,i} , \qquad g^i(x) \cdot g_j(x) = \delta^i_j . \tag{C.18}$$

Here a comma stands for partial derivative with respect to the coordinates ξ^i. The geometric meaning of the natural base vectors is simple: these are the vectors tangent to the curvilinear coordinate lines and they vary from point to point.

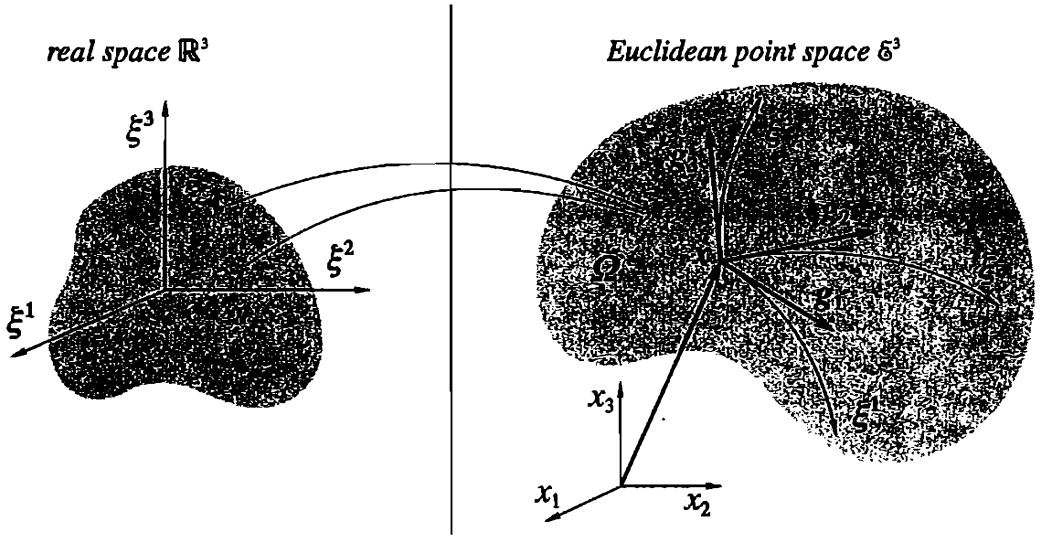


Fig. 5

The natural base vectors defined by (C.18) are necessarily linearly independent. Accordingly, they span at each point $x \in \mathscr{E}$ the three-dimensional vector space,

denoted by $T_x\mathcal{E}$ and called the tangent space. The underlying Euclidean parallelism of the ambient space allows to identify this space with the translation space . In effect, $T_x\mathcal{E}$ is just the space E attached to the underlying point.

Metric tensor. From (C.6) the differential line element and its length are obtained in the form $d\boldsymbol{x} = \boldsymbol{g}_i d\xi^i$ and $ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = g_{ij} d\xi^i d\xi^j$, where the components of the metric tensor are defined in the standard manner

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j , \qquad g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j . \tag{C.19}$$

In order to ensure that the underlying transformation of the coordinates is non-singular and orientation preserving we require that $g \equiv \det g_{ij} > 0$. All other relevant relations follow from the above relations with the standard operation of rising and lowering indices.

Partial derivatives. In a given coordinate system in a domain Ω, a mapping $\boldsymbol{\phi} : \Omega \to \mathcal{F}$ may be considered as a function of the curvilinear coordinates, $\boldsymbol{\phi}(\xi^i) = \boldsymbol{\phi}(\boldsymbol{x}(\xi^i))$. If $\boldsymbol{\phi}$ is differentiable at a point $\boldsymbol{x} \in \Omega$, we may define partial derivatives in the direction of the natural base vectors:

$$\partial_i \boldsymbol{\phi}(\boldsymbol{x}) = \big(\nabla \boldsymbol{\phi}(\boldsymbol{x})\big)\boldsymbol{g}_i(\boldsymbol{x}) = \frac{\partial \boldsymbol{\phi}}{\partial \xi^i}(\boldsymbol{x}) . \tag{C.20}$$

It is of importance to note that the existence of all partial derivatives does not imply that $\boldsymbol{\phi}$ is differentiable. However, if partial derivatives exist and are continuous at a point $\boldsymbol{x}$, then the field $\boldsymbol{\phi}$ is differentiable at this point and the gradient can be computed pointwise according to the rule

$$\nabla \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\phi}_{,i}(\boldsymbol{x}) \otimes \boldsymbol{g}^i(\boldsymbol{x}) , \qquad \boldsymbol{\phi}_{,i}(\boldsymbol{x}) \equiv \frac{\partial \boldsymbol{\phi}(\boldsymbol{\varphi}(\xi))}{\partial \xi^i} . \tag{C.21}$$

The partial derivatives obey the usual rule of differentiation, in particular, the Leibnitz rule.

Christoffel symbols. Assuming that the position vector $\boldsymbol{x}(\xi^i)$ is twice differentiable with respect to the curvilinear coordinates, the partial derivatives of the natural base vectors can be expressed as linear combination of the vectors themself:

$$\boldsymbol{g}_{i,j} = \Gamma_{kij} \boldsymbol{g}^k = \Gamma^k_{ij} \boldsymbol{g}_k , \qquad \boldsymbol{g}^i{}_{,j} = -\Gamma^i_{jk} \boldsymbol{g}^k . \tag{C.22}$$

The coefficients in these expressions are called the Christoffel symbols of the first and second kind, respectively. They are defined by

$$\Gamma_{kij} = \boldsymbol{g}_k \cdot \boldsymbol{g}_{i,j} = \boldsymbol{x}_{,k} \cdot \boldsymbol{x}_{,ij}\,, \qquad \Gamma^k_{ij} = -\boldsymbol{g}^k \cdot \boldsymbol{g}_{i,j} = -g^{kl}\boldsymbol{x}_{,l} \cdot \boldsymbol{x}_{,ij}\,. \tag{C.23}$$

Since the order of partial differentiation satisfies the rule $\boldsymbol{x}_{,ij} = \boldsymbol{x}_{,ji}$, a suitable combination of these relations makes it possible to express the Christoffel symbols in terms of the partial derivatives of the components of the metric tensor

$$\Gamma_{kij} = \frac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k})\,, \qquad \Gamma^k_{ij} = g^{kl}\Gamma_{lij}\,. \tag{C.24}$$

Moreover, it can be shown that

$$\Gamma^k_{kj} = g^{kl}\Gamma_{klj} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\xi^j}\,. \tag{C.25}$$

It is worth noting that the Christoffel symbols are not the components of a tensor. If they were, such a tensor would vanish in Cartesian coordinates, since the base vectors are constant. This is a contradiction, since a tensor is independent of its component representation.

Divergence of a tensor field. In classical context the divergence of a differentiable vector field $\boldsymbol{u}: \Omega \to E$ is defined pointwise to be the unique scalar $Div\boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}$ such that

$$Div\boldsymbol{u}(\boldsymbol{x}) = tr(\nabla\boldsymbol{u}(\boldsymbol{x}))\,. \tag{C.26}$$

With the help of (C.26) one then defines the divergence of a differentiable tensor field $\boldsymbol{T}: \Omega \to E \otimes E$ to be the vector field, which is characterized pointwise by

$$\big(Div\boldsymbol{T}(\boldsymbol{x})\big) \cdot \boldsymbol{k} = Div\big(\boldsymbol{T}(\boldsymbol{x})\boldsymbol{k}\big)\,, \tag{C.27}$$

where $\boldsymbol{k} \in E$ is any constant vector.

In general, let $\boldsymbol{\Phi}: \Omega \to F \otimes E$ be a field differentiable at a point $\boldsymbol{x} \in \Omega$. Then so is the field $\boldsymbol{\Phi}^T: \Omega \to E \otimes F$. Moreover, for every constant vector $\boldsymbol{\kappa} \in F$ the field $\boldsymbol{\Phi}^T\boldsymbol{\kappa}: \Omega \to E$ is differentiable at the point $\boldsymbol{x} \in \Omega$ and $\nabla(\boldsymbol{\Phi}^T\boldsymbol{\kappa})(\boldsymbol{x}) \in E \otimes E$ so that its trace is well defined. With this in mind we introduce the following definition:

The divergence of a field $\boldsymbol{\Phi}: \Omega \to F \otimes E$ at a point $x \in \Omega$ is the unique element $Div\boldsymbol{\Phi}(x) \in F$ of the vector space F, which satisfies the following condition

$$\big(Div\boldsymbol{\Phi}(x)\big) \cdot \boldsymbol{\kappa} = tr\big(\nabla(\boldsymbol{\Phi}^T\boldsymbol{\kappa})(x)\big), \qquad \forall\, \boldsymbol{\kappa} \in F. \tag{C.28}$$

It is seen that for the divergence to exists it is enough that $\boldsymbol{\Phi}$ is a field of class C^k, $k \geq 1$. Clearly, the definition (C.28) contains as special cases the two definitions (C.27) and (C.26), the former case by setting $F = \mathbb{R}$ and taking into account that $\mathbb{R} \otimes E = E$, the latter case is obvious.

The divergence is the first order differential operator, i.e. it is a map of the space of differentiable fields $C^k(\Omega, F \otimes E)$, $k \geq 1$, into the space of vector fields $C^{k-1}(\Omega, F)$ with the following property: for any differentiable fields $\boldsymbol{\Phi}, \boldsymbol{\Psi}: \Omega \to F \otimes E$ and each real number $\lambda \in \mathbb{R}$,

$$Div(\lambda\boldsymbol{\Phi} + \boldsymbol{\Psi}) = \lambda(Div\boldsymbol{\Phi}) + Div\boldsymbol{\Psi}. \tag{C.29}$$

This rule holds for any linear combination of differentiable fields.

The product rule. It will be necessary to compute the derivative of a product of two fields possibly of different kind. Various products have one property in common, the bilinearity. Therefore, in order to establish a product rule, which is valid in all cases of interest, we shall denote by

$$\boldsymbol{\beta}: F \times E \to F', \qquad (\boldsymbol{\phi}, \boldsymbol{u}) \to \boldsymbol{\beta}\{\boldsymbol{\phi}, \boldsymbol{u}\}, \tag{C.30}$$

a bilinear map, i.e. linear in both arguments separately. For example the tensor product of vectors, the inner product of vectors or the action of a tensor on a vector are all of the form (C.30).

Let $\boldsymbol{\phi} \in C^k(\Omega, F)$ and $\boldsymbol{u} \in C^k(\Omega, E)$ be differentiable fields. Then their product

$$\boldsymbol{\sigma}(x) = \boldsymbol{\beta}\{\boldsymbol{\phi}(x), \boldsymbol{u}(x)\} \tag{C.31}$$

is a field on Ω of the same class, i.e. $\boldsymbol{\sigma} \in C^k(\Omega, F')$. Moreover, at every point $x \in \Omega$ the following rule of differentiation holds[1]

$$\big(\nabla\boldsymbol{\sigma}(x)\big)\boldsymbol{k} = \boldsymbol{\beta}\big\{\boldsymbol{\phi}(x), \nabla\boldsymbol{u}(x)\boldsymbol{k}\big\} + \boldsymbol{\beta}\big\{\nabla\boldsymbol{\phi}(x)\boldsymbol{k}, \boldsymbol{u}(x)\big\}, \qquad \forall\, \boldsymbol{k} \in E. \tag{C.32}$$

[1] Gurtin [1972].

This theorem applied to the tensor product yields

$$(\nabla(\boldsymbol{\phi}\otimes\boldsymbol{u}))\boldsymbol{k}=\boldsymbol{\phi}\otimes(\nabla\boldsymbol{u})\boldsymbol{k}+(\nabla\boldsymbol{\phi})\boldsymbol{k}\otimes\boldsymbol{u}\,. \tag{C.33}$$

Covariant derivative. Every tensor field $\boldsymbol{\Phi}\in C(\Omega,F\otimes E)$ can be represented in the form

$$\boldsymbol{\Phi}=\boldsymbol{\phi}_i\otimes\boldsymbol{g}^i=\boldsymbol{\phi}^i\otimes\boldsymbol{g}_i\,. \tag{C.34}$$

Assuming that this field is differentiable, its gradient can be computed applying the formula

$$\begin{aligned}\nabla\boldsymbol{\Phi}&=(\boldsymbol{\phi}_i\otimes\boldsymbol{g}^i)_{,j}\otimes\boldsymbol{g}^j=(\boldsymbol{\phi}_{i,j}\otimes\boldsymbol{g}^i+\boldsymbol{\phi}_i\otimes\boldsymbol{g}^i{}_{,j})\otimes\boldsymbol{g}^j\\&=(\boldsymbol{\phi}^i\otimes\boldsymbol{g}_i)_{,j}\otimes\boldsymbol{g}^j=(\boldsymbol{\phi}^i{}_{,j}\otimes\boldsymbol{g}_i+\boldsymbol{\phi}^i\otimes\boldsymbol{g}_{i,j})\otimes\boldsymbol{g}^j\,.\end{aligned} \tag{C.35}$$

Making use of the formulae for the partial derivatives of the base vectors the gradient of the field (C.34) is obtained in the form

$$\nabla\boldsymbol{\Phi}=\boldsymbol{\Phi}_{,j}\otimes\boldsymbol{g}^j=\boldsymbol{\phi}_i{}_{;j}\otimes\boldsymbol{g}^i\otimes\boldsymbol{g}^j=\boldsymbol{\phi}^i{}_{;j}\otimes\boldsymbol{g}_i\otimes\boldsymbol{g}^j\,, \tag{C.36}$$

where a semicolon stands for the so-called covariant derivative, which is defined by the formulae

$$\boldsymbol{\phi}_{i;j}=\boldsymbol{\phi}_{i,j}-\Gamma^k_{ij}\boldsymbol{\phi}_k\,,\qquad \boldsymbol{\phi}^i{}_{;j}=\boldsymbol{\phi}^i{}_{,j}+\Gamma^i_{kj}\boldsymbol{\phi}^k\,. \tag{C.37}$$

The same considerations can be applied to fields of any kind. For subsequent considerations it will suffice to consider the tensor field $\boldsymbol{\Psi}\in C(\Omega,F\otimes E\otimes E)$, which can be represented in the form

$$\boldsymbol{\Psi}=\boldsymbol{\psi}_{ij}\otimes\boldsymbol{g}^i\otimes\boldsymbol{g}^j=\boldsymbol{\psi}^{ij}\otimes\boldsymbol{g}_i\otimes\boldsymbol{g}_j\,. \tag{C.38}$$

Assuming that this field is differentiable, its gradient can be obtained in the form

$$\nabla\boldsymbol{\Psi}=\boldsymbol{\psi}_{ij;k}\,\boldsymbol{g}^i\otimes\boldsymbol{g}^j\otimes\boldsymbol{g}^k=\boldsymbol{\psi}^{ij}{}_{;k}\,\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}^k\,, \tag{C.39}$$

where

$$\boldsymbol{\psi}_{ij;k}=\boldsymbol{\psi}_{ij,k}-\Gamma^j_{lk}\boldsymbol{\psi}_{ij}-\Gamma^j_{lk}\boldsymbol{\psi}_{ij}\,,\qquad \boldsymbol{\psi}^{ij}{}_{;k}=\boldsymbol{\psi}^{ij}{}_{,k}+\Gamma^i_{lk}\boldsymbol{\psi}^{lj}+\Gamma^j_{lk}\boldsymbol{\psi}^{il}\,. \tag{C.40}$$

By direct calculation we find that the covariant derivatives of the natural base vectors vanish identically

$$\boldsymbol{g}_i;_j = \boldsymbol{g}^j;_j = \boldsymbol{0}. \tag{C.41}$$

By implication, the covariant derivatives of the components of the metric tensor and the permutation tensor associated with the given coordinate system vanish also identically:

$$g_{ij};_k = g^{ij};_k = 0, \qquad \varepsilon_{ijk};_l = \varepsilon^{ijk};_l = 0. \tag{C.42}$$

When the above formulae are applied to the second order tensor field, which can be represented in the form

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{t}_j \otimes \boldsymbol{g}^j = T_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j \\ &= \boldsymbol{t}^j \otimes \boldsymbol{g}_j = T^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j , \end{aligned} \tag{C.43}$$

we obtain the following form for the gradient

$$\begin{aligned} \nabla \boldsymbol{T} &= \boldsymbol{t}_j;_k \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k = T_{ij};_k \boldsymbol{g}^i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k \\ &= \boldsymbol{t}^j;_k \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k = T^{ij};_k \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k , \end{aligned} \tag{C.44}$$

where

$$\begin{aligned} \boldsymbol{t}_j;_k &= \boldsymbol{t}_j,_k - \Gamma^l_{jk}\boldsymbol{t}_l , & \boldsymbol{t}^j;_k &= \boldsymbol{t}^j,_k + \Gamma^j_{lk}\boldsymbol{t}^l , \\ T_{ij};_k &= \boldsymbol{T}_{ij},_k - \Gamma^j_{lk}T_{ij} - \Gamma^j_{lk}T_{ij} , & T^{ij};_k &= T^{ij},_k + \Gamma^i_{lk}T^{lj} + \Gamma^j_{lk}T^{il} . \end{aligned} \tag{C.45}$$

Formulae for divergence. The results of the previous section enable us to obtain useful formulae for the calculation of the divergence operator in a given coordinate system. For a field $\boldsymbol{\Phi} \in C(\Omega, F \otimes E)$ the following rules hold:

$$Div\boldsymbol{\Phi} = g^{ij}\boldsymbol{\phi}_i;_j = \boldsymbol{\phi}^k;_k . \tag{C.46}$$

Thus taking into account (C.37) we may express (C.46) in the form containing no covariant differentiation

$$Div\boldsymbol{\Phi} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,\boldsymbol{\phi}^k\right),_k . \tag{C.47}$$

For a second order tensor field with the formulae for the covariant derivative given by (C.45) we obtain

$$\begin{aligned} Div\boldsymbol{T} &= g^{jk}\boldsymbol{t}_j;_k = \boldsymbol{t}^j;_j \\ &= g^{jk}T_{ij};_k \boldsymbol{g}^i = T^{ij};_j \boldsymbol{g}_i . \end{aligned} \tag{C.48}$$

Appendix D

Curves, surfaces and surface fields

In this appendix $\mathscr{E}$ denotes the three-dimensional Euclidean point space and E the three-dimensional Euclidean vector space (translation space of $\mathscr{E}$). Moreover, we shall assume that a particular Cartesian coordinate system in $\mathscr{E}$ has been fixed once and for all.

Preliminary remarks. There are several relevant concepts of curve and surface, just as there are several concepts of line and surface integrals. These concepts are not only related to one another, but they are also related to the solution of many problems in mathematics and mechanics. Classically, in differential geometry one considers only curves and surfaces in the three-dimensional Euclidean space defined as smooth mappings. Such curves and surfaces may be considered as differentiable manifolds of dimension one and two, respectively. However, the definition of curves and surfaces as smooth mappings has certain drawbacks. In particular, there is an inevitable restriction on the types of singularities that can occur, and there is an a priori restriction on the topological complexity. An alternative to curves and surfaces as smooth mappings is provided by rectifiable currents, one-dimensional and two-dimensional oriented curves and surfaces of the geometric measure theory. The relevant functions need not be smooth but merely Lipschitzian. Roughly speaking, a rectifiable set is arbitrarily close (in the sense of measure) to being a differentiable manifold and yet may have a wide class of singularities, i.e. non-manifold points. For example, a subset of the space illustrated in Fig. 1, resembling the familiar soap bubble, is not a surface in the sense of classical differential geometry because of the singular circle, where three spherical segments meet. However, it is a two-rectifiable set, because we can remove an arbitrary small neighborhood of the singular set and have the three segments of spherical surfaces left (two- dimensional manifolds).

There is also an essential difference between curve theory and surface theory. For a given curve there is a natural parametrization and this parametrization can be used to describe every point on the curve. For a surface there is no natural geometric parametrization. In fact, often it is not possible to find a parametrization

that describes the whole surface in a unique way. Accordingly, a general theory of surfaces entails also a much deeper connection with topology and measure theory than for curves.

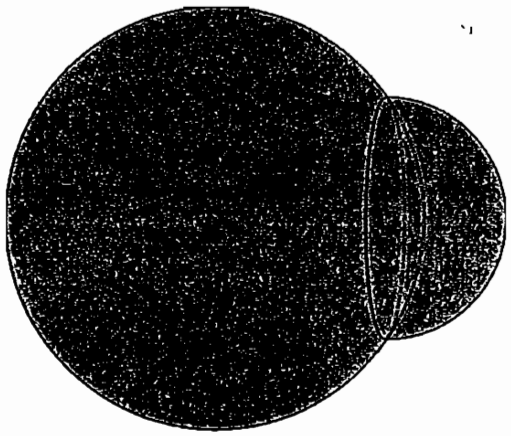


Fig. 1

Parametric curves. A *parametric curve* in the space $\mathcal{E}$ is defined as a continuous, not identically constant map $\gamma: I \to \mathcal{E}$ of an interval $I \equiv [a,b] \subset \mathbb{R}$. In other words, a parametric curve is defined by a continuous vector-valued function of a real argument (Fig. 2):

$$\boldsymbol{x}(t) = \boldsymbol{x}(\gamma(t)) = x_k(t)\boldsymbol{e}_k\,, \qquad t \in I \equiv [a,b]. \tag{D.1}$$

In general, a map $\gamma: I \to \mathcal{E}$ of a curve need not be one-to-one. It has to be noted that a parametric curve is defined as a map and not as a subset of the space $\mathcal{E}$. A curve as a set in $\mathcal{E}$ can be defined as the image set $\Gamma = \gamma(I)$. In this case $\gamma: I \to \mathcal{E}$ is called a *parametric representation* of the curve Γ.

The curve Γ can be a very complicated set, in fact, there are curves which fill out a cube (such a curve is called a Peano curve or a space-filling curve). Moreover, it is obvious there are infinitely many maps $\gamma: I \to \mathcal{E}$, all of which should be considered as representations of the same curve Γ. This can be made precise in the following way. Let two parametric curves $\gamma_1: [a_1,b_1] \to \mathcal{E}$ and $\gamma_2: [a_2,b_2] \to \mathcal{E}$ be given and let $\tau: [a_1,b_1] \to [a_2,b_2]$ be a topological map (homeomorphism). Then one defines the distance between two parametric curves in the sense of Fréchet. Two parametric curves with zero Fréchet distance are regarded as identical, being

merely two representations of the same curve Γ. In other words, a curve Γ is defined to be a maximal class of Fréchet equivalent maps.

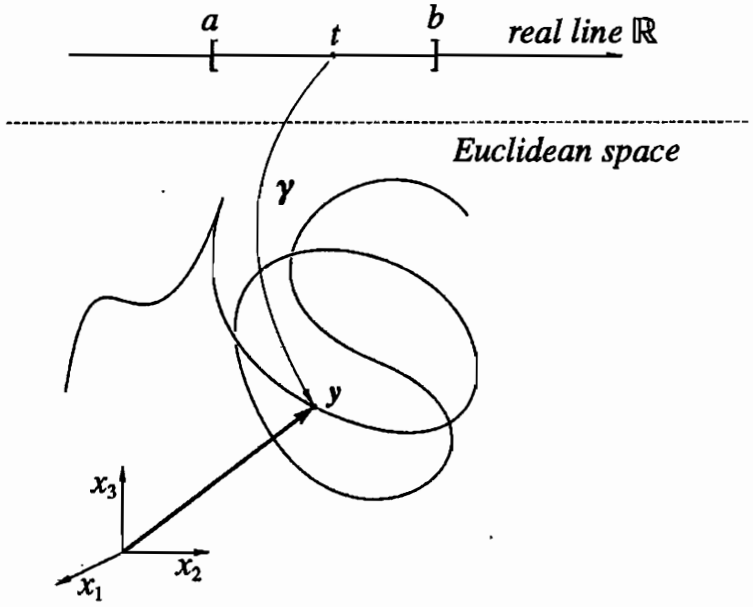


Fig. 2

Using the concept of equivalence, we can assign a precise mathematical meaning to the intuitive idea of "transversing" a curve in a given direction. Two parametric curves are said to have the same orientation in the sense of Fréchet, if there exists an orientation preserving (i.e. strictly increasing) homeomorphic map τ such that $\gamma_2 = \gamma_1 \circ \tau$. An oriented curve is then precisely a maximal class of equally oriented maps in the sense of Fréchet. A *path* or an *oriented curve* in the space $\mathscr{E}$ is a class of parametric curves, any two of which are equivalent and orientation preserving.

A *closed curve* is a parametric curve $\gamma: I \to \mathscr{E}$ with the property that $\gamma(a) = \gamma(b)$, i.e. the terminal point of the curve coincides with the initial point. If $\gamma(a) \neq \gamma(b)$, a parametric curve is called an *arc* with endpoints $\gamma(a)$ and $\gamma(b)$. If $\gamma: I \to \mathscr{E}$ is one-to-one, then the parametric curve is called a *Jordan curve* or a *simple curve* and it is called a *Jordan arc* if $\gamma(a) \neq \gamma(b)$ (Fig. 3).

A curve Γ is said to be of class C^k, if there exits a parametric representation $\gamma: I \to \mathscr{E}$ of class C^k (k-times continuously differentiable with respect to the parameter). A curve Γ is called *regular* of class C^k, $k \geq 1$, if there exits a

parametric representation $\gamma: I \to \mathcal{E}$ of class C^k and $\dot{\gamma}(t) \neq \mathbf{0}$ for all $t \in I = [a,b]$. Here and in the sequel a superimposed dot stands for the derivative with respect to the parameter. The condition $\dot{\gamma}(t) \neq \mathbf{0}$ ensures the existence of the tangent vector at every point of the regular curve. A regular curve of class C^1 is called *smooth*. A regular curve of class C^k, $k \geq 2$, is called a *differential geometric curve*.

two-dimensional Euclidean space (plane)

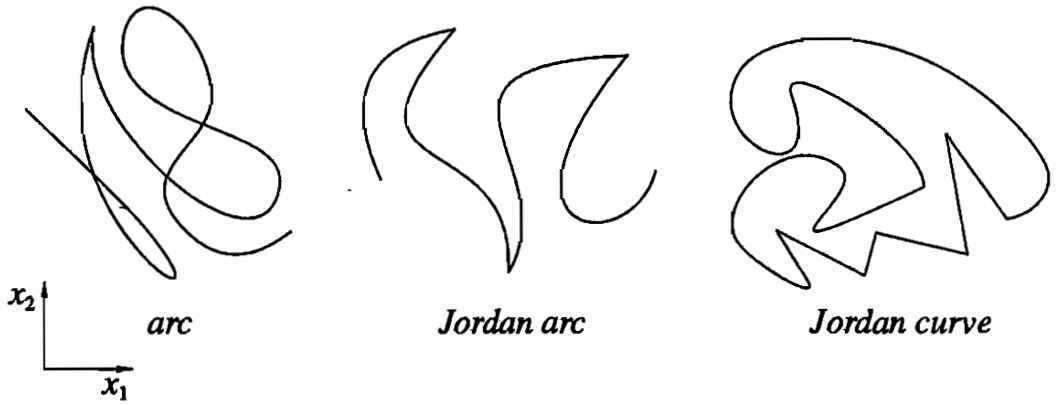


Fig. 3

A curve Γ is called piecewise regular of class C^k, if there exits its parametric representation $\gamma: I \to \mathcal{E}$ and a partition Δ of the interval $I = [a,b]$,

$$\Delta: \{a = t_1 < t_2 < \ldots < t_{n-1} < t_n = b\}, \tag{D.2}$$

such that the restriction of γ to each subinterval (t_i, t_{i+1}) is a function of class C^k and $\dot{\gamma}(t) \neq \mathbf{0}$ for every $t \in (t_i, t_{i+1})$. Thus the points $t_k \in I$ are the only possible points, where the *k-th* derivative might not exist or where it vanishes (Fig. 4).

Given a piecewise regular curve Γ of class C^k, $k \geq 1$, with the parametric representation $\gamma: I \to \mathcal{E}$, the new parameter s along Γ, called the arc length parameter, is defined by

$$s(t) = \int_a^t \sqrt{\dot{x}(\lambda) \cdot \dot{x}(\lambda)}\, d\lambda. \tag{D.3}$$

The function $s = s(t)$ is increasing except for at most finitely many points. Accordingly, the inverse function is also monotonously increasing and piecewise

of class C^k. Using this function we obtain an equivalent parametric representation of the curve Γ in the form $\boldsymbol{x}(t(s))$, called natural representation.

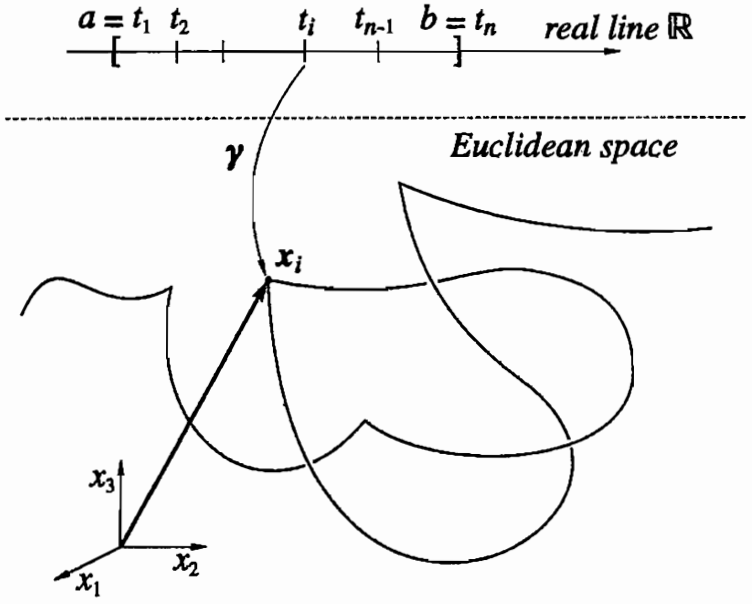


Fig. 4

The path integral or the integral of a vector field $\boldsymbol{u}: \mathcal{E} \to E$ along the path (i.e. the oriented curve Γ) is defined by

$$\int_\Gamma \boldsymbol{u}(\boldsymbol{x})\, ds = \int_a^b \boldsymbol{u}(\boldsymbol{x}(t))\, \|\dot{\boldsymbol{x}}(t)\|\, dt\,. \tag{D.4}$$

If a curve Γ is only piecewise smooth or a field $\boldsymbol{u}$ is piecewise continuous, we can still define the path integral by

$$\int_\Gamma \boldsymbol{u}(\boldsymbol{x})\, ds = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \boldsymbol{u}(\boldsymbol{x}(t))\, \|\dot{\boldsymbol{x}}(t)\|\, dt\,. \tag{D.5}$$

The above two definitions apply also to tensor fields of any order.

Parametric surfaces. In the most general case, a parametric surface M in the space $\mathcal{E}$ may be defined to be a map of some subset $\Pi \subset \mathbb{R}^2$ of the real plane into $\mathcal{E}$:

$$\boldsymbol{\rho}: \Pi \to \mathcal{E}\,, \qquad (\xi^\beta) \to \boldsymbol{y} = \boldsymbol{\rho}(\xi^\beta)\,, \tag{D.6}$$

or simply $y = \rho(\xi)$, $\xi \equiv (\xi^\beta)$. The set Π is then called the parameter domain. Let us note that a parametric surface M is defined to be a map and not a point set $\rho(\Pi) \subset \mathcal{E}$, which is the image of the parameter domain under the map ρ. Moreover, to make it clear that the concept of a parametric surface M depends on both the parameter domain Π and the map ρ, one often writes $M = (\rho, \Pi)$.

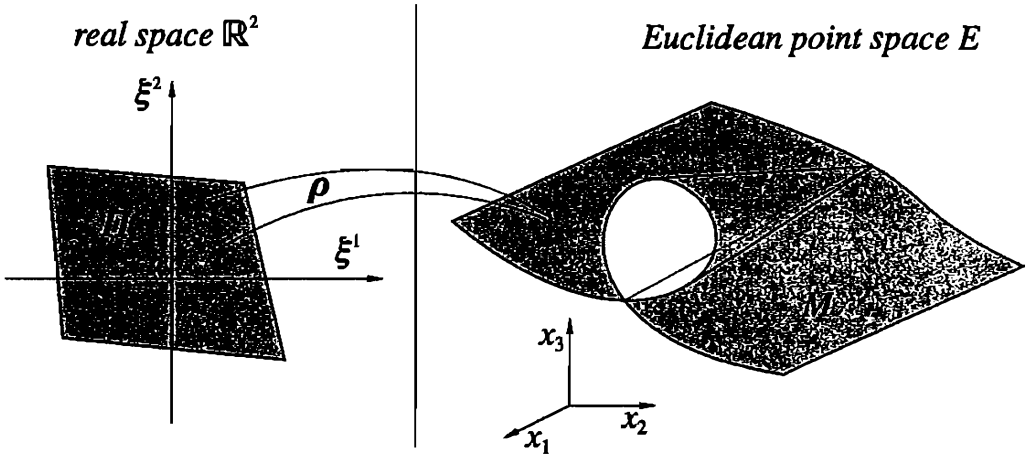


Fig. 5

It is also apparent that without some restrictions on both, the parameter domain Π and the map ρ, the above definition would be useless. By imposing additional conditions, we can make the shape of a surface defined by (D.6) to conform with our intuition, at least locally. Even then, however, we cannot rule out self-intersections (Fig. 5), self-tangencies (Fig. 6), coverings and certain other unexpected properties.

In general, the map (D.6) is assumed to be continuous, non-constant, and single valued, but not necessarily one-to-one. That is, each point $\xi \in \Pi$ is mapped into one and only one point $y = \rho(\xi)$ of the image set $\rho(\Pi)$. However, any point of $y \in \rho(\Pi)$ may be the image of more than one point of Π, even infinitely many. There are even more possibilities in choosing the parameter set. In general, it is taken to be a connected subset of the real plane. Usually, a parameter set Π is taken to be a bounded domain (i.e. a bounded, open and connected set) or a compact set bounded by a finite number of disjoint piecewise smooth Jordan curves.

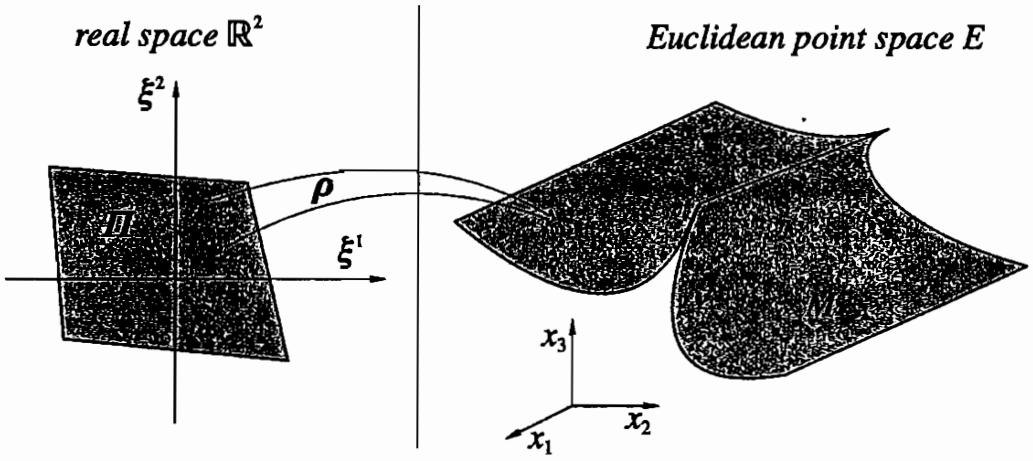


Fig. 6

Equivalent parametrizations of a surface. The distance between two parametric surfaces $M=\rho(\Pi)$ and $M'=\rho'(\Pi')$ with topologically equivalent parameter domains is defined in the same way as the distance between two curves. Two parametric surfaces with zero Fréchet distance are merely two parametrizations of the same surface and can be regarded as identical. In this sense, a surface is a maximal class of Fréchet equivalent parametric surfaces. A necessary, but by no means sufficient, condition for $\rho:\Pi\to\mathcal{E}$ and $\rho':\Pi'\to\mathcal{E}$ to be two parametrizations of the same surface is that the two parameter domains Π and Π' be topologically equivalent, i.e. there exists a homeomorphic map

$$\varphi:\Pi\to\Pi', \qquad \xi\to\xi'=\varphi(\xi) \tag{D.7}$$

of Π onto Π', and, in the case of compact parameter domains, that the images be identical.

In spite of its appearance as a point set in space, a surface inherits its topological properties from its parameter domain. This is indeed characteristic for the notion of a parametric surface. Moreover, without some restrictions the intersections and other types of singularities of surfaces in the space could be very complicated.

Boundary of a surface. Some surfaces have boundaries and other do not. For example, a hemisphere has a boundary consisting of its equatorial rim. An entire

sphere, an ellipsoid, and the surface of a cube are examples of surfaces without boundary.

Let a parametric surface $M=(\rho,\Pi)$ be given. If the parameter set Π is taken to be a bounded domain (i.e. a bounded, open and connected set), then M is called a surface without boundary or an open surface (Fig. 7).

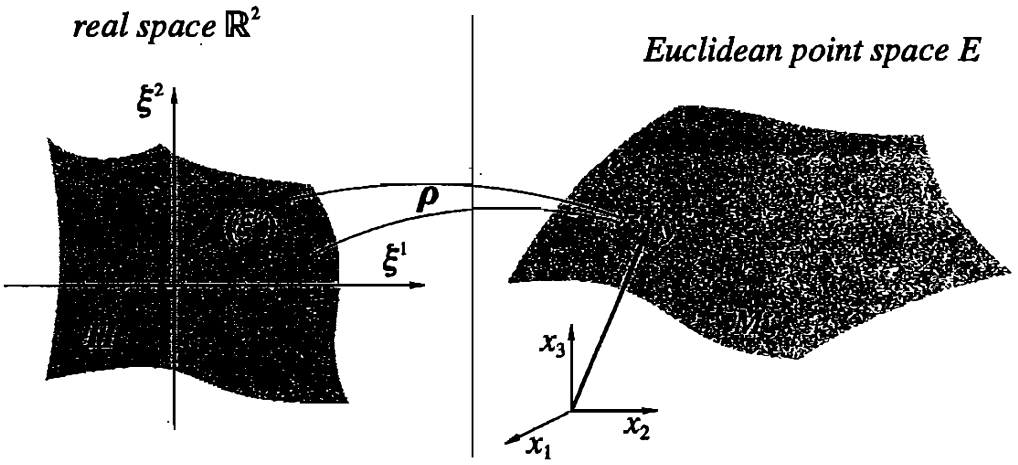


Fig. 7

If the parameter set Π is taken to be a compact set bounded by a finite number of disjoint piecewise smooth Jordan curves γ_k, $k=1,2,\ldots,n$, then the boundary of M is said to consist of the images $\Gamma_k=\rho(\gamma_k)$ of γ_k (Fig. 8).

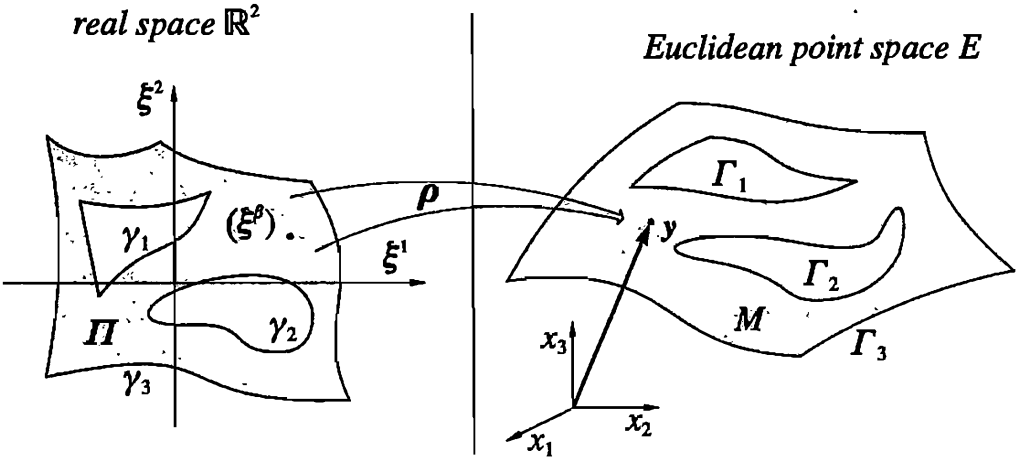


Fig. 8

In general, the boundary of a parametric surface $M=(\rho,\Pi)$ is defined to be the set, denoted by ∂M, of accumulation points of all sequences $\{y_n\}$, $y_n=\rho(\xi_n)$, $n=1,2,\ldots$, where the sequences $\{\xi_n\}$ converge to the boundary $\partial\Pi$ of the parameter set Π.

Tangent plane to a parametric surface. If the representation $\rho:\Pi\to\mathcal{E}$ of a parametric surface M is a differentiable map at a point $\xi\in\Pi$, then its differential is the linear map $D\rho(\xi):\mathbb{R}^2\to E=T_y\mathcal{E}$. The parametrization ρ of M is called regular at a point $\xi\in\Pi$, if the tangent map $D\rho(\xi)$ has maximal rank two. Let $\xi\in\Pi$ be a regular point. We then define the tangent space T_yM to the surface M at a point $y=\rho(\xi)$ to be the image of $\mathbb{R}^2$ under the linear map $D\rho(\xi)$. Geometrically, the tangent space T_yM is represented by a tangent plane (Fig. 9). It is also clear that T_yM is the two-dimensional subspace of the translation space $E=T_y\mathcal{E}$ attached to the point $y\in M$. Moreover, the inner product on E induces the inner product on the tangent space T_yM, which is defined as the restriction of the inner product on E to T_yM. As a result we have the following decomposition of the space $E=T_y\mathcal{E}$:

$$E=T_y\mathcal{E}=T_yM\oplus T_yM^{\perp}, \tag{D.8}$$

where $T_yM^{\perp}$ denotes the orthogonal complement of the tangent space.

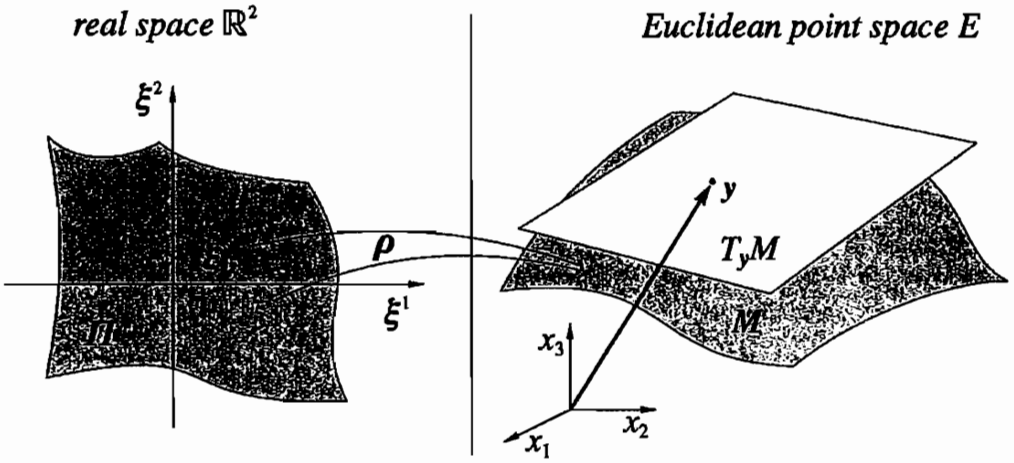


Fig. 9

Assuming that the parametrization of M is fixed, we shall simply write $y = y(\xi^\beta)$. Then at a regular point $y \in M$ one defines the natural base vectors and the reciprocal base vectors by

$$a_\beta = y_{,\beta}\,, \qquad a^\alpha \cdot a_\beta = \delta^\alpha_\beta\,. \tag{D.9}$$

Since M is assumed to be regular the natural base vectors as well as the reciprocal base vectors are linearly independent and they span the tangent space $T_y M$ at each point of M (Fig. 10).

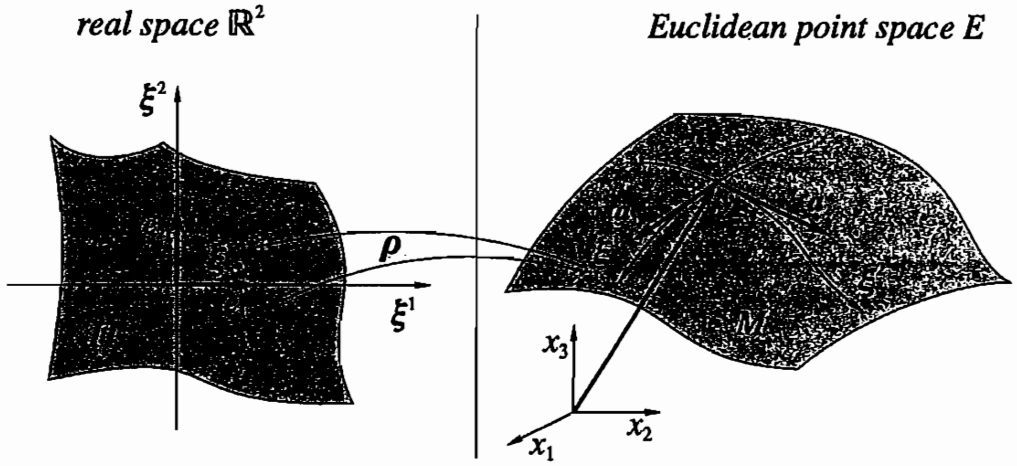


Fig. 10

In the following we shall usually assume that all considered surfaces are regular, except possibly at isolated points.

Piecewise smooth surfaces. Let $\Pi \subset \mathbb{R}^2$ be a simple regular region, i.e. an open connected and bounded set, whose boundary $\partial\Pi$ consists of a finite number of piecewise smooth simple closed curves (of class C^n, $n \geq 1$), and let $\Omega \subset \mathbb{R}^2$ be an open domain such that $\Pi \cup \partial\Pi \subset \Omega$. Let

$$\rho : \Omega \to \mathscr{E}\,, \qquad (\xi^\beta) \to y = \rho(\xi^\beta) \tag{D.10}$$

be a one-to-one map of class C^k, $k \geq 1$. Then a set $M = \rho(\Pi \cup \partial\Pi) \subset \mathscr{E}$ is called a simple smooth surface element of class C^k (Fig. 11). The boundary of M is taken to be $\partial M = \rho(\partial\Pi)$ and the map (D.10) is called a simple parametrization of the surface M.

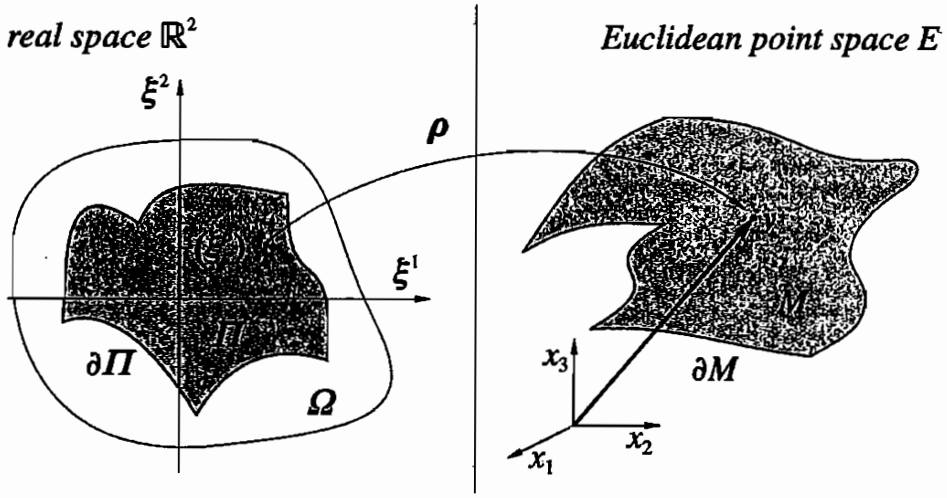


Fig. 11

A regular surface may be taken to be a continuous surface which is piecewise smooth. Roughly speaking, a piecewise smooth surface consists of portions of smooth surfaces joined together. Such a surface may have sharp edges and corners. The surface of a cube or other polyhedron is a simple example of a piecewise smooth surface.

A piecewise smooth surface $M \subset \mathcal{E}$ is a union of a finite number of smooth surface elements $M_{(k)}$, $k = 1,2,\ldots,n$, provided that (Fig. 12):

(i) No two distinct $M_{(k)}$ have common interior points, $\operatorname{int} M_{(k)} \cap \operatorname{int} M_{(l)} = \varnothing$ for all $k \neq l$.

(ii) The intersection $\partial M_{(k)} \cap \partial M_{(l)}$, $k \neq l$ of the boundaries of two distinct elements $M_{(k)}$ and $M_{(l)}$ is either empty or a single point, which is a vertex for both, or a piecewise smooth arc $\Gamma_{(k,l)} = \partial M_{(k)} \cap \partial M_{(l)}$, which is an edge for both.

(iii) The boundaries of any three or more distinct elements $M_{(k)}$ have at most one point in common, which is the vertex for all elements.

(iv) Any two points of M can be joined by a path in M.

(v) The union of all arcs, each of which is on the boundary of only one of the $M_{(k)}$, forms a finite number of disjoint, piecewise smooth, simple closed curves.

The set of points in (v) constitutes the boundary ∂M of M. If this set is empty, then M is called a piecewise smooth surface without boundary.

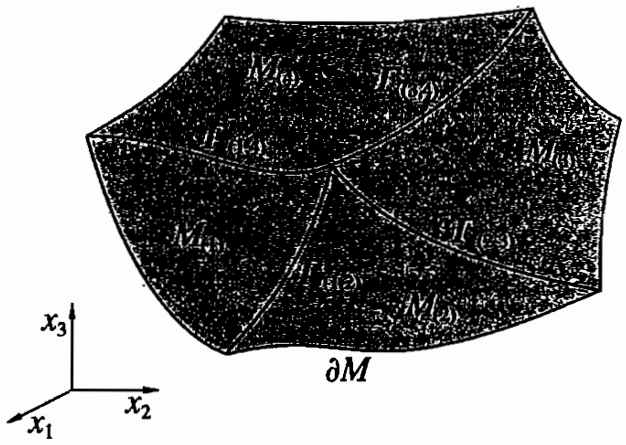


Fig. 12

The term edge here refers to one of the finite number of regular arcs comprising the boundary of a regular surface element. The term vertex is a point, at which two edges meet. If all the edges of a regular surface belong each to two of its surface elements, the surface is a closed regular surface. Note that a regular surface (and hence a closed regular surface) is necessarily both connected and bounded.

Orientation of a surface. Let M be a smooth surface. Then at each point of M the tangent space $T_y M$, geometrically represented by the tangent plane, is well defined. A smooth surface M is called orientable, if there exists a continuous field $\boldsymbol{n}: M \to E$ of unit vectors such that $\boldsymbol{n}(y) \in T_y M^{\perp}$ is the normal vector at each point $y \in M$. Each smooth orientable surface possesses exactly two orientations, $\boldsymbol{n}(y)$ or $-\boldsymbol{n}(y)$, each of which is the negative of the other (Fig. 13). It is intuitively clear that smooth surfaces such as spheres, ellipsoids, tori, etc. are all orientable. However, there are smooth surfaces, for which there is no way to choose a continuous unit normal over the entire surface. The orientation of the surface is definitely not a local property. Locally, every smooth surface is orientable.

A piecewise smooth surface has edges, and so a continuous unit normal vector field cannot be defined over the entirety of such a surface. However, a piecewise smooth surface M admits a partition into a finite number of smooth surface elements $M_{(k)}$. Each $M_{(k)}$ can be oriented and to each boundary $\partial M_{(k)}$ can be given

a positive orientation. Suppose that $\Gamma_{(k,l)} = \partial M_{(k)} \cap \partial M_{(l)}$ is a smooth arc which is the common boundary of two surface elements $M_{(k)}$ and $M_{(l)}$. If the positive direction of $\Gamma_{(k,l)}$ as part of the boundary $M_{(k)}$ is the opposite of the positive direction of $\Gamma_{(k,l)}$ as part of the boundary $M_{(l)}$ for all arcs $\Gamma_{(k,l)}$, we say that the piecewise smooth surface M is an orientable one (Fig. 14). This condition ensures the possibility of defining the unit normal vector on each smooth surface element.

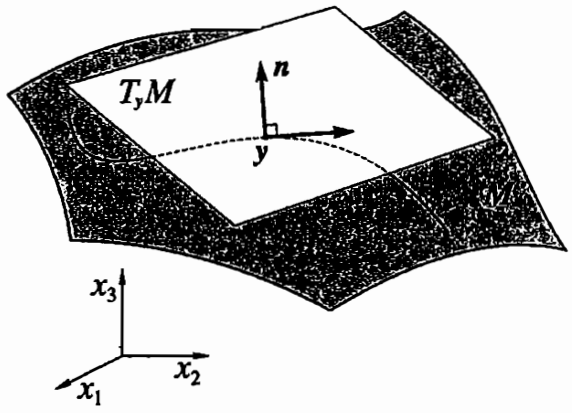


Fig. 13

It can be shown that if a piecewise smooth surface M is orientable according to one partition into smooth surface elements, then it is orientable according to any other such partition.

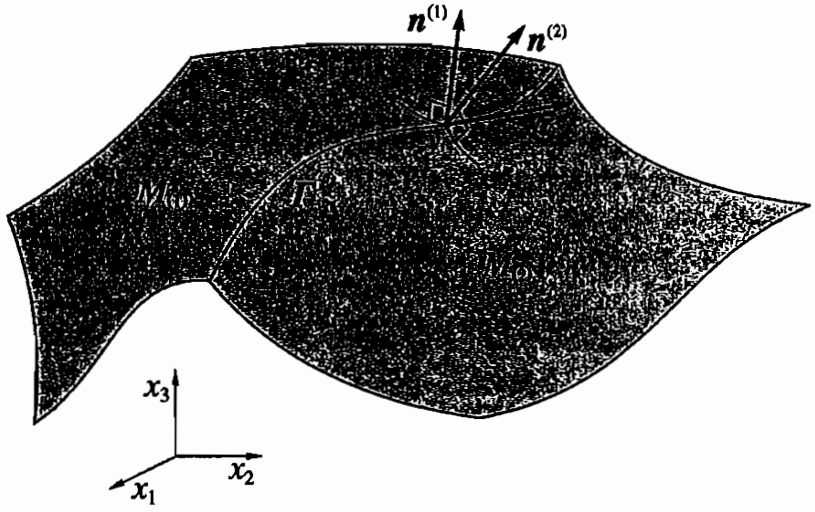


Fig. 14

Projection and inclusion operators. At each regular point for every $y \in M$ of a given surface, the decomposition (D.8) makes it possible to define a canonical injection $I(y): T_y M \to E$, called the inclusion operator, which maps every vector $u \in T_y M$ into itself considered as the element of E. Similiarly, the orthogonal projection operator along the normal vector $n(y)$ is defined to be a map $P(y): E \to T_y M$, which assigns to every spatial vector at $y \in M$ its tangential component, i.e. $P(y)u \in T_y M$ for every vector $u \in E = T_y \mathcal{E}$.

The inclusion map $I(y) \in E \otimes T_y M$ has to be distinguished from both the unit tensor (identity map) $\mathbf{1}(y) \in T_y M \otimes T_y M$ and the unit tensor $1 \in E \otimes E$. We have

$$u \cdot I(y)v = I(y)^T u \cdot v = P(y)u \cdot v\,, \qquad u \in E\,,\ v \in T_y M\,. \tag{D.11}$$

Moreover,

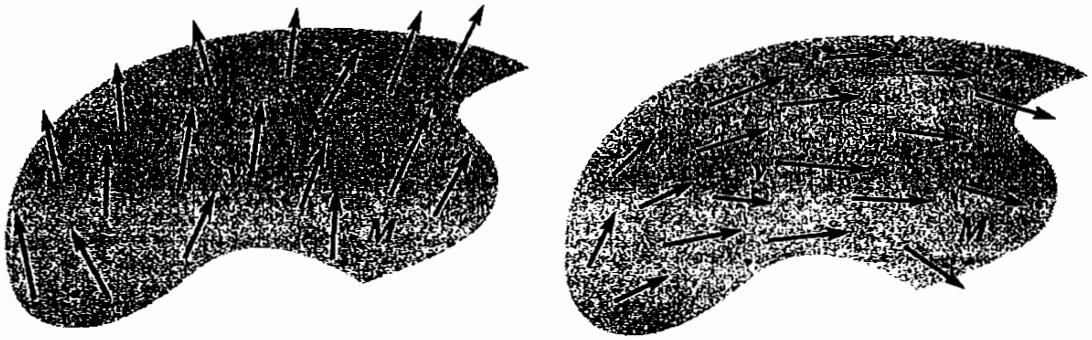
$$I(y)^T = P(y)\,, \qquad P(y)I(y) = \mathbf{1}(y)\,. \tag{D.12}$$

Fields over a surface. Given a surface $M \subset \mathcal{E}$, the term F-valued surface field will generally signify a map $\phi: M \to F$, which assigns to every point $y \in M$ an element $\phi(y) \in F$ of a finite dimensional inner product vector space F (usually the set $\mathbb{R}$ of real numbers or the translation space E of $\mathcal{E}$). This term applies also for surface fields with values in the Euclidean point space $\mathcal{E}$, whose translation space is E and for fields, whose domain is any subset of a given surface M. We shall always assume that M is an oriented and connected surface (but not necessarily simply connected) with or without boundary. In general, M need not be smooth.

We shall denote by $C(M,F)$ a set of all F-valued fields over a given surface M. A few special fields will be our main concern. A real function on M is a map $\phi: M \to \mathbb{R}$, and $u: M \to E$ will be called a spatial vector field over the given surface. The sets of all such fields will be denoted by $C(M,\mathbb{R})$ and $C(M,E)$, respectively.

If M is an open (without boundary), smooth surface so that the tangent space $T_y M$ is well defined at every point of M, we define a tangential vector field on M as a map, which assigns to every point $y \in M$ an element $v(y) \in T_y M$ of the tangent space at that point (Fig. 15). We write $C(T_y M)$ for the set of all such fields. This notation is not incidental. In the terminology of differential geometry, a tangential vector field over a manifold is a cross section of the tangent bundle

$TM = \bigcup_{y \in M} T_y M$. Analogously, $C(E \otimes T_y M)$ will denote a set of all surface fields, whose values at a point $y \in M$ is a tensor $S(y) \in E \otimes T_y M$, i.e. a linear map of the tangent space $T_y M$ at that point into the vector space E.



spatial vector field *tangential vector field*

Fig. 15

Differentiable surface fields. Our aim is now to introduce the concept of continuity, differentiability and differential operators for surface fields. This is not at all an easy task, since the domain of surface fields lack a vector space structure. Moreover, it is intuitively clear that the regularity of surface fields can depend on the regularity of a given surface itself. There are two main ways to introduce appropriate concepts.

We shall first consider the case, when M is an open surface. Let us note that this means the same as to consider a field over an open subsurface of M, not necessarily an open one. If M is an open surface, then there exists a parametric representation of M, i.e. a continuous map $\rho : \Pi \to \mathcal{E}$. Given a field $\phi : M \to F$ on M, the composed map (Fig. 16)

$$\tilde{\phi} \equiv \phi \circ \rho : \Pi \to F, \qquad \xi \to \tilde{\phi}(\xi) = \phi(\rho(\xi)), \tag{D.13}$$

is called the representative of the given field ϕ. With the help of (D.13) we can tray to transfer the concept of continuity of fields over the surface to the well defined concept of continuity of fields over domains in the real vector space $\mathbb{R}^2$. And indeed this idea works well in the case of differential geometric surfaces:

Definition 1. Let M be a regular, open surface of class C^r, $r \geq 1$, so that there exists a parametric representation $\rho : \Pi \to \mathcal{E}$ of M, which is a diffeomorphism of

class C^r. A field $\phi: M \to F$ is said to be differentiable of class C^k, $k \leq r$, at a point $y \in M$ if the representative (D.13) of ϕ is differentiable of class C^k at the point $\rho^{-1}(y) \in \Pi$. A field $\phi: M \to F$ is said to be differentiable of class C^k on M, if the representative (D.13) of ϕ is differentiable of class C^k at every point of Π.

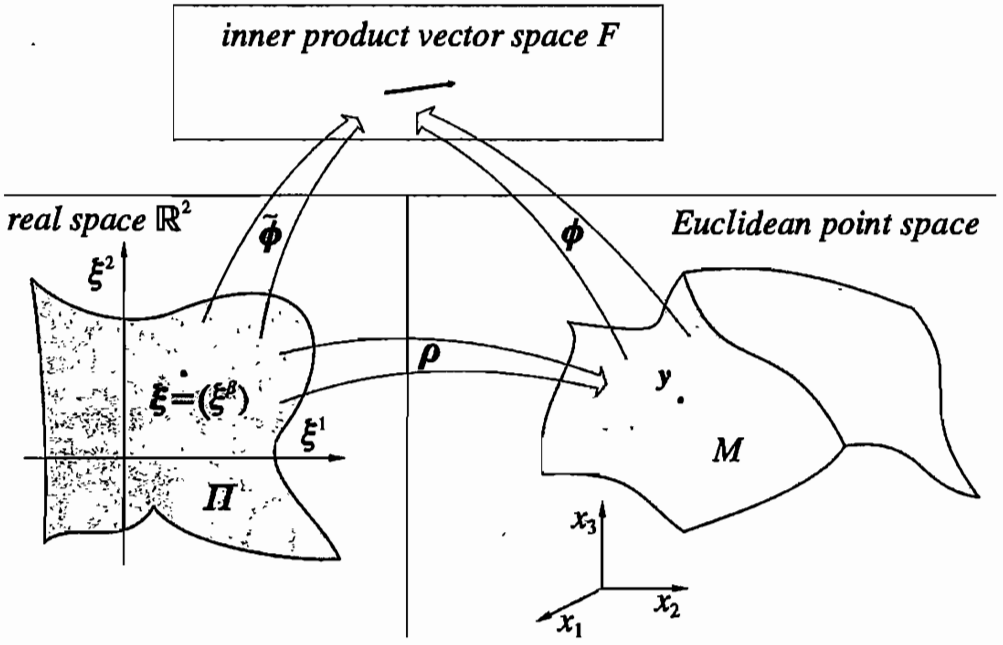


Fig. 16

Definition 1. Let M be a regular, open surface of class C^r, $r \geq 1$, so that there exists a parametric representation $\rho: \Pi \to \mathcal{E}$ of M, which is a diffeomorphism of class C^r. A field $\phi: M \to F$ is said to be differentiable of class C^k, $k \leq r$, at a point $y \in M$ if the representative (D.13) of ϕ is differentiable of class C^k at the point $\rho^{-1}(y) \in \Pi$. A field $\phi: M \to F$ is said to be differentiable of class C^k on M, if the representative (D.13) of ϕ is differentiable of class C^k at every point of Π.

Recalling that a regular surface M of class C^r, $r \geq 1$, is defined as an equivalence class of diffeomorphic parametric representations of class C^r, it becomes clear that this definition is independent of the parametrization of M. Indeed, let $\rho_1: \Pi_1 \to M$ and $\rho_2: \Pi_2 \to M$ be two parametric representations of class C^r of a given surface M (see Fig. 17). Then there exists a diffeomorphism $\rho_{12} \equiv \rho_2^{-1} \circ \rho_1: \Pi_1 \to \Pi_2$ of class C^r. Denoting by $\tilde{\phi}_1: M \to F$ and $\tilde{\phi}_2: M \to F$, respectively, the representative

of a given field $\phi: M \to F$, we have $\tilde{\phi}_2 \equiv \tilde{\phi}_1 \circ \rho_{12}^{-1}$. Moreover, since the composition of two diffeomorphisms of a given class is again a diffeomorphism of the same class, we see that $\tilde{\phi}_1: M \to F$ is differentiable of a given class if and only if $\tilde{\phi}_2: M \to F$.

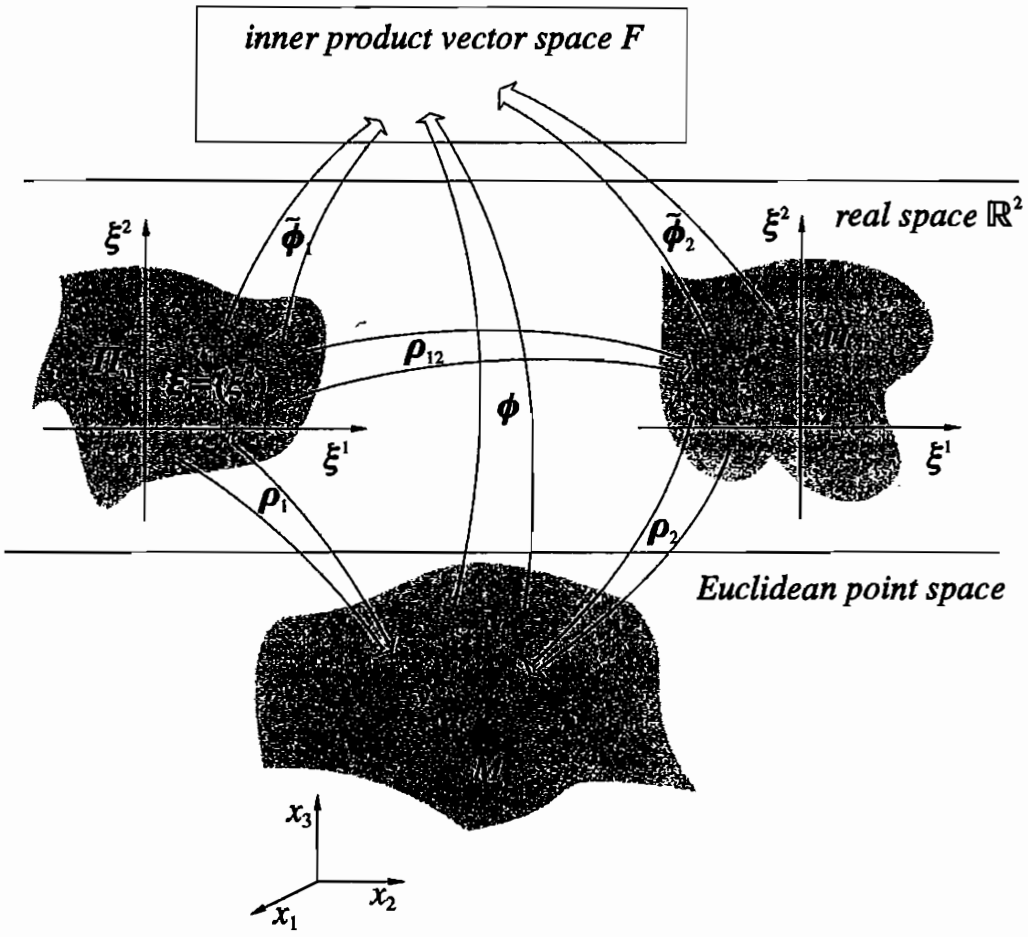


Fig. 17

A set of all differentiable fields of class C^k on M will be denoted by $C^k(M,F)$. Let us recall here that a field is said to be of class C^k, if the derivatives up to and including the order $k-1$ are differentiable and the k–th derivative is continuous. In the same manner we define analytical fields and we write $C^\omega(M,F)$ for the sets of all such fields. A differentiable field of class C^0 is just a continuous field. Note that in this case the parametric representation of M is a homeomorphism.

The concept of continuity and differentiability introduced above is very convenient but too restrictive for many reasons. First of all, it applies only in the case of differential geometric surfaces. In a more general case, it is not possible to show that the continuity and differentiability of a surface field is independent of the parametric representation of the given surface. Moreover, it is intuitively clear that even if a surface M is not smooth, it may be possible to define a field on M in a continuous manner (Fig. 18). Therefore as an alternative to the Definition 1 we may introduce the following definition.

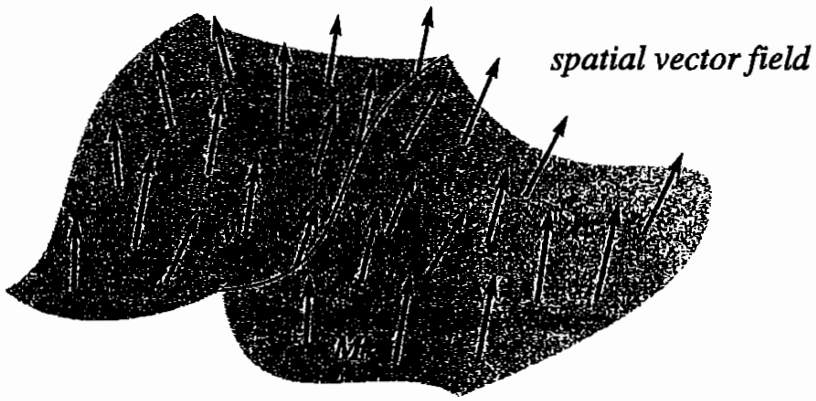


Fig. 18

Definition 2. A field $\boldsymbol{\phi}: M \to F$ over a given surface M is said to be differentiable of class C^k, if there exists an open set $\Omega \subset \mathcal{E}$ containing M, i.e. $M \subset \Omega$, and a differentiable field $\hat{\boldsymbol{\phi}}: \Omega \to F$ of class C^k on B such that $\boldsymbol{\phi}$ is a restriction of $\hat{\boldsymbol{\phi}}$ to M, that is $\boldsymbol{\phi} = \hat{\boldsymbol{\phi}}_{|M}$.

A continuous and analytical field in the sense of this definition is defined in the same manner. Let us note that we do not require here that M is a differential geometric surface, it need not be smooth nor even regular.

Certainly, it may be possible that if M is not smooth but piecewise smooth, there can exist a field $\hat{\boldsymbol{\phi}}: \Omega \to F$, whose restriction to M will be a continuous or even differentiable field $\boldsymbol{\phi}: M \to F$.

The above definition, unlike the previous one, has another important property: it can be extended to rectifiable sets, i.e. to geometric measure surfaces.

Surface gradient. Let M be an open, regular surface of class C^r with $r \geq 1$ large enough for any operation required. Thus the tangent space $T_y M$ at each point $y \in M$ is well defined. Let $\gamma : I \to M$ be a smooth curve on M, where $I \subset \mathbb{R}$ is an open interval containing zero.

Given a differentiable field $\phi : M \to F$ on M, the restriction of ϕ to the curve γ becomes a F-valued function of the real argument, $\phi(\gamma(t))$, whose derivative with respect to a parameter $t \in I$ is well defined. Setting $y = \gamma(0)$, the tangent vector to this curve is $\tau(y) = \dot{\gamma}(0)$. We define the directional derivative of a field $\phi : M \to F$ at the point $y \in M$ and in the direction of the vector $\tau(y) \in T_y M$ by $\dot{\phi}(y) = \dot{\phi}(\gamma(0))$. A field $\phi : M \to F$ is said to be differentiable at a point $y \in M$ if there exists a linear map (tensor) $\nabla_s \phi(y) : T_y M \to F$, called the surface gradient of ϕ, such that

$$\dot{\phi}(y) = \nabla_s \phi(y) \tau(y), \tag{D.14}$$

for every smooth curve $\gamma : I \to M$. The surface gradient of a given field is unique whenever it exists. Moreover, if $\phi : M \to F$ and $\psi : M \to F$ are differentiable fields so is the field $\lambda\phi + \psi$ for every real number , and we have

$$\nabla_s(\lambda\phi + \psi) = \lambda \nabla_s \phi + \nabla_s \psi. \tag{D.15}$$

More general, if a scalar field $\psi : M \to \mathbb{R}$ and a field $\phi : M \to F$ are differentiable, then so is the field $\psi\phi : M \to F$ and

$$\nabla_s(\psi\phi) = \psi \nabla_s \phi + \phi \otimes \nabla_s \psi. \tag{D.16}$$

The surface gradient of a scalar field ϕ is the unique tangential vector field $\nabla_s \phi$.

Partial derivatives. Let M denote a regular surface of class C^r with r large enough for any operation required. Usually, we shall assume that $r \geq 3$. Moreover, assuming that the parametrization of M is fixed.

The inner product on each tangent space $T_y M$ gives rise to the metric on the surface, i.e. the 2-covariant, symmetric, and proper non-degenerated tensor field, that is a bilinear real-valued map $a : T_y M \times T_y M \to \mathbb{R}$. The metric tensor makes the surface M into a Riemannian manifold.

The metric tensor $\boldsymbol{a}(y) \in T_y M \otimes T_y M$ in the given parametrization of M can be expressed in the classical form with components defined by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta , \qquad a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta , \qquad a = det a_{\alpha\beta} > 0 . \tag{D.17}$$

Now the unit normal vector $\boldsymbol{n}$ can be expressed in the following form

$$\boldsymbol{n} = \frac{1}{2} \epsilon^{\alpha\beta} \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta . \tag{D.18}$$

As simple implications of the definitions (D.9) and (D.17) we obtain the classical formulae

$$\boldsymbol{a}_\alpha = a_{\alpha\beta} \boldsymbol{a}^\beta , \qquad \boldsymbol{a}^\alpha = a^{\alpha\beta} \boldsymbol{a}_\beta , \qquad a^{\alpha\lambda} a_{\lambda\beta} = \delta^\alpha_\beta . \tag{D.19}$$

Given a parametric representation $\boldsymbol{\rho} : \Pi \to M$ of M, we can use the natural base vectors to define the directional derivatives along the coordinate curves:

$$\partial_\beta \boldsymbol{\phi}(y) = \big(\nabla_s \boldsymbol{\phi}(y)\big) \boldsymbol{a}_\beta(y) . \tag{D.20}$$

Taking further into account that the representative of the given field is given by $\tilde{\boldsymbol{\phi}}(\boldsymbol{\xi}) = \boldsymbol{\phi}\big(\boldsymbol{\varphi}(\boldsymbol{\xi})\big)$, from the chain rule we have

$$\partial_\beta \boldsymbol{\phi}(y) = \frac{\partial_\beta \tilde{\boldsymbol{\phi}}(\boldsymbol{\xi})}{\partial \xi^\beta} , \qquad y = \boldsymbol{\varphi}(\boldsymbol{\xi}) . \tag{D.21}$$

Thus the directional derivatives along the coordinate curves are the partial derivatives of the representative of the field. The partial derivatives obey the usual rule of differentiation, in particular, the Liebnitz rule, i.e.

$$\partial_\beta (\boldsymbol{\phi} \otimes \boldsymbol{\psi}) = (\partial_\beta \boldsymbol{\phi}) \otimes \boldsymbol{\psi} + \boldsymbol{\phi} \otimes (\partial_\beta \boldsymbol{\psi}) , \tag{D.22}$$

for any two differentiable fields $\boldsymbol{\phi} : M \to F$ and $\boldsymbol{\phi} : M \to F$.

If the partial derivatives (D.20) exist and are continuous at a point $y \in M$, then the field $\boldsymbol{\phi} : M \to F$ is differentiable at this point and its gradient can be computed according to

$$\nabla_s \boldsymbol{\phi} = \partial_\beta \boldsymbol{\phi} \otimes \boldsymbol{a}^\beta . \tag{D.23}$$

It must be stressed here that partial derivatives can exist, but a given field may not be differentiable at a point. Therefore, the formula (D.23) cannot be taken as the definition of the surface gradient.

Surface covariant derivative. A tangential vector field $\boldsymbol{u} \in C(TM)$ over a regular surface M of class C^r is said to be of C^k-class, $k \geq r \geq 1$, if the real-valued component functions u_β and hence u^β are of C^k-class for some and hence every parametrization of the surface M.

The surface gradient of a tangential vector field is defined as above. Let us note that the surface gradient of a tangential vector field will be a spatial tensor field, in general. The tangential derivative of a tangential vector field $\boldsymbol{u} \in C(TM)$ is the unique tangential tensor field $D\boldsymbol{u} \in C(TM \otimes TM)$ defined by

$$D\boldsymbol{u} = \boldsymbol{P}(\nabla_s \boldsymbol{u}). \tag{D.24}$$

If the surface M is given in parametric form, the tangential field can be expressed in the component form $\boldsymbol{u} = u_\beta \boldsymbol{a}^\beta$ and its tangential gradient is given by

$$D\boldsymbol{u} = u_{\alpha|\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta . \tag{D.25}$$

If M is of class C^2 or higher then the Gauss map $\boldsymbol{n} : M \to S^2$ is differentiable and the curvature tensor $\boldsymbol{b}(\boldsymbol{y}) \in T_y M \otimes T_y M$ is defined by

$$\boldsymbol{b} = -D\boldsymbol{n} = -\boldsymbol{P}(\nabla_s \boldsymbol{n}). \tag{D.26}$$

For the chosen parametrization of M the curvature tensor can be expressed in the form $\boldsymbol{b} = b_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$ with the components defined by

$$b_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{n} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{n}_{,\beta} , \qquad b^\alpha_\beta = a^{\alpha\lambda} b_{\lambda\beta} = a_{\alpha\lambda} b^{\lambda\beta} . \tag{D.27}$$

Divergence of a surface field. Given a differentiable tangential field $\boldsymbol{u} \in C^k(TM)$, the surface divergence is a scalar field $Div_s \boldsymbol{u} \in C^{k-1}(M)$ defined pointwise by

$$Div_s \boldsymbol{u}(\boldsymbol{y}) = tr\big(D\boldsymbol{u}(\boldsymbol{y})\big). \tag{D.28}$$

Note that $D\boldsymbol{u} \in C^{k-1}(TM \otimes TM)$ is a tangential tensor field and hence the trace operation is well defined.

For all differentiable tangential fields $\boldsymbol{u}, \boldsymbol{w} \in C^k(TM)$ and every real number $\lambda \in \mathbb{R}$, we have

$$Div_s(\lambda \boldsymbol{u} + \boldsymbol{w}) = \lambda Div_s \boldsymbol{u} + Div_s \boldsymbol{w} . \tag{D.29}$$

Taking the trace of (D.25) we have

$$Div_s \boldsymbol{u} = a^{\alpha\beta} u_{\alpha|\beta} = u^{\beta}{}_{|\beta} . \tag{D.30}$$

The divergence of a differentiable field $\boldsymbol{\Phi} \in C^k(F \otimes TM)$ is the unique F-valued field, $Div_s \boldsymbol{\Phi} \in C^{k-1}(M, F)$, defined pointwise by

$$\big(Div_s \boldsymbol{\Phi}(y)\big) \cdot \boldsymbol{v} = Div_s\big((\boldsymbol{\Phi}(y))^T \boldsymbol{v}\big), \tag{D.31}$$

for every constant vector $\boldsymbol{v} \in F$.

Let us note that $\boldsymbol{\Phi}(y)^T \boldsymbol{v} \in T_y M$ at every point y so that $\boldsymbol{\Phi}^T \boldsymbol{v} \in C^{k-1}(TM)$ is a differentiable tangential field of the same class. Thus the right-hand side of (D.31) is well defined. As simple implication of the linear property of the surface gardient operator we have

$$Div_s(\lambda \boldsymbol{\Phi} + \boldsymbol{\Psi}) = \lambda Div_s \boldsymbol{\Phi} + Div_s \boldsymbol{\Psi} , \tag{D.32}$$

for all differentiable fields $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in C^k(F \otimes TM)$ and every real number.

If the surface M is given in the parametric form, the field $\boldsymbol{\Phi} \in C^k(F \otimes TM)$ can be expressed in the form $\boldsymbol{\Phi} = \boldsymbol{\phi}_\beta \otimes \boldsymbol{a}^\beta$. Then

$$\boldsymbol{\Phi}^T \boldsymbol{v} = (\boldsymbol{a}^\beta \otimes \boldsymbol{\phi}_\beta)\boldsymbol{v} = \boldsymbol{a}^\beta (\boldsymbol{\phi}_\beta \cdot \boldsymbol{v}) \tag{D.33}$$

and

$$Div_s(\boldsymbol{\Phi}^T \boldsymbol{v}) = a^{\alpha\beta}(\boldsymbol{\phi}_{\alpha|\beta} \cdot \boldsymbol{v}) = (a^{\alpha\beta} \boldsymbol{\phi}_{\alpha|\beta}) \cdot \boldsymbol{v} . \tag{D.34}$$

Hence

$$Div_s \boldsymbol{\Phi} = a^{\alpha\beta} \boldsymbol{\phi}_{\alpha|\beta} = \boldsymbol{\phi}^{\beta}{}_{|\beta} . \tag{D.35}$$

Differential identities. Let a scalar field $\phi \in C^k(M)$, spatial vector fields $\boldsymbol{u}, \boldsymbol{v} \in C^k(M, E)$ and a tensor field $\boldsymbol{S} \in C^k(E \otimes TM)$ be given, $k \geq 1$. The following identities hold:

$$\begin{aligned} \nabla_s(\phi\psi) &= \phi\nabla_s\psi + \psi\nabla_s\phi, \\ \nabla_s(\phi\boldsymbol{u}) &= \phi\nabla_s\boldsymbol{u} + \boldsymbol{u}\otimes\nabla_s\phi, \\ \nabla_s(\boldsymbol{u}\cdot\boldsymbol{w}) &= (\nabla_s\boldsymbol{u})^T\cdot\boldsymbol{w} + (\nabla_s\boldsymbol{w})^T\cdot\boldsymbol{u}. \end{aligned} \tag{D.36}$$

$$\begin{aligned} div_s(\phi\boldsymbol{u}) &= \phi\, div_s\boldsymbol{u} + \boldsymbol{u}\cdot\nabla_s\phi, \\ div_s(\phi\boldsymbol{S}) &= \phi\, div_s\boldsymbol{S} + \boldsymbol{S}(\nabla_s\phi), \\ div_s(\boldsymbol{S}^T\boldsymbol{u}) &= (div_s\boldsymbol{S})\cdot\boldsymbol{u} + \boldsymbol{S}\cdot\nabla_s\boldsymbol{u}. \end{aligned} \tag{D.37}$$

Singular curves for surface fields. Let Γ be a smooth (e.g. C^1-class) curve on a regular surface M of class C^r with $r \geq 1$ enough large (for example), which divides M into two parts $M^{(-)}$ and $M^{(+)}$ having the common boundary along the curve Γ (Fig. 19). Let $\boldsymbol{\phi}: M \to F$ be a continuous field on M except possibly at points belonging to the curve Γ. We also require that the field $\boldsymbol{\phi}$ has finite limits,

$$\boldsymbol{\phi}^{(\pm)}(y) = \lim_{z\to y}\boldsymbol{\phi}(z), \qquad z \in M^{(\pm)}, \tag{D.38}$$

at all points $y \in \Gamma$ taken from the positive and negative sides, that is from paths entirely within $M^{(-)}$ and $M^{(+)}$, respectively. Both limits $\boldsymbol{\phi}^{(\pm)}$ are smooth fields along the curve Γ. The difference of the limits,

$$[\![\boldsymbol{\phi}]\!](y) = \boldsymbol{\phi}^{(+)}(y) - \boldsymbol{\phi}^{(-)}(y), \tag{D.39}$$

is called jump along the curve Γ.

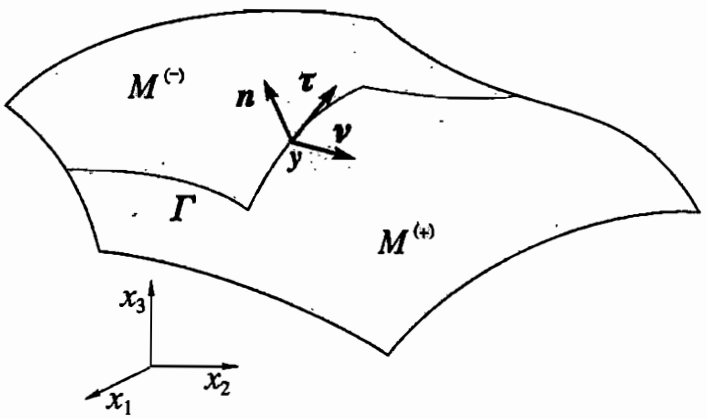


Fig. 19

If the field $\boldsymbol{\phi}$ is continuous across the curve Γ, then $[\![\boldsymbol{\phi}]\!]=\mathbf{0}$. If a field $\boldsymbol{\phi}$ together with its surface gradients up to order k–1 suffer no jumps along the curve Γ, but its partial derivatives of order k or higher have jumps across Γ, then the curve Γ is called a singular curve of order k for the field $\boldsymbol{\phi}$. The strongest singularity is of order zero, in which $\boldsymbol{\phi}$ is discontinuous along the curve Γ.

The fundamental result, upon which the theory of singular curves rests, is known as Hadamard's lemma, according to which the chain rule for the differentiation holds on the singular curves:

$$\partial_\nu[\![\boldsymbol{\phi}]\!]=[\![\nabla_s\boldsymbol{\phi}]\!]\boldsymbol{\nu}\,,\qquad \partial_\tau[\![\boldsymbol{\phi}]\!]=[\![\nabla_s\boldsymbol{\phi}]\!]\boldsymbol{\tau}\,. \tag{D.40}$$

This asserts that the jump in the tangential derivative is the tangential derivative of the jump. As simple implication we obtain

$$[\![\nabla_s\boldsymbol{\phi}]\!]=(\partial_\nu[\![\boldsymbol{\phi}]\!])\otimes\boldsymbol{\nu}+(\partial_\tau[\![\boldsymbol{\phi}]\!])\otimes\boldsymbol{\tau}\,. \tag{D.41}$$

This result, which is known as Maxwell's theorem, expresses the fact that the jump in the surface gradient of a continuous field is normal to the singular curve.

Repeated application of Hadamard's lemma gives the same rule for the higher derivatives.

Appendix E

Surface divergence theorems

Green's theorem in the plane. The classical Green's theorem is an extension to the plane (i.e. the two-dimensional space $\mathbb{R}^2$) of the fundamental theorem of integral calculus, which states that if $[a,b]\subset\mathbb{R}$ is an interval and $f:[a,b]\to\mathbb{R}$ is a continuously differentiable function, then

$$\int_a^b f'(x)dx = f(b)-f(a). \tag{E.1}$$

As extension of this theorem to the plane one considers a sufficently "nice" region $\Pi\subset\mathbb{R}^2$ with a positively oriented boundary $\partial\Pi$. An example of such a "nice" region is shown in Fig. 1. Let $f_1(\xi^1,\xi^2)$ and $f_2(\xi^1,\xi^2)$ be two real-valued functions defined and continuous on $cl\Pi=\Pi\cup\partial\Pi$. Let us further assume that the partial derivatives of both functions exist and are bounded in the interior of Π. Then

$$\iint_\Pi\left(\frac{\partial f_2}{\partial\xi^1}-\frac{\partial f_1}{\partial\xi^2}\right)d\xi^1 d\xi^2 = \int_{\partial\Pi}\left(f_1 d\xi^1 + f_2 d\xi^2\right), \tag{E.2}$$

whenever the double integral exists. This result is usually attributed to G. Green and it is known as the classical Green's theorem.

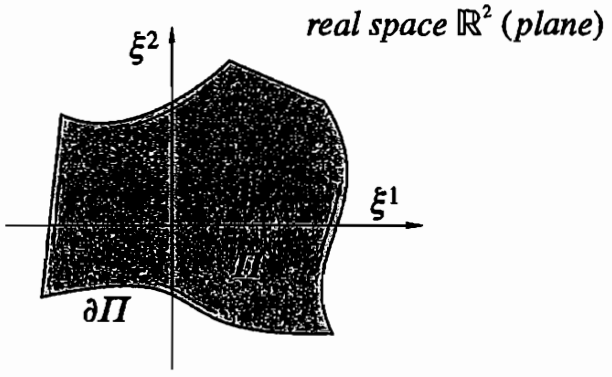


Fig. 1

The proof of Green's theorem presents no difficulties when Π is a rectangle, since then we can express the double integral and the line integral in (E.2) in terms of Riemann integrals. This method works also for many other regions such as disks or polygons. By a different approach Green's theorem (E.2) can be proved for every region in $\mathbb{R}^2$ bounded by a rectifiable Jordan curve (such as shown in Fig. 1).

In this appendix we state and outline the prove of a number of versions of the surface divergence theorem relating the integral of the divergence of a surface field on a given surface M to the line integral along its boundary. A surface divergence theorem requires the use of surface integrals. A surface integral can be thought of as a two-dimensional analogue of a line integral. However, before we discuss surface integrals, we must come to some understanding as to what we mean by a surface area.

Classical area formulae. Let $M \subset \mathcal{E}$ be an oriented simple surface element, i.e. the image $M = \boldsymbol{\rho}(\Pi)$ of a simple region $\Pi \subset \mathbb{R}^2$ under a smooth one-to-one map $\boldsymbol{\rho} : \Pi \to \mathcal{E}$. Then the position vector $\boldsymbol{y} = \boldsymbol{\rho}(\xi^\beta)$ of M is continuously differentiable in the parameter domain Π and $\boldsymbol{\rho}_{,1} \times \boldsymbol{\rho}_{,2} \neq \boldsymbol{0}$ at every point $\boldsymbol{\xi} \in \Pi$, $\boldsymbol{\xi} \equiv (\xi^\beta)$. Under this regularity condition the area of the surface element M is given by the classical formula of differential geometry:

$$area(M) = \iint_\Pi \| \boldsymbol{\rho}_{,1} \times \boldsymbol{\rho}_{,2} \| \, d\xi^1 d\xi^1 . \tag{E.3}$$

In this context one defines the oriented and the scalar differential area elements $\boldsymbol{da}$ and da, respectively, by the formulae

$$\boldsymbol{da} = (\boldsymbol{\rho}_{,1} \times \boldsymbol{\rho}_{,2}) d\xi^1 d\xi^2 , \qquad da = \| \boldsymbol{\rho}_{,1} \times \boldsymbol{\rho}_{,2} \| \, d\xi^1 d\xi^2 . \tag{E.4}$$

Recalling the definition of the unit normal vector $\boldsymbol{n}$ for a given parametric representation of the surface M, we have

$$\sqrt{a} = \sqrt{\det a_{\alpha\beta}} = (\boldsymbol{\rho}_{,1} \times \boldsymbol{\rho}_{,2}) \cdot \boldsymbol{n} > 0 . \tag{E.5}$$

Then the formula (E.3) can be written as

$$area(M) = \iint_\Pi \sqrt{a(\xi^1, \xi^2)} \, d\xi^1 d\xi^2 . \tag{E.6}$$

If we write $da(\boldsymbol{y}) = da(\boldsymbol{\rho}(\boldsymbol{\xi}))$ and $da(\boldsymbol{y}) = da(\boldsymbol{\rho}(\boldsymbol{\xi}))$, then the classical area formula (E.3) can be expressed in the form

$$area(M) = \iint_M n(y) \cdot da(y) = \iint_M da(y). \tag{E.7}$$

The classical area formula (E.7) can be applied to compute the area of more general, but not all, differential geometric surfaces.

Let M be a piecewise smooth regular surface and let a partition of M be given, i.e. M is given as the union $M = \bigcup_{k=1}^{n} M_{(k)}$ of a finite number of smooth surface elements $M_{(k)}$. We then define the area of M to be given by

$$area(M) = \sum_{k=1}^{n} \iint_{M(k)} da(y), \tag{E.8}$$

where the integrals on the right-hand side of (E.8) need to be computed according to the classical formula (E.7).

Area of a general surface. While the definition (E.7) and the more general definition (E.8) of the area of a given surface are entirely satisfactory as far as they go, they are open to at least two fundamental objections:

1) The formula (E.7) should be rather proved on the basis of the previously stated definition of a surface area than taken as definition.
2) More general surfaces need to be considered.

It is much more difficult to define the area of a general surface than the length of a general curve. A thorough discussion of the theory of surface area would exceed the scope of this work.

In the last century the general opinion was that, for a reasonable surface, the supremum of the area of all inscribed elementary polyhedral surfaces is finite and that the area of the surface could then be defined as that supremum. This was shown to be false by H.A. Schwarz (1880) and independently by G. Peano (1882). The fundamental error was due to the unverified assumption that, for any three points on a surface converging to a common limit point, the planes passing through the three points converge to the tangent plane of the surface at that limit point.

Since the surfaces, considered in the last century, were always regular surfaces, this fallacy in the definition of surface area never led to incorrect results for specific cases. Schwarz's discovery gave rise to extensive investigations of surfaces and the notion of surface area. Of the many definitions of area, which

have been proposed, two have been proved to be particularly useful: the definition of the "Lebesgue content" of a Fréchet surface and Hausdorff's generalization of Caratheodory's notion of the two-dimensional measure of a point set in space.

Let $M=(\rho,\Pi)$ be a given parametric surface. If the position vector $y=\rho(\xi^\beta)$ of M is continuously differentiable in the interior of the parameter domain Π, then the area integral is defined by

$$I(M)=\iint_{\mathrm{int}\,\Pi}\|\rho_{,1}\times\rho_{,2}\|\,d\xi^1 d\xi^1\,. \tag{E.9}$$

In general, this formula must be used with great caution, even if the integral is well defined. J.W.T. Youngs showed that every surface has a parametrization, for which the components of the vector product exist almost everywhere and are integrable, but for which the surface area integral has the value zero rather than the correct value of the area.

Surface integral. The main idea of a surface integral is to transfer the concept of integration of functions over surfaces to the integration of functions over domains in the real space $\mathbb{R}^2$. This approach works particularly nicely for orientable differential geometric surfaces.

Let M be an oriented simple surface, i.e. the image $M=\rho(\Pi)$ of a simple region $\Pi\subset\mathbb{R}^2$ under a smooth one-to-one map $\rho:\Pi\to\mathcal{E}$. If $\phi: M\to\mathbb{R}$ is a real-valued continuous function on M, we define the integral of ϕ over M to be

$$\iint_M\phi(y)\,da(y)=\iint_\Pi\phi(\rho(\xi^1,\xi^2))\sqrt{a(\xi^1,\xi^2)}\,d\xi^1 d\xi^2\,. \tag{E.10}$$

It can be shown that this definition is independent of the parametrization of the surface.

If M is a piecewise smooth surface M and $\phi: M\to\mathbb{R}$ is a continuous function on each smooth surface element $M_{(k)}$, then the integral of ϕ over M is defined by

$$\iint_M\phi\,da=\sum_{k=1}^n\iint_{M_{(k)}}\phi\,da\,. \tag{E.11}$$

For example, the integral over the surface of the cube may be expressed as the sum of the integrals over the six sides.

In the same manner one defines the integral of a function $\phi: M \to \mathbb{R}$ along a given smooth curve Γ on the surface M.

$$\int_\Gamma \phi(y)\,dl(y) = \int_a^b \phi(t)\alpha(t)\,dt\,, \qquad \alpha(t) = \sqrt{\dot\rho(t)\cdot\dot\rho(t)}\,. \tag{E.12}$$

The formulae (E.11) and (E.12) apply also to vector-valued fields over a given surface

$$\iint_M \phi(y)\,da(y) = \iint_\Pi \phi(\xi^1,\xi^2)\sqrt{a(\xi^1,\xi^2)}\,d\xi^1 d\xi^2\,. \tag{E.13}$$

In the following, when considering integrals of fields over surfaces and curves in the Euclidean point space $\mathcal{E}$, we shall write da and dl for the area measure and length measure, respectively.

Basic properties of surface integrals. From the definition (E.13) of the surface integral we can deduce some fundamental properties. These properties are essentially the same as for the integrals of functions over regions in the plane. In the following, F and F' will denote any finite dimensional inner product vector spaces.

Let $\phi, \psi: M \to F$ be integrable fields over a surface M, and let $\lambda \in \mathbb{R}$ be any real number. Then $\lambda\phi + \psi$ is a F-valued integrable field on M, and

$$\iint_M (\lambda\phi + \psi)\,da = \lambda \iint_M \phi\,da + \iint_M \psi\,da\,. \tag{E.14}$$

The transposition is the linear map of the tensor space $F' \otimes F$ into the tensor space $F \otimes F'$. The trace is the linear map of $F \otimes F$ into the set of real numbers. Both are continuous (they are algebraic functions). As a result we have the following useful formulae.

If a field $\Psi: M \to F' \otimes F$ is integrable over a surface M, then so is the field $\Psi^T: M \to F \otimes F'$. Moreover,

$$\iint_M \Psi^T\,da = \left(\iint_M \Psi\,da\right)^T. \tag{E.15}$$

If a field $\Psi: M \to F \otimes F$ is integrable over a surface M, then so is the field $tr\Psi: M \to \mathbb{R}$. Moreover,

$$\iint_M tr\Psi\,da = tr\left(\iint_M \Psi\,da\right). \tag{E.16}$$

By the same arguments, if $w \in C(M,E)$ is an integrable vector field, so is the skew tensor field $W \in C(M, E \wedge E)$, whose axial vector is w, and

$$ad\left(\iint_M w\, da\right) = \iint_M W\, da\,, \qquad W = adw\,. \tag{E.17}$$

In particular, for any two integrable vector fields u and w we have

$$ad\left(\iint_M u \times w\, da\right) = -\iint_M u \wedge w\, da\,. \tag{E.18}$$

Classical surface divergence theorem. In classical form the surface divergence theorem is essentially a generalization of Green's theorem for simple regions to smooth simple surfaces. It asserts that for a given tangential vector field $u \in C(TM)$ over a surface M,

$$\iint_M div_s u\, da = \int_{\partial M} u \cdot \nu\, dl\,, \tag{E.19}$$

where $\nu(y) \in T_y M$, $y \in \partial M$, denote the outward unit normal vector to the boundary curve lying in the tangent plane to the surface.

Let us note that the theorem (E.19) consists of two things, a surface and a surface field, and both must be sufficiently regular for the theorem to hold. However, it is by no means an easy task to make precise under which regularity assumptions this theorem holds. Fortunately, for a quite wider class of surfaces and fields this problem has a definite answer.

Let M be a regular surface of class C^r, $r \geq 1$, given in a parametric form $\rho : \Pi \to \mathcal{E}$, where $\Pi \subset \mathbb{R}^2$ denotes a closed subset of the plane which is homeomorphic to the disk and whose boundary $\partial\Pi$ is a piecewise smooth closed curve (Fig. 2). If we consider $\partial\Pi$ as a parametrized curve in the plane, then $\rho : \partial\Pi \to \mathcal{E}$ parametrizes a piecewise smooth curve ∂M, a consistently oriented boundary of M. Let further $u \in C^k(TM)$, $k \geq r$, be a tangential vector field on M, admitting an extension of the same class to the closure of M. Then the theorem (E.19) holds.

The proof is rather simple. Since the surface M is given in the parametric form, the field u can be represented in the form $u = u^\beta a_\beta$. Taking further into account that $Div_s u = u^\beta{}_{|\beta}$ and $u \cdot \nu = u^\beta \nu_\beta$, the theorem (E.19) can be written as

$$\iint_M u^\beta{}_{|\beta}\, da = \int_{\partial M} u^\beta \nu_\beta\, dl\,. \tag{E.20}$$

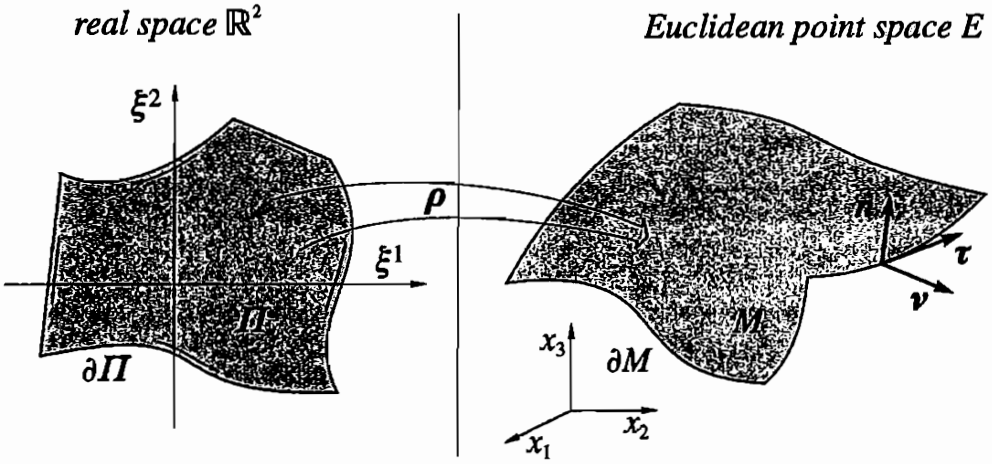


Fig. 2

Since $da = \sqrt{a}\, d\xi^1 d\xi^2$ and the divergence of the field $\boldsymbol{u}$ is given by

$$Div_s \boldsymbol{u} = u^\beta{}_{|\beta} = \frac{1}{\sqrt{a}} \left(\sqrt{a}\, u^\beta\right)_{,\beta}, \tag{E.21}$$

the left-hand side of (E.20) is equal to

$$\iint_M u^\beta{}_{|\beta}\, da = \iint_\Pi \left(\sqrt{a} u^\beta\right)_{,\beta} d\xi^1 d\xi^2 = \iint_\Pi \left(\frac{\partial\left(\sqrt{a} u^1\right)}{\partial \xi^1} + \frac{\partial\left(\sqrt{a} u^2\right)}{\partial \xi^2} \right) d\xi^1 d\xi^2 \tag{E.22}$$

by virtue of the definition (E.10) of the surface integral. The right-hand side of (E.20) can be written in the form

$$\begin{aligned} \int_{\partial M} u^\beta \nu_\beta\, dl &= \int_{\partial M} \epsilon_{\alpha\beta} u^\alpha \tau^\beta\, dl = \int_{\partial M} \epsilon_{\alpha\beta} u^\alpha \frac{d\xi^\beta}{ds} ds = \int_{\partial \Pi} \epsilon_{\alpha\beta} u^\alpha\, d\xi^\beta \\ &= \int_{\partial \Pi} \left(\sqrt{a} u^2 d\xi^1 - \sqrt{a} u^1 d\xi^2\right). \end{aligned} \tag{E.23}$$

Setting

$$f_1 = \sqrt{a} u^2, \qquad f_2 = -\sqrt{a} u^1, \tag{E.24}$$

in (E.22) and (E.23) we see that the surface divergence theorem (E.19) holds by virtue of the classical Green theorem (E.2) for two dimensions.

The surface divergence theorem can be proved also for more general simple smooth surfaces as well as for some, but not all, surfaces that are not simple.

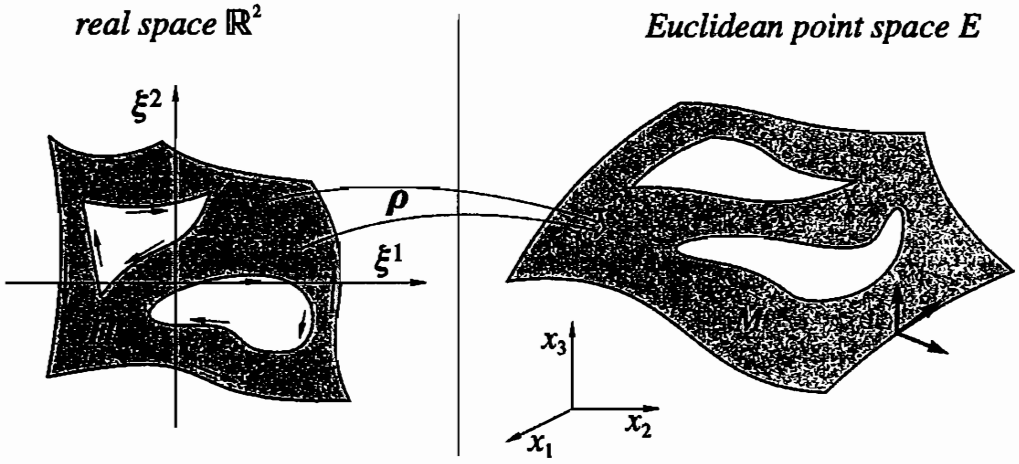


Fig. 3

If a parameter set Π is multiply connected with a finite number of holes, then a smooth one-to-one image $M = \rho(\Pi)$ will contain the same number of holes as Π (Fig. 3). In order to prove the theorem (E.19) for such surfaces we can use exactly the same type of argument as in the previous section, except that we employ Green's theorem for multiply connected plane regions. For example, if Π has n holes and if the boundary curves Γ_k, $k = 1,2,\ldots,n$, are transversed in the positive directions, the theorem (E.19) still applies with the boundary integral given by

$$\int_{\partial M} \boldsymbol{u} \cdot \boldsymbol{\nu}\, dl = \sum_{k=1}^{n} \int_{\Gamma_{(k)}} \boldsymbol{u} \cdot \boldsymbol{\nu}\, dl\,. \tag{E.25}$$

The sphere shown in Fig. 4 is an orientable smooth surface, which is not a simple surface but the union of two simple parametric surfaces (hemispheres). In this case, like for all orientable smooth surfaces without boundary, the theorem (E.19) still can be proved. Moreover, in such cases the boundary term vanishes. This will become clear from the theorem, which we shall prove later on.

There exists also a wide class of smooth surfaces, for which the theorem (E.19) fails to hold. Two classes of such surfaces are shown in Fig. 5 and Fig. 6. Their

common characteristic is the lack of suitable regularity properties at the boundaries.

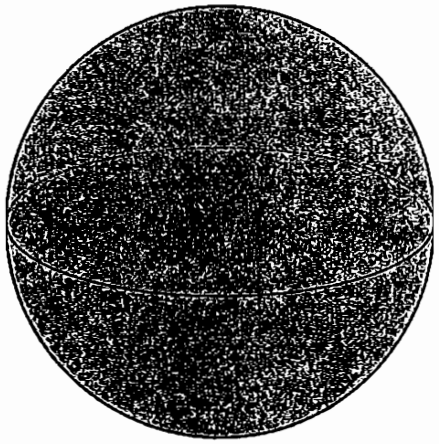


Fig. 4

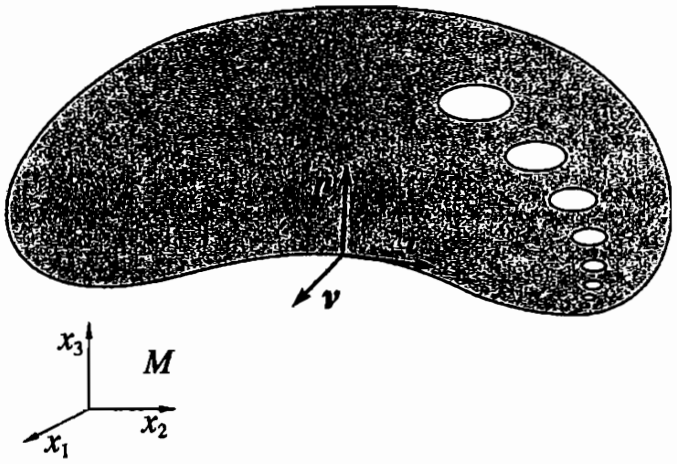


Fig. 5

On the other hand, if an underlying surface satisfies the regularity assumptions of the theorem (E.19), then this theorem can be extended to more general surface fields.

Theorems. Let S be a field of class C^k, $m \geq 1$, such that $S(y) \in E \otimes T_y M$ at each point $y \in M$. Then

$$\int_{\partial M} S\nu \, dl = \iint_M Div_s S \, da \,. \tag{E.26}$$

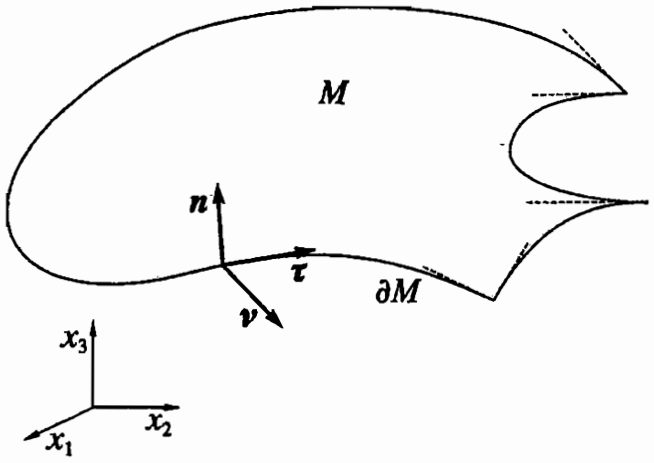


Fig. 6

More general, let F be any finite dimensional inner product vector space. Then for every tensor field $\boldsymbol{\Phi}$ of class C^k, $m \geq 1$, such that $\boldsymbol{\Phi}(y) \in F \otimes T_y M$, we have

$$\int_{\partial M} \boldsymbol{\Phi} \boldsymbol{\nu}\, dl = \iint_M Div_s \boldsymbol{\Phi}\, da\,. \tag{E.27}$$

Since the theorem (E.26) is just a special case of (E.27) upon taking $F = E$, we need only to prove the latter theorem.

We first note that for every constant vector $\boldsymbol{\kappa} \in F$, $\boldsymbol{\Phi}^T \boldsymbol{\kappa}$ is a tangential field of the same class. Hence the theorem (E.19) can be applied

$$\iint_M Div_s(\boldsymbol{\Phi}^T \boldsymbol{\kappa})\, da = \int_{\partial M} \boldsymbol{\Phi}^T \boldsymbol{\kappa} \cdot \boldsymbol{\nu}\, dl = \int_{\partial M} \boldsymbol{\kappa} \cdot \boldsymbol{\Phi} \boldsymbol{\nu}\, dl\,. \tag{E.28}$$

Recalling next that the divergence of the field is defined by $(Div_s \boldsymbol{\Phi}) \cdot \boldsymbol{\kappa} = Div_s(\boldsymbol{\Phi}^T \boldsymbol{\kappa})$, for every constant vector $\boldsymbol{\kappa} \in F$, we have

$$\iint_M Div_s(\boldsymbol{\Phi}^T \boldsymbol{\kappa})\, da = \iint_M \boldsymbol{\kappa} \cdot (Div_s \boldsymbol{\Phi})\, da\,. \tag{E.29}$$

Since $\boldsymbol{\kappa}$ is a constant vector, we have

$$\int_{\partial M} \boldsymbol{\kappa} \cdot \boldsymbol{\Phi} \boldsymbol{\nu}\, dl = \boldsymbol{\kappa} \cdot \left(\int_{\partial M} \boldsymbol{\Phi} \boldsymbol{\nu}\, dl \right). \tag{E.30}$$

By virtue of (E.29) and (E.30) the theorem (E.27) follows.

If the surface M is given in the parametric form, then the field $\boldsymbol{\Phi}$ can be rerpresented in the form $\boldsymbol{\Phi} = \boldsymbol{\phi}^{\beta} \otimes \boldsymbol{a}_{\beta}$ and $Div_s \boldsymbol{\Phi} = \boldsymbol{\phi}^{\beta}{}_{|\beta}$. Then the theorem (E.27) can be written as

$$\int_{\partial M} \boldsymbol{\phi}^{\beta} \nu_{\beta}\, dl = \iint_{M} \boldsymbol{\phi}^{\beta}{}_{|\beta}\, da\,. \tag{E.31}$$

Theorems. Let M be a regular surface of class C^r, $r \geq 1$. Let $\boldsymbol{u} \in C^k(M,E)$ be a spatial vector field and $S \in C^k(E \otimes TM)$ a tensor field, $k \geq r$, both having extensions of the same class to the closure of M. Then

$$\begin{aligned}
&\int_{\partial M} \boldsymbol{u} \otimes S\boldsymbol{v}\, dl = \iint_{M} \left(\boldsymbol{u} \otimes (Div_s S) + (\nabla_s \boldsymbol{u}) S^T\right) da\,,\\
&\int_{\partial M} \boldsymbol{u} \bullet S\boldsymbol{v}\, dl = \iint_{M} \left((Div_s S) \bullet \boldsymbol{u} + S \bullet \nabla_s \boldsymbol{u}\right) da\,,\\
&\int_{\partial M} \boldsymbol{u} \wedge S\boldsymbol{v}\, dl = \iint_{M} \left(\boldsymbol{u} \wedge (Div_s S) + (\nabla_s \boldsymbol{u}) S^T - S(\nabla_s \boldsymbol{u})^T\right) da\,,\\
&\int_{\partial M} \boldsymbol{u} \times S\boldsymbol{v}\, dl = \iint_{M} \left(\boldsymbol{u} \times (Div_s S) - ad^{-1}((\nabla_s \boldsymbol{u}) S^T - S(\nabla_s \boldsymbol{u})^T)\right) da\,.
\end{aligned} \tag{E.32}$$

Again we can actually prove more general theorems, of which (E.32) are special cases.

Proposition. Let $\boldsymbol{v} \in C^k(M,F)$ be a vector field and $\boldsymbol{\Phi} \in C^k(F \otimes TM)$ a tensor field, $k \geq r$, both having extensions of the same class to the closure of M. Then

$$\begin{aligned}
&\int_{\partial M} \boldsymbol{v} \otimes \boldsymbol{\Phi}\boldsymbol{v}\, dl = \iint_{M} \left(\boldsymbol{v} \otimes (Div_s \boldsymbol{\Phi}) + (\nabla_s \boldsymbol{v}) \boldsymbol{\Phi}^T\right) da\,,\\
&\int_{\partial M} \boldsymbol{v} \bullet \boldsymbol{\Phi}\boldsymbol{v}\, dl = \iint_{M} \left((Div_s \boldsymbol{\Phi}) \bullet \boldsymbol{v} + \boldsymbol{\Phi} \bullet \nabla_s \boldsymbol{v}\right) da\,,\\
&\int_{\partial M} \boldsymbol{v} \wedge \boldsymbol{\Phi}\boldsymbol{v}\, dl = \iint_{M} \left(\boldsymbol{v} \wedge (Div_s \boldsymbol{\Phi}) + (\nabla_s \boldsymbol{v}) \boldsymbol{\Phi}^T - \boldsymbol{\Phi}(\nabla_s \boldsymbol{v})^T\right) da\,.
\end{aligned} \tag{E.33}$$

Let us note here that since $\boldsymbol{v}$ is a F-valued field, $\boldsymbol{v}(\boldsymbol{y}) \in F$, hence $\nabla_s \boldsymbol{v}(\boldsymbol{y}) \in F \otimes T_{\boldsymbol{y}} M$ at each point $\boldsymbol{y} \in M$. Moreover, $\boldsymbol{\Phi}(\boldsymbol{y}) \in F \otimes T_{\boldsymbol{y}} M$ and hence $\boldsymbol{\Phi}(\boldsymbol{y})^T \in T_{\boldsymbol{y}} M \otimes F$. Thus

$$\nabla_s \boldsymbol{v}(\boldsymbol{y}) \boldsymbol{\Phi}(\boldsymbol{y})^T \in F \dot{\otimes} F \tag{E.34}$$

is well defined.

Taking the trace of both sides of $(E.33)_1$ we obtain $(E.33)_2$. The transposition of both sides of $(E.31)_1$ yields

$$\int_{\partial M} \boldsymbol{\Phi}\boldsymbol{v} \otimes \boldsymbol{v}\, dl = \iint_{M} \left((Div_s \boldsymbol{\Phi}) \otimes \boldsymbol{v} + \boldsymbol{\Phi}(\nabla_s \boldsymbol{v})^T\right) da\,. \tag{E.35}$$

The difference of both sides of $(E.33)_1$ and (E.35) yields $(E.33)_3$. Moreover, both sides of $(E.33)_3$ are necessarily skew-symmetric tensors. Hence, taking F to be the three-dimensional Euclidean vector space E, we may compute the associated axial vectors. Having in mind that

$$\int_{\partial M} \boldsymbol{u} \wedge \boldsymbol{S}\boldsymbol{v}\, dl = -\int_{\partial M} ad(\boldsymbol{u} \times \boldsymbol{S}\boldsymbol{v})\, dl = -ad\left(\int_{\partial M} \boldsymbol{u} \times \boldsymbol{S}\boldsymbol{v}\, dl\right) \tag{E.36}$$

as implication of (E.36) we obtain $(E.32)_4$.

Piecewise smooth fields. In the previous section we have proved a number of surface divergence theorems under the assumption that all fields are smoothly differentiable over the entire surface M, which in turn was assumed to be a smooth, regular surface.

Let M be a regular surface of class C^r, $r \geq 1$, with a boundary ∂M consistently oriented with M. Let $\Gamma \subset M$ be a piecewise smooth curve, which divides M into two parts $M^{(-)}$ and $M^{(+)}$. We give Γ the orientation induced by $\partial M^{(-)}$, i.e. the orientation Γ inherits as part of the boundary of $M^{(-)}$ (Fig. 7).

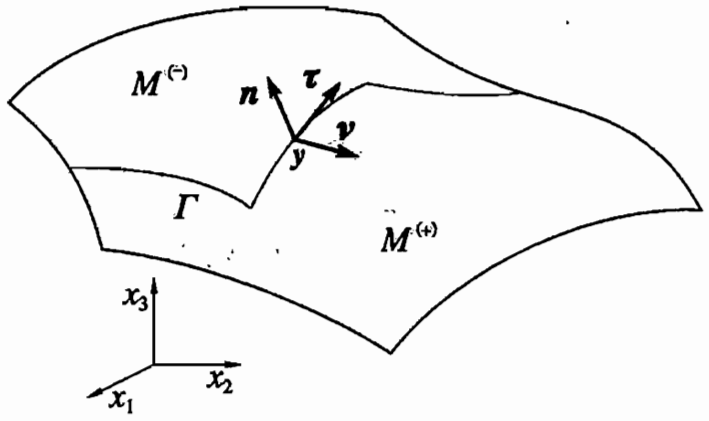


Fig. 7

Let us consider now a field $\boldsymbol{\Phi} \in C(F \otimes TM)$, which may have Γ as a discontinuous curve. Let us further assume that the field $\boldsymbol{\Phi}$ is differentiable in the interior of $M^{(-)}$ and $M^{(+)}$ and has finite limits at the curve Γ. Then the integral of $\boldsymbol{\Phi}$ over the entire surface M is the sum of integrals over the two parts:

$$\iint_M Div_s \boldsymbol{\Phi}\, da = \iint_{M^{(-)}} Div_s \boldsymbol{\Phi}\, da + \iint_{M^{(+)}} Div_s \boldsymbol{\Phi}\, da\,. \tag{E.37}$$

By virtue of the surface divergence theorem (E.27) applied separately to $M^{(-)}$ and $M^{(+)}$, we have

$$\begin{aligned} \iint_{M^{(-)}} Div_s \boldsymbol{\Phi}\, da &= \int_{\partial M^{(-)}\setminus\Gamma} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl + \int_{\Gamma} \boldsymbol{\Phi}^{(-)}\boldsymbol{\nu}^{(-)}\, dl\,, \\ \iint_{M^{(+)}} Div_s \boldsymbol{\Phi}\, da &= \int_{\partial M^{(+)}\setminus\Gamma} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl + \int_{\Gamma} \boldsymbol{\Phi}^{(+)}\boldsymbol{\nu}^{(+)}\, dl\,, \end{aligned} \tag{E.38}$$

where the limits are defined by

$$\boldsymbol{\phi}_{\nu}^{(-)} = \boldsymbol{\Phi}^{(-)}\boldsymbol{\nu}^{(-)} = \boldsymbol{\Phi}^{(-)}\boldsymbol{\nu}\,, \qquad \boldsymbol{\phi}_{\nu}^{(+)} = \boldsymbol{\Phi}^{(+)}\boldsymbol{\nu}^{(+)} = -\boldsymbol{\Phi}^{(+)}\boldsymbol{\nu}\,. \tag{E.39}$$

Adding the results of (E.38), we obtain

$$\begin{aligned} \iint_{M} Div_s \boldsymbol{\Phi}\, da &= \iint_{M^{(-)}} Div_s \boldsymbol{\Phi}\, da + \iint_{M^{(+)}} Div_s \boldsymbol{\Phi}\, da \\ &= \int_{\partial M^{(-)}\setminus\Gamma} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl + \int_{\Gamma} \boldsymbol{\phi}_{\nu}^{(-)}\, dl + \int_{\partial M^{(+)}\setminus\Gamma} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl - \int_{\Gamma} \boldsymbol{\phi}_{\nu}^{(+)}\, dl \\ &= \int_{\partial M} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl + \int_{\Gamma} \left(\boldsymbol{\phi}_{\nu}^{(-)} - \boldsymbol{\phi}_{\nu}^{(+)}\right) dl\,. \end{aligned} \tag{E.40}$$

The minus sign in the second line integral is due to the difference in the orientation of Γ taken as the boundary of $M^{(-)}$ and $M^{(+)}$. In this way we have shown that

$$\iint_{M\setminus\Gamma} Div_s \boldsymbol{\Phi}\, da = \int_{\partial M} \boldsymbol{\Phi}\boldsymbol{\nu}\, dl - \int_{\Gamma} [\![\boldsymbol{\phi}_{\nu}]\!]\, dl\,, \tag{E.41}$$

where the jump across the curve Γ is defined by

$$\begin{aligned} [\![\boldsymbol{\phi}_{\nu}]\!] &= \boldsymbol{\phi}_{\nu}^{(+)} + \boldsymbol{\phi}_{\nu}^{(-)} = \boldsymbol{\Phi}^{(+)}\boldsymbol{\nu}^{(+)} + \boldsymbol{\Phi}^{(-)}\boldsymbol{\nu}^{(-)} \\ &= -\boldsymbol{\Phi}^{(+)}\boldsymbol{\nu} + \boldsymbol{\Phi}^{(-)}\boldsymbol{\nu} = -(\boldsymbol{\Phi}^{(+)} - \boldsymbol{\Phi}^{(-)})\boldsymbol{\nu}\,, \end{aligned} \tag{E.42}$$

or

$$[\![\boldsymbol{\phi}_{\nu}]\!] = -[\![\boldsymbol{\Phi}]\!]\boldsymbol{\nu}\,. \tag{E.43}$$

Exactly in the same way together with the use of standard differential identities, we can prove a number of theorems for more general tensor fields. However, we shall see below that the theorem (E.41) is a special case of a more general one.

Pachwork surface divergence theorems. Let M be a piecewise smooth surface. Then the edge set Γ is defined as the union of all smooth curves $\Gamma_{(k,l)}$ on the common boundary between two regular surface elements $M_{(k)}$ and $M_{(l)}$ (see Appendix D):

$$M = \bigcup_{k=1}^{n} M_{(k)}, \qquad \Gamma = \bigcup_{k,l=1}^{n} \Gamma_{(k,l)}. \tag{E.44}$$

At every point $y \in \Gamma_{(k,l)}$ the two unit normals $n^{(k)}$ and $n^{(l)}$ are defined as limits of the adjoining surface elements. Let $\nu^{(k)}$ and $\nu^{(l)}$ denote the outward unit normal vectors in the tangent planes $T_y M^{(k)}$ and $T_y M^{(k)}$, respectively, as shown in Fig. 8.

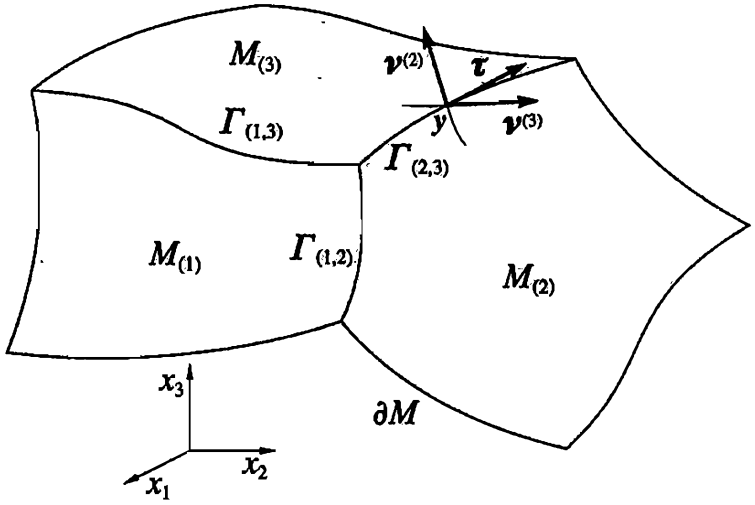


Fig. 8

Let $\Phi \in C(F \otimes TM)$ be a field on M that may have Γ as a discontinuous curve but otherwise differentiable in the interior of each smooth surface element $M_{(l)}$, i.e it is of class C^n, $n \geq 1$, in the interior $int\, M_{(k)}$. We shall also assume that the field Φ has finite limits at the curve Γ:

$$\phi_{\nu}^{(k)}(y) = \Phi^{(k)}(y)\nu^{(k)}(y) = \lim_{z \to y,\, z \in M_{(k)}} \Phi(z)\nu(z). \tag{E.45}$$

Then the integral of Φ over M can be expressed as sum of the integrals over all regular parts:

$$\iint_{M \setminus \Gamma} Div_s \Phi \, da = \sum_{k=1}^{n} \iint_{M_{(k)}} Div_s \Phi \, da. \tag{E.46}$$

Applying the surface divergence theorem (E.27) to each smooth surface element we have

$$\iint_{M_{(k)}} Div_s \Phi \, da = \int_{\partial M_{(k)} \setminus \Gamma} \Phi \nu \, dl + \int_{\Gamma} \Phi^{(k)} \nu^{(k)} \, dl. \tag{E.47}$$

Adding the results we obtain

$$\int_{\partial\Pi} \boldsymbol{\Phi}\boldsymbol{v}\, dl = \iint_{\Pi\setminus\Gamma} Div_s\boldsymbol{\Phi}\, da - \int_{\Gamma} [\![\boldsymbol{\Phi}\boldsymbol{v}]\!]\, dl, \tag{E.48}$$

where the jumps are defined by

$$[\![\boldsymbol{\Phi}\boldsymbol{v}]\!] = \boldsymbol{\Phi}^{(k)}\boldsymbol{\nu}^{(k)} + \boldsymbol{\Phi}^{(l)}\boldsymbol{\nu}^{(l)}. \tag{E.49}$$

It is seen now that if the surface M is smooth, but the field suffers jump discontinuity across the curve Γ, then the theorem (E.48) reduces to the theorem (E.39).

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