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Whitham modulation theory and direct methods for nonlinear dispersive waves

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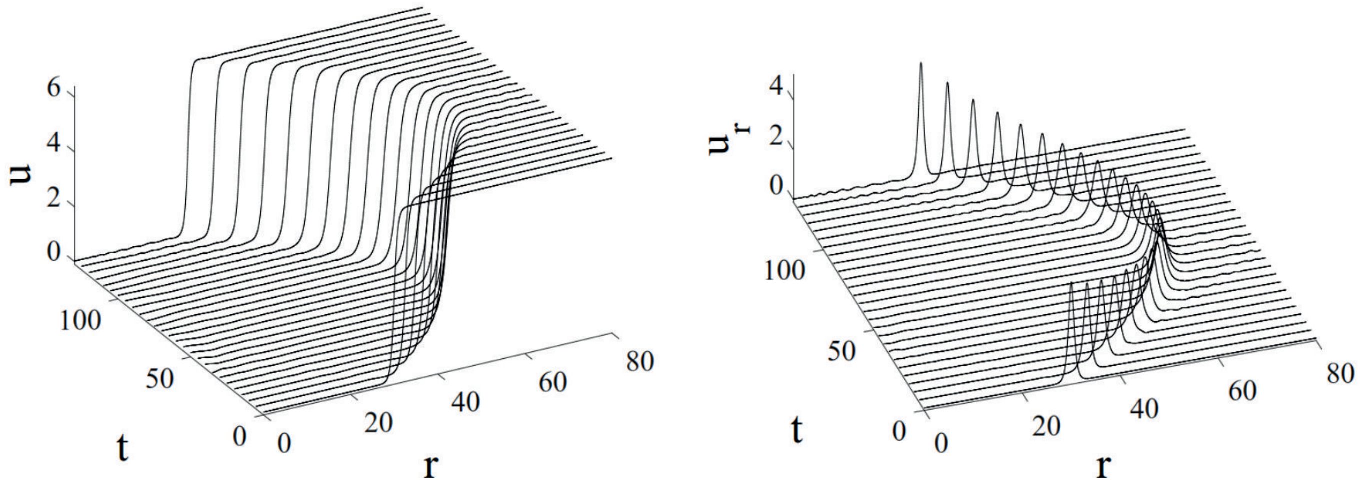

(a) Solution $u(r,t)$ (b) Derivative $u_r(r,t)$

Dissertation

Whitham modulation theory and direct methods for nonlinear dispersive waves

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to my parents

Abstract

This dissertation is contributed to the study of nonlinear dispersive waves by using the Whitham modulation theory and the Hirota direct method and its relevant techniques. The Whitham modulation equations can be derived with the aid of the variational-asymptotic method. Accordingly, the amplitude modulation equations for the Korteweg-de Vries equation, the Boussinesq equation and the slope modulation equations for the multi-dimensional sine-Gordon equation are obtained. Subsequently, the asymptotic modulation solutions describing the amplitude of the train of solitons and the single positon at large time are obtained. The direct methods are applied to seek the multi-soliton solution, the positon solution and the complexiton solution. The modulation solution for the sine-Gordon equation is obtained to describe the slope of the solution and the predictor-corrector scheme is used to integrate the equation numerically. The comparison between the modulation solutions and exact or numerical solutions shows excellent agreement and validates the modulation theory.

I would like to express my gratitude to my first supervisor Prof. Khanh Chau Le for his academic mentoring. I would like to thank my second supervisor Prof. Klaus Hackl for his helpful advices. Lastly I would like to dedicate this dissertation to my parents who have been constantly supporting me.

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1. Supervisor: Prof. Khanh Chau Le
2. Supervisor: Prof. Klaus Hackl

Contents

Literature review

The starting point of the soliton theory and the nonlinear dispersive waves is the discovery of solitary wave by Scott Russel during his experiments in the Union Canal, connecting Edinburgh and Glassgow [1]. This discovery arouse a great controversy surrounding the mathematicians about if his observation had been included in the linear wave theory. It took more than half a century that the phenomenon was finally justified by the celebrated paper by Korteweg and de Vries [2]. They derived the equation of shallow water waves traveling in one direction, which was later named after them, without being aware of the works by Boussinesq [3, 4]. It is understandable since the communication in academia at the time was not as convenient as in our electronic era. For the historical interest, we refer to a comprehensive review of the origin of the Korteweg-de Vries (KdV) equation and the relevant Boussinesq equation which was done at the Korteweg-de Vries institute, University of Amsterdam, by de Jager [5].

Although these two equations had been derived for a long time, it was only until the discovery of the so-called inverse scattering transform (IST) for the nonlinear Schödinger equation and the similar formulation of the KdV equation in the inverse scattering spirit, the shallow water wave equations can be treated thoroughly [6–9]. Shortly after that several evolution equations in physics and water waves were revisited by using the IST. A list of equations that have been treated by the IST can be found in the classic book by Ablowitz who has been one of the leading researchers in the field of nonlinear dispersive waves [10]. Although this method provides us a tool for fully integrating the nonlinear evolution equations, it requires a dramatically large background in functional analysis and complex analysis. Most importantly, it is only applicable to the integrable systems. In our context the integrability is understood in the sense that there exists a Lax pair for the evolution equation to reduce the nonlinear problem to the generalized linear eigenvalue problem [10]. In this sense the question of integrability can be only answered on a case-by-case basis and it implies that the method has its own limits. In addition to this, a full range of solutions subject to arbitrary initial conditions can be achieved only with the aid of the more general inverse spectral methods that are frequently reducible to Riemann-Hilbert problems (see a short introduction in [11]). This should be remarked here that all these achievements had been stimulated by a fascinating finding by Zabusky and Kruskal in their numerical attempt of simulating the interaction of solitons in a collisionless plasma [12]. The particle-like behavior of this "clean" interaction gave a strong indication that multi-soliton solution exists. The suffix *-on* was also coined due to this particle-like interaction. Almost around the same time, in an independent research Hirota from Japan developed an effective technique that involves only symbolic calculation of differentiation to extract the multi-soliton solution governed by the KdV equation [13]. It turned out that many equations that admit the inverse scattering formulation can also be dealt with by his method. The Japanese research group has then developed it, which is nowadays called the Hirota direct method (HDM), and successfully applied it to solve several physical equations. The outcome of this promising technique is the fruitful results that are unified in a classic book by Hirota [14] in which one can find a list of equations that have been solved for multi-soliton solution. It should be noted that the formulation obtained by using the HDM can be written in the determinant form and thus can be interpreted as a special case extracted from the IST. It is argued that the HDM is still a formal method and hence only applicable to the integrable systems in the above sense. More strictly speaking, it can be applied to the equations that can be put into the bilinear form proposed by Hirota through some clever transformation. This transformation can be found by using the (modified) homogeneous balance method [15–18]. As the soliton solitons play important role in many applications of physics, there have been several analytical techniques devised to treat the time-dependent partial differential equations such as the $\tanh$-method [19], the $\exp$-function method [20], the variation-iteration method [21, 22], the Wronskian formulation method and so on. Although the method of Wronskian solution was probably first introduced by Satsuma[1] based on a fruitful discussion of the relationships between the IST, the Bäcklund transformation and an infinite number of conservation laws [23, 24], the method was later spread more worldwide by Freeman and Nimmo who exhibited the proof in detail and the construction of solution [25–27]. A generalization and short summary of a typical procedure to extract solutions from this technique can be found in [28]. It should be emphasized here that though several mathematical tools have been introduced, most of the credit should go to only two effective techniques, that is, the inverse scattering transform and the Hirota direct method. It is rather fair to claim the superiority of these over all the last-mentioned mathematical tools as they can be derived from these two "parental" methods. In fact, any variants based on the exponential functions work towards their advantage provided that we have some priori knowledge of the solution in that it can be expressed as a combination of the exponential functions. By applying these last-mentioned methods, we automatically restrict ourselves to the smaller branches of the general techniques IST and HDM. We conclude the paragraph by emphasizing that the above-mentioned methods are only for the integrable systems or for low-order solution of non-integrable systems.

Alternative to the analytical methods is the numerical treatment. There is a substantial literature of numerical studies of wave equations. Apparently, the finite difference method is the first candidate for the partial differential equations and hence for the wave equations. The method has been developed for a long time and a good reference would be directed to a compact but informative book by LeVeque [29]. Perhaps the most appealing to engineers is the finite element method. Among the very first numerical studies of wave equations in this direction, we refer the works done by Winther [30], Sanz-Serna and Christie [31] Argyris and Hasse [32], Carey, Jiang and Shen [33, 34]. There are certainly several numerical methods invented later based on these two large categories. Provided that we have a system of differential equations, either we work with its differential form on a grid of collocation points or an equivalent variational formulation is derived and solved with the Galerkin-based methods. Nevertheless, the author finds that the most effective technique at the moment would be the spectral methods shortly discussed below.

An ingenious idea for treating the non-integrable equations of nonlinear dispersive waves was introduced by Whitham. He was concerned with analyzing the wavetrains whose wave properties such as amplitude, wavenumber and any associated mean values may vary in space and time. But the work was initially not so attractive to mathematicians due to its nature of formality [35]. One year after this ground-breaking paper, Luke published an article justifying Whitham's idea in which the communication between them was clearly indicated [36]. Luke borrowed the perturbation procedure devised by Kuzmak and extended it to a simple partial differential equation in order to explain how Whitham modulation theory is valid in a "formal" perturbation expansion at the first approximation [37]. In the

[1] It is up to my best knowledge.

first appearance the Whitham modulation theory was derived by averaging the solutions governed by the set of appropriate conservation equations over the phase variable. The resultant set of equations are then called the modulation equations, and by intuitive argument they govern the evolution of the wave characteristics. In this piece of work Whitham also obtained the Riemann invariant form of the modulation equations for the KdV equation and argued that this helped to generalize the notion of group velocity in the linear wave theory to the characteristic velocities in the nonlinear theory. Probably he did not expect that his transformation of the KdV modulation equations to the corresponding Riemann invariant form contributed to different branches of applied mathematics in several aspects [38]. He recognized after some skillful calculation that three linear combinations of the roots of the cubic polynomial obtained in the first integral form the Riemann invariants. Later with the aid of the finite-gap integration method which was first invented to solve the periodic KdV initial problem, one can also find such Riemann invariants [39] in the averaged equations. In any case, finding the Riemann invariants of a certain integrable system is not trivial at all. At the time Whitham wrote the first paper introducing this technique he already suspected that there should be a variational formulation for the whole procedure by basing his argument on the beautiful work by Emmy Noether regarding the relation between the invariant of the Lagrangian density with respect to some variable and the corresponding conservation equation [40, 41]. This was finally realized by the appearance of his important paper that outlined how the procedure using the differential formulation can be bypassed by using the relevant variational formulation [42]. The modulation theory was subsequently used to study the water waves [43,44] and to reproduce Benjamin's theory of instability [45,46]. The modulation theory eliminated the cumbersome assumption in the formal expansion proposed by Benjamin based on priori knowledge and a great deal of experience. A few years later Whitham tailored all the crucial ideas to present a unified approach and to justify his method by the so-called two-timing variational principles [47]. Since then the Whitham modulation theory has proved its advantage in action. The modulation equations for the single phase solution of the KdV equation was then generalized to the multiple phase wavetrains by using the similar technique [48]. After the advent of the IST, the modulation equations for N phase solution were subsequently justified by rigorous functional and complex analysis. The latter method can be actually employed to derive the modulation equations for any equation with an inverse scattering solution and apparently this is considered mathematically rigorous.

Although Whitham realized at the time that the modulation equations for the KdV equation possessed a centred, simple wave solution, he did not think such solution is physically meaningful in the context of slowly varying wavetrain as it is generated in accordance with a jump initial condition. Much later after his notice the simple wave solution was published in a milestone paper by Gurevich and Pitaevskii [49] in which they showed that it corresponded to an undular bore. The undular bore consisted of a modulated wavetrain with a solitary zone at leading edge and a linear dispersive waves at trailing edge. The work raised a great interest of mathematicians and physicists as the undular bore is considered as dispersive shock waves. This work stimulated the joint work between Whitham and Fornberg on an investigation of a wide range of nonlinear dispersive waves by using the so-called pseudospectral (spectral) method [50]. The outcome of this collaboration helped to validate the modulation theory from the numerical point of view. The paper not only put the spectral methods into action for the partial differential equations but also discussed several theoretical aspects. Briefly speaking, the spectral methods are based on the discrete Fourier transform, which is nothing else but the truncation of the Fourier series, or a grid of spectral collocation points and possess a "spectral" accuracy of approximation in discretization [51, 52]. The discrete Fourier transform can be effectively computed using the Fast Fourier Transform (FFT) introduced by Cooley and Tukey[2] [53], so that the spectral methods have surged as a powerful tool for numerical investigation of time-dependent partial differential equations. After such a wide and thorough investigation, the hydrodynamics of dispersive shock waves has been studied intensively so that it has stood solely as an interesting subject. We refer the readers to the review article [55] in the special issue dedicated to the Whitham modulation theory being applied to dispersive hydrodynamics [56].

Whitham formulated the modulation equations and emphasized that this could be considered as a generalization of the Krylov-Bogoliubov averaging technique for the ordinary differential equations to the partial differential equations [57, 58]. However, it required an ingenious observation of Whitham to end up with the modulation theory for the wave equations admitting the periodic uniform wavetrain. In the Krylov-Bogoliubov approach the small parameter enters the equation through the coefficient of equation or slow variation of external input that is usually indicated explicitly by its dependence on slow parameter. On the other hand, such a small parameter in Whitham's mind lies in the initial condition. The initial wavetrain has the wave characteristics such as the wavenumber, wave frequency, and amplitude varying slowly at a much larger scale than the local oscillation of the wave. It is widely accepted that the variational-asymptotic method (VAM) is equivalent to the perturbation theory in the sense that the former is handled in a variational formulation, the latter in the differential formulation. Indeed, the VAM has been employed to reproduce several classical results in structural mechanics [59] continuum mechanics and fluid mechanics [60]. Consequently, the Whitham modulation theory can be effectively reflected in the variational-asymptotic analysis [61]. By comparing the procedure sketched in [47] and [61], one can easily recognize that the two approaches are completely equivalent.

Last but not least, the standard textbook by Whitham that has remained its tremendous influence on the linear and nonlinear wave theories is referred here [62]. Despite the fact that there has been a huge amount of materials on the soliton and wave theories, this textbook is widely accepted by applied mathematicians and physicists that it is the best reference up to date for both study and research purposes. The textbook by Kamchatnov focusing on the modulation theory using different mathematical ideas and tools should also be highlighted here [63]. The history as well as scientific achievements of Whitham (13th December, 1927 –26th January, 2014) can be found in the biographical article [64].

[2]The credit went to Cooley and Tukey for their bringing the FFT to the computation community, but Gauss was actually the first one who discovered this effective computation [54].

1 Modulation theory for nonlinear dispersive waves

1.1 Motivation and preliminary knowledge

The rise of the so-called soliton solution of the Korteweg-de Vries equation describing the traveling of uni-directed shallow water wave is extremely interesting. This kind of wave does not change its shape while it propagates in time. Indeed, this helps to answer many important questions in physics as well as to give reasonable explanation to physical phenomena. Backing to the history, we may understand better why it is the case. In fact, the solitary wave was observed and reported by John Scott Russel in 1834 and he called it the wave of translation at the time. While conducting experiments to determine the most efficient design for canal boats, he noticed the existence of wave with very long tail propagating without changing its form. The phenomenon was intricately described in his report and an excerpt is quoted here.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated round and prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300–450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [2–3 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

However, backing to the time, the mathematical tools were still limited to develop a theory explaining his observations. It took until the 1870s before an exposition was supplemented. Joseph Boussinesq mentioned Scott Russell's work in his papers [3, 4] and through these works the observations were for the first time admitted as true. Very later after his description, Korteweg and de Vries did not mention John Scott Russell's name at all in their celebrated paper in 1895 but they did quote Boussinesq's achievement. Despite the fact that the paper by Korteweg-de Vries in 1895 was not the first theoretical treatment of this subject that had raised a great controversy within the academic forum, it is widely accepted as the milestone in the history of soliton theory. Very much later the time the Korteweg-de Vries equation was firstly introduced, with the advent of the modern computers Norman Zabusky and Martin Kruskal could set up the numerical experiment to discover the surprising behaviors of the solitary waves. Despite of using a quite primitive finite difference method, they succeeded to find two important features of the soliton solutions governed by the KdV equation. On the one hand, many solitons can travel in one direction with unchanged speeds and amplitudes before and after their interaction, but on the other hand, they experience slight phase shifts after collisions. The phase shift can be interpreted as follows. The solitons slightly change their positions as compared to those as if they would have not experienced collisions. The observation suggests that the wavetrain behaves very much like photons in quantum mechanics and hence the name soliton is coined with the suffix "on" following the adjective "solitary". Immediately after these findings the mathematicians were extremely attracted to the evolution equations that may lead to soliton solutions and once again a foundation for analytical treatment of the soliton theory was established in the same decade. The credits were awarded to two efficient analytical methods which were developed independently almost around the same time by two groups, namely the inverse scattering transform and the Hirota direct method. Since then many scientists have revisited several challenging problems in the past and have obtained new exact solutions in the closed form. The existence of solitons helps physicists and engineers to explain many physical phenomena [65–71] and occasionally some biology mechanisms [72–74].

1.1.1 Solitary and periodic waves

Let us first go into the topic and introduce the relevant terminology by working on the example of the Korteweg-de Vries equation [2, 63]

$$\zeta_t + \sqrt{gh}\left(1 + \frac{3}{2}\frac{\zeta}{h}\right)\zeta_x + \frac{1}{6}h^2\sqrt{gh}\zeta_{xxx} = 0, \tag{1.1}$$

where ζ is the amplitude of the wave surface measured from the free surface, g the gravity constant and h the depth from the free surface to the basin. In Fig. 1.1 the sketch of setting for this equation is shown. This equation was derived in theoretical investigation

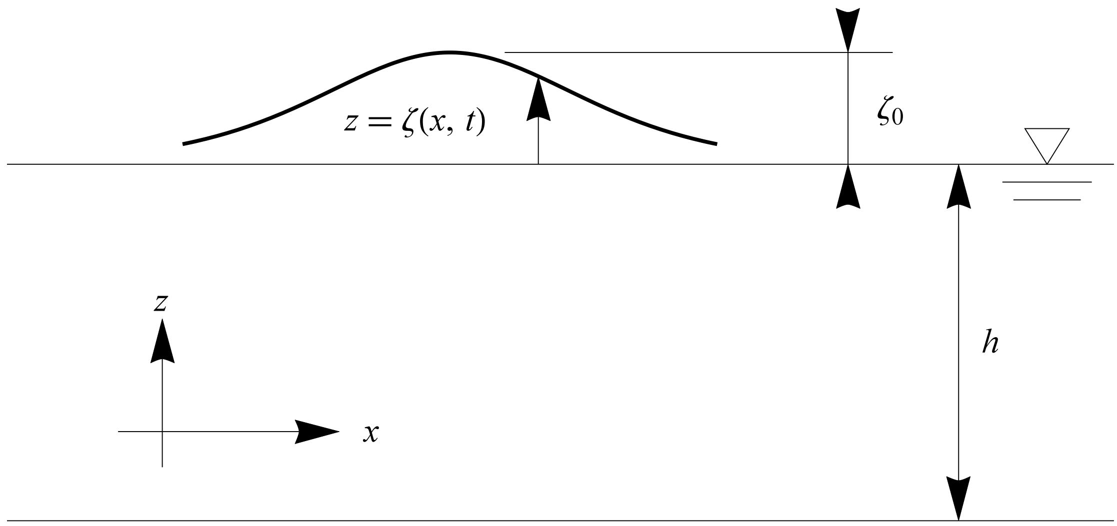


Figure 1.1: Definition sketch of the description of water shallow wave governed by the Korteweg-de Vries equation.

of nonlinear waves propagating on the surface of shallow water by Diederik Johannes Korteweg and Gustav de Vries. In the linear limit of infinitesimal amplitude ζ it can be linearly approximated to

$$\zeta_t + \sqrt{gh}\left(\zeta_x + \frac{1}{6}h^2\zeta_{xxx}\right) = 0,$$

where the linearization is taken about the mean value $\bar{\zeta} = 0$ and thus the nonlinear term $\zeta\zeta_x$ is neglected. At small but finite values of the amplitude it takes into account the nonlinear effects as well as the cubic order dispersive effect caused by the last term. To carry on the investigation, let us transform Eq. (1.1) to the dimensionless form

$$\eta_T + \frac{3}{2}\eta\eta_X + \frac{1}{6}\eta_{XXX} = 0$$

through the following change of variables

$$X = \frac{x - \sqrt{gh}\,t}{h}, \quad T = \sqrt{\frac{g}{h}}\,t, \quad \eta = \frac{\zeta}{h}. \tag{1.2}$$

The first variable reflects the alteration of the frame of reference so that the term $\zeta_t + \sqrt{gh}\zeta_x$ is interminably incorporated. Then after the scale of variables as follows

$$\mu = bX, \quad \nu = aT, \quad u = c\eta,$$

we obtain

$$au_\nu + \frac{3}{2}bcuu_\mu + \frac{1}{6}b^3u_{\mu\mu\mu} = 0,$$

in which the arbitrary constant parameters a, b, c can be chosen so that the coefficients acquire their desired values. Backing to the history of several discoveries on this equation, the most frequent choice, probably the most standard, is

$$a = \frac{9}{16}, \quad b = \frac{3}{2}, \quad c = \frac{3}{2}.$$

Upon abuse of denotation of independent variable leaves the canonical form of the KdV equation as follows

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.3}$$

We seek the solution of this equation in form of a uniform wave packet

$$u(x,t) = \varphi(\xi), \quad \xi = x - ct,$$

where c is the velocity of propagating wave. Upon substitution of this Ansatz into Eq. (1.3), it is reduced to an ordinary differential equation (ODE)

$$\varphi_{\xi\xi\xi} = c\varphi_\xi - 6\varphi\varphi_\xi$$

with an obvious first integral

$$\varphi_{\xi\xi} = -g + cu - 3\varphi^2,$$

where g is an integration constant. Multiplying this equation by φ_ξ and integrating the obtained equation, the next first integral is derived as follows

$$\frac{1}{2}\varphi_\xi^2 = -\varphi^3 + \frac{1}{2}c\varphi^2 - g\varphi + h,$$

where h is another integration constant. In the most general case both constants are nonzero and the last equation can be cast to

$$\varphi_\xi^2 = 2(\alpha - \varphi)(\varphi - \beta)(\varphi - \gamma),$$

where the new parameters α, β, γ are the zeros of the cubic equation

$$p(\varphi) = \varphi^3 - \frac{1}{2}c\varphi^2 + g\varphi - h = 0,$$

and they are ordered according to $\gamma \le \beta \le \alpha$. The Viet's theorem implies that the zeros and the coefficients of this cubic polynomial are related by

$$c = 2(\alpha + \beta + \gamma), \quad g = \alpha\beta + \beta\gamma + \gamma\alpha, \quad h = \alpha\beta\gamma. \tag{1.4}$$

In the first integral

$$\frac{1}{2}\varphi_\xi^2 + \varphi^3 - \frac{1}{2}c\varphi^2 + g\varphi = h$$

the function $q(\varphi) = \varphi^3 - c\varphi^2/2 + g\varphi$ plays the role of the potential function whose typical plot is shown in Fig. 1.2. The corresponding phase portrait helping us to examine different

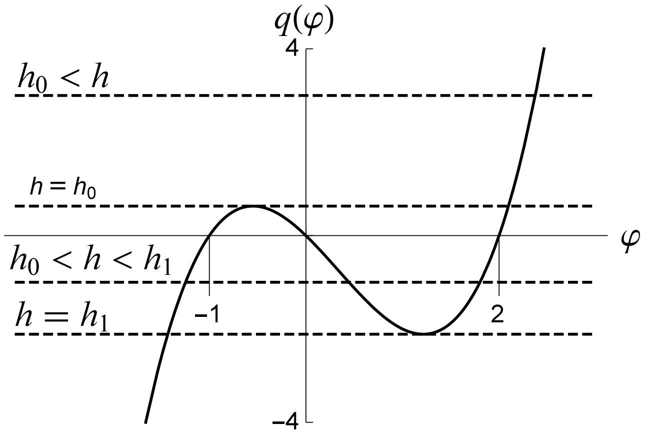


Figure 1.2: The pseudo-potential function exhibits one maximum and one minimum with appropriate choice of its coefficients.

cases of solution is plotted in Fig. 1.3. For bounded solutions all zeros must be real and the periodic solution must oscillate between two of them, which turns out $\beta \le \varphi \le \alpha$ in this case. Thus the closed orbit corresponds to the case of three single real zeros, the open orbit to the case of two complex conjugate zeros and the separatrix to the case of one single and one double real zeros. Since the open orbits do not correspond to the physically meaningful wave solution, we focus ourselves on either the closed orbit or the separatrix. Then the standard integration yields the solution expressed in terms of the Jacobi elliptic function (cf. [75]) as follows

$$\varphi(\xi) = \beta + (\alpha - \beta)\mathrm{cn}^2\left(\sqrt{\frac{\alpha - \gamma}{2}}\xi, m\right), \quad m = \frac{\alpha - \beta}{\alpha - \gamma}.$$

This result was first obtained by Korteweg and de Vries [2] and owning to its similarity to the 'cosinusoidal' solution of linear equation the term 'cnoidal wave' was coined according to them. Now the identity

$$m[1 - \mathrm{cn}^2(z, m)] + \mathrm{dn}^2(z, m) = 1$$

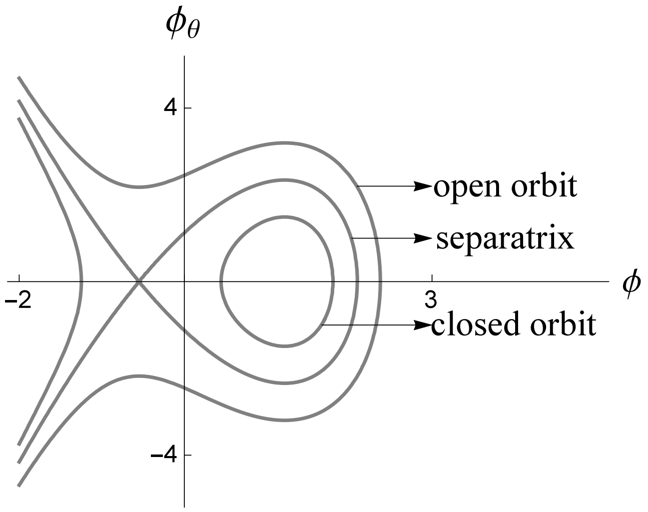


Figure 1.3: Phase portrait of the KdV equation gives us the information about the periodic solution.

and a substitution back to the wave variable give

$$u(x,t) = (\alpha - \gamma)\mathrm{dn}^2\left[\frac{\alpha-\gamma}{2}(x-ct), m\right] + \gamma. \tag{1.5}$$

It is rather practical to introduce some preliminary notions associated with the wave propagation phenomenon via objective example. According to the phase portrait, the altitude of the cnoidal wave varies only between the two larger zeros. Thus, one possibility of the amplitude definition is

$$2a = u_{\max} - u_{\min} = \alpha - \beta,$$

which allows us to rewrite the solution in the form

$$u(x,t) = \frac{2a}{m}\mathrm{dn}^2\left[\sqrt{\frac{a}{m}}(x-ct), m\right] + \gamma.$$

The wavelength λ is defined as the distance between two nearest wave crests, namely the two neighboring maxima or minima. Due to the fact that the period of the elliptic function $\mathrm{dn}(z,m)$ is $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind, it can be found immediately from this solution formula that

$$\lambda = \frac{2K(m)}{\sqrt{a/m}} = 2\sqrt{\frac{m}{a}}K(m).$$

One specific cnoidal wave and its wavelength are illustrated in Fig. 1.4 whereas its corresponding contour plot is shown in Fig. 1.5. It is rather difficult to demonstrate the period with respect to time of the solution by using the static figures, yet it can be easily found with simple argument. Since the solution has been assumed as a function of one variable $\xi = x - ct$, the infinitesimal differentials of three variables ξ, x and t are related by

$$\mathrm{d}x = \mathrm{d}\xi, \quad \mathrm{d}x = -c \times \mathrm{d}t.$$

Thus, one particle in the wave packet will return to its original state after the instant of time

$$\tau = \frac{1}{c}\int_0^\lambda \mathrm{d}x = \frac{\lambda}{c}$$

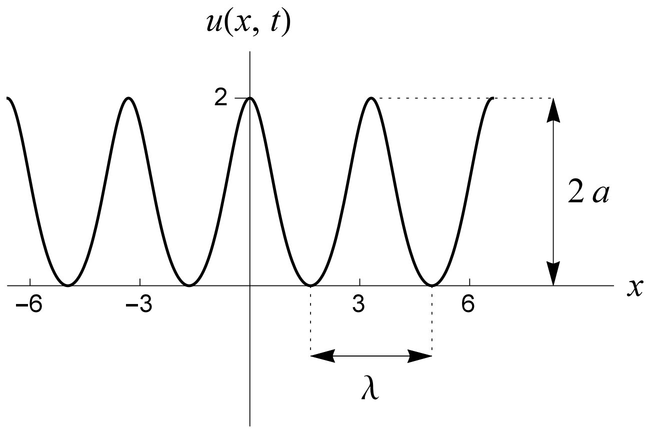


Figure 1.4: Cnoidal wave and its wavelength.

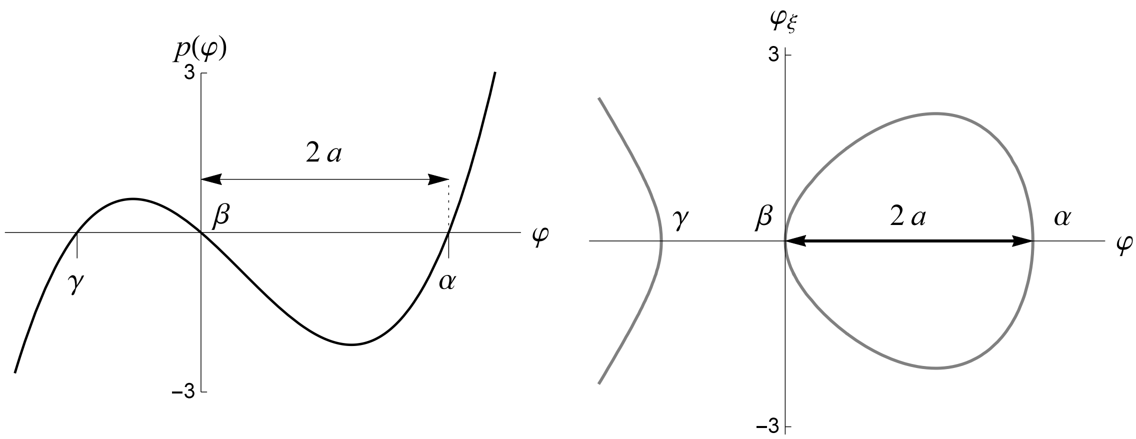


Figure 1.5: The potential function $q(\varphi)$ and the contour plot of the cnoidal wave corresponding to the zero energy level.

so that we can define the frequency as

$$\omega = \frac{2\pi}{\tau} = c \times \frac{2\pi}{\lambda}.$$

The above argument is visualized in Fig. 1.6. If we define the phase of the wave, the wave number and the frequency by

$$\theta = kx - \omega t, \quad k = \frac{2\pi}{\lambda}, \quad \omega = ck,$$

and the new dependent variable ϕ by

$$\phi(\theta) = \varphi(\theta/k) - \gamma,$$

then we can recognize that the solution is 2π-periodic with respect to its phase variable

$$u(x,t) = \phi(\theta(x,t)) + \gamma, \quad \theta = kx - \omega t.$$

This formulation is crucial in description of methodology for the modulation theory. As we have introduced a set of three parameters a, m, c to present the solution instead of three zeros, there must be relations between them. It is easy to check that they are given as follows

$$\alpha = \frac{2a}{m} + \gamma, \quad \beta = 2a\left(\frac{1}{m} - 1\right) + \gamma, \quad c = 4a\left(\frac{2}{m} - 1\right) + 6\gamma.$$

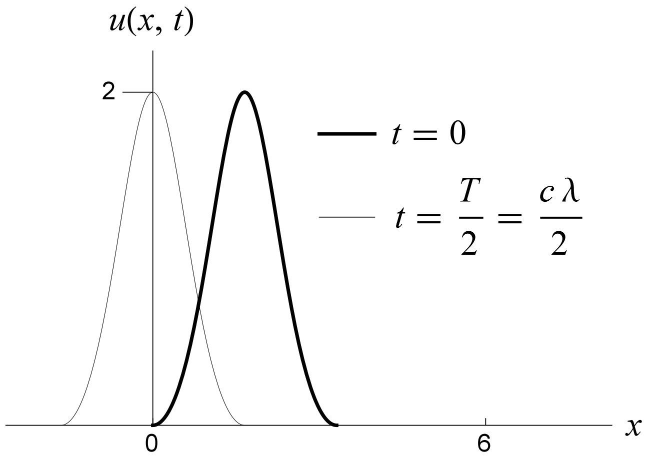


Figure 1.6: The cnoidal wave plotted only in one wavelength traveling in time.

Additionally, the dispersion relation reads

$$\omega = ck = \left[4a\left(\frac{2}{m} - 1\right) + 6\gamma\right] k = \Omega(a, k),$$

where the modulus m can be expressed in terms of k and a, that is $m = m(k, a)$. Thus, we have just established the most important and typical feature of nonlinear waves: The dispersion relation involves not only the wave vector and the frequency but also the amplitude of the wave. In fact, this is similar to the nonlinear vibrations where the frequency depends also on the amplitude. We investigate in the following two limiting cases.

If the amplitude of the wave is small $a \ll 1$ and $m \to 0$ but the wave number remains finite, then we have the approximate period $2K(m) \approx \pi$. In this limit we have

$$\text{dn}^2(z, m) = 1 - m\,\text{sn}^2(z, m) = 1 - m \sin^2 z + O(m^2)$$

and the solution (1.5) transforms in this linear approximation to

$$u(x, t) \approx a \cos\left[2\sqrt{\frac{a}{m}}(x - ct)\right] + \frac{2a}{m} + \gamma - a.$$

The additive constant is equal to

$$\frac{2a}{m} + \gamma - a = \alpha - \gamma + \gamma - \frac{\alpha - \beta}{2} = \frac{\alpha + \beta}{2},$$

while the dispersion relation can be approximated by

$$\omega \approx 2\gamma k \approx -4\frac{\pi^2}{\lambda^2}k = -k^3.$$

Thus, we recover the dispersion relation of the linearized KdV equation

$$u_t + u_{xxx} = 0.$$

There is another interesting limiting case when $m \to 1$. In this case the KdV equation possesses the solitary wave solution or soliton that decay exponentially as x goes to infinity. In the limit $m \to 1$ or equivalently $\gamma \to \beta$ the above solution becomes

$$u(x, t) = 2a\,\text{sech}^2[\sqrt{a}(x - ct)] + \gamma,$$

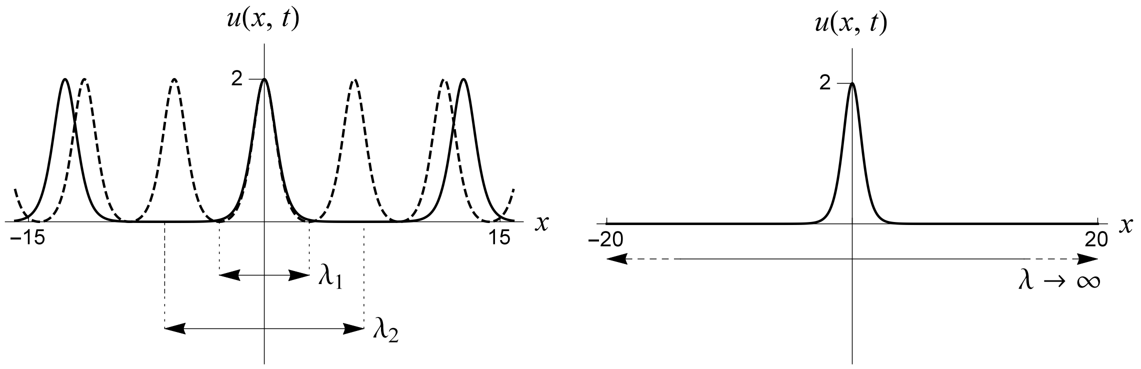


Figure 1.7: The periodic solution with large finite wavelength (left) approaches the solitary wave with infinite wavelength (right).

which corresponds to the separatrix on the phase portrait. Indeed, when $m \to 1$, the cubic equation has one single zero $\varphi_1 = \alpha$ and one double zero $\varphi_2 = \varphi_3 = \beta$ and the first integral reduces to

$$\varphi_\xi^2 = 2(\alpha - \varphi)(\varphi - \beta)^2,$$

whose integration yields the above soliton solution. Note that the wavelength λ in this limit tends to infinity, which is completely expected due to the phase portrait. The envelope of wave starts from zero at infinity $\xi \to -\infty$, achieves its maximum at some finite $\xi = \xi_0$ but never reaches zero except at another infinity $\xi \to \infty$. A visualized explanation can be found in Fig. 1.7 where the periodic solution with finite but large wavelength is flatten out so that its shape approaches that of soliton solution with infinite wavelength. The dispersion relation in this case is given by

$$\omega \approx (4ak + 6\beta)k$$

taking into account non-zero basement value β. With the assumption of zero basement, that is $\beta = \gamma = 0$, it is reduced to $\omega \approx 4ak$.

1.1.2 Waves of kink-type and breather

Wave propagation finds its application not only to fluid mechanics but also to many other physical fields. For instance, the Klein–Gordon equation

$$u_{tt} - u_{xx} + \Phi'(u) = 0 \tag{1.6}$$

occurs frequently in quantum mechanics and material science. Let us sketch a dynamical discrete model introduced by Frenkel and Kontorova to describe the nearest-neighbor interaction in the chain of atoms. The system comprises an infinite chain of particles interacting to each other via an source of potential correlating them. This kind of correlation can be mechanically represented by a discrete system of springs of stiffness k attached to the particles which uniformly occupy a one-dimensional space in the equilibrium state. In addition, the chain is subject to an external on-site periodic potential $\Phi(x)$ so that the potential acting on the n^{th} particle is taken as $\Phi(x_n)$. The distance of between the two neighboring particles is designated by a_0 while the period of the potential $\Phi(x)$ by a_s. In Fig. 1.8 the entire model is schematically sketched. The kinetic energy of the system is

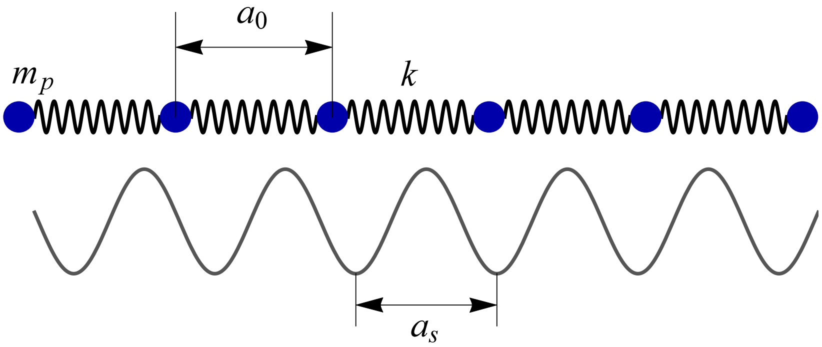


Figure 1.8: Frenkel-Kontorova model: a chain of particles interacting to each other under an external potential field.

$$K = \frac{m_p}{2} \sum_n \left(\frac{\mathrm{d}x_n}{\mathrm{d}t} \right)^2, \tag{1.7}$$

where m_p is the mass of each individual particle. The potential energy is the sum of two parts as follows

$$U = U_{\text{int}} + U_{\text{ext}}, \tag{1.8}$$

in which the first part accounting for the interaction of the nearest neighbors in the chain and the second part for the external potential energy are given by

$$U_{\text{int}} = \frac{k}{2} \sum_n (x_{n+1} - x_n)^2, \quad U_{\text{ext}} = \sum_n \Phi(x_n). \tag{1.9}$$

The model designed according to equations (1.7)–(1.9) are justified under the two following assumptions:

1. The particles are restrictively moved only in one direction.
2. The correlation between the particles only considers the nearest-interaction rule. That is, one particular particle is only exposed to the interaction of its one left and one right particle so that

$$U_{\text{int}} = \sum_n V_{\text{int}}(x_{n+1} - x_n),$$

where V_{int} is the potential energy stored in the interaction between two particles. Expanding this potential function into a Taylor series and keeping only up to the second order, the stiffness of the spring is $k = V''(0)$.

From the Largrange function of this system

$$L = \frac{m_p}{2} \sum_n \left(\frac{\mathrm{d}x_n}{\mathrm{d}t} \right)^2 - \frac{k}{2} \sum_n (x_{n+1} - x_n)^2 - \sum_n \Phi(x_n),$$

it follows the equations of motion

$$m_p \frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} - k(x_{n+1} - 2x_n + x_{n-1}) - \Phi'(x_n) = 0, \quad \forall n.$$

It is not difficult to normalize this equation to obtain the a system with unit mass $m_p = 1$ via some transformation. If the continuum assumption is exploited instead of the discrete setting, after a normalization of the mass m_p to unit we arrive at

$$v_{tt} - kv_{xx} + \Phi'(v) = 0, \tag{1.10}$$

where $v = v(x,t)$ is the displacement of the particle at coordinate x at the time instant t. We may say the "x^{th} particle" vibrates in time. This equation with $k = 1$ is called the Klein-Gordon equation. If the external potential is characterized according to the harmonic law

$$\Phi(v) = 1 - \cos\left(\frac{2\pi}{a_s}v\right),$$

the last equation takes the form

$$v_{tt} - kv_{xx} + \frac{2\pi}{a_s}\sin\left(\frac{2\pi}{a_s}v\right) = 0,$$

which is called the sine-Gordon (SG) equation. Under the change of variables

$$X = \frac{2\pi}{a_s\sqrt{k}}x, \quad T = \frac{2\pi}{a_s}t, \quad u = \frac{2\pi}{a_s}v,$$

the SG equation can be transformed to its canonical form

$$u_{tt} - u_{xx} + \sin u = 0, \tag{1.11}$$

where the denotations x and t have been reused. This equation, encountered in the problems of moving dislocations in crystals [65] and long Josephson junctions [66], has attracted attention from both physicists and mathematicians owning to its integrability by the Hirota direct method or optionally by the inverse scattering transform. According to these works, the exact multiple-soliton solutions exists and they are later classified into the kink-type category due to their special geometrical form. Not so long later, the so-called Bäcklund transformation method was separately developed to enable one to construct several other kinds of solutions such as standing breather, moving breather, and collision between breathers and kink-type solutions. The above-mentioned solution methods for the sine–Gordon equation can be fully found in the works by Hirota [76], Ablowitz et. al [77], and Rodd and Bullough [78]. In this context we shall list these solutions and discuss their properties.

Kink-type solution If we look for solution of Eq. (1.11) in the form of traveling wave

$$u(x,t) = \phi(\xi), \quad \xi = x - ct,$$

then it is reduced to

$$(c^2 - 1)\phi_{\xi\xi} + \sin\phi = 0.$$

Its obvious first integral is given by

$$\frac{1}{2}(c^2 - 1)\phi_\xi^2 + 1 - \cos\phi = h, \tag{1.12}$$

where the potential function $\Phi(\phi) = 1 - \cos\phi$ has been chosen. Two cases may happen: the subsonic regime with $c < 1$ and the supersonic regime with $c > 1$. The first integral in two cases can be rewritten in the form

$$\frac{1}{2}m\phi_\xi^2 + \Phi_\mp(\phi) = h, \quad \Phi_\mp(\phi) = \mp(1 - \cos\phi),$$

where we have used the positive constant $m > 0$. Since the phase portraits corresponding to these two forms differ from each other by a shift in ϕ-axis by a constant π (see figures 1.9 and 1.10), we shall restrict ourselves to the subsonic regime. According to the phase portrait, the solution may behave in three different manners in response to the close orbit, the open orbit and the separatrix. Using the theorem of inverse function, we may rewrite

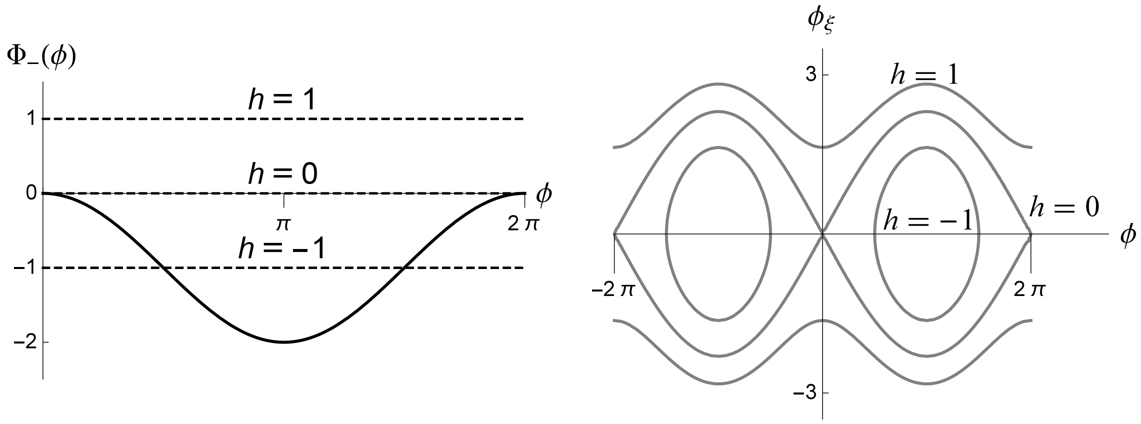


Figure 1.9: Phase portrait of the wave solution governed by the sine–Gordon equation in the subsonic regime.

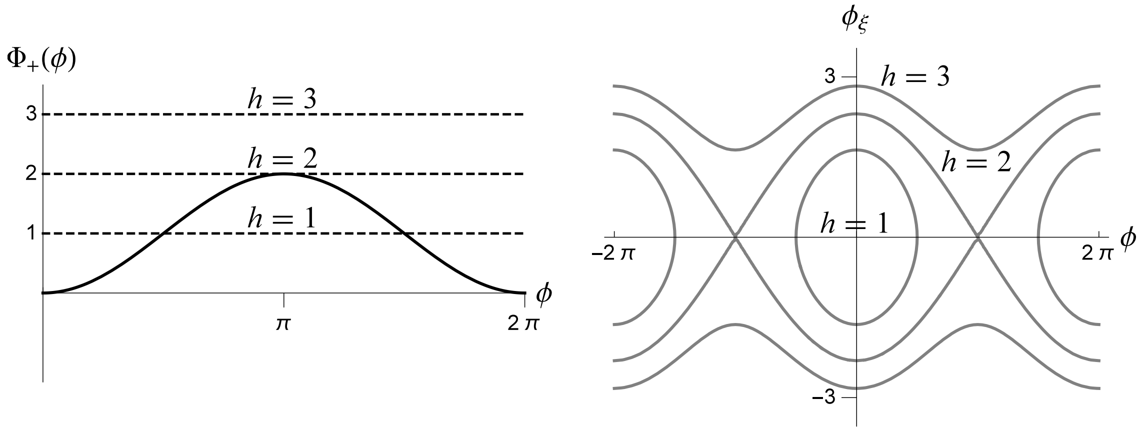


Figure 1.10: Phase portrait of the wave solution governed by the sine–Gordon equation in the supersonic regime.

the first integral for the subsonic case as follows

$$\xi'(\phi) = \pm\left[\frac{2}{1 - c^2}(h + 1 - \cos\phi)\right]^{-1/2}.$$

Integrating this equation, we obtain

$$\xi(\phi) = \pm\int_{\phi_0}^{\phi}\sqrt{\frac{1 - c^2}{2(h + 1 - \cos\psi)}}\mathrm{d}\psi + \xi_0, \tag{1.13}$$

where the initial condition $\phi(\xi_0) = \phi_0$ is taken into account. If $-2 < h < 0$, we are on the closed orbit and the periodic solution is obtained. On the other hand, if $h > 0$ the right-hand side of the last equation is a monotonic function of ϕ and thus the inverse function theorem allows us to recover the original solution ϕ. The closed form of solution is not trivial in both cases. Nevertheless, the more interesting solution occurs when the limit of separatrix is considered. As such, without loss of generality we choose $\phi(0) = 0$ and let h tend to zero in Eq. 1.13 to obtain

$$\xi(\phi) = \pm \int \frac{\sqrt{1-c^2}}{2\sin(\phi/2)} \mathrm{d}\phi = \pm\sqrt{1-c^2}\log\left(\tan\frac{\phi}{4}\right).$$

Solving this transcendental equation for ϕ and substituting $\xi = x - c\,t$ back, we obtain

$$u_{\pm}(x,t) = 4\arctan\left[\exp\left(\pm\frac{x-ct}{\sqrt{1-c^2}}\right)\right].$$

The solution u_+ is called the kink solution while the other u_- is named the anti-kink solution. As reflected from the information of the phase portrait, the kink increases from 0 at $\xi \to -\infty$ to 2π at $\xi \to \infty$ and the anti-kink exhibits the opposite behavior. In Fig. 1.11 these two solutions are plotted, demonstrating this argument. As compared to the solitary wave of

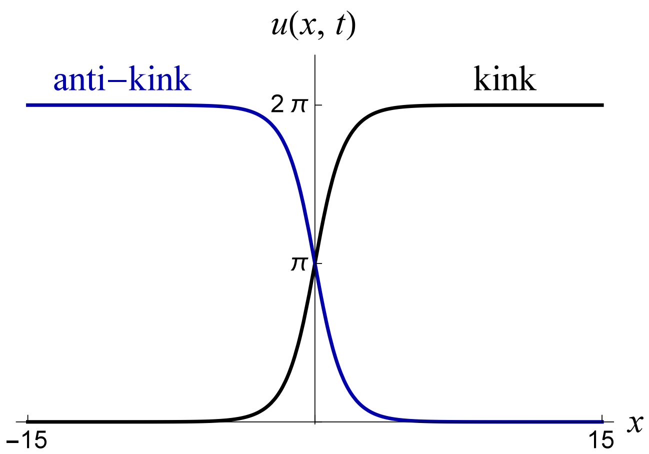


Figure 1.11: Kink and anti-kink solution of the sine–Gordon equation.

the KdV equation, the appearance of the kink-type solution is distinguishable. Though the soliton must flatten itself out to zero on both sides at infinity, the kink does flatten out to a non-zero constant which is the multiple of 2π either on one side or both sides. This character leads to the idea that after the solution increases from 0 to 2π, it may keep climbing up to a multiple of 2π and stay there at infinity. This is actually the case according to Hirota's discovery of the multi-kink solution in his celebrated work [76]. A wave packet of three kinks is shown together with its slope in Fig. 1.12.

In addition, the kink-type solution of the sine–Gordon equation differs from the solitary wave of the KdV equation by the possibility of bi-directed propagation. This can be realized by comparing their first integrals (1.4) and (1.12). Whereas in the KdV case the velocity c appears with power of one, it appears in the sine–Gordon case in the quadratic manner. In Fig. 1.13 a packet of two groups of kinks and anti-kinks separating and traveling to the left and to the right, respectively, is shown. The solutions of kink type can be used to model geometrically the propagation process of the opening crack in crystal materials. When we extract a small portion of the wave packet comprising of one kink and one anti-kink, it is

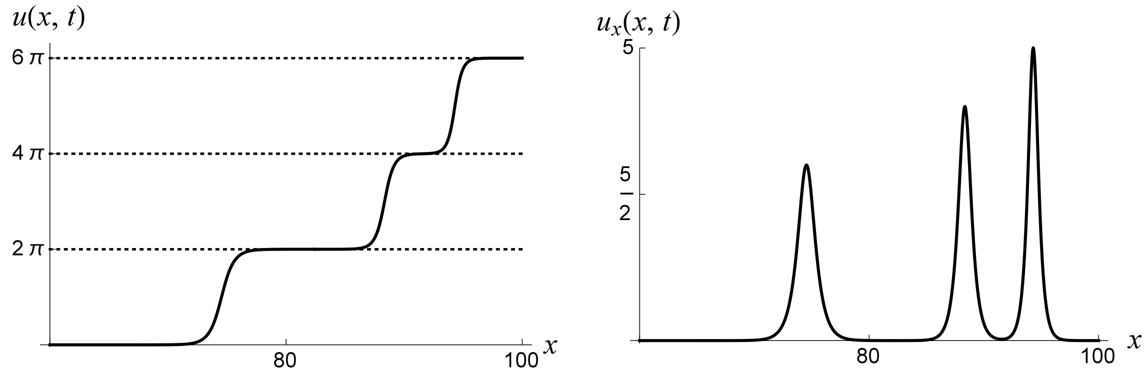


Figure 1.12: A 3-kink solution (left) to the sine–Gordon equation and its slope (right) travel in time.

easier to investigate the characteristics of the solution. In Fig. 1.14 the derivative of such portion of solution with respect to the spatial variable, which is called the slope, is illustrated at two time instants. From this figure it seems that the slope of the wave packet obeys some asymptotic law.

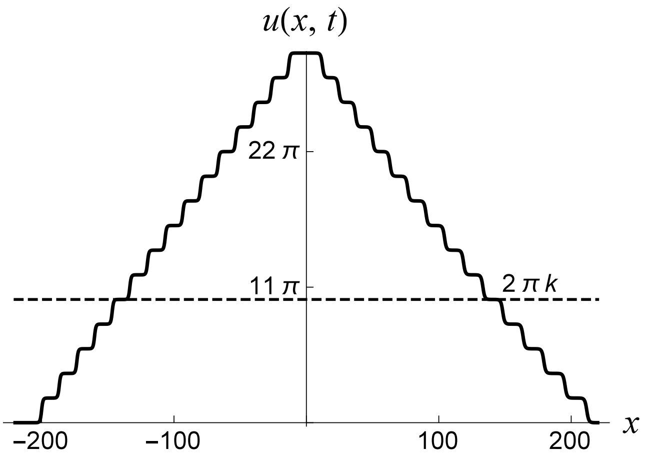


Figure 1.13: The opening of crack in materials made of crystal can be described by the propagation of kink and anti-kink waves in a certain finite domain.

1.2 Idea of modulation of wave packet

From the above well-known examples, we see that the modulations of strongly nonlinear waves with finite amplitude cannot be approximated by linear wave packets. For instance, we are considering the evolution of a wave packet governed by the KdV equation whose initial disturbance can be described by the modulated periodic wave (1.5), where the parameters a, m, and γ defining the solution are some functions of x. The question arises naturally: How do they evolve with time? In such problem, two possibilities may occur as follows.

- If in the initial state the parameters a, m, γ change significantly in one wavelength $\lambda \sim \sqrt{m/a}$, then we would not expect that the evolving wave can be captured by the modulated periodic wave.

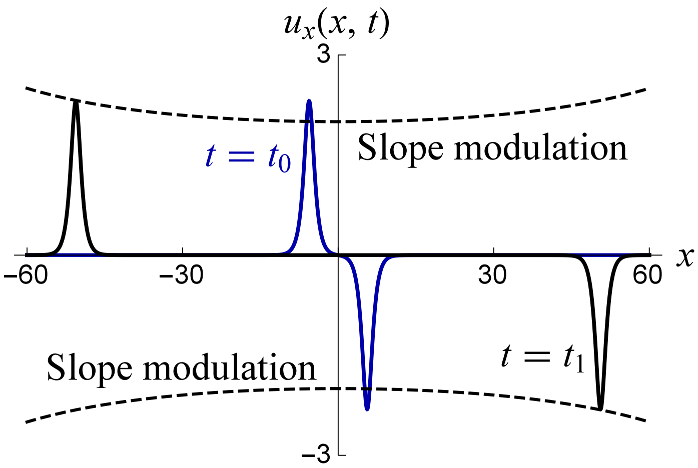


Figure 1.14: Slope of a kink-antikink solution and its modulation at large time.

- On the contrary, if the initial modulation is rather slow, that is the parameters a, m, γ change little within one wavelength, then one may expect that modulation remains slow up to a certain time, and consequently the evolution can be described by a slow change of these parameters.

In the second scenario, since these parameters vary slowly in one wavelength, we can consider them as slow functions of the space and time coordinates. Our aim is to calculate how these parameters evolve in time. The modulation scale is a larger scale in which the wave properties can be effectively described. We can notice the existence of two scales of distance and time in the modulation problem. On the one hand, the phase variable oscillates fast in one wavelength $\lambda = 2\pi/k$ and one period $\tau = 2\pi/\omega$ and on the other hand, the parameters characterizing the periodic solution evolve slowly. This observation suggests the method of approach. Averaging the equations of motion over the fast variable results in the new equations with only the slow variables and it is reasonable to anticipate that these averaged equations govern the slow evolution of relevant parameters. Therefore, this method is categorized as the multi-scaled method. This mentality is credited to Whitham, who introduced the idea in his seminal work [35, 42]. Apparently, the procedure bears a resemblance to the Krylov-Bogoliubov method or WKB theory, which was initially applied to nonlinear ordinary differential equations with the presence of small parameters.

One of useful applications of the modulation theory is the amplitude modulation for nonlinear dispersive waves. Indeed, the equation of amplitude modulation for the KdV equation was first derived by Whitham to describe the evolution of the parameters associated with the periodic wave solution. Let us touch the idea by revisiting the solution formula (1.5) and assuming that three zeros α, β and γ are three slowly varying functions of x and t in one wavelength and one period, that is

$$\alpha = \alpha(x, t), \quad \beta = \beta(x, t), \quad \gamma = \gamma(x, t).$$

Then the dependence of $c = c(x, t)$ and $m = m(x, t)$ on x and t is realized by the their algebraic relations with these parameters. At the next step substitution of these functions into Eq. (1.5) in replacement of the corresponding constants yields the non-uniform wave packet governed by the KdV equation. Then this approach enables us to keep track of the evolution of the whole packet up to some certain time and coordinate as long as the evolution of the governing parameters are known. One straightforward application of the described procedure is to produce the asymptotic envelopes or the amplitudes of the non-uniform wave packet. In Fig. 1.15 the plot of a non-uniform wave packet governed by the KdV equation with its envelopes being marked with bold line is shown. From this figure we can see

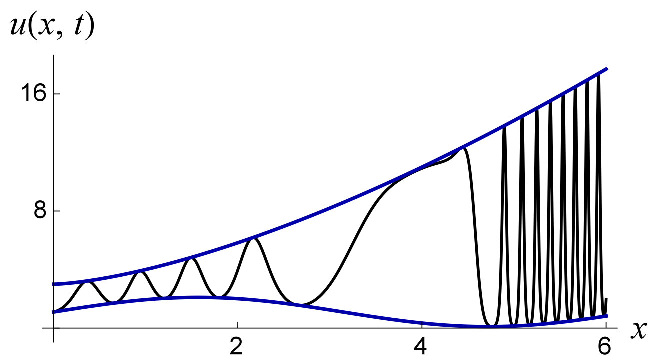


Figure 1.15: A nonuniform wave packet governed by the KdV equation with its envelopes.

immediately that the upper and lower envelopes α, β change slowly in one wavelength and so does the amplitude. Furthermore, it is recognized that the wave number k varies slowly in one "local" wavelength. The similar observation of slow variance of the frequency and the amplitude within one period can be obtained if we stay at one fixed coordinate to follow the vibration of the particle. Such observation is well explained in Fig. 1.16 where the same wave packet is plotted with respect to time instead of the space coordinate. It is by now clear that two distinct scales are detectable in our problem. The characteristic sizes Λ, T are associated with the period of the amplitude and any average quantities of the wave packets while the other sizes λ, τ with the fast oscillation of the wave. Returning back to the

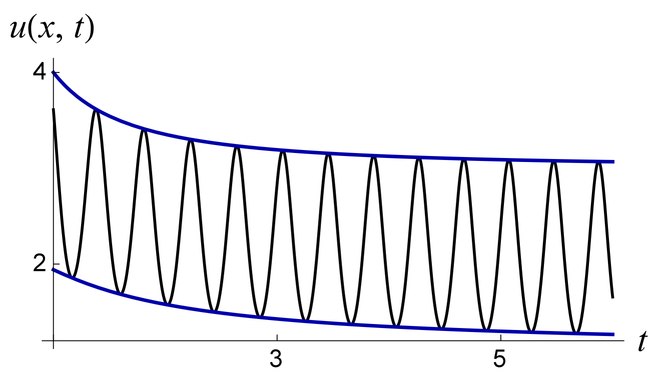


Figure 1.16: The vibration of one particle in the nonuniform wave packet governed by the KdV equation with its amplitude varying slowly in time.

important solution of the KdV equation, namely the solitary wave, one may wonder whether we can describe the amplitude of a packet of several solitons if there is such one. In the case of uniform wave comprising only one soliton the wave amplitude can be depicted by only one constant, which is demonstrated in Fig. 1.17. Yet the amplitude modulation must be a slowly varying function of x and t if the train consists of multiple solitons with different amplitudes and propagating velocities. In Fig. 1.18 the wave packet of multiple solitons of the KdV equation during and after the collisions is plotted in separate frames. Though it is difficult to describe the amplitude of the packet during the interaction time, it can be figured out that there seems to be an amplitude modulation for the same packet with well-separated solitons. For the shallow-water it is reasonable to deal with the solitary waves with the zero basement since the altitude of the wave can be measured from the non-disturbance water surface. It is convenient to drop the factor in front of the last definition of amplitude and use only a for the amplitudes of solitons. Our aim is to derive the equations governing the evolution of the amplitude modulation $a = a(x,t)$ as the slow function of x and t and to solve these equations for the asymptotic solutions. The theory of slope modulation can be also derived using this identical technique.

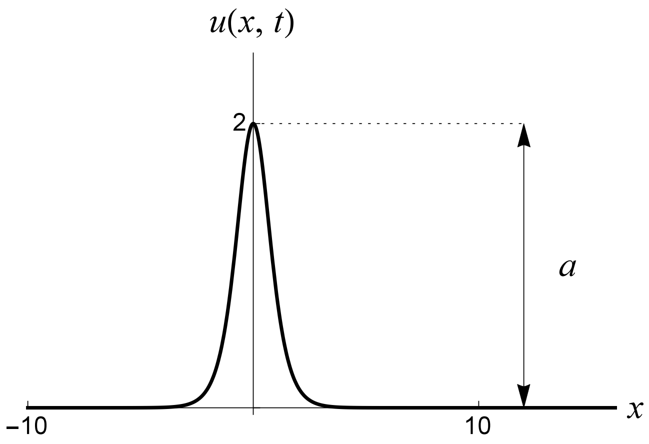


Figure 1.17: One soliton with constant amplitude travels in time.

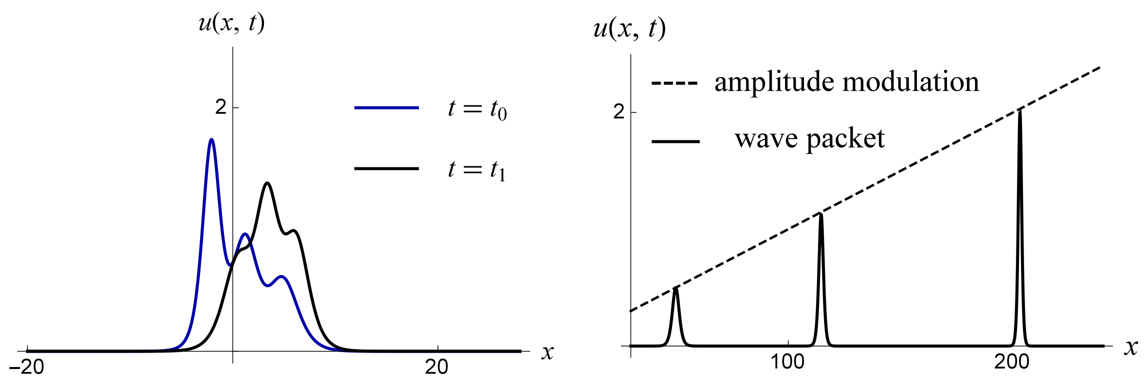


Figure 1.18: Three solitons governed by KdV equation interact with each other (left) and then separate at large time (right).

1.3 Variational-asymptotic method

The modulation equations derived by Whitham can also be reconstructed by the so-called variational asymptotic method (VAM). The method has been applied to many dynamical systems such as vibrations of shells and rods [59], contiuum mechanics and fluid dynamics [60]. Very recently the method has also been utilized to develop the theory of modulations for nonlinear dispersive waves [61]. As the name suggests, the first ingredient of the method is the possibility of formulating the problem or reformulating the governing differential equations in a variational statement. The trend of dealing with problems using the variational approach has been increasing over decades owing to two advantages.

(a) It reduces the difficulty of taking care of several orders of approximation, sometimes the heuristic ones.

(b) The knowledge of physical properties and their interpretation are contained in the compact variational formulation that enables one to work with the problem mathematically.

On account of these benefits one may say that the variational approach helps us less prone to making unnecessary mistakes during the course of derivation. In the following we shall present the method and how to apply it to the modulation of nonlinear dispersive waves step by step.

Problem setting Consider the variational problem in form of the Hamilton's variational principle: Find the extremal of the action functional

$$I[u_i(\mathbf{x},t)] = \iint\limits_R L(u_i, u_{i,\alpha}, u_{i,t})\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{1.14}$$

where $R = V \times (t_0, t_1)$ is any finite fixed region in $(d+1)$-dimensional space-time. We assume that u_i are prescribed at the boundary ∂R. Since there are many wave variables and coordinates being incorporated within our problem, we use here the comma to denote the partial differentiation as well as to separate the indices associated with the field variables and with the coordinate variables. The Latin index corresponds to the field variables and the Greek index to the coordinates. We look for the extremal of this variational problem in form of a slowly varying wave packet

$$u_i = \phi_i(\theta(\mathbf{x},t), \mathbf{x}, t),$$

where $\theta = \theta(\mathbf{x},t)$ plays the role of the phase, while $\theta_{,\alpha}$ and $-\theta_{,t}$ correspond to the wave vector k_α and the frequency ω, respectively, ϕ_i are 2π-periodic functions with respect to θ. Within one period of ϕ_i the wavelength λ and the period τ are defined as the largest constants in the inequalities

$$|\theta_{,\alpha}| \le \frac{2\pi}{\lambda}, \quad |\theta_{,t}| \le \frac{2\pi}{\tau}.$$

Analogously, the characteristic length- and time-scales Λ and T of changes of the wave vector k_α, the frequency ω and $\phi_i(\theta,\mathbf{x},t)|_{\theta=\text{const}}$ are defined as the largest constants in the inequalities

$$|\theta_{,\alpha\beta}| \le \frac{2\pi}{\lambda\Lambda}, \quad |\theta_{,\alpha t}| \le \frac{2\pi}{\lambda T}, \quad |\theta_{,t\alpha}| \le \frac{2\pi}{\tau\Lambda}, \quad |\theta_{,tt}| \le \frac{2\pi}{\tau T},$$
$$|\partial_\alpha \phi_i| \le \frac{\overline{\phi_i}}{\Lambda}, \quad |\partial_t \phi_i| \le \frac{\overline{\phi_i}}{T}, \quad |\phi_{i,\theta}| \le \overline{\phi_i},$$

where

$$\partial_\alpha \phi_i = \frac{\partial \phi_i}{\partial x_\alpha}\Big|_{\theta=\text{const}}, \quad \partial_t \phi_i = \frac{\partial \phi_i}{\partial t}\Big|_{\theta=\text{const}}.$$

Fundamental hypotheses In order that the modulated wave remains good approximate at large time, two important hypotheses are proposed as follows.

(a) The assumption of slow variance of the wave vector and the frequency in one wavelength and one period is kept. It is assumed further that $\phi_i(\theta,\mathbf{x},t)|_{\theta=\text{const}}$ changes slowly in such characteristic scale.

(b) The second assumption relating the two characteristic scales are proposed as follows

$$\lambda \ll \Lambda, \quad \tau \ll T.$$

So we have two small parameters as the ratios λ/Λ and τ/T.

Strip problem The partial derivatives of u_i are given by

$$u_{i,\alpha} = \phi_{i,\theta}\theta_{,\alpha} + \partial_\alpha \phi_i, \quad u_{i,t} = \phi_{i,\theta}\theta_{,t} + \partial_t \phi_i.$$

Due to the second hypothesis, they can be approximately replaced by

$$u_{i,\alpha} = \phi_{i,\theta}\theta_{,\alpha}, \quad u_{i,t} = \phi_{i,\theta}\theta_{,t}.$$

Keeping in the action functional (1.14) the asymptotically principal terms, we obtain in the first approximation

$$I_0[\phi] = \iint_R L(\phi_i, \phi_{i,\theta}\theta_{,\alpha}, \phi_{i,\theta}\theta_{,t})\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Now we decompose the domain R into the $(d+1)$-dimensional strips bounded by the d-dimensional phase surfaces (see Fig. 1.19)

$$\theta = 2\pi n, \quad n \in \mathbb{Z}.$$

The integral over R can then be replaced by the sum of the integrals over the strips

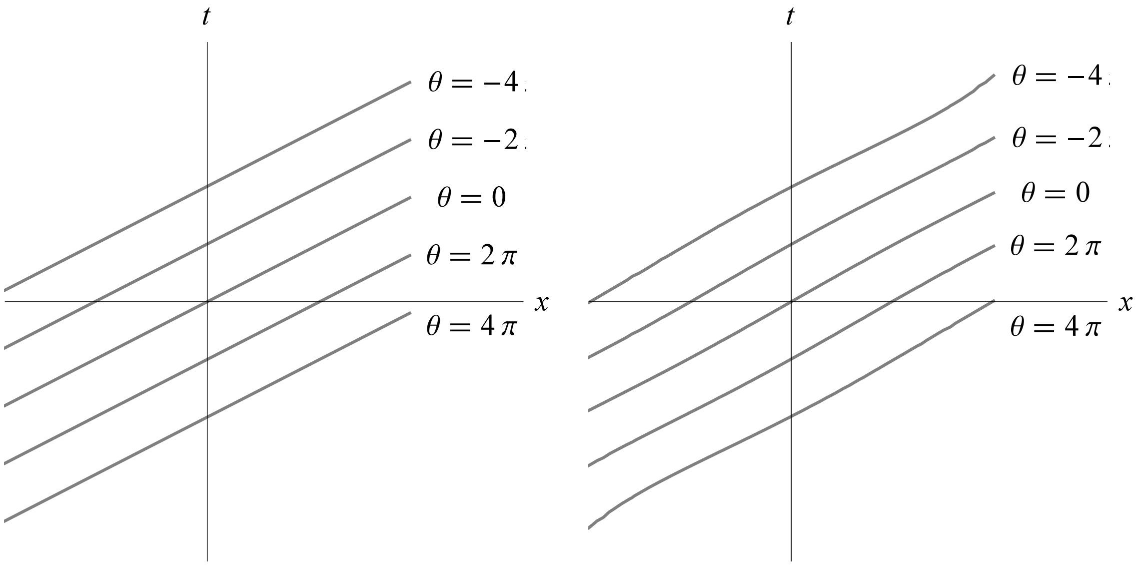


Figure 1.19: The domain of consideration is decomposed into many strips bounded by the curves $\theta = 2\pi n$, $n \in \mathbb{Z}$. The left figure corresponds to the case of uniform wave, while the right to the case of nonuniform wave.

$$\iint_R L\mathrm{d}\mathbf{x}\mathrm{d}t = \sum_{\text{strips}} \iint L(\phi_i, \phi_{i,\theta}\theta_{,\alpha}, \phi_{i,\theta}\theta_{,t})\kappa\, \mathrm{d}\theta\mathrm{d}\zeta, \tag{1.15}$$

where ζ_α are the coordinates along the phase surfaces $\theta = $ const, and κ is the Jacobian transformation from x_α, t to θ, ζ_α. In the first approximation we may regard κ, $\theta_{,\alpha}$ and $\theta_{,t}$ in each strip as independent from θ. Hence, we obtain in each strip at the first step of the variational-asymptotic method, that is in the limit $\lambda/\Lambda \to 0, \tau/T \to 0$, the same problem: Find the extremal of the functional

$$\bar{I}_0[\phi_i] = \frac{1}{2\pi}\int_0^{2\pi} L(\phi_i, \phi_{i,\theta}\theta_{,\alpha}, \phi_{i,\theta}\theta_{,t})\mathrm{d}\theta \tag{1.16}$$

among 2π-periodic functions $\phi_i(\theta)$. According to the first hypothesis, the quantities $k_\alpha = \theta_{,\alpha}$ and $\omega = -\theta_t$ deviate negligibly within one strip, so they may be regarded as constants in this functional (compare two figures 1.20 and 1.21). Taking advantage of this consideration into account, we can rewrite the above functional in the more convenient form

$$\bar{I}_0[\phi_i] = \frac{1}{2\pi}\int_0^{2\pi} L(\phi_i, k_\alpha \phi_{i,\theta}, -\omega\phi_{i,\theta})\mathrm{d}\theta.$$

Varying this functional, using the integration by parts making use of the periodicity condi-

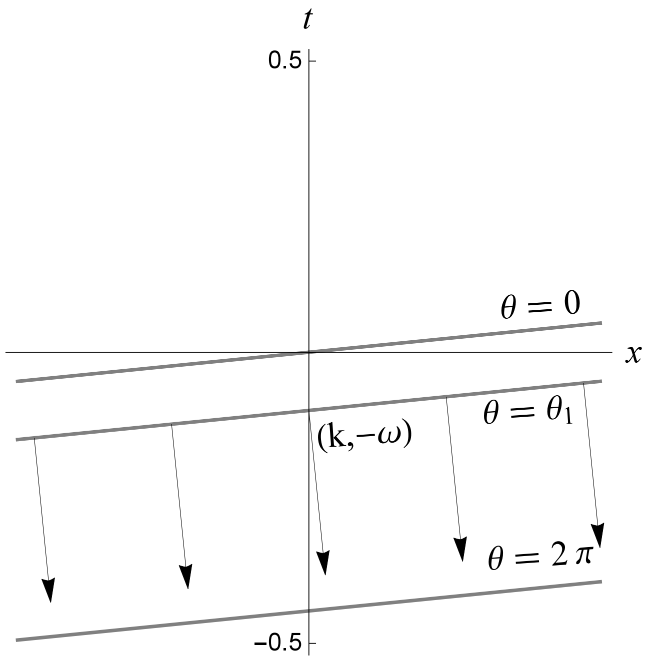


Figure 1.20: The wave vector and the frequency are strictly constants throughout the whole domain in the formulation of the uniform wave packet.

tions, we obtain

$$\delta\bar{I} = \frac{1}{2\pi}\int_0^{2\pi}[L_i - k_\alpha(L_{i\alpha})_\theta + \omega(L_{it})_\theta]\delta\phi_i\,\mathrm{d}\theta = 0,$$

where the subscripts denotes the partial derivatives of L with respect to its arguments according to the rule

$$L_i = \frac{\partial L}{\partial u_i}, \quad L_{i\alpha} = \frac{\partial L}{\partial u_{i,\alpha}}, \quad L_{it} = \frac{\partial L}{\partial u_{i,t}}.$$

The Euler-Lagrange equations then read

$$L_i - k_\alpha(L_{i\alpha})_\theta + \omega(L_{it})_\theta = 0. \tag{1.17}$$

As standardized, one typical first integral is obtained by multiplying both sides by $\phi_{j,\theta}$ and then integrating the obtained equation with respect to θ as follows

$$\int L_i\mathrm{d}\phi_j + \int[\omega(L_{it})_\theta - k_\alpha(L_{i\alpha})_\theta]\phi_{j,\theta}\mathrm{d}\theta = h_j.$$

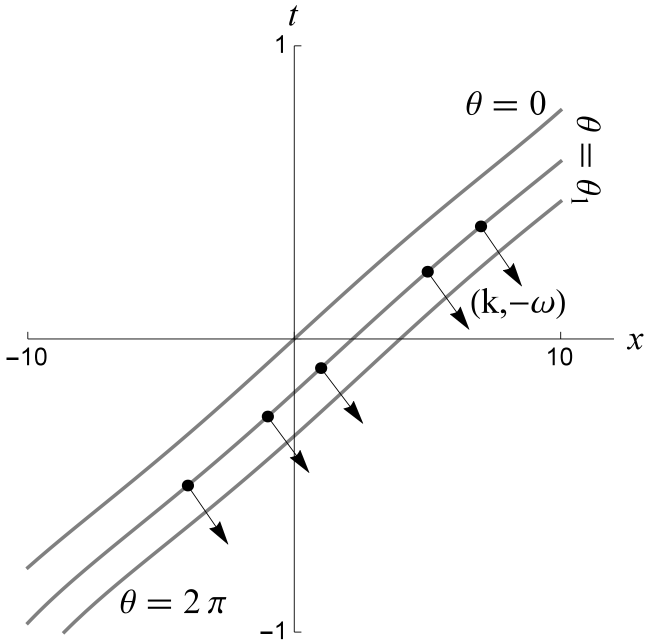


Figure 1.21: The wave vector and the frequency can be regarded as constants throughout each strip in the formulation of the nonuniform wave packet.

Provided that the original Lagrangian does not contain any cross terms between u_i and its derivatives $u_{i,\alpha}$, $u_{i,t}$, the functions $L_{i\alpha}$ and L_{it} do not depend on ϕ_i so that the first integral can be reduced further to

$$\int L_i \mathrm{d}\phi_j + \sum_m \int \left(\omega \frac{\partial L_{it}}{\partial \phi_{m,\theta}} - k_\alpha \frac{\partial L_{i\alpha}}{\partial \phi_{m,\theta}} \right) \phi_{j,\theta} \mathrm{d}(\phi_{m,\theta}) = h_j. \tag{1.18}$$

The associated Euler-Lagrange equations (1.17) are a system of n nonlinear second-order ordinary differential equations which naturally admits the same number of the first integrals. There exists $2n$ integration constants in the solutions, half of which is determined from the periodicity conditions on $\phi_i(\theta)$ and the other half can be determined by specifying the n constants entering the above first integrals. Nevertheless, it is not always convenient to employ these integration constants as slowly modulated parameters in the non-uniform wavetrain. Such choice of parameters depends on the phase portraits and obviously on our specific interests. We call this variational problem *strip problem.*

Modulation equations Let us denote $\mathcal{L}$ the value of the functional (1.16) at its extremal and call it the average Lagrangian. It is a function of h_i, k_α and ω

$$\mathcal{L} = \mathcal{L}(h_i, k_\alpha, \omega),$$

where h_i are the parameters we have chosen in the strip problem. The original sum (1.15), as $\lambda/\Lambda \to 0$ and $\tau/T \to 0$, can be approximately replaced by the integral

$$\iint_R \mathcal{L}(h_i, k_\alpha, \omega) \mathrm{d}\mathbf{x} \mathrm{d}t,$$

with k_α and ω being regarded as functions of x and t at this step. Taking into account that the wave vector and the frequency are defined through the common phase variable $\theta = \theta(\mathbf{x}, t)$, the first variation of this functional with respect to h_i and θ gives

$$\frac{\partial \mathcal{L}}{\partial h_i} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \theta_{,t}} + \frac{\partial}{\partial x_\alpha}\frac{\partial \mathcal{L}}{\partial \theta_{,\alpha}} = 0.$$

It is however more practical to deal with the system of first-order differential equations involving the more directly relevant characteristic quantities of waves such the wave vector $\mathbf{k} = (k_\alpha)$ and the frequency ω. For this reason we cast this system to

$$\frac{\partial \mathcal{L}}{\partial h_i} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \omega} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{k}} = 0, \tag{1.19}$$

which must be complemented by the compatibility conditions

$$k_{\alpha,t} + \omega_{,\alpha} = 0, \quad k_{\alpha,\beta} - k_{\beta,\alpha} = 0.$$

The first equation $(1.19)_1$ expresses the solvability conditions for the strip problem leading to the nonlinear dispersion relation, while the second equation $(1.19)_2$ is equivalent to the equation of energy propagation. It is helpful to put here a comment that the solvability conditions are nothing else but the conditions of normalizing the periods of $\phi_i(\theta)$ to specific constants which are all equal to the factor 2π in our consideration.

At this point, let us take the chance to introduce the notion of the amplitudes a_i and the maximal slopes p_i according to

$$a_i = \max_{\theta \in [0,2\pi]} |\phi_i|, \quad p_i = \max_{\theta \in [0,2\pi]} |\phi_{i,\theta}|.$$

These notions will be intensively investigated in the subsequent chapters. If the amplitudes a_i are chosen as the governing parameters instead of the energy levels h_i, then the first equation of system (1.19) can be engendered to

$$\frac{\partial \mathcal{L}}{\partial a_i} = 0,$$

and the second equation is called *the equation of amplitude modulation*. Analogously, if the maximal slopes are used, the solvability conditions take the form

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0,$$

and the other equation is called *the equation of slope modulation*.

Remark The carefully described procedure can be actually generalized to the system whose Lagrangian contains higher-order derivatives. Furthermore, it could be used to handle with the "‘nearly periodic” solution with slight modification and of course change of interpretation. Although some modifications of the theory could be addressed here, it simply irritates the readers to generalize the method without any feasible application to the realistic example, in shorter words, with lack of motivation. Thus we shall postpone the discussion till the point we have to.

Bibliographical acknowledgement

This chapter is aimed at providing the preliminary knowledge and a mathematical tool for later investigation. Since the modulation theory has a long path of development, it must be brought out that the author borrowed several sources of material scattered in the published works by Whitham, Kamchatnov, Le and Nguyen [42,61–63]. More details and applications of the modulation theory can be found in these standard textbooks. It is not surprising that a part of presentation here is subject to that done in the fore-mentioned publications.

2 Direct methods for evolution equations

The aim of this Thesis is to construct the theory of amplitude (or slope) modulation of nonlinear dispersive waves and to compare the obtained results with the exact solutions of the corresponding evolution equation. For integrable systems the exact solutions can be found either by the inverse scattering transform or by the direct methods. Although the inverse scattering transform allows one to integrate the evolution equations more thoroughly than the direct method, it requires a substantial background in mathematics such as complex analysis and functional analysis. On the contrary, the Hirota's method asks for quite minimal knowledge of mathematics that mostly involves only differentiation and simple algebra with some skillful computation. In this chapter we shall present some direct methods for the exact integration of evolution equations based on three analytical techniques

- Modified homogeneous balance method,
- Hirota's bilinear differential operators,
- Wronskian determinant.

The first one helps us to recognize the balance between the nonlinear and dispersive effects in some sense which is still mysterious to the author. Subsequently, using the differential operators which were credited to Hirota, the original equation is transformed to the so-called bilinear form. More details on this subject can be found in the classic textbook. As a result, we expect that it is more feasible to solve the bilinear differential equation in replacement of the original nonlinear equation. The relevant references will be noted in the end of the chapter.

2.1 Multiple collisions of solitons

Being consistent with the mentality of the first chapter, we examine one objective example before coming to the outline of the solution method. Then we shall apply it to several evolution equations.

2.1.1 Multiple-soliton of the Korteweg-de Vries equation

Due to the significant historical importance, we start first with the most well-known prototype of soliton equation, namely the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.1}$$

Modified homogeneous balance method In order to balance the nonlinear and dispersive terms, the solution will be sought in the form

$$u = \partial_x^n \varphi(\theta(x,t)) + a, \tag{2.2}$$

where $\varphi = \varphi(\theta)$ will be found later using one ad hoc condition and a is an arbitrary constant. The order n of derivative with respect to x is chosen in such a way that the "total" orders of derivatives of the nonlinear term uu_x and of the dispersive term u_{xxx} are equal, that is

$$n + (n+1) = n + 3 \quad \Rightarrow \quad n = 2.$$

Upon substitution of this Ansatz into Eq. (2.1), a quite complicated differential equation is obtained

$$\begin{aligned}
&(6\varphi_{\theta\theta}\varphi_\theta^{(3)} + \varphi_\theta^{(5)})\theta_x^5 + (18\varphi_{\theta\theta}^2 + 6\varphi_\theta\varphi_\theta^{(3)} + 10\varphi_\theta^{(4)})\theta_x^3\theta_{xx} \\
&+(\theta_t\theta_x^2 + 6a\theta_x^3 + 15\theta_x\theta_{xx}^2 + 10\theta_x^2\theta_{xxx})\varphi_\theta^{(3)} + (18\theta_x\theta_{xx}^2 + 6\theta_x^2\theta_{xxx})\varphi_\theta\varphi_{\theta\theta} \\
&+(2\theta_t\theta_{xt} + \theta_t\theta_{xx} + 18a\theta_x\theta_{xx} + 10\theta_{xx}\theta_{xxx} + 5\theta_x\theta_{xxxx})\varphi_{\theta\theta} + 6\theta_{xx}\theta_{xxx}\varphi_\theta^2 \\
&+(\theta_{xxt} + 6a\theta_{xxx} + \theta_{xxxxx})\varphi_\theta = 0.
\end{aligned} \tag{2.3}$$

The most important heuristic condition is that the coefficient of θ_x^5 vanishes

$$6\varphi_{\theta\theta}\varphi_\theta^{(3)} + \varphi_\theta^{(5)} = 0.$$

It should be emphasized that this coefficient is collected from only the nonlinear and the highest-order derivative terms. We hope that this equation is solvable so that the explicit expression for $\varphi(\theta)$ is revealed. It is the case indeed when we integrate this equation consecutively. By choosing the zero integration constants, this gives

$$\varphi(\theta) = 2\log\theta. \tag{2.4}$$

Using this simple solution, we can express the multiplication of its derivatives in terms of the single derivatives as follows

$$\varphi_{\theta\theta}^2 = -\frac{1}{3}\varphi_\theta^{(4)}, \quad \varphi_\theta\varphi_\theta^{(3)} = -\frac{2}{3}\varphi_\theta^{(4)}, \quad \varphi_\theta\varphi_{\theta\theta} = -\varphi_\theta^{(3)}, \quad \varphi_\theta^2 = -2\varphi_{\theta\theta}.$$

Substituting these results into Eq. (2.3), we obtain

$$P_1(\theta)\varphi_\theta^{(3)} + P_2(\theta)\varphi_{\theta\theta} + P_3(\theta)\varphi_\theta = 0, \tag{2.5}$$

where the coefficients are given by

$$\begin{aligned}
P_1 &= 6a\theta_x^3 + \theta_t\theta_x^2 + 4\theta_x^2\theta_{xxx} - 3\theta_x\theta_{xx}^2, \\
P_2 &= 18a\theta_x\theta_{xx} - 2\theta_{xx}\theta_{xxx} + 5\theta_x\theta_{xxxx} + 2\theta_x\theta_{xt} + \theta_t\theta_{xx}, \\
P_3 &= 6a\theta_{xxx} + \theta_{xxt} + \theta_{xxxxx}.
\end{aligned} \tag{2.6}$$

It is not clear at the first sight that Eq. (2.5) is of the potential-field form

$$\partial_x G(\theta, \varphi_\theta(\theta)) = 0,$$

in which we assign the name "potential-field" for the comfort of the subsequent procedures. This name comes from the high school knowledge. We call a vector field $\mathbf{F}$ potential field if it can be generated through the gradient of a certain function ϕ as $\mathbf{F} = \nabla\phi$. To make this form realizable, one feasible trick is to expand the following expression

$$(A\varphi_{\theta\theta} + B\varphi_\theta)_x = A\theta_x\varphi_\theta^{(3)} + A_x\varphi_{\theta\theta} + B\theta_x\varphi_{\theta\theta} + B_x\varphi_\theta,$$

and to compare it with Eq. (2.5). By performing such comparison, we end up with

$$P_1 = A\theta_x, \quad P_2 = A_x + B\theta_x, \quad P_3 = B_x,$$

where the expressions A, B are given by

$$A = 6a\theta_x^2 + \theta_t\theta_x + 4\theta_x\theta_{xxx} - 3\theta_{xx}^2, \quad B = 6a\theta_x + \theta_{xt} + \theta_{xxxx}.$$

Thus, we have just arrived at the equation

$$(A\varphi_{\theta\theta} + B\varphi_\theta)_x = 0.$$

An integration of this equation with respect to x yields

$$A\varphi_{\theta\theta} + B\varphi_\theta = q(t),$$

where $q(t)$ is a function depending only on time variable. Using Eq. (2.4), this equation can be rewritten in a more explicit form

$$B\theta - A = \theta_{xt}\theta - \theta_x\theta_t + 6a(\theta\theta_{xx} - \theta_x^2) + \theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2 = \theta^2\frac{q(t)}{2}. \quad (2.7)$$

Bilinear form It is a great achievement of Ryogo Hirota when he discovered that the KdV equation can be transformed to the bilinear differential equation by devising the bilinear differential operators. The operators act on two differentiable functions $f = f(x,t)$ and $g = g(x,t)$ according to the rule

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \xi}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}\right)^n f(x,t)g(\xi,\tau)\Big|_{\xi\to x,\tau\to t}. \quad (2.8)$$

Some examples are given below

$$\begin{aligned} D_x f \cdot g &= f_x g - f g_x, \\ D_x^2 f \cdot g &= f_{xx} g - 2f_x g_x + f g_{xx}, \\ D_x D_t f \cdot g &= f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \\ D_x^3 f \cdot g &= f_{xxx} g - 3f_{xx} g_x + 3f_x g_{xx} - f g_{xxx}, \\ D_x^4 f \cdot g &= f_{xxxx} g - 4f_{xxx} g + 6f_{xx} g_{xx} - 4f_x g_{xxx} + f g_{xxxx}. \end{aligned} \quad (2.9)$$

It is interesting that the formulations of these operators are more or less like the power expansions of the form $(f-g)^n$ inasmuch as the power of each multiplicator are replaced by the corresponding order of differentiation. To clarify this statement, we may examine the similarity of two expansions

$$(f-g)^n = \sum_{k=0}^n C_n^k f^{n-k} g^k = C_n^0 f^n g^0 - C_n^1 f^{n-1} g + \cdots + (-1)^n C_n^n f^0 g^n,$$

and

$$D_x^n f \cdot g = \sum_{k=0}^n C_n^k \partial_x^{n-k} f \times \partial_x^k g = C_n^0 \partial_x^n f\, g - C_n^1 \partial_x^{n-1} f\, g + \cdots + (-1)^n C_n^n f\, \partial_x^n g,$$

where C_n^k, $0 \le k \le n$, are the binomial coefficients defined by

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Most importantly, the definition (2.8) provides the bilinear differential operators, that is

$$D_x^m D_t^n[(\alpha f_1 + \beta f_2)\cdot g] = \alpha D_x^m D_t^n f_1 \cdot g + \beta D_x^m D_t^n f_2 \cdot g,$$
$$D_x^m D_t^n[f\cdot(\alpha g_1 + \beta g_2)] = \alpha D_x^m D_t^n f \cdot g_1 + \beta D_x^m D_t^n f \cdot g_2.$$

Now, if we set $f = g = \theta$ in Eq. (2.9), we get the following relevant formulas

$$D_x D_t \theta\cdot\theta = 2(\theta_{xt}\theta - \theta_x\theta_t),$$
$$D_x^2\theta\cdot\theta = 2(\theta\theta_{xx} - \theta_x^2),$$
$$D_x^4\theta\cdot\theta = 2(\theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2).$$

Thus, Eq. (2.7) is engendered in a more elegant form

$$D_x(D_t + 6aD_x + D_x^3)\theta\cdot\theta = \theta^2 q(t).$$

When $q = 0$, this equation is called the bilinear form of the KdV equation or simply the KdV bilinear equation. From now on we shall call the operator defined by Eq. (2.8) the D-operator.

Useful properties of Hirota's operators It is appropriate to list here a few properties that are beneficial for the method of solution.

(a) The normal derivatives and the D-operators can be connected by

$$D_x^m D_t^n \theta\cdot 1 = D_x^m D_t^n 1\cdot\theta = \frac{\partial^{n+m}\theta}{\partial x^n t^m}. \tag{2.10}$$

(b) Action of the D-operators on the exponential functions is given by

$$\begin{aligned} D_x^m D_t^n(\exp\eta_1\cdot\exp\eta_2) &= (k_1 - k_2)^m(\omega_1 - \omega_2)^n\exp(\eta_1+\eta_2),\\ \eta_i &= k_i x + \omega_i t + \delta_i, \quad i = 1, 2. \end{aligned} \tag{2.11}$$

The second can be generalized by representing it in the form of polynomials with arguments being the bilinear operators. Indeed, let P be a polynomial of two variables

$$P(X,T) = \sum a_{mn}X^mT^n,$$

where the range of summation can be freely determined up to our specific problems, then we can apply the formal polynomial of two arguments D_x and D_t to the exponential functions to obtain

$$P(D_x, D_t)(\exp\eta_1\cdot\exp\eta_2) = P(k_1 - k_2, \omega_1 - \omega_2)\exp(\eta_1 + \eta_2).$$

Soliton solutions of the bilinear equation For the soliton solutions, it suffices to solve the bilinear equation

$$D_x(D_t + 6aD_x + D_x^3)\theta\cdot\theta = 0. \tag{2.12}$$

We shall deal with this equation by using the perturbation technique. It is worth giving a short remark here. It is not quite accurate to claim the application of either the perturbation method or the perturbation theory since there exist no small parameters in our current differential equation, which could be explained clearer during the course of derivation. Nevertheless, following the spirit of the perturbation method, we expand θ in the power series of the form

$$\theta = \epsilon^0 + \epsilon^1\theta_1 + \epsilon^2\theta_2 + \cdots = \sum_{i=0}^{\infty} \epsilon^i\theta_i, \tag{2.13}$$

where ϵ could be considered as small parameter though it is certainly not, and the functions $\theta_i = \theta_i(x,t)$ must be subsequently determined. Substituting this Ansatz into Eq. (2.12), equating the coefficients of ϵ^i, $i \geq 0$, to zero, a sequence of equations for solving θ_i is obtained as follows

$$\begin{aligned}
\epsilon^0 &: \quad D_x(D_t + 6aD_x + D_x^3)(1\cdot 1) = 0,\\
\epsilon^1 &: \quad D_x(D_t + 6aD_x + D_x^3)(1\cdot\theta_1 + \theta_1\cdot 1) = 0,\\
\epsilon^2 &: \quad D_x(D_t + 6aD_x + D_x^3)(1\cdot\theta_2 + \theta_1\cdot\theta_1 + \theta_2\cdot 1) = 0,\\
\epsilon^3 &: \quad D_x(D_t + 6aD_x + D_x^3)(1\cdot\theta_3 + \theta_1\cdot\theta_2 + \theta_2\cdot\theta_1 + \theta_3\cdot 1) = 0,
\end{aligned}$$

and so on. Obviously, we may write the system in the general form as

$$D_x(D_t + 6aD_x + D_x^3)\Big(\sum_{i=0}^{m} \theta_i\cdot\theta_{m-i}\Big) = 0, \quad m \geq 0.$$

Using Eq. (2.10), the equation for ϵ^1 is transformed to the linear ordinary differential equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + 6a\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)\theta_1 = 0.$$

It is well-established in the theory of ODE that this linear equation admits the solution of the exponential form

$$\theta_1 = \exp\eta_1, \quad \eta_1 = k_1x + \omega_1t + \delta_1,$$

which, upon substitution into the above equation, implies

$$\omega_1 + 6ak_1 + k_1^3 = 0.$$

Recalling the relation between the wave number and the frequency, we may also name this equation the dispersion relation. Then the equation for ϵ^2 can be rearranged in such a way that the unknown functions remain on the left-hand side and the known functions are moved to the other side

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + 6a\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)\theta_2 = -D_x(D_t + 6aD_x + D_x^3)\theta_1\cdot\theta_1.$$

By using the second property (2.11), the right-hand side produces zero so that we are allowed to choose $\theta_2 = 0$ in this equation. Apparently, the other vanishing functions $\theta_i = 0, i \geq 3$, make all the coefficients of ϵ^i equal to zero and hence the series (2.13) is truncated into the finite sum

$$\theta = 1 + \epsilon\theta_1.$$

Since the parameter ϵ can be absorbed into the initial phase δ_1, we finally arrive at the solution of the bilinear equation as follows

$$\theta = 1 + \exp(k_1x + \omega_1t + \delta_1), \quad \omega_1 + 6ak_1 + k_1^3 = 0.$$

The absorption of ϵ into the arbitrary initial phase also implies that this parameter is not necessarily small. Substituting this result into two equations (2.2) and (2.4), the solitary wave is obtained as follows

$$u(x,t) = 2\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a = \frac{2k_1^2 \exp(\eta_1)}{(1+\exp\eta_1)^2} + a = \frac{k_1^2}{2}\mathrm{sech}^2\frac{\eta_1}{2} + a,$$

which is called the one-soliton. We see that the function given by Eq. (2.4) is a bridge between the solution to the bilinear equation and the original solution to the KdV equation. For its crucial role we call the function $\varphi(\theta)$ in our context the 'pivot' function. Excluding the constant a dictating the basement of soliton, the 1-soliton solution has two arbitrary parameters k_1 and δ_1. The former determines the amplitude of the soliton, and the latter its initial position or initial phase.

By accepting that 1-soliton is governed by two independent parameters, it is expected that the n-soliton can be fully characterized by $2n$ independent parameters. Indeed, if we look back at the entire procedure, we see that the left-hand side of each equation for the power ϵ^i can always be managed to transformed to the linear ODE. This fact suggests us to use the linear superposition principle for the solution at the first step

$$\theta_1 = \exp\eta_1 + \exp\eta_2, \quad \eta_i = k_i x + \omega_i t + \delta_i, \quad i = 1,2.$$

Once again, in order that the first step is completed, the dispersion relations must be fulfilled, that is

$$\omega_i + 6ak_i + k_i^3 = 0, \quad i = 1,2.$$

Continuing the perturbation procedure, the second term θ_2 cannot be zero in this case but the third one is. Indeed, let us write down the next step

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + 6a\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)\theta_2 = -D_x(D_t + 6aD_x + D_x^3)[\exp\eta_1\cdot\exp\eta_2 + \exp\eta_2\cdot\exp\eta_1],$$

which is cast down to

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + 6a\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right)\theta_2 = -(k_1-k_2)[(\omega_1-\omega_2)+6a(k_1-k_2)+(k_1-k_2)^3]\exp(\eta_1+\eta_2).$$

It is natural to try the solution of the form

$$\theta_2 = \gamma_{12}\exp(\eta_1+\eta_2),$$

where γ_{12} is a constant. Upon substitution of this Ansatz into the above equation and elimination of the common factor $\exp(\eta_1+\eta_2)$, the coefficient γ_{12} is obtained as the following ratio

$$\gamma_{12} = -\frac{(k_1-k_2)\,[(\omega_1-\omega_2) + 6a(k_1-k_2) + (k_1-k_2)^3]}{(k_1+k_2)\,[(\omega_1+\omega_2) + 6a(k_1+k2) + (k_1+k_2)^3]} = -\frac{(k_1-k_2)^2}{(k_1+k_2)^2}.$$

The other equations are automatically fulfilled with $\theta_i = 0, i \geq 3$, and the exact solution is given by

$$\theta = 1 + \exp\eta_1 + \exp\eta_2 + \gamma_{12}\exp(\eta_1+\eta_2),$$

where the powers of ϵ disappear according to the same argument as above. The solution (2.2) obtained from this function describes the interaction of two solitons. The possibility of deriving the coefficient γ_{12} from the algebraic structure of the bilinear equation is remarkable. To clarify this point, let us introduce the polynomial

$$P(X,T) = X(T + 6aX + X^3)$$

and rewrite the KdV bilinear equation using the formal polynomial as follows

$$P(D_x, D_t)\theta \cdot \theta = D_x(D_t + 6aD_x + D_x^3)\theta \cdot \theta = 0.$$

Then the coefficient can be represented by using this formal polynomial as

$$\gamma_{12} = -\frac{P(k_1 - k_2, \omega_1 - \omega_2)}{P(k_1 + k_2, \omega_1 + \omega_2)}.$$

This observation will be extensively used.

By conducting the perturbation procedure up to many higher-order solutions, we obtain the inductive formula for generating the n-soliton solution of the KdV equation. Omitting the detailed calculations, we write down only the final result

$$u(x,t) = 2\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a,$$
$$\theta = 1 + \sum_{n=1}^{N} \sum_{C_N^n} \left[\prod_{k<l}^{(n)} \gamma(i_k, i_l) \right] \exp(\eta_{i_1} + \cdots + \eta_{i_n}),$$
$$\eta_i = k_i x + \omega_i t + \delta_i, \quad \omega_i = -6ak_i - k_i^3,$$
$$\gamma(i,j) = -\frac{(k_i - k_j)[\omega_i - \omega_j + 6a(k_i - k_j) + (k_i - k_j)^3]}{(k_i + k_j)[\omega_i + \omega_j + 6a(k_i + k_j) + (k_i + k_j)^3]} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}.$$

In this solution formula C_N^n indicates all possible combination of n elements from the set of N elements

$$\Omega_N = \{j \in \mathbb{N}, 1 \le j \le N\},$$

and $\prod_{k<l}^{(n)}$ is the product of all possible combinations of these taken n elements. Note that the solution provided here is different from what we have known due to the appearance of the arbitrary constant a (cf. [13]). It is interesting that the n-soliton solution provided above gives more flexibility in determining the velocity of each soliton which is determined by

$$c_i = -\omega_i / k_i = 6a + k_i^2.$$

Thus, one soliton propagates to the left if $6a + k_i^2 < 0$, to the right if $6a + k_i^2 > 0$, and stands still otherwise. To illustrate this argument, we pick up a 3-soliton solution generated with the following parameters: $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $a = -2/3$. Accordingly, the velocities of three respective solitons are $c_1 = -3$, $c_2 = 0$, $c_3 = 5$, so two of them propagate in opposite directions whereas the other does not propagate at all. In Fig. 2.1 such exact solution is plotted at three different time instants to make the explanation visible.

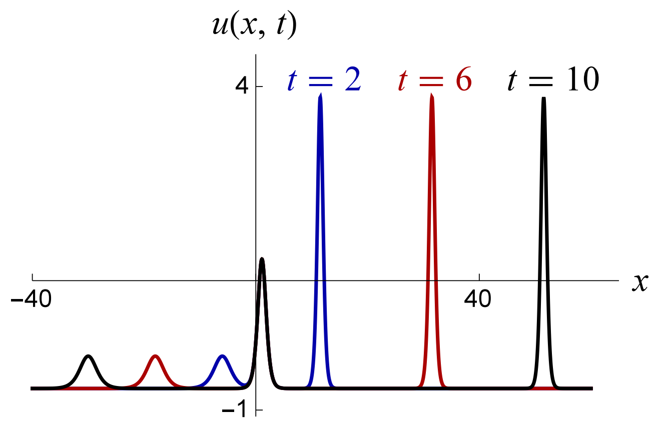


Figure 2.1: Three solitons, governed by KdV equation, propagate in time.

Summary of the solution method As said clearly before, we are ready to recapitulate the method of solution. Let us first describe the modified homogeneous balance method using the general form of the evolution equation

$$\Phi(u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0. \tag{2.14}$$

Step 1 The solution is sought in the form

$$u(x,t) = \frac{\partial^{m+n}}{\partial x^m t^n}\varphi(\theta(x,t)) + a,$$

where a is an arbitrary constant, m and n are two positive integers. The values of these integers should be determined in such a way that the "total" derivative of the nonlinear term is in balance with that of the term containing the highest-order derivative.

Step 2 Substituting this Ansatz into Eq. (2.14), we obtain

$$\Psi\left(\theta_x^M \mathcal{P}(\varphi), P_{M-1}(\theta)\varphi_\theta^{(M-1)}, P_{M-2}(\theta)\varphi_\theta^{(M-2)}, \ldots, P_1(\theta)\varphi_\theta\right) = 0, \tag{2.15}$$

where M is the highest derivative present after the substitution. The coefficient $\mathcal{P}(\varphi)$ is collected only from parts of the nonlinear and the highest-order derivative terms. It is proposed that the pivot function $\varphi = \varphi(\theta)$ is found with the heuristic condition

$$\mathcal{P}(\varphi) = 0.$$

To carry on the method, it must be assumed that this equation is an integrable ODE and the zero integration constants can be chosen for the simplicity. Such choice is hoped to be sufficient at least for the soliton solution.

Step 3 With the explicit pivot function in hand we can compute its all involved derivatives and relate their products to the single derivatives so that Eq. (2.15) can be reduced to

$$\Psi\left(P_N(\theta)\varphi_\theta^{(N)}, P_{N-1}(\theta)\varphi_\theta^{(N-1)}, \ldots, P_1(\theta)\varphi_\theta\right) = 0, \tag{2.16}$$

where N is the highest order that still remains. Then recognition of the equation in the potential-field form

$$\partial_x^r \partial_t^s (A_q \varphi_\theta^{(q)} + \cdots + A_1\varphi_\theta + A_0\varphi) = 0 \tag{2.17}$$

allows one to integrate it with respect x and t to obtain

$$A_q\varphi_\theta^{(q)} + \cdots + A_1\varphi_\theta + A_0\varphi = 0, \tag{2.18}$$

where the integration functions have been chosen to be all zero. A good strategy to perform it is to "run backward" rather than "forward". As the matter of fact, if this step is doable, then expansion of Eq. (2.17) must coincide with Eq. (2.16). Then by equating the corresponding coefficients of $\varphi_\theta^{(i)}$, the expressions for A_i can be derived.

Step 4 Using the explicit formula for $\varphi(\theta)$, Eq. (2.18) is transformed to the PDE containing only the phase function $\theta(x,t)$ as follows

$$\Lambda(\theta_x, \theta_t, \theta_{tt}, \theta_{xt}, \theta_{xx}, \ldots) = 0. \tag{2.19}$$

It is probably the most essential step to rearrange the entire equation in such a way that its bilinear form is disclosed

$$\mathcal{B}(D_t, D_x, D_t^2, D_xD_t, D_x^2, \ldots)(\theta\cdot\theta) = 0. \tag{2.20}$$

On the other hand, it should be noted that the transformation from Eq. (2.19) to Eq. (2.20) is not a must as long as we can solve the former equation using some particular techniques.

Step 5 In the last step it is assumed that the bilinear equation (2.20) is derivable. Hence the perturbation technique shall work towards its advantage to find the exact solution as the finite truncation of the series in the small parameter. To this end, we provide here the multi-soliton solution to the original evolution equation (2.14) using the inductive formula as follows

$$\begin{aligned} u &= \frac{\partial^{m+n}}{\partial x^m t^n}\varphi(\theta(x,t)) + a, \\ \theta &= 1 + \sum_{n=1}^{N}\sum_{C_N^n}\left[\prod_{k<l}^{(n)}\gamma(i_k, i_l)\right]\exp(\eta_{i_1} + \cdots + \eta_{i_n}), \\ \eta_i &= k_ix + \omega_it + \delta_i, \quad \mathcal{B}|_{D_t\to\omega_i, D_x\to k_i} = 0, \\ \gamma(i,j) &= -\frac{\mathcal{B}|_{D_t\to(\omega_i-\omega_j), D_x\to(k_i-k_j)}}{\mathcal{B}|_{D_t\to(\omega_i+\omega_j), D_x\to(k_i+k_j)}}, \end{aligned} \tag{2.21}$$

where replacements of the arguments of $\mathcal{B}$ with the real values are indicated by the arrows, and the summation and the product were clearly explained above.

2.1.2 Multiple-soliton of the scalar Boussinesq equation

The general scalar Boussinesq (BSQ) equation is given in the form

$$v_{tt} + \alpha v_{xx} + \beta(v^2)_{xx} + \gamma v_{xxxx} = 0, \tag{2.22}$$

where α, β and γ are three constants. It is usually distinguished as *good* and *bad* ones based on the well-posedness of the linearized equation. If $\gamma > 0$, the equation is classified as the good BSQ equation since its linearized version is known as a well-posed equation. In contrast, if $\gamma < 0$, the linearized equation is ill-posed and called the bad BSQ equation. Indeed, two distinct variants of the BSQ equation

$$u_{tt} + (u^2)_{xx} \pm u_{xxxx} = 0$$

can be obtained under the changes of variables

$$v(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\beta} u(x, \sqrt{\pm\gamma}t),$$

in which the plus sign corresponds to the good BSQ equation with $\gamma > 0$ and the minus sign to the bad BSQ equation with $\gamma < 0$. Since the treatments of both equations for the soliton solutions are more or less similar, we shall focus ourselves on the good BSQ equation, namely

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0. \tag{2.23}$$

Without much explanation we apply the above outlined procedure and present the derivation step by step.

Step 1 We propose the Ansatz

$$u(x,t) = \varphi(\theta(x,t))_{xx} + a, \tag{2.24}$$

in which the order of derivative n comes from the balance of total derivatives of the nonlinear term and the dispersive term, that is

$$2n + 2 = n + 4 \quad \Rightarrow \quad n = 2.$$

Step 2 Substituting this Ansatz into Eq. (2.23) and collecting the coefficient of θ_x^6, the ODE for determining the pivot function is obtained as follows

$$2\varphi_{\theta\theta\theta}^2 + 2\varphi_{\theta\theta}\varphi_\theta^{(4)} + \varphi_\theta^{(6)} = 0.$$

Solving this equation with zero integration constants, we obtain

$$\varphi(\theta) = 6\log\theta \quad \Rightarrow \quad u(x,t) = 6\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a.$$

Step 3 Substituting this Ansatz into Eq. (2.23), it is expanded to

$$A\theta_x^2\varphi_\theta^{(4)} + C\varphi_\theta^{(3)} + D\varphi_{\theta\theta} + B_{xx}\varphi_\theta = 0, \tag{2.25}$$

where

$$\begin{aligned}
A &= \theta_t^2 + 2a\theta_x^2 - 3\theta_{xx}^2 + 4\theta_x\theta_{xxx},\\
C &= \theta_x^2\theta_{tt} + 4\theta_t\theta_x\theta_{xt} + \theta_t^2\theta_{xx} + 12a\theta_x^2\theta_{xx} - 3\theta_{xx}^3 + 9\theta_x^2\theta_{xxxx},\\
D &= 2\theta_{xt}^2 + \theta_{tt}\theta_{xx} + 6a\theta_{xx}^2 + 2\theta_t\theta_{xxt} + 2\theta_x\theta_{xtt} + 3\theta_{xx}\theta_{xxxx}\\
&\quad + 8a\theta_x\theta_{xxx} + 6\theta_x\theta_{xxxxx} - 2\theta_{xxx}^2,\\
B &= \theta_{tt} + 2a\theta_{xx} + \theta_{xxxx}.
\end{aligned}$$

Direct calculation shows that

$$C = A\theta_{xx} + 2A_x\theta_x + B\theta_x^2, \quad D = A_{xx} + 2B_x\theta_x + B\theta_{xx}.$$

Introducing a new dependent variable $\psi = \varphi_\theta$ and substituting the above relations into Eq. (2.25), we find that it can be transformed to

$$(A\psi_\theta + B\psi)_{xx} = 0.$$

Step 4 Integrating this equation with respect to x twice and using the definition $\psi = 6/\theta$, it is explicitly written in the form

$$\theta\theta_{tt} - \theta_t^2 + 2a(\theta\theta_{xx} - \theta_x^2) + \theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2 = \frac{1}{2}\theta^2(p(t)x + q(t)).$$

Finally, the bilinear form of the Boussinesq equation has just been established as follows

$$(D_t^2 + 2aD_x^2 + D_x^4)\theta \cdot \theta = 0, \tag{2.26}$$

in which $p(t)$ and $q(t)$ are chosen to be identically zero.

Step 5 In analogy with the KdV case, the general formula for constructing the n-soliton is given by

$$\begin{aligned} u(x,t) &= 6\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a, \\ \theta &= 1 + \sum_{n=1}^{N}\sum_{C_N^n}\left[\prod_{k<l}^{(n)}\gamma(i_k, i_l)\right]\exp(\eta_{i_1} + \cdots + \eta_{i_n}), \\ \eta_i &= k_i x + \omega_i t + \delta_i, \quad \omega_i^2 + 2ak_i^2 + k_i^4 = 0, \\ \gamma(i,j) &= -\frac{(\omega_i - \omega_j)^2 + 2a(k_i - k_j)^2 + (k_i - k_j)^4}{(\omega_i + \omega_j)^2 + 2a(k_i + k_j)^2 + (k_i + k_j)^4} = \frac{(c_i - c_j)^2 - 3(k_i - k_j)^2}{(c_i - c_j)^2 - 3(k_i + k_j)^2}. \end{aligned} \tag{2.27}$$

Similarly to the argument in the last section, the velocity of each soliton is given by

$$c_i = -\omega_i/k_i = \epsilon_i\sqrt{-(2a + k_i^2)}, \tag{2.28}$$

where ϵ_i taking value 1 or -1 describe the propagating direction.

Unlike the KdV case, the addition of constant a in Ansatz (2.24) in case of the BSQ equation is essential. It helps not only to adjust the velocity of each soliton but also to make the existence of solution with exponential tail possible. Indeed, let us assume that $a = 0$, $p = q = 0$ so that the bilinear form (2.26) reduces to that derived in [79]

$$(D_t^2 + D_x^4)\theta \cdot \theta = 0.$$

The dispersion relation in this case becomes $\omega_i^2 + k_i^4 = 0$, which yields only the roots ω_i in complex plane provided that the wave numbers k_i are real. This argument is reinforced by following the construction of solutions in [79] with the aid of Wronskian formulation. The soliton solution with exponential tail could not be extracted by using this formulation.

A 2-soliton solution corresponding to the parameters $k_1 = 1$, $k_2 = 2$, $\epsilon_1 = 1$, $\epsilon_2 = -1$, $a = -3$ is plotted in Fig. 2.2. Two solitons first approach to each other from two opposite sides. They experience the interaction near the origin and reflect to inverse their propagating directions. Note that one soliton is constantly subject to singularity and that both solitons change their amplitudes and widths after interaction. Even though the singularity of solution is beyond both the knowledge and physical interpretation of the author, this kind of solutions is still reported in case it might become helpful in the future research. Actually, it might be not appropriate to call this a soliton due to its singularity.

Another amazing phenomenon is reported here on 2-soliton when the parameters are chosen to satisfy the condition $\gamma(1,2) = 0$. We visualize the behavior of solution with $k_1 = 1$, $k_2 = 2$, $a = -2$ in Fig. 2.3 for illustration. Two solitons start to propagate with the equal

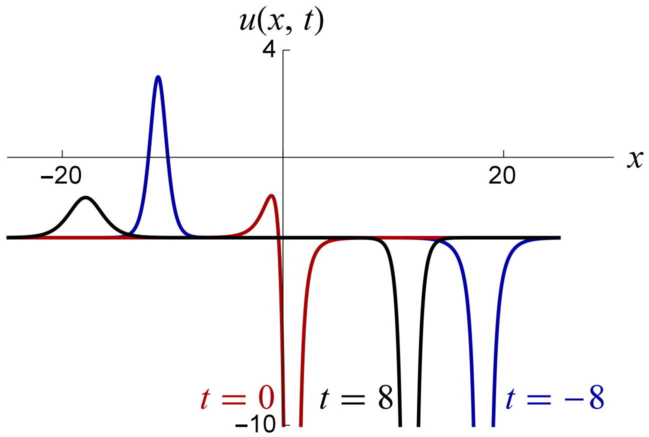


Figure 2.2: Two solitons, governed by the BSQ equation, approach to each other and experience reflection after interaction.

velocity $c = \sqrt{3}$ from two sides in opposite directions. They approach and blend into each other instead of reflection so that they could pile up towards a standing bell-shaped pulse at large time. Accordingly, this solution must converge to a certain function $f(x)$ as t tends to infinity. In our specific example this limit function can be found to be

$$f(x) = 6\mathrm{sech}^2(x) - 2.$$

It is not so surprising that $f(x)$ is the "standing wave" solution of the BSQ equation, that is $f_t = 0$ and

$$(f^2)_{xx} + f_{xxxx} = 0.$$

Note also that the velocity formula (2.28) does not strictly hold when it is applied to compute c_2.

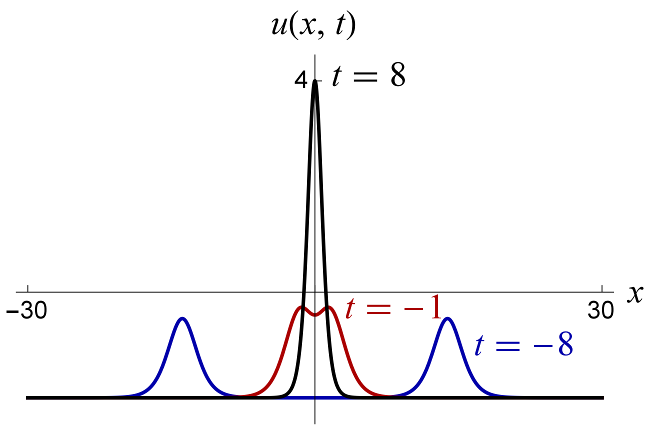


Figure 2.3: Two solitons, governed by the BSQ equation, approach to each other in opposite directions and pile up together at large time.

Mathematical justification of the n-soliton solution It is probably a right moment to justify the recursive formula for the n-soliton solution. Since both formulas for the multi-soliton solution of the KdV equation and of the BSQ equation possess the identical algebraic structure, it suffices to verify one of them. We shall prove the validity of the solution (2.27) by direct substitution and mathematical induction. In order to use the bilinear form (2.26), we rewrite the phase function as follows

$$\theta(x,t) = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{(N)} \mu_i \mu_j \varphi(i,j) + \sum_{i=1}^{N} \mu_i \eta_i \right),$$

where $\sum_{\mu=0,1}$ represents the summation over all possible values of μ_i and

$$\varphi(i,j) = \log \frac{(c_i - c_j)^2 - 3(k_i - k_j)^2}{(c_i - c_j)^2 - 3(k_i + k_j)^2}.$$

Substituting this Ansatz into Eq. (2.26), we obtain

$$\sum_{\mu=0,1} \sum_{\nu=0,1} \left[\left(\sum_{i=1}^{N} \omega_i(\mu_i - \nu_i) \right)^2 + 2a\left(\sum_{i=1}^{N} k_i(\mu_i - \nu_i) \right)^2 + \left(\sum_{i=1}^{N} k_i(\mu_i - \nu_i) \right)^4 \right]$$
$$\times \exp\left[\sum_{i<j}^{(N)} \varphi(i,j)(\mu_i\mu_j + \nu_i\nu_j) + \sum_{i=1}^{N} (\mu_i + \nu_i)\eta_i \right] = 0.$$

Inspecting this equation, the following observations are recorded.

Observation 1 It can be seen that the left-hand side of the equation is simply a polynomial of the variables $\exp \eta_i$ of maximum order $2N-2$ since the coefficients of the terms of order $2N-1$ or $2N$ vanish.

Observation 2 Since any permutations of indices leave the equation unchanged, we examine only the coefficients of the factors

$$\exp\left(\sum_{i=1}^{n} \eta_i + \sum_{i=n+1}^{m} 2\eta_i \right).$$

Hereby, let us denote this coefficient $D(n;m)$. Then we have

$$D(n;m) = \sum_{\mu=0,1} \sum_{\nu=0,1} \text{cond}(\mu,\nu) h(\omega,k,\mu,\nu) g(\omega,k,\mu,\nu),$$
$$h(\omega,k,\mu,\nu) = \left(\sum_{i=1}^{N} \omega_i(\mu_i - \nu_i) \right)^2 + 2a\left(\sum_{i=1}^{N} k_i(\mu_i - \nu_i) \right)^2 + \left(\sum_{i=1}^{N} k_i(\mu_i - \nu_i) \right)^4,$$
$$g(\omega,k,\mu,\nu) = \exp\left[\sum_{i<j}^{(N)} \varphi(i,j)(\mu_i\mu_j + \nu_i\nu_j) \right],$$

where the factor $\text{cond}(\mu,\nu)$ indicates the summation is performed in accordance with the following conditions

$$\begin{aligned} &\mu_i + \nu_i = 1, \quad i = 1,\ldots,n, \\ &\mu_i + \nu_i = 2, \quad i = n+1,\ldots,m, \\ &\mu_i + \nu_i = 0, \quad i = m+1,\ldots,N. \end{aligned} \tag{2.29}$$

Besides, the dependent variables ω, k, μ and ν without indices represent their respective vectors of components. Under conditions (2.29) the previously introduced coefficient $D(n;m)$

can be rewritten as follows

$$D(n;m) = \sum_{\mu_1,\ldots,\mu_n=0,1} \hat{h}(\omega,k,\mu)\hat{g}(\omega,k,\mu),$$

$$\hat{h}(\omega,k,\mu) = \Big(\sum_{i=1}^{n} \omega_i(-1+2\mu_i)\Big)^2 + 2a\Big(\sum_{i=1}^{n} k_i(-1+2\mu_i)\Big)^2 + \Big(\sum_{i=1}^{n} k_i(-1+2\mu_i)\Big)^4,$$

$$\hat{g}(\omega,k,\mu) = \exp\Big\{\sum_{i<j}^{(n)} \varphi(i,j)\frac{1}{2}[1+(1-2\mu_i)(1-2\mu_j)]\Big\}.$$

By denoting

$$\sigma_i = 1-2\mu_i,$$

which take values ± 1, these functions can be simplified to the shorter forms

$$\hat{h}(\omega,k,\sigma) = \Big(\sum_{i=1}^{n} \sigma_i\omega_i\Big)^2 + 2a\Big(\sum_{i=1}^{n} \sigma_i k_i\Big)^2 + \Big(\sum_{i=1}^{n} \sigma_i k_i\Big)^4,$$

$$\hat{g}(\omega,k,\sigma) = \exp\Big[\sum_{i<j}^{(n)} \varphi(i,j)\frac{1}{2}(1+\sigma_i\sigma_j)\Big].$$

Noting that

$$\begin{aligned}\varphi(i,j)\frac{1}{2}(1+\sigma_i\sigma_j) &= \frac{1}{2}(1+\sigma_i\sigma_j)\log\frac{(c_i-c_j)^2-3(k_i-k_j)^2}{(c_i-c_j)^2-3(k_i+k_j)^2}\\ &= \log[(c_i-c_j)^2-3(\sigma_i k_i-\sigma_j k_j)^2] - \log[(c_i-c_j)^2-3(k_i+k_j)^2],\end{aligned}$$

we can factorize a constant out of the summation in $D(n;m)$ and rewrite it as

$$D(n;m) = \text{const}\times\hat{D}(k_1,\ldots,k_n),$$

$$\hat{D}(k_1,\ldots,k_n) = \sum_{\sigma=\pm 1} \hat{h}(\omega,k,\sigma)p(c,k,\sigma),$$

where the polynomial p and the constant are given by

$$p(c,k,\sigma) = \prod_{i<j}^{(n)} [(c_i-c_j)^2-3(\sigma_i k_i-\sigma_j k_j)^2],$$

$$\text{const} = 1/\prod_{i<j}^{(n)} [(c_i-c_j)^2-3(k_i+k_j)^2].$$

According to *observation 2* and the above results, we only need to prove that the following identities hold true

$$\hat{D}(k_1,\ldots,k_n) = 0 \quad \text{for } n=1,\ldots,N.$$

At the end of the task we shall use the mathematical induction. Since the parameters ϵ_i in the definition of c_i are only responsible for the direction of wave propagation but do not affect the procedure of proof, we can assume, for simplicity, that all ϵ_i take the positive value 1. If k_i is expressed in terms of c_i as

$$k_i = \sqrt{-(2a+c_i^2)},$$

the function $\hat{D}$ can be considered as a polynomial of variables $c_1, \ldots, c_n$, that is

$$\hat{D} = \hat{D}(c_1, \ldots, c_n).$$

It is recognized that $\hat{D}$ possesses the following three properties.

(a) $\hat{D}$ is a symmetric polynomial of $c_1, \ldots, c_n$.

(b) If $c_1 = \sqrt{-2a}$, then

$$\hat{D}(c_1, c_2, \ldots, c_n)\big|_{c_1=\sqrt{-2a}} = \hat{D}(c_2, \ldots, c_n) \times 2\prod_{j=2}^{n}[(c_1 - c_j)^2 - 3k_j^2].$$

(c) If $c_1 = c_2$, then

$$\hat{D}(c_1, \ldots, c_n)\big|_{c_1=c_2}$$
$$= \hat{D}(c_3, c_4, \ldots, c_n) \times 24k_1^2 \prod_{j=3}^{n}[(c_1 - c_j)^2 - 3(k_1 - k_j)^2][(c_1 - c_j)^2 - 3(k_1 + k_j)^2].$$

Firstly, it is straightforward to verify that

$$\hat{D}(c_1, \ldots, c_n) = 0 \tag{2.30}$$

holds true for $n = 1$ and $n = 2$ by direct inspection. Indeed, let us write these polynomials explicitly in the following.

For $n = 1$

$$\hat{D}(c_1) = c_1^2 \times [-(2a + c_1^2)] + 2a \times [-(2a + c_1^2)] + [-(2a + c_1^2)]^2.$$

For $n = 2$

$$\begin{aligned}\hat{D}(c_1, c_2) = 2[(\omega_1 - \omega_2)^2 + 2a(k_1 - k_2)^2 + (k_1 - k2)^4] \times [(c_1 - c_2)^2 - 3(k_1 + k_2)^2] \\ + 2[(\omega_1 + \omega_2)^2 + 2a(k_1 + k_2)^2 + (k_1 + k_2)^4] \times [(c_1 - c_2)^2 - 3(k_1 - k_2)^2].\end{aligned}$$

Thus, we have $\hat{D}(c_1) = \hat{D}(c_1, c_2) = 0$. With the help of properties (a), (b), (c) it is seen that $\hat{D}$ can be factorized by a polynomial

$$\prod_{i<j}^{(n)}(c_i - c_j)^2 \prod_{i=1}^{n}(c_i^2 + 2a)$$

of degree $n(n-1) + 2n$. The definition of $\hat{D}$, on the other hand, shows that its degree is $n(n-1) + 4$. As a result, we have just proved that $\hat{D}$ vanishes for n and the mathematical induction implies the validity of identity (2.30).

2.1.3 Different solutions of the Kaup-Boussinesq equations

This section revisits the system of Kaup-Boussinesq equations by applying the proposed modified homogeneous balance method to achieve the reduced form of equation. The Kaup-Boussinesq model is described by

$$\begin{aligned}h_t + (hu)_x + u_{xxx} = 0, \\ u_t + h_x + uu_x = 0.\end{aligned} \tag{2.31}$$

After completing the first and the second steps in the algorithm, the solutions are sought in the form

$$h(x,t) = \varphi(\theta(x,t))_{xx} + a, \quad u(x,t) = \varphi(\theta(x,t))_x + b, \quad \varphi(\theta) = 2\log\theta. \tag{2.32}$$

Coming back to system (2.31) and using the above transformation, we expand the left-hand sides of two equations as follows

$$\begin{aligned} h_t + (hu)_x + u_{xxx} &= (\theta_x^2\theta_t + b\theta_x^3 + \theta_x^2\theta_{xx})\varphi_\theta^{(3)} \\ &\quad + (2\theta_x\theta_{xt} + \theta_{xx}\theta_t + 3b\theta_x\theta_{xx} + a\theta_x^2 + 2\theta_x\theta_{xxx} + \theta_{xx}^2)\varphi_{\theta\theta} \\ &\quad + (\theta_{xxt} + a\theta_{xx} + b\theta_{xxx} + \theta_{xxxx})\varphi_\theta, \\ u_t + h_x + uu_x &= (\theta_x\theta_t + \theta_x\theta_{xx} + b\theta_x^2)\varphi_{\theta\theta} + (\theta_{xt} + b\theta_{xx} + \theta_{xxx})\varphi_\theta. \end{aligned} \tag{2.33}$$

Let us denote

$$A = \theta_t + b\theta_x + \theta_{xx}, \quad B = \theta_t + b\theta_x + a\theta + \theta_{xx}, \tag{2.34}$$

and rewrite equations (2.33) as

$$A\theta_x^2\varphi_\theta^{(3)} + (\theta_{xx}A + 2A_x\theta_x + a\theta_x^2)\varphi_{\theta\theta} + B_{xx}\varphi_\theta = 0, \quad A\theta_x\varphi_{\theta\theta} + A_x\varphi_\theta = 0.$$

Using the chain rule of differentiation and taking the following relation into account

$$B_{xx} = A_{xx} + a\theta_{xx},$$

this system can be transformed to

$$(A\varphi_\theta + a\varphi)_{xx} = 0, \quad (A\varphi_\theta)_x = 0 \quad \Rightarrow \quad (A\varphi_\theta)_x = 0, \quad (a\varphi)_{xx} = 0.$$

Integrating the last two equations and substituting the definition (2.34) back into the obtained equations, we can write them explicitly as

$$\begin{aligned} 2(\theta_t + b\theta_x + \theta_{xx}) &= \theta\lambda(t), \\ 2a\log\theta &= \alpha(t)\,x + \beta(t), \end{aligned} \tag{2.35}$$

where α, β, and λ are three arbitrary functions of one variable t. We see immediately that the first equation is a linear partial differential equation with variable coefficient while the second is a simple transcendental equation. The Hirota's method does not enter the game, yet the method of solution is straightforward and borrowed from the theory of linear differential equation.

Trivial solution In case $a \neq 0$, Eq. $(2.35)_2$ gives

$$\theta(x,t) = \exp\left[\frac{\alpha(t)x + \beta(t)}{2a}\right],$$

which yields directly

$$h(x,t) = a, \quad u(x,t) = \alpha(t)/a + b.$$

These trivial solutions are not of our interest as they cannot simulate the propagation process. For this reason we restrict ourselves to the case $a = 0$ in the next investigations.

Soliton solution In the most simplified case $\alpha = \beta = \lambda = 0$, the system is reduced to

$$\theta_t + b\theta_x + \theta_{xx} = 0, \quad a = 0. \tag{2.36}$$

It is notable that this result is derived in [80] although the intermediate steps are different. In that paper the author applied directly the standard homogeneous balance method to obtain a system of five equations for θ and realized that they could be significantly reduced to one unified and equivalent equation by integrating some of the equations in the system. In contrast, the steps done in this work are in some ways around. Equation (2.36) is obtained by integrating the original equations in terms of θ directly.

The linear equation $(2.36)_1$ can be solved easily since it accepts exponential function as solution and apparently the principle of superposition. So, we seek the solution in the form

$$\theta = \sum_{i=1}^{N} A_i \exp(k_i x + \omega_i t + \delta_i),$$

where A_i, δ_i, $i = 1, \ldots, N$, are arbitrary constants. Substituting it into Eq. $(2.36)_1$ and taking the linear independence property of these exponential functions into account, a system of algebraic equations is derived as follows

$$\omega_i + bk_i + k_i^2 = 0, \quad i = 1, \ldots, N. \tag{2.37}$$

The wave numbers k_i can be chosen in complex plane in this solution procedure. For the solution with exponential tail these numbers must be chosen to be real. In such circumstance we obtain the kink-type solution for u and the solitary solution for h whose visualization can be found in Fig. 2.4.

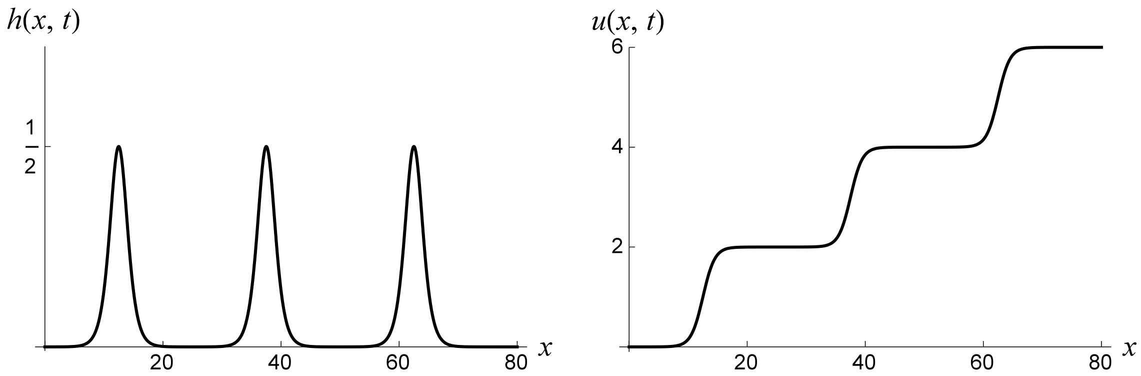


Figure 2.4: The solution for u is of kink-type whereas the solution for h is of soliton-type.

Periodic solutions Since the trigonometric functions are expressible in terms of exponential functions with complex exponent, we can come up with the periodic solution by considering the purely imaginary values $k_j = \mathrm{i}\kappa_j$, where κ_j are arbitrary real numbers. Using Eq. (2.37), the solution reads

$$\theta = \sum_{j=1}^{N} A_j \exp(\kappa_j^2 t)\{\cos[\kappa_j(x - bt)] + \mathrm{i}\sin[\kappa_j(x - bt)]\}.$$

The real-valued version of this solution can be obtained by choosing the complex conjugate pairs of constants A_j and A_j^*. Thus, we may write the real-valued solution in the form

$$\theta = \sum_{j=1}^{N} A_j \exp(\kappa_j^2 t) \cos[\kappa_j(x - bt) + \phi_j],$$

where A_j and ϕ_j are arbitrary real numbers.

Rational solutions We investigate now the circumstance in which $a = 0$ and $\lambda(t) \neq 0$ and derive the rational solutions up to any certain order. In this case the system (2.35) takes the form

$$2(\theta_t + b\theta_x + \theta_{xx}) = \theta\lambda(t), \quad \alpha(t)x + \beta(t) = 0.$$

The second equation implies immediately that $\alpha(t) = \beta(t) = 0$. Thus, it suffices to solve the partial differential equation

$$2(\theta_t + b\theta_x + \theta_{xx}) = \theta\lambda(t). \tag{2.38}$$

We propose to seek its solution in the form

$$\theta(x,t) = \sum_{i=1}^{N} p_i(x)q_i(t), \tag{2.39}$$

where $N > 1$ is a finite positive integer, $p_i(x) = x^{i-1}$ are the monomials, and $q_i(t)$ will be determined later. Upon substitution of Eq. (2.39) into Eq. (2.38), we obtain

$$\sum_{i=1}^{N} p_i(x)[2q_i'(t) - q_i(t)\lambda(t)] + \sum_{i=1}^{N} 2[bp_i'(x) + p_i''(x)]q_i(t) = 0.$$

Taking into account the two identities

$$p_i'(x) = (i-1)p_{i-1}(x), \quad p_i''(x) = (i-1)(i-2)p_{i-2}(x),$$

we manage to recognize this equation as a linear combination of p_i in the following manner

$$\begin{aligned}
&\sum_{i=1}^{N-2} p_i(x)[2q_i'(t) + 2ibq_{i+1}(t) + 2i(i+1)q_{i+2}(t) - \lambda(t)q_i(t)] \\
&+ p_{N-1}(x)[2q_{N-1}'(t) + 2b(N-1)q_N(t) - \lambda(t)q_{N-1}(t)] \\
&+ p_N(x)[2q_N'(t) - \lambda(t)q_N(t)] = 0.
\end{aligned}$$

Thus, the linear independence property of p_i implies the system of equations

$$\begin{aligned}
q_i'(t) + biq_{i+1}(t) + i(i+1)q_{i+2}(t) - \frac{\lambda(t)}{2}q_i(t) = 0, &\quad i = 1, \dots, N-2, \\
q_{N-1}'(t) + b(N-1)q_N(t) - \frac{\lambda(t)}{2}q_{N-1}(t) = 0, & \\
q_N'(t) - \frac{\lambda(t)}{2}q_N(t) = 0. &
\end{aligned} \tag{2.40}$$

Note that this system has not been obtained from the result given in [80]. It can be exactly integrated for any integer value N by applying the method of integrating factor iteratively starting from the last equation for $q_N(t)$ and ending at the first equation for $q_1(t)$. The integrating factor for all these equations is given by

$$\psi(t) = \exp\left[-\int \frac{\lambda(t)}{2}\mathrm{d}t\right] \quad \Rightarrow \quad \psi'(t) = -\frac{\lambda(t)}{2}\psi(t).$$

Multiplying all the equations (2.40) by this common factor and introducing the new variables $Q_i(t) = q_i(t)\psi(t)$, this system can be transformed to

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_i(t) + biQ_{i+1}(t) + i(i+1)Q_{i+2}(t) = 0, \quad i = 1, \dots, N-2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{N-1}(t) + b(N-1)Q_N(t) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q_N(t) = 0.$$

The solutions can be expressed in the iterative formulas

$$Q_N(t) = -A_N, \quad Q_{N-1}(t) = b(N-1)A_N t + A_{N-1},$$
$$Q_i(t) = -\int [ibQ_{i+1}(t) + i(i+1)Q_{i+2}(t)]\mathrm{d}t, \quad i = N-2, \dots, 1.$$

Substituting Eq. (2.39) into the transformation (2.32), we can rewrite the original solutions in the form of rational functions as follows

$$h(x,t) = 2\frac{\sum_{i=1}^{N} p_i''(x)Q_i(t) \times \sum_{i=1}^{N} p_i(x)Q_i(t) - \left(\sum_{i=1}^{N} p_i'(x)Q_i(t)\right)^2}{\left(\sum_{i=1}^{N} p_i(x)Q_i(t)\right)^2},$$
$$u(x,t) = 2\frac{\sum_{i=1}^{N} p_i'(x)Q_i(t)}{\sum_{i=1}^{N} p_i(x)Q_i(t)} + b.$$

From this formulation, we see that even though Eq. (2.38) contains the arbitrary function $\lambda(t)$, the solutions found in this form turn out independent of $\lambda(t)$ and are simply rational solutions. The rational solution of order four is computed in the following using the above formulas with $N = 4$, $A_3 = 1$ and other constants of integration being zero

$$h(x,t) = -4\frac{b^2t^2 - 2btx + 2t + x^2}{(b^2t - 2btx - 2t + x^2)^2}, \quad u(x,t) = \frac{4(x - bt)}{b^2t^2 - 2btx - 2t + x^2} + b.$$

The rational solution does not convey the physical meaning itself, so its visualization does not bring any benefits. Nevertheless, it plays a part in the next examination.

Interaction of different types of solutions From the above analysis the linearity of equation (2.38) allows us to use the principle of superposition to construct the interaction of different types of solutions. Besides, since all the rational solutions can be constructed independently of $\lambda(t)$, we need to consider only the case $\lambda(t) = 0$ in this context. To this end, we write its solution as follows

$$\theta = \theta_1 + \theta_2 + \theta_3,$$
$$\theta_1 = \sum_{i=1}^{N_1} A_i \exp[k_i x - (bk_i + k_i^2)t],$$
$$\theta_2 = \sum_{i=1}^{N_2} B_i \exp(\kappa_i^2 t)\cos[\kappa_i(x - bt) + \phi_i],$$
$$\theta_3 = \sum_{i=1}^{N_3} C_i\, p_i(x)Q_i(t).$$

Before proceeding further, we should notice from this solution formula that the solutions given in form of a combination of different kinds are all singular. The singularities is due to the vanishing of the denominator in the final closed-form solution. In dynamics of fluid they cannot be associated with any real phenomena. Unfortunately, up to the best of author's knowledge, it has not been explained in which physical contexts these solutions can be found meaningful. Apparently, one might think of the instability of wavetrains in such scenarios but we are not sure of the mechanism behind it. Nevertheless, the mathematical method is still thoroughly detailed in case that they might be applicable in other mathematical contexts.

Rational-periodic interaction One interesting solution rises up from this form is the positon solution oscillating with different frequencies in fast variables, namely the phases. It can be derived from $\theta = \theta_2 + \theta_3$. In Fig. 2.5 the positon solution with three different frequencies are plotted. It is generated by using the following phase function

$$\theta(x,t) = \sum_{i=1}^{3} \exp(\kappa_i^2 t) \cos[\kappa_i(x-t)] - x + t + 1,$$

where $\kappa_1 = 2$, $\kappa_2 = 3$, $\kappa_3 = 5$ and $b = 1$. The part responsible for the rational solution partially dictates the envelope of the positon. It is easier to explain this statement by examining the simple 1-positon which can be derived from the phase function

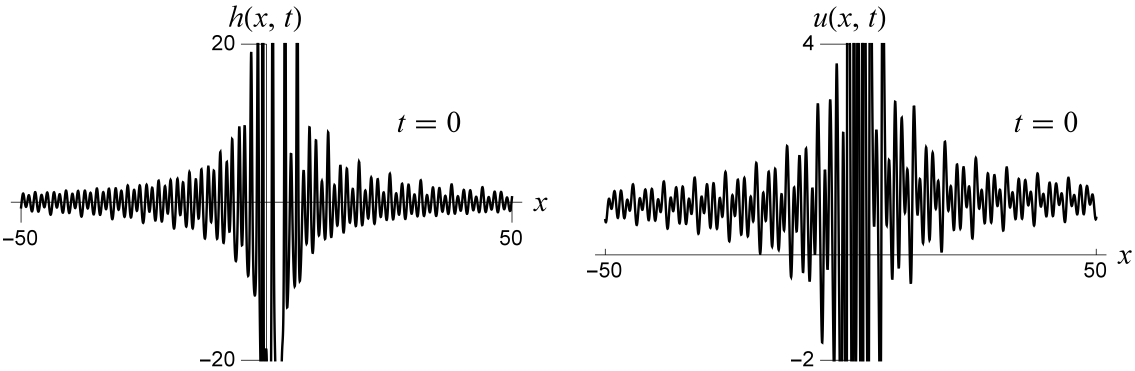


Figure 2.5: Positon solution with triple-frequency of the Kaup-Boussinesq system.

$$\theta(x,t) = B_1 \exp(\kappa^2 t) \cos\xi(x,t) + \chi(x,t),$$
$$\xi(x,t) = \kappa(x - bt), \quad \chi(x,t) = A_1 - A_2(x - bt).$$

Using these denotations, the solutions are given by

$$h(x,t) = -\frac{2[A_2^2 + \kappa^2 B_1^2 \mathrm{e}^{2\kappa^2 t} + \kappa^2 B_1 \mathrm{e}^{\kappa^2 t}\chi(x,t)\cos\xi(x,t) + 2\kappa A_2 B_1 \mathrm{e}^{\kappa^2 t}\sin\xi(x,t)]}{[\chi(x,t) + B_1 \mathrm{e}^{\kappa^2 t}\cos\xi(x,t)]^2},$$
$$u(x,t) = -\frac{2[A_2 + \kappa B_1 \mathrm{e}^{\kappa^2 t}\sin\xi(x,t)]}{\chi(x,t) + B_1 \mathrm{e}^{\kappa^2 t}\cos\xi(x,t)} + b.$$

By "ignoring" the oscillation due to the harmonic functions, we may obtain the approximate envelopes of $h(x,t)$ and $u(x,t)$ in accordance with

$$h_e(x,t) = -\frac{2[A_2^2 + \kappa^2 B_1^2 \mathrm{e}^{2\kappa^2 t} + \mathrm{e}^{\kappa^2 t}(\kappa^2 B_1 \chi(x,t) + 2\kappa A_2 B_1)]}{[\chi(x,t) + B_1 \mathrm{e}^{\kappa^2 t}]^2},$$
$$u_e(x,t) = -\frac{2[A_2 + \kappa B_1 \mathrm{e}^{\kappa^2 t}]}{\chi(x,t) + B_1 \mathrm{e}^{\kappa^2 t}} + b.$$

All these arguments are illustrated in Fig. 2.6, where the dashed curves indicate the envelopes and the entering parameters in the solutions are all chosen to be one.

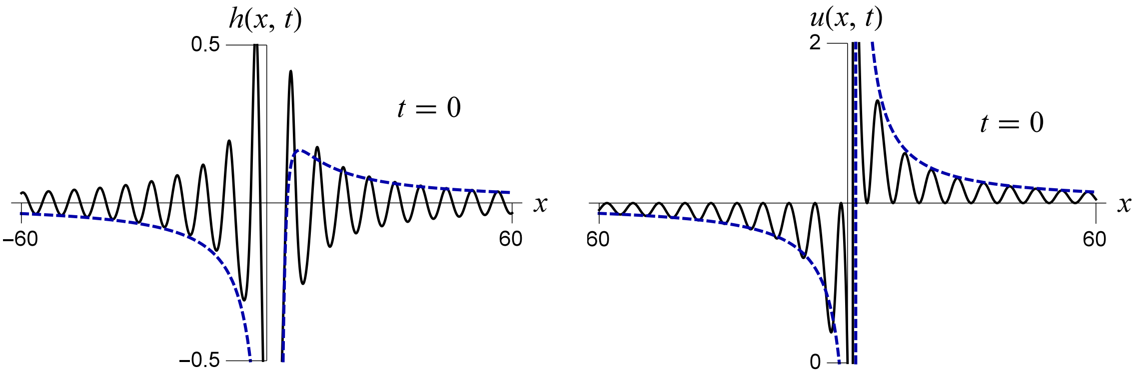


Figure 2.6: Positon solution of the Kaup-Boussinesq system versus its envelopes.

Rational-solitary interaction An interaction between rational and soliton solutions can be derived from the phase function $\theta = \theta_1 + \theta_3$. In principle we can freely choose the parameter b, but one of the solutions corresponding to $b = 0$ provides a quite interesting behavior. In Fig. 2.7 we illustrate the interaction between 1-soliton and 1-order rational solutions

$$h(x,t) = -\frac{2\mathrm{e}^t[\mathrm{e}^t + \mathrm{e}^x(x-3)]}{[\mathrm{e}^x - \mathrm{e}^t(x-1)]^2}, \quad u(x,t) = \frac{2(\mathrm{e}^x - \mathrm{e}^t)}{\mathrm{e}^x - \mathrm{e}^t(x-1)},$$

which are derived from the phase function $\theta = \mathrm{e}^{x-t} - x + 1$. It can be seen, also from the figure, that the solutions behave asymptotically as hyperbolic functions at large time in accordance with

$$\lim_{t\to\infty} h(x,t) = -\frac{2}{(x-1)^2}, \quad \lim_{t\to\infty} u(x,t) = \frac{2}{x-1}.$$

It means that at large time the propagating waves tend to a non-trivial stationary configuration. This phenomenon does not happen for the nonzero parameter b.

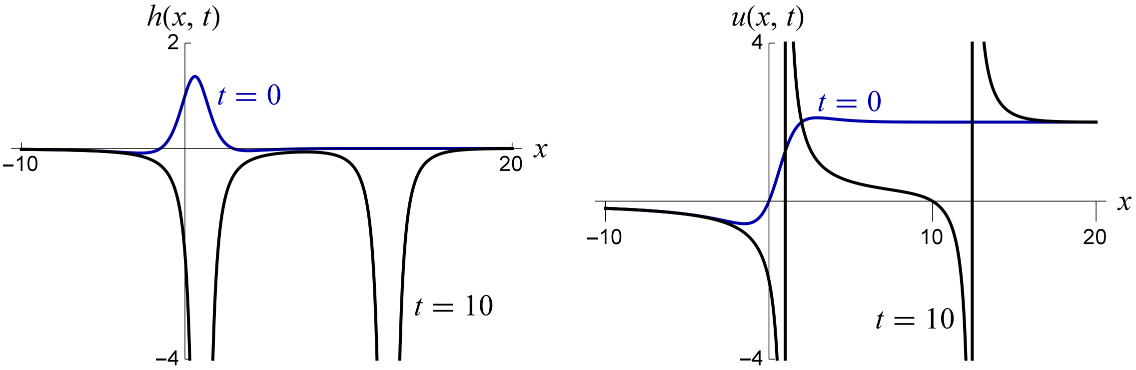


Figure 2.7: Interaction between 1-soliton and 1-order rational solution of the Kaup-Boussinesq system.

Periodic-solitary interaction In addition, the interaction between the periodic solution and the soliton can be constructed using the phase function $\theta = \theta_1 + \theta_2$. In Fig. 2.8 such an interaction between 1-soliton and 1-order periodic solution is plotted using the following formulas

$$h(x,t) = -2\frac{\mathrm{e}^{7t}[4\mathrm{e}^{2x}\sin(t-x) - 3\mathrm{e}^{2x}\cos(t-x) + \mathrm{e}^{7t}]}{[\mathrm{e}^{7t}\cos(t-x) + \mathrm{e}^{2x}]^2},$$

$$u(x,t) = 2\frac{\mathrm{e}^{7t}\sin(t-x) + 2\mathrm{e}^{2x}}{\mathrm{e}^{7t}\cos(t-x) + \mathrm{e}^{2x}} + 1,$$

where the phase function $\theta = \mathrm{e}^{2x-6t} + \mathrm{e}^{t}\cos(t-x)$ has been used.

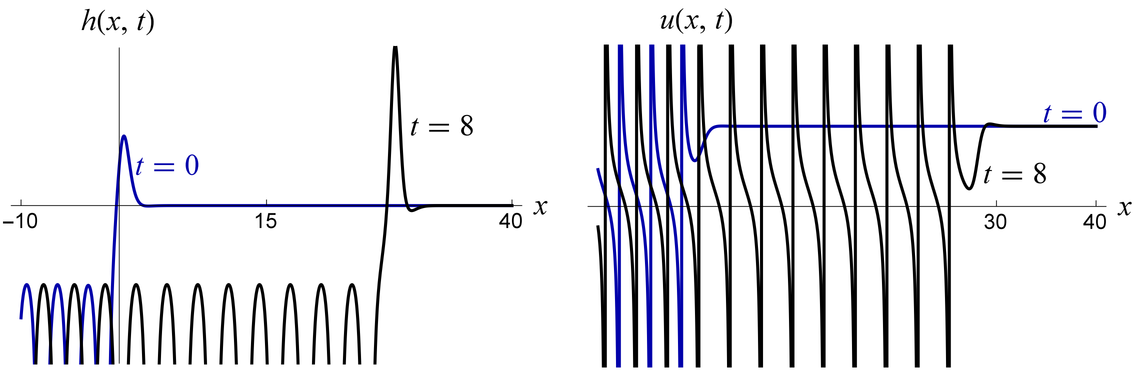


Figure 2.8: Interaction between 1-soliton and 1-order periodic solution of the Kaup-Boussinesq system.

2.1.4 Boussinesq Benjamin–Ono equation

Besides the Benjamin-Ono equation [81–83], the Boussinesq Benjamin-Ono (BBO) equation finds its application in several models of algebraic Rossby solitary waves [84]. In spite of its importance in fluid dynamics, this equation is quite new and has not received much attention from researchers. The conservative BBO equation is given by

$$v_{tt} + a_1 v_{xx} + a_2 (v^2)_{xx} + a_3 \mathrm{H} v_{xxx} = 0, \tag{2.41}$$

where H is the well-known Hilbert transform

$$\mathrm{H}v(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(z,t)}{x-z} \mathrm{d}z.$$

Applying the change of both dependent and independent variables

$$v = \lambda u + \gamma, \quad X = \alpha x, \quad T = \beta t,$$

to Eq. (2.41), it can be transformed to

$$u_{TT} + \frac{\alpha^2}{\beta^2}(a_1 + 2a_2\gamma) u_{XX} + a_2 \lambda \frac{\alpha^2}{\beta^2}(u^2)_{XX} + a_3 \lambda \frac{\alpha^3}{\beta^2} \mathrm{H} u_{XXX} = 0.$$

We choose $a_1 + 2a_2\gamma = 0$ so that the term involving u_{XX} is eliminated. To simplify it further, we may choose

$$a_2 \lambda \frac{\alpha^2}{\beta^2} = 1, \quad a_3 \lambda \frac{\alpha^3}{\beta^2} = -1.$$

The last condition is set for the purpose of solution method and is equivalent to $\alpha a_3 / a_2 = -1$. As such we shall consider the BBO equation in the simplified form

$$u_{tt} + (u^2)_{xx} + \mathrm{H} u_{xxx} = 0, \tag{2.42}$$

where the Hilbert transform now takes the form

$$\mathrm{H}u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(z,t)}{z-x} \mathrm{d}z.$$

Bi-trilinear form of the BBO equation Following Satsuma and Ishimori's proposal [83], we use the variable transformation

$$u(x,t) = \mathrm{i}\frac{\partial}{\partial x}\log\frac{g(x,t)}{f(x,t)} = \mathrm{i}\frac{g_x f - g f_x}{gf}, \tag{2.43}$$

where "i" is the imaginary unit and f (g) can be written as an infinite or finite product of $x - z_n$ $(x - z'_n)$ for z_n (z'_n) in the upper (lower)-half complex plane. The zeros of f and g should not necessarily be simple. The Hilbert transform can then be computed in accordance with

$$\mathrm{H}\left[\mathrm{i}\frac{\partial}{\partial x}\log\frac{g}{f}\right] = -\frac{\partial}{\partial x}\log(gf). \tag{2.44}$$

Substituting equations (2.43) and (2.44) into Eq. (2.42), integrating the obtained equation with respect to x and taking the integration constant to be zero, we obtain

$$\mathrm{i}\frac{\partial^2}{\partial t^2}\left[\log\frac{g}{f}\right] + (u^2 + \mathrm{H}u_x)_x = 0.$$

By using D-operators, it is ready to compute

$$\frac{\partial^2}{\partial t^2}\left[\log\frac{g}{f}\right] = \frac{D_t^2 g\cdot g}{2g^2} - \frac{D_t^2 f\cdot f}{2f^2}, \quad u^2 + \mathrm{H}u_x = -\frac{D_x^2 g\cdot f}{gf}.$$

These expressions and the formula $\partial_x(a/b) = D_x a\cdot b/b^2$ transform the above equation to

$$\frac{\mathrm{i}}{2}\left(\frac{D_t^2 g\cdot g}{g^2} - \frac{D_t^2 f\cdot f}{f^2}\right) - \frac{D_x[D_x^2(g\cdot f)\cdot(gf)]}{(gf)^2} = 0.$$

Upon multiplication of both sides by $(gf)^2$, we arrive at the bi-trilinear form of the BBO equation

$$\frac{\mathrm{i}}{2}(f^2 D_t^2 g\cdot g - g^2 D_t^2 f\cdot f) - D_x[D_x^2(g\cdot f)\cdot(gf)] = 0. \tag{2.45}$$

Note that this equation is not a bilinear equation due to the presence of f^2 and g^2. However, it is sufficient for us to find the 1-order periodic solution by using an appropriate solution Ansatz and henceforth the soliton solution.

Exact 1-order periodic solution Applying the perturbation technique, we shall solve the bi-trilinear equation for exact solution by using the simplest expansion

$$f = 1 + \alpha f_1, \quad g = 1 + \beta g_1.$$

Taking the bi-linearity of D-operators into account, it is ready to compute three terms in Eq. (2.45) as follows

$$\begin{aligned}
P_1 &= f^2 D_t^2 g\cdot g = (1 + 2\alpha f_1 + \alpha^2 f_1^2)[\beta D_t^2(1\cdot g_1 + g_1\cdot 1) + \beta^2 D_t^2(g_1\cdot g_1)],\\
P_2 &= g^2 D_t^2 f\cdot f = (1 + 2\beta g_1 + \beta^2 g_1^2)[\alpha D_t^2(1\cdot f_1 + f_1\cdot 1) + \alpha^2 D_t^2(f_1\cdot f_1)],\\
P_3 &= D_x[D_x^2(g\cdot f)\cdot(gf)] = \alpha D_x(D_x^2 1\cdot f_1)\cdot 1 + \beta D_x(D_x^2 g_1\cdot 1)\cdot 1\\
&\quad + \alpha\beta[D_x(D_x^2 g_1\cdot f_1)\cdot 1 + D_x(D_x^2 g_1\cdot 1)\cdot f_1 + D_x(D_x^2 1\cdot f_1)\cdot g_1]\\
&\quad + \alpha^2 D_x(D_x^2 1\cdot f_1)\cdot f_1 + \beta^2 D_x(D_x^2 g_1\cdot 1)\cdot g_1\\
&\quad + \alpha^2\beta[D_x(D_x^2 g_1\cdot f_1)\cdot f_1 + D_x(D_x^2 1\cdot f_1)\cdot(g_1 f_1)]\\
&\quad + \alpha\beta^2[D_x(D_x^2 g_1\cdot f_1)\cdot g_1 + D_x(D_x^2 g_1\cdot 1)\cdot(g_1 f_1)]\\
&\quad + \alpha^2\beta^2 D_x(D_x^2 g_1\cdot f_1)\cdot(g_1 f_1).
\end{aligned}$$

Up to this point let us assume the condition $D_x^n g_1 \cdot f_1 = 0$, which can be fulfilled by choosing $f_1 = g_1 = \phi$ as the exponential function of a phase variable. Taking advantage of this choice, we have

$$P_1 - P_2 = 2(\beta - \alpha)(1 - \alpha\beta\phi^2)D_t^2(\phi \cdot 1),$$
$$P_3 = (\alpha + \beta)[D_x^3\phi \cdot 1 + (\alpha + \beta)D_x(D_x^2\phi \cdot 1) \cdot \phi + \alpha\beta\, D_x(D_x^2\phi \cdot 1) \cdot \phi^2].$$

Observing the structure of these expressions, it is appropriate to choose

$$\phi = \exp\theta, \quad \theta = \mathrm{i}(kx - \omega t + \gamma),$$

where k and ω are two real constants that should satisfy some relation determined later and γ is an arbitrary phase constant. Substituting this Ansatz into Eq. (2.45) and using the above calculation, the bi-trilinear form is dramatically reduced to

$$\mathrm{i}[(\alpha - \beta)\omega^2 + (\alpha + \beta)k^3](\exp\theta - \alpha\beta\exp 3\theta) = 0.$$

This equation holds true if and only if

$$(\alpha - \beta)\omega^2 + (\alpha + \beta)k^3 = 0. \tag{2.46}$$

With the explicit expressions of g and f we write down the original solution

$$u(x,t) = \frac{k(\alpha - \beta)[(1 + \alpha\beta)\cos\xi + \alpha + \beta + \mathrm{i}(1 - \alpha\beta)\sin\xi]}{[1 + \alpha\beta\cos 2\xi + (\alpha + \beta)\cos\xi]^2 + [\alpha\beta\sin 2\xi + (\alpha + \beta)\sin\xi]^2},$$

where $\xi = kx - \omega t + \gamma$. This solution is a complex-valued function and it is strictly real if the imaginary part vanishes, requiring

$$1 - \alpha\beta = 0 \quad \Rightarrow \quad \beta = 1/\alpha. \tag{2.47}$$

Under this condition the real solution then reads

$$u(x,t) = \frac{k(\alpha - 1/\alpha)(\alpha + 1/\alpha + 2\cos\xi)}{(1 + \alpha^2 + 2\alpha\cos\xi)(1 + 1/\alpha^2 + 2/\alpha\cos\xi)} = \frac{k(\alpha^2 - 1)}{1 + \alpha^2 + 2\alpha\cos\xi}, \tag{2.48}$$

where the constant α is found by solving the system (2.46)–(2.47) as follows

$$\alpha = \left[\frac{1 - k/c^2}{1 + k/c^2}\right]^{1/2}.$$

Rational solution There are two ways of obtaining rational soliton solution. Firstly, the solutions g and f of the bi-trilinear form are sought in the polynomial form. Secondly, in the limit $k \to 0$, the periodic solution converges to the soliton solution, which is the periodic solution with infinite wavelength. In the following we shall implement the first method and then demonstrate that the second method works out with the coincided result.

In order for 1-soliton solution we choose $g = f^*$ and rewrite Eq. (2.45) as

$$\frac{\mathrm{i}}{2}[f^2 D_t^2 f^* \cdot f^* - f^{*2} D_t^2 f \cdot f] - D_x[(D_x^2 f^* \cdot f) \cdot (f^* f)] = 0.$$

This equation can be envisaged as

$$\mathrm{Im}(f^{*2} D_t^2 f \cdot f) - 2D_x[\mathrm{Re}(f^* f_{xx}) - f_x f_x^*] \cdot f f^* = 0, \tag{2.49}$$

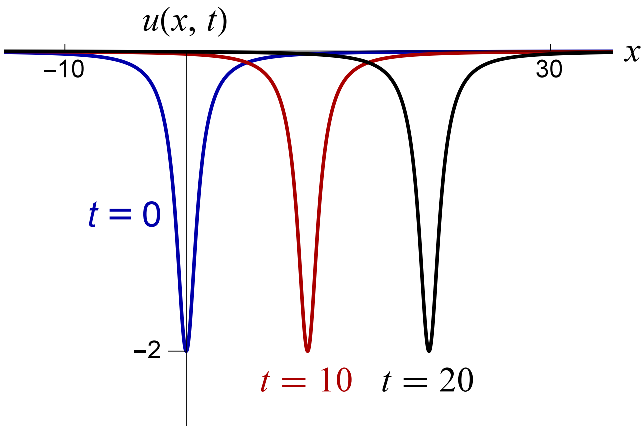


Figure 2.9: Soliton solution governed by the BBO equation.

where the operators Re and Im pick up the real and imaginary parts of a complex-valued function, respectively. From this representation we see that the bi-trilinear form is a partial differential equation in the real plane. We shall seek the solution in the form

$$f(x,t) = \mathrm{i}\chi(x,t) + \delta, \quad \chi(x,t) = kx - \omega t + \gamma.$$

Substituting this Ansatz into Eq. (2.49) and noting that $f_{xx} = f_{tt} = 0$, it is simplified to

$$-4\chi(x,t)(\delta\omega^2 + k^3) = 0,$$

which is fulfilled if and only if

$$\delta\omega^2 + k^3 = 0 \quad \Rightarrow \quad \delta = -\frac{k}{c^2},$$

where $c = \omega/k$ is the phase velocity. Now that we can write down the original solution using the above results as follows

$$u(x,t) = \frac{2k\delta}{\delta^2 + \chi(x,t)^2} = -\frac{2c^2}{1 + c^4(x-ct)^2}, \tag{2.50}$$

where the initial phase γ has been set to zero. This solution is plotted in Fig. 2.9 at different time instant. Interestingly, the soliton solution in this case decays at the infinity in the rational manner, that is

$$u \sim \frac{1}{x^2} \quad \text{as } x \to \infty.$$

Solitary wave as the periodic solution in the long wave limit The soliton solution, which turns out the rational solution, can also be obtained by letting the wave number k in the periodic solution tend to zero but still keeping the phase velocity c constant. To this end, we assume $k \ll 1$ and $c = O(1)$ in the following analysis. With the constant

$$\alpha = \tanh\frac{\psi}{2} = \left[\frac{1 - k/c^2}{1 + k/c^2}\right]^{1/2}$$

being substituted into Eq. (2.48) we may rewrite the periodic solution characterized by the two parameters k and c as follows

$$u(x,t) = \frac{k\,\mathrm{sech}\psi}{1 + \tanh\psi\cos\xi}, \quad \xi = k(x - ct) + \gamma, \tag{2.51}$$

where

$$\text{sech}\psi = -\frac{1-\alpha^2}{1+\alpha^2} = -\frac{k}{c^2}, \quad \tanh\psi = \frac{2\alpha}{1+\alpha^2} = \sqrt{1-\frac{k^2}{c^4}}.$$

Employing the assumption of small wave number and choosing $\gamma = \pi$, we may expand

$$\tanh\psi = 1 - k^2/2c^4 + O(k^4), \quad \cos\xi = -[1 - k^2(x-ct)^2/2] + O(k^4).$$

Substituting these into Eq. (2.51), we can approximate the periodic solution by

$$u(x,t) = -\frac{2/c^2}{1/c^4 + (x-ct)^2} + O(k^2),$$

which recovers the rational solitary solution given by Eq. (2.50) in the long wave limit $k \to 0$. Unfortunately, the multi-soliton solution has not been found yet due to the complexity of the "nearly" tri-bilinear expression of the BBO equation. The solution is needed for the comparison with the amplitude modulation of a train of many solitons.

2.2 Wronskian solution of the bilinear equation

The Wronskian determinant is a handy tool in mathematics, especially in linear differential equations. It allows us to check if a set of functions are functionally independent. It turns out that the Wronskian determinant is also a powerful technique to construct the exact solutions to the bilinear equations. In fact, using the inverse scattering transform, the researchers can easily find that many solutions of the well-known evolution equations such as the nonlinear Schrödinger (NLS) equation [7], the KdV equation [8] and the BSQ equation [9] can be expressed in terms of Wronskian determinant of a certain set of functions. It is not surprising that these functions must satisfy some specific conditions that are usually represented in form of the eigenvalue problem. However, it was completely astonished that the technique is itself so robust that it enables the finding of many other different kinds of solutions. The work done by considering the Wronskian technique as a research subject is even less than that done by jamming it into the method of inverse scattering transform. For these reasons this section is dedicated to the direct method based on the technique of using the Wronskian determinant. Nevertheless, it is not quite accurate to claim its advantages over the fore-mentioned direct methods for two reasons. First, the Wronskian formula is aimed at solving the bilinear expression of the corresponding equation. Second, the construction of such formula requires some special observations and algebra. Before going into detail, we adopt here the compact notation introduced first by Freeman and Nimmo

$$(\widehat{N-1}) = (\widehat{N-1}, \Phi) = W(\phi_1, \phi_2, \ldots, \phi_N) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix},$$

where the superscripts denote the order of differentiation with respect to x according to

$$\phi_i^{(0)} = \phi_i, \quad \phi_i^{(j)} = \frac{\partial^j}{\partial x^j}\phi_i, \quad j \geq 1,\ 1 \leq i \leq N.$$

2.2.1 Wronskian formulation for the KdV equation

The reasons to start with the KdV equation in explaining a new technique are by now crystal clear. It is the key in the development of several analytical methods dealing with the wave equations. Besides, since it possesses the elegant bilinear form according to Hirota, it apparently becomes the testing sample for the researchers to explore new findings applicable to many other equations. The solution determined by a Wronskian determinant is called the Wronskian solution. For instance, the function

$$\theta = W(\phi_1, \phi_2, \ldots, \phi_N)$$

solving Eq. (2.12) is called the Wronskian solution to the KdV bilinear equation, and the function defined by

$$u = 2\frac{\partial^2}{\partial x^2} \log W(\phi_1, \phi_2, \ldots, \phi_n) + a$$

solves Eq. (2.1) and hence is called the Wronskian solution to the KdV equation.

Sufficient conditions on Wronskian solutions According to the above exposition, the main task in the Wronskian technique is to find the appropriate conditions on the set of functions

$$\Phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$$

to guarantee that its Wronskian expression solves the corresponding bilinear equation. A special technique based on the combination of the so-called pfaffians, the Laplace expansions of determinants, Plücker relations and the Jacobi identities is outlined by Hirota's scientific group to seek such conditions [14]. Though one might think that the procedure is breathtakingly carried out, the sufficient conditions can be actually found in many cases by trial and error. Indeed, the author has made several efforts looking for such conditions by trial and error strategy. To start with, we go directly to the sufficient conditions summarized in the following statement.

Statement If the set of functions $\phi_i = \phi_i(x,t), 1 \leq i \leq N$, fulfill the two sets of conditions

$$\begin{aligned} -\phi_{i,xx} &= \sum_{j=1}^{N} \lambda_{ij}(t)\phi_j, & 1 \leq i \leq N, \\ \phi_{i,t} &= -4\phi_{i,xxx} + \gamma(t)\phi_i - 6a\phi_{i,x}, & 1 \leq i \leq N, \end{aligned} \tag{2.52}$$

where $\Lambda(t) = (\lambda_{ij}(t))$ is a matrix of arbitrary differentiable real functions of t and $\gamma(t)$ itself is also a real function of t, then $\theta = (\widehat{N-1})$ is the Wronskian solution to the bilinear KdV equation.

We shall not prove this statement and postpone the task to the next example where we study the scalar BSQ equation. Instead, it is more helpful and practical to examine the properties of these two linear systems of second-order and third-order partial differential equations. Solving the above system is itself already an interesting mathematical problem. We list in the following three key observations that reduce the scale of the problem to the more reasonable one.

Observation 1 Differentiating the first equation and the second of the system (2.52) once with respect to t and twice with respect to x, respectively, we obtain the compatibility conditions $\phi_{i,xxt} = \phi_{i,txx}$ in the form

$$\sum_{j=1}^{N} \lambda_{ij,t}\phi_j = 0, \quad 1 \le i \le N.$$

The Wronskian determinant $W(\phi_1, \phi_2, \ldots, \phi_N)$ vanishes if the set Φ is linearly dependent, which means that there exists at least one entry λ_{ij} with $\lambda_{ij,t} \neq 0$.

Observation 2 Applying the integrating factor

$$f(t) = \exp\left(-\int \gamma(t)\mathrm{d}t\right) \quad \Rightarrow \quad f'(t) = -\gamma(t)f(t)$$

to the second equation of system (2.52), we can transform it to the simpler one

$$\psi_{i,t} = -4\psi_{i,xxx} - 6a\psi_{i,x}, \quad \psi_i = f(t)\phi_i.$$

Multiplying the first condition by the same integrating factor, it can be seen that its form does not change, that is

$$\psi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}(t)\psi_j.$$

From the fundamental property of determinant the resultant Wronskian solution remains the same

$$u(x,t) = 2[\log W(\Phi)]_{xx} + a = 2[\log W(\Psi)]_{xx} + a,$$

where $\Psi = f(t)\Phi$.

Observation 3 If the coefficient matrix Λ is similar to another matrix $\Lambda_P = (\mu_{ij})$ by an invertible constant matrix P in the sense that

$$\Lambda = P^{-1}\Lambda_P P, \quad \Lambda_P = P\Lambda P^{-1},$$

then the new set of functions $\Psi = P\Phi$ solves the same sufficient conditions with the matrix Λ being replaced with Λ_P, that is

$$-\Psi_{xx} = \Lambda_P\Psi, \quad \Psi_t = -4\Psi_{xxx} + \gamma(t)\Psi - 6a\Psi_x.$$

Additionally, the new set produces the same solution to the KdV equation

$$u(x,t;\Lambda) = 2[\log W(\Phi)]_{xx} + a = 2[\log W(\Psi)]_{xx} + a = u(x,t;\Lambda_P).$$

We have the following direct consequences.

Consequence 1 According to the two first observations, we see that even though the Wronskian solutions to the bilinear equation can be different, the associated solutions to the KdV equation can still be identical. Thus, it suffices to consider only the reduced case of constant matrix $\Lambda = (\lambda_{ij})$ and vanishing function γ. In formulation we have scaled the condition equations (2.52) to the simpler and yet "equivalent" system

$$\begin{aligned} -\phi_{i,xx} &= \sum_{j=1}^{N} \lambda_{ij}\phi, && 1 \le i \le N, \\ \phi_{i,t} &= -4\phi_{i,xxx} - 6a\phi_{i,x}, && 1 \le i \le N. \end{aligned} \tag{2.53}$$

Consequence 2 Since any constant matrix can be reduced to its Jordan form through a similarity transformation represented by an invertible matrix, we may exhaustively investigate the solution by considering only the Jordan form of Λ which has two types of blocks as follows

$$\Lambda_r = \begin{pmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{pmatrix}_{k_i \times k_i}, \quad \Lambda_c = \begin{pmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{pmatrix}_{l_i \times l_i},$$

where

$$A_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first block Λ_r accounts for the real eigenvalues of the matrix Λ while the second block Λ_c for the complex eigenvalues.

Outline of the solution method As we have seen before in the Kaup–Boussinesq case, it is always of advantage to degenerate the larger-scale system of equations to the smaller and to solve for different kinds of solutions separately. It is possible to solve the reduced system (2.53) with Λ being in the Jordan form by recursion process. Either when Λ has the real eigenvalues, we solve the corresponding equations one by one or when the complex eigenvalues happen, we solve the corresponding equations pair by pair. Let us explain this strategy in more detail in the two below cases.

Real eigenvalue We consider the simple block

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix},$$

and the corresponding equations $(2.53)_1$ in the form

$$-\phi_{1,xx} = \lambda\phi_1, \quad -\phi_{2,xx} = \phi_1 + \lambda\phi_2.$$

The first equation is decoupled and can be solved separately, then the second can be solved with ϕ_1 being considered as known.

Complex eigenvalue As expected, the second simple block is

$$\Lambda = \begin{pmatrix} A & 0 \\ I & A \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and we can write immediately the associated equations with this block

$$\begin{aligned} -\phi_{1,xx} &= \alpha\phi_1 - \beta\phi_2, & -\phi_{3,xx} &= \alpha\phi_3 - \beta\phi_4 + \phi_1, \\ -\phi_{2,xx} &= \beta\phi_1 + \alpha\phi_2, & -\phi_{4,xx} &= \beta\phi_3 + \alpha\phi_4 + \phi_2, \end{aligned}$$

The first two equations are considered as decoupled from the entire system and can be solved as in pair. The remaining equations are subsequently solved with ϕ_1 and ϕ_2 being known.

The above analysis allows us to divide the original problem into two sub-problems, that is to solve the representative systems of non-homogeneous differential equations in either of the two following forms.

Sub-problem A

$$-\phi_{xx} = \lambda\phi + f, \quad \phi_t = -4\phi_{xxx} - 6a\phi_x, \tag{2.54}$$

Sub-problem B

$$\begin{aligned} -\phi_{1,xx} &= \alpha\phi_1 - \beta\phi_2 + f_1, \\ -\phi_{2,xx} &= \beta\phi_1 + \alpha\phi_2 + f_2, \\ \phi_{i,t} &= -4\phi_{i,xxx} - 6a\phi_{i,x}, \quad i = 1, 2, \end{aligned} \tag{2.55}$$

where λ, α, β are real constants, f, f_1 and f_2 are three given functions satisfying the evolution equation

$$f_t = -4f_{xxx} - 6af_x.$$

The last condition is self-explained in the above analysis because we have not mentioned the evolution conditions on the decoupled solutions. Since both sub-problems (2.54) and (2.55) are linear differential equations, the method of solution is standardized and can be directly applied. The full solution procedure will definitely take breathtaking actions and hence waste the length of the report if we take into account the inhomogeneous effect. Instead, we shall specifically consider the homogeneous sub-problems and treat them thoroughly for different cases of eigenvalues.

Solutions of sub-problem A

Zero eigenvalue In this case the homogeneous version of Eq. (2.54) reduces to

$$\phi_{xx} = 0, \quad \phi_t = -4\phi_{xxx} - 6a\phi_x.$$

Integrating the first equation with respect to x twice, substituting the obtained solution into the second equation and equating the coefficients of x^i, $i = 0, 1$, we obtain

$$\phi(x, t) = c_1(t)x + c_0(t),$$

where c_1 and c_0 satisfy the equations

$$c_{1,t} = 0, \quad c_{0,t} = -6ac_1.$$

Thus, we have the final solution

$$\phi(x, t) = A(x - 6at) + B, \tag{2.56}$$

where A and B are two arbitrary constants.

Positive eigenvalue The homogeneous Eq. (2.54) can be written as

$$\phi_{xx} + \alpha^2\phi = 0, \quad \phi_t = -4\phi_{xxx} - 6a\phi_x, \quad \alpha^2 = \lambda > 0.$$

The first equation yields the harmonic solution with the eigenfrequency $\sqrt{\lambda}$

$$\phi(x, t) = c(t)\cos(\alpha x) + s(t)\sin(\alpha x).$$

Substituting this equation into the evolution equation, collecting the coefficients of two linearly independent functions $\cos(\alpha x)$ and $\sin(\alpha x)$, we obtain the equations for $c(t)$ and $s(t)$ as follows

$$c'(t) + 2a(3a - 2\alpha^2)s(t) = 0, \quad s'(t) - 2a(3a - 2\alpha^2)c(t) = 0.$$

It is easy to solve this system to obtain the final solution as

$$\phi(x,t) = [A\sin(\kappa t) + B\cos(\kappa t)]\sin(\alpha x) + [A\cos(\kappa t) - B\sin(\kappa t)]\cos(\alpha x), \tag{2.57}$$

where $\kappa = 2\alpha(3a - 2\alpha^2)$ and A, B are two arbitrary constants.

Negative eigenvalue We rewrite the representative equations in this case as follows

$$\phi_{xx} - \alpha^2\phi = 0, \quad \phi_t = -4\phi_{xxx} - 6a\phi_x, \quad \alpha^2 = -\lambda > 0.$$

The standard integration of the first equation yields

$$\phi(x,t) = p(t)\exp(\alpha x) + q(t)\exp(-\alpha x).$$

The similar operations to the above give the evolution equations

$$p'(t) + 2\alpha(3a + 2\alpha^2)p(t) = 0, \quad q'(t) - 2\alpha(3a + 2\alpha^2)q(t) = 0.$$

The system is decoupled and can be solved by the method of separation of variables. After obtaining the explicit expressions for $p(t)$ and $q(t)$, we write down the final solution

$$\phi(x,t) = A\exp(\alpha x - \kappa t) + B\exp(\kappa t - \alpha x), \tag{2.58}$$

where $\kappa = 2\alpha(3a + 2\alpha^2)$, A and B are two arbitrary constants.

Solution of sub-problem B In analogous manner, we first consider the system

$$-\phi_{1,xx} = \alpha\phi_1 - \beta\phi_2, \quad -\phi_{2,xx} = \beta\phi_1 + \alpha\phi_2.$$

Elimination of the unknown ϕ_2 from this system leads to the four-order differential equation

$$\phi_{1,xxxx} + 2\alpha\phi_{1,xx} + (\alpha^2 + \beta^2)\phi_1 = 0.$$

It is interesting that this equation is even in its orders of differentiation and so is its characteristic equation

$$\lambda^4 + 2\alpha\lambda^2 + (\alpha^2 + \beta^2) = 0.$$

The roots are given by

$$\lambda_1^{\pm} = \Delta \pm \mathrm{i}\delta, \quad \lambda_2^{\pm} = -\Delta \pm \mathrm{i}\delta,$$

where the squares of two introduced constants are

$$\Delta^2 = \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2}, \quad \delta^2 = \frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2}.$$

Therefore, the solution ϕ_1 is the linear combination of four linearly independent functions that can be summarized as

$$\phi_1 = [f_1(t)\cos(\delta x) + f_2(t)\sin(\delta x)]\exp(\Delta x) + [f_3(t)\cos(\delta x) + f_4(t)\sin(\delta x)]\exp(-\Delta x).$$

Using Eq. (2.55)$_3$, the evolution equations for the coefficients f_i, $i = 1, \ldots, 4$, can be deduced easily as follows

$$
\begin{aligned}
&f_1'(t) + 2(3a - 6\delta^2\Delta + 2\Delta^3)f_1(t) - 4(\delta^3 - 3\delta\Delta^2)f_2(t) = 0,\\
&f_2'(t) + 4(\delta^3 - 3\delta\Delta^2)f_1(t) + 2(3a - 6\delta^2\Delta + 2\Delta^3)f_2(t) = 0,\\
&f_3'(t) + 2(3a + 6\delta^2\Delta - 2\Delta^3)f_3(t) - 4(\delta^3 - 3\delta\Delta^2)f_4(t) = 0,\\
&f_4'(t) + 4(\delta^3 - 3\delta\Delta^2)f_3(t) + 2(3a + 6\delta^2\Delta - 2\Delta^3)f_4(t) = 0.
\end{aligned}
$$

Denoting two new real constants

$$\mu = -2(3a - 6\delta^2\Delta + 2\Delta^3), \quad \nu = -4(\delta^3 - 3\delta\Delta^2),$$

this system can be rewritten in the more compact form

$$\begin{pmatrix} f_i'(t) \\ f_j'(t) \end{pmatrix} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} f_i(t) \\ f_j(t) \end{pmatrix}, \quad \{i,j\} = \{\{1,2\},\{3,4\}\}.$$

Solving this system by the separation of variables yields

$$\begin{pmatrix} f_i(t) \\ f_j(t) \end{pmatrix} = \begin{pmatrix} A_i \\ A_j \end{pmatrix} \exp(\mathbf{M}\,t), \quad \mathbf{M} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}.$$

The evolution operator $\exp(\mathbf{M}\,t)$ is defined by

$$\exp(\mathbf{M}\,t) = \exp(\mu t) \begin{pmatrix} \cos\nu t & -\sin\nu t \\ \sin\nu t & \cos\nu t \end{pmatrix},$$

in which μ and ν are the real and imaginary parts of the eigenvalues of the matrix $\mathbf{M}$, respectively. Recalling that ϕ_1 and ϕ_2 are coupled into one system of differential equations, ϕ_2 should be covered from one of the original equations. It is convenient to compute ϕ_2 according to

$$\phi_2 = \frac{1}{\beta}(\phi_{1,xx} + \alpha\phi_1).$$

Thus, we have just completed the solution method for the case of complex eigenvalues.

Different solutions and their interaction We come to the point to present several solutions using the preceding theoretical results. Let us go from simplicity to complication by considering first 1-order solution that is generated with only determinant of first order. Using the formulas (2.56)–(2.58), we can write immediately three solutions of the KdV equation

$$
\begin{aligned}
u_r(x,t) &= a - \frac{2A^2}{[B - A(x - 6at)]^2},\\
u_p(x,t) &= a - \frac{2(A^2 + B^2)\alpha^2}{[A\cos(\alpha x - \kappa_p t) + B\sin(\alpha x - \kappa_p t)]^2},\\
u_s(x,t) &= a + \frac{8AB\alpha^2\exp[2(\alpha x + \kappa_s t)]}{[A\exp(2\alpha x) + B\exp(2\kappa_s t)]^2},
\end{aligned}
$$

where $\kappa_p = 2\alpha(3a - 2\alpha^2)$ and $\kappa_s = 2\alpha(3a + 2\alpha^2)$. These three solutions are called the rational solution, the periodic solution and the soliton solution, respectively.

Recalling the superposition principle of solutions to the linear differential equations, we may wonder if there is some similar mechanism on the bilinear differential equations. If yes, the natural question arises how it should be superposed. The answer turns out simple and lies in the algebraic structure of the Wronskian determinant. Let say, the fundamental functions ϕ_r, ϕ_p, ϕ_s and ϕ_c generate the rational, periodic, soliton and complexiton solutions, respectively, then the "superposition" is constructed by the Wronskian determinant of a set of these functions. For demonstration the interaction solutions are given in the short list below. Since it is not beneficial to write the final solution to the KdV equation

$$u(x,t) = 2\frac{WW_{xx} - W_x^2}{W^2} + a, \tag{2.59}$$

in its full form, only the Wronskian solutions to the bilinear KdV equation are provided.

Rational-periodic interaction

$$\begin{aligned} W(\phi_r, \phi_p) &= \chi_c(x,t)\cos(\alpha x - \kappa_p t) + \chi_s(x,t)\sin(\alpha x - \kappa_p t), \\ \chi_c(x,t) &= \alpha A_1 B_2(x - 6at) + \alpha B_1 B_2 - A_1 A_2, \\ \chi_s(x,t) &= -\alpha A_1 A_2(x - 6at) - \alpha A_2 B_1 - A_1 B_2. \end{aligned}$$

Rational-solitary interaction

$$\begin{aligned} W(\phi_r, \phi_s) &= \chi_+(x,t)\exp(\alpha x - \kappa_s t) + \chi_-(x,t)\exp(\kappa_s t - \alpha x), \\ \chi_+(x,t) &= \alpha A_1 A_2(x - 6at) + \alpha A_2 B_1 - A_1 A_2, \\ \chi_-(x,t) &= -\alpha A_1 B_2(x - 6at) - \alpha B_1 B_2 - A_1 B_2. \end{aligned}$$

Periodic-solitary interaction

$$\begin{aligned} W(\phi_p, \phi_s) = {} & \alpha(A_1A_2 - A_2B_1)f_1(x,t) - \alpha(B_1B_2 + A_1B_2)f_2(x,t) \\ & + \alpha(A_1A_2 + A_2B_1)f_3(x,t) - \alpha(B_1B_2 - A_1B_2)f_4(x,t), \end{aligned}$$

$$\begin{aligned} f_1(x,t) &= \exp(\alpha x - \kappa_s t)\cos(\alpha x - \kappa_p t), \quad & f_3(x,t) &= \exp(\alpha x - \kappa_s t)\sin(\alpha x - \kappa_p t), \\ f_2(x,t) &= \exp(\kappa_s t - \alpha x)\cos(\alpha x - \kappa_p t), \quad & f_4(x,t) &= \exp(\kappa_s t - \alpha x)\sin(\alpha x - \kappa_p t). \end{aligned}$$

The KdV solutions derived from these Wronskian solutions are plotted in three Figures 2.10, 2.11, 2.12 corresponding to the rational-periodic interaction, the rational-solitary interaction and the periodic-solitary interaction, respectively.

And we can obviously construct many other interactions including also the complexiton solution according to the simple rule

$$\theta = W(\Phi_r, \Phi_s, \Phi_p, \Phi_c),$$

where Φ_r, Φ_p, Φ_s, Φ_c are four sets of functions corresponding to the rational, periodic, soliton and complexiton solutions, respectively. The derivation of the closed form of solution becomes the matter of lengthy symbolic computation.

Before closing this section, we present here one Wronskian solution that is going to be utilized in the subsequent chapter

$$W(x,t) = \sin(2T) - \kappa X, \quad T = \kappa(x + 4\kappa^2 t), \quad X = T_\kappa = x + 12\kappa^2 t, \quad a = 0.$$

Upon its substitution into the solution formula (2.59), it follows that [1]

$$u(x,t) = \frac{32\kappa^2(\kappa X\cos T - \sin T)\sin T}{(\sin 2T - 2\kappa X)^2}. \tag{2.60}$$

Note that this solution was obtained by Matveev [85, 86].

[1] We especially use this solution form because the article [91] refers to it.

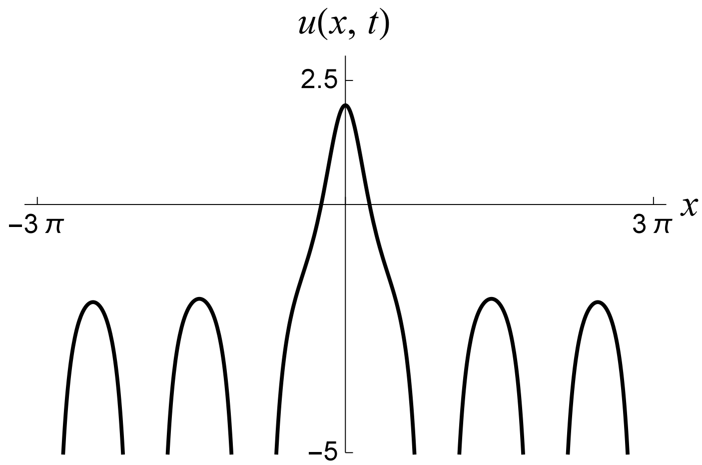


Figure 2.10: Interaction between rational and periodic solutions of the KdV equation.

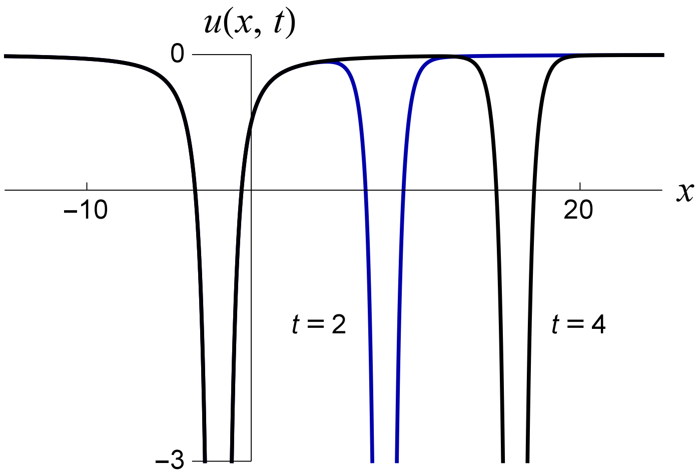


Figure 2.11: Interaction between rational and solitary solutions of the KdV equation.

2.2.2 Wronskian formulation for the BSQ equation

In this section we shall deal with the "bad" version of the BSQ equation

$$u_{tt} + (u^2)_{xx} - u_{xxxx} = 0, \tag{2.61}$$

which, through the transformation

$$u(x,t) = -6(\log\theta(x,t))_{xx} + a = -6\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a,$$

is engendered to the bilinear equation

$$(D_t^2 + 2aD_x^2 - D_x^4)\theta\cdot\theta = 0. \tag{2.62}$$

Since the entire spirit of the Wronskian technique has been thoroughly demonstrated in the last paragraphs, let us take this case as a chance to give proofs wherever the statements are not trivial.

Sufficient conditions on the Wronskian solution Let us denote $\epsilon = \pm 1$. The function $\theta = (\widehat{N-1})$ solves the bilinear BSQ equation (2.62) with vanishing a

$$(D_t^2 - D_x^4)\theta\cdot\theta = 0 \tag{2.63}$$

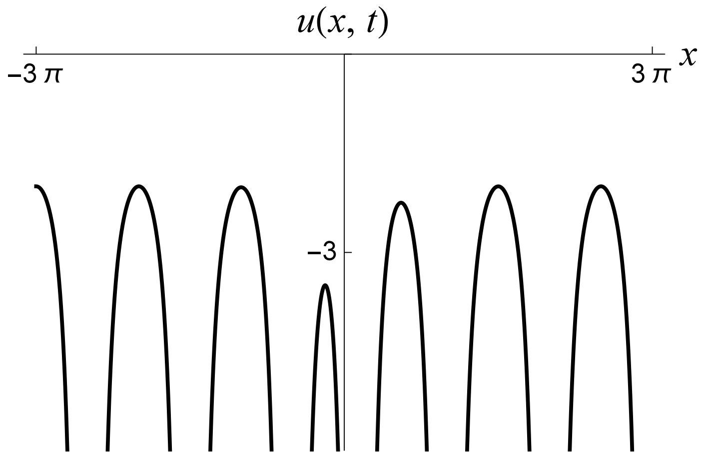


Figure 2.12: Interaction between periodic and solitary solutions of the KdV equation.

provided that the set of functions ϕ_i satisfies the following linear system of equations

$$\begin{aligned} \phi_{k,xxx} &= \sum_{j=1}^{N} \lambda_{kj}(t)\phi_j, \qquad 1 \le k \le N, \\ \phi_{k,t} &= \epsilon\delta\phi_{k,xx}, \qquad\qquad 1 \le k \le N, \end{aligned} \tag{2.64}$$

where λ_{kj} are arbitrary functions of t and δ is a complex constant which can be determined during the course of derivation.

Proof We expand Eq. (2.63) and rewrite it in the form

$$\theta(\theta_{tt} - \theta_{xxxx}) - \theta_t^2 - 3\theta_{xx}^2 + 4\theta_x\theta_{xxx} = 0. \tag{2.65}$$

To facilitate the presentation of our proof, let us introduce a short useful denotation. From now on $(\widehat{N-k})$ will represent the consecutive ordered columns $\Phi^{(0)}, \ldots, \Phi^{(N-k)}$. Thus, this denotation occupies a matrix with N rows and $N-k+1$ columns. Then, it still needs other $k-1$ columns to fill in a square matrix. For instance, the denotation $(\widehat{N-3}, N-1, N)$ designates a N-order square matrix. The following proof will take advantage of these notations. Verification of the above statement as well as determination of the constant δ is completed by direct substitution. Firstly, the derivatives of θ with respect to x can be worked out according to

$$\begin{aligned} \theta_x &= (\widehat{N-2}, N), \\ \theta_{xx} &= (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1), \\ \theta_{xxx} &= (\widehat{N-4}, N-2, N-1, N) + 2(\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2), \\ \theta_{xxxx} &= (\widehat{N-5}, N-3, N-2, N-1, N) + 3(\widehat{N-4}, N-2, N-1, N+1) \\ &\quad + 2(\widehat{N-3}, N, N+1) + 3(\widehat{N-3}, N-1, N+2) + (\widehat{N-2}, N+3). \end{aligned}$$

Secondly, the derivatives of θ with respect to t are computed using condition $(2.64)_2$ as follows

$$\begin{aligned} \theta_t &= \epsilon[-\delta(\widehat{N-3}, N-1, N) + \delta(\widehat{N-2}, N+1)], \\ \theta_{tt} &= \delta^2(\widehat{N-5}, N-3, N-2, N-1, N) + 2\delta^2(\widehat{N-3}, N, N+1) \\ &\quad - \delta^2\left[(\widehat{N-3}, N-1, N+2) + (\widehat{N-4}, N-2, N-1, N+1) - (\widehat{N-2}, N+3)\right]. \end{aligned}$$

It is obvious that the determinant equality

$$\sum_{k=1}^{N}\begin{vmatrix} \text{Row}(M,1) \\ \cdots \\ \text{Row}(M,k)_{xxx} \\ \cdots \\ \text{Row}(M,N) \end{vmatrix} = \sum_{j=1}^{N} \left| \ \text{Col}(M,1),\ldots,\text{Col}(M,j)_{xxx},\ldots,\text{Col}(M,N)\ \right|$$

holds true. In this formula M is a matrix of functions, $\text{Row}(M,k)$ and $\text{Col}(M,j)$ denote the k^{th} row and the j^{th} column of this matrix, respectively. Applying this identity to two matrices $A=(\widehat{N-1})$ and $B=(\widehat{N-2},N)$, we obtain

$$\begin{aligned}\sum_{k=1}^{N}\lambda_{kk}(t)(\widehat{N-1}) &= (\widehat{N-4},N-2,N-1,N) \\ &\quad -(\widehat{N-3},N-1,N+1)+(\widehat{N-2},N+2), \\ \sum_{k=1}^{N}\lambda_{kk}(t)(\widehat{N-2},N) &= (\widehat{N-5},N-3,N-2,N-1,N) \\ &\quad -(\widehat{N-3},N,N+1)+(\widehat{N-2},N+3).\end{aligned}$$

With the aid of the above results it is ready to compute that

$$\begin{aligned}&\theta(\theta_{tt}-\theta_{xxxx}) \\ &=3(\delta^2-1)(\widehat{N-1})(\widehat{N-3},N,N+1) \\ &\quad -(\delta^2+3)(\widehat{N-1})[(\widehat{N-4},N-2,N-1,N+1)+(\widehat{N-3},N-1,N+2)] \\ &\quad +(\delta^2-1)\sum_{k=1}^{N}\lambda_{kk}(t)(\widehat{N-1})(\widehat{N-2},N),\end{aligned}$$

$$\begin{aligned}-\theta_t^2-3\theta_{xx}^2 &= -(\delta^2+3)(\widehat{N-3},N-1,N)^2-(\delta^2+3)(\widehat{N-2},N)^2 \\ &\quad +2(\delta^2-3)(\widehat{N-3},N-1,N)(\widehat{N-2},N+1), \\ 4\theta_x\theta_{xxx} &= 12(\widehat{N-2},N)(\widehat{N-3},N-1,N+1)+4\sum_{k=1}^{N}\lambda_{kk}(t)(\widehat{N-1})(\widehat{N-2},N).\end{aligned}$$

Substituting these formulas into the left-hand side of Eq. (2.65) and requiring

$$\delta^2+3=0, \quad \text{or} \quad \delta=\mathrm{i}\sqrt{3},$$

it is reduced to

$$\begin{aligned}&\theta(\theta_{tt}-\theta_{xxxx})-\theta_t^2-3\theta_{xx}^2+4\theta_x\theta_{xxx} \\ &=-12(\widehat{N-1})(\widehat{N-3},N,N+1)+12(\widehat{N-2},N)(\widehat{N-3},N-1,N+1) \\ &\quad -12(\widehat{N-3},N-1,N)(\widehat{N-2},N+1) \\ &=-6\begin{vmatrix} \widehat{N-3} & 0 & N-2 & N-1 & N & N+1 \\ 0 & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix}.\end{aligned}$$

The last determinant identically vanishes and hence the proof is completed.

Once again, two fundamental properties on the sufficient conditions are reported.

Observation 1 In order to make system (2.64) solvable, the compatibility conditions

$$\phi_{k,xxxt} = \phi_{k,txxx}, \quad 1 \le k \le N,$$

must be taken into account. Differentiating the first equation of system (2.64) and the second equation with respect to t and x, respectively, we obtain

$$\phi_{k,xxxt} = \sum_{j=1}^{N} \lambda_{kj,t}\phi_j + \lambda_{kj}\phi_{j,t}, \quad \phi_{k,txxx} = \epsilon i\sqrt{3} \times \partial_x^5 \phi_k = \sum_{j=1}^{N} \lambda_{kj}\epsilon i\sqrt{3}\phi_{j,xx} = \sum_{j=1}^{N} \lambda_{kj}\phi_{j,t}.$$

The compatibility conditions imply

$$\sum_{j=1}^{N} \lambda_{kj,t}\phi_j = 0, \quad 1 \le k \le N.$$

Thus, provided that the coefficient matrix $\Lambda = (\lambda_{kj})$ is precisely dependent on t, that is, $\lambda_t \ne 0$, the Wronskian determinant $W(\phi_1, \ldots, \phi_N)$ is identically equal to zero. Taking this fact into account, it suffices to deal with the matrix $\Lambda = (\lambda_{kj})$ of constant coefficients.

Observation 2 If the coefficient matrix Λ is similar to another matrix $\Lambda_P = (\mu_{kj})$ under an invertible constant matrix P, then the new set of functions $\Psi = P\Phi$ solves the same sufficient conditions with the matrix Λ being replaced by Λ_P, that is

$$\Psi_{xxx} = \Lambda_P \Psi, \quad \Psi_t = \epsilon\, i\sqrt{3}\Psi_{xx}.$$

Even though the Wronskian determinants in two cases are distinguishable, the Wronskian solutions to the BSQ equation (2.63) are the same

$$u(x,t;\Lambda) = -6[\log W(\Phi)]_{xx} = -6[\log W(\Psi)]_{xx} = u(x,t;\Lambda_P).$$

The second observation following the KdV case is redundant here. With the analogous argument to that of the last section the direct consequence of this property is that our linear problem can be decomposed into two sub-problems

Sub-problem A

$$\phi_{xxx} = \lambda\phi + f, \quad \phi_t = \epsilon\, i\sqrt{3}\phi_{xx}, \tag{2.66}$$

Sub-problem B

$$\begin{aligned} \phi_{1,xxx} &= \alpha\phi_1 - \beta\phi_2 + f_1, \\ \phi_{2,xxx} &= \beta\phi_1 + \alpha\phi_2 + f_2, \\ \phi_{k,t} &= \epsilon\, i\sqrt{3}\phi_{k,xx}, \quad k = 1, 2, \end{aligned} \tag{2.67}$$

where λ, α, β are three arbitrary real constants, f, f_1, f_2 are three given functions satisfying the evolution equation

$$g_t = \epsilon\, i\sqrt{3}g_{xx}.$$

In contrary to the last strategy of presentation, we will not try to solve the homogeneous sub-problems but the non-homogeneous problems through illustration of concrete examples. In fact, repetition of the "sibling mathematical exercises" may lead to tedious reading.

Solution corresponding to zero eigenvalues As a first example let us study the simplest case of the system (2.66)

$$\phi_{xxx} = 0, \quad \phi_t = \epsilon \mathrm{i}\sqrt{3}\phi_{xx}.$$

The general solution of the first equation reads

$$\phi(x,t) = c_1(t)x^2 + c_2(t)x + c_3(t).$$

Substituting this into the second equation and using the linear independence of monomials x^k, we obtain the conditions

$$c_{1,t} = 0, \quad c_{2,t} = 0, \quad c_{3,t} = \epsilon \mathrm{i}2\sqrt{3}c_1,$$

which can be easily integrated for the solutions

$$c_1(t) = c_{10}, \quad c_2(t) = c_{20}, \quad c_3(t) = \epsilon \mathrm{i}2\sqrt{3}c_{10}t + c_{30},$$

where c_{10}, c_{20}, and c_{30} are three arbitrary constants. The rational solution of Eq. (2.61) is then given by

$$u(x,t) = 6\frac{2c_{10}^2(x^2 - \epsilon \mathrm{i}2\sqrt{3}t) + 2c_{10}c_{20}x + c_{20}^2 - 2c_{10}c_{30}}{[c_{10}(x^2 + \epsilon \mathrm{i}2\sqrt{3}t) + c_{20}x + c_{30}]^2}.$$

For a higher-order rational solution we examine the following system

$$\begin{aligned} &\phi_{0,xxx} = 0, \qquad \phi_{0,t} = \epsilon \mathrm{i}\sqrt{3}\phi_{0,xx},\\ &\phi_{1,xxx} = \phi_0, \qquad \phi_{1,t} = \epsilon \mathrm{i}\sqrt{3}\phi_{1,xx},\\ &\phi_{2,xxx} = \phi_1, \qquad \phi_{2,t} = \epsilon \mathrm{i}\sqrt{3}\phi_{2,xx}. \end{aligned} \tag{2.68}$$

The equation for ϕ_0 yields

$$\phi_0(x,t) = c_{10}(x^2 + \epsilon \mathrm{i}2\sqrt{3}t) + c_{20}x + c_{30}.$$

Then using this general solution, the second equation can be integrated to give

$$\phi_1(x,t) = \int_0^x \int_0^\xi \int_0^\eta \phi_0(\nu,t)\mathrm{d}\nu \mathrm{d}\eta \mathrm{d}\xi + d_1(t)x^2 + d_2(t)x + d_3(t).$$

In an analogous way to the above example, the evolution equation for ϕ_1 is used to determine the evolution of the coefficients $d_k = d_k(t)$

$$d_{1,t} = \epsilon\frac{\mathrm{i}\sqrt{3}}{2}c_{20}, \quad d_{2,t} = \epsilon \mathrm{i}\sqrt{3}c_{30} - 6c_{10}t, \quad d_{3,t} = \epsilon \mathrm{i}2\sqrt{3}d_1.$$

Integrating this system, we obtain

$$\begin{aligned} d_1(t) &= \epsilon\frac{\mathrm{i}\sqrt{3}}{2}c_{20}t + d_{10},\\ d_2(t) &= -3c_{10}t^2 + \epsilon \mathrm{i}\sqrt{3}c_{30}t + d_{20},\\ d_3(t) &= -\frac{3}{2}c_{20}t^2 + \epsilon \mathrm{i}2\sqrt{3}d_{10}t + d_{30}. \end{aligned}$$

Similarly, we can solve the two last equations to obtain

$$\phi_2(x,t) = \int_0^x \int_0^\xi \int_0^\eta \phi_1(\nu,t)\mathrm{d}\nu\mathrm{d}\eta\mathrm{d}\xi + g_1(t)x^2 + g_2(t)x + g_3(t),$$

where g_1, g_2 and g_3 are three functions of variable t satisfying the system

$$\begin{aligned}
g_1'(t) &= \frac{1}{2}[-3c_{30}t + \epsilon\,\mathrm{i}\sqrt{3}(d_{20} - 3c_{10}t^2)],\\
g_2'(t) &= \frac{1}{2}[-12d_{10}t + \epsilon\,\mathrm{i}\sqrt{3}(2d_{30} - 3c_{20}t^2)],\\
g_3'(t) &= 2\epsilon\,\mathrm{i}\sqrt{3}g_1(t).
\end{aligned}$$

It is easy to integrate these equations to obtain

$$\begin{aligned}
g_1(t) &= \frac{1}{2}\left[-\frac{3}{2}c_{30}t^2 + \epsilon\,\mathrm{i}\sqrt{3}(d_{20}t - c_{10}t^3)\right] + g_{10},\\
g_2(t) &= \frac{1}{2}\left[-6d_{10}t^2 + \epsilon\,\mathrm{i}\sqrt{3}(2d_{30}t - c_{20}t^3)\right] + g_{20},\\
g_3(t) &= \epsilon\,\mathrm{i}\sqrt{3}\left(-\frac{1}{2}c_{30}t^3 + 2g_{10}t\right) - 3\left(\frac{1}{2}d_{20}t^2 - \frac{1}{4}c_{10}t^4\right) + g_{30}.
\end{aligned}$$

For a particular choice of parameters

$$\epsilon = 1, \quad c_{10} = 2, \quad c_{20} = c_{30} = d_{10} = d_{20} = d_{30} = g_{10} = g_{20} = g_{30} = 0,$$

the Wronskian solution is given by

$$W(\Phi) = -108t^6 + \mathrm{i}36\sqrt{3}t^5x^2 + 9t^4x^4 - \mathrm{i}\frac{6\sqrt{3}}{5}t^3x^6 - \frac{39}{140}t^2x^8 + \mathrm{i}\frac{\sqrt{3}}{100}tx^{10} + \frac{x^{12}}{2800}.$$

We suppress its presentation due to the complication of the final solution $u(x,t)$. It is self-convinced that the solution is easily extracted as long as its Wronskian function is given. If we cut down the order of the above solution by considering only two functions ϕ_0 and ϕ_1 satisfying two equations $(2.68)_1$–$(2.68)_2$ and follow the described procedure up to the second step, the following Wronskian solution can be obtained

$$W(\phi_0,\phi_1) = x^6/5 + \mathrm{i}2\sqrt{3}tx^4 - 12t^2x^2 - \mathrm{i}24\sqrt{3}t^3.$$

The final solution then reads

$$u(x,t) = \frac{36(\mathrm{i}2400\sqrt{3}t^5 + 6000t^4x^2 - \mathrm{i}400\sqrt{3}t^3x^4 + 120t^2x^6 - \mathrm{i}10\sqrt{3}tx^8 - x^{10})}{(120\sqrt{3}t^3 - \mathrm{i}60t^2x^2 - 10\sqrt{3}tx^4 + \mathrm{i}x^6)^2}.$$

It is clear that the N-order rational solution can be constructed in a systematic way by considering the coefficient matrix Λ with N zero eigenvalues. One of many possibilities is to solve the following system of ordinary differential equations

$$\begin{aligned}
&\phi_{0,xxx} = 0, && \phi_{0,t} = \epsilon\mathrm{i}\sqrt{3}\phi_{0,xx},\\
&\phi_{k,xxx} = \phi_{k-1}, && \phi_{k,t} = \epsilon\mathrm{i}\sqrt{3}\phi_{k,xx}, \qquad k = 1,\ldots,N-1.
\end{aligned}$$

Solution corresponding to non-zero eigenvalues The solution rather than the rational solution can be obtained with the coefficient matrix Λ having non-zero eigenvalues. This is due to the appearance of exponential part in the Wronskian solution, which might reveal positon or complexiton solution. As a first illustration let us take the simplest example

$$\phi_{xxx} = \delta^3\phi, \quad \phi_t = \epsilon \mathrm{i}\sqrt{3}\phi_{xx},$$

where δ is a given complex constant. The characteristic equation of the first equation $\mu^3 = \delta^3$ has three roots

$$\mu_1 = \delta, \quad \mu_2 = \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)\delta, \quad \mu_3 = \left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right)\delta.$$

Then it follows the general solution

$$\phi(x,t) = \nu_1(t)\mathrm{e}^{\mu_1 x} + \nu_2(t)\mathrm{e}^{\mu_2 x} + \nu_3(t)\mathrm{e}^{\mu_3 x}.$$

Similarly to the case of zero eigenvalues, the second equation requires

$$\nu_{k,t} = \epsilon\,\mathrm{i}\sqrt{3}\mu_i^2\nu_k(t), \quad k = 1,2,3,$$

which engenders

$$\nu_k(t) = \nu_{k0}\mathrm{e}^{\epsilon\,\mathrm{i}\sqrt{3}\mu_i^2 t}, \quad k = 1,2,3,$$

where ν_{10}, ν_{20}, and ν_{30} are three arbitrary complex constants. Combining these results, we are ready to write the Wronskian solution

$$\phi(x,t) = \sum_{k=1}^{3} \nu_{k0}\exp(\epsilon\,\mathrm{i}\sqrt{3}\mu_k^2 t + \mu_k x),$$

and the solution of the BSQ equation

$$u(x,t) = -6\frac{\sum_{k<j}^{3} \nu_{k0}\nu_{j0}(\mu_k - \mu_j)^2\exp[(\mu_k + \mu_j)x + \epsilon\mathrm{i}\sqrt{3}(\mu_k^2 + \mu_j^2)t]}{\sum_{k=1}^{3} \nu_{k0}\exp[\mu_k(x + \epsilon\,\mathrm{i}\sqrt{3}\mu_k t)]}, \tag{2.69}$$

where $\sum_{k<j}^{3}$ indicates that the summation is performed over three pairs of indices $(1,2)$, $(1,3)$ and $(2,3)$. Observing the above formula, some real as well as the complex solutions of BSQ equation (2.61), which can be constructed with different specific sets of parameters $\{\delta, \epsilon, \nu_{10}, \nu_{20}, \nu_{30}\}$, are presented in the following.

Real solutions It is interesting that the real soliton and negaton solutions can be extracted from formula (2.69). For 1-soliton we choose

$$\delta = \mathrm{i}, \quad \epsilon = 1, \quad \nu_{10} = 0, \quad \nu_{20} = 1, \quad \nu_{30} = 1,$$

then the solution (2.69) reads

$$u(x,t) = -\frac{18\mathrm{e}^{\sqrt{3}x+3t}}{(1+\mathrm{e}^{\sqrt{3}x+3t})^2} = -\frac{9}{1+\cosh(\sqrt{3}x+3t)},$$

which is nothing else but 1-soliton. On the other hand, a slight change of the initial constants

$$\delta = \mathrm{i}, \quad \epsilon = 1, \quad \nu_{10} = 0, \quad \nu_{20} = 1, \quad \nu_{30} = -1$$

yields the negaton solution

$$u(x,t) = \frac{18\mathrm{e}^{\sqrt{3}x+3t}}{(1-\mathrm{e}^{\sqrt{3}x+3t})^2} = -\frac{9}{1-\cosh(\sqrt{3}x+3t)}.$$

Complex solutions Generally, the solution formula (2.69) produces complex solutions. In the following we provide some examples. Similarly to the case of real solutions, for

$$\delta = 1, \quad \epsilon = 1, \quad \nu_{10} = 0, \quad \nu_{20} = 1, \quad \nu_{30} = 1,$$

the solution reads

$$u(x,t) = \frac{18\mathrm{e}^{\mathrm{i}\sqrt{3}x+3t}}{(1+\mathrm{e}^{\mathrm{i}\sqrt{3}x+3t})^2} = \frac{9}{1+\cosh(\mathrm{i}\sqrt{3}x+3t)},$$

while choosing $\nu_{10} = 1$ and keeping others unchanged, the solution becomes rather complicated

$$\begin{aligned}
&u(x,t) = P(x,t)/Q(x,t),\\
&P(x,t) = 18\mathrm{e}^{\mathrm{i}\sqrt{3}x+3t} - 9(1-\mathrm{i}\sqrt{3})\mathrm{e}^{\frac{3}{2}[(1+\mathrm{i}\sqrt{3})x+(3+\mathrm{i}\sqrt{3})t]} - 9(1+\mathrm{i}\sqrt{3})\mathrm{e}^{\frac{1}{2}[(3+\mathrm{i}\sqrt{3})x+(3+\mathrm{i}3\sqrt{3})t]},\\
&Q(x,t) = \left(1+\mathrm{e}^{\mathrm{i}\sqrt{3}x+3t} + \mathrm{e}^{\frac{1}{2}[(3+\mathrm{i}\sqrt{3})x+(3+\mathrm{i}3\sqrt{3})t]}\right)^2.
\end{aligned}$$

This variant of bad BSQ equation admits the complex-valued solutions as a result of the special structure of the sufficient conditions involving the imaginary unit. It is suspected that there may be other sufficient conditions that produce the real solutions in a more direct fashion. Though such conditions are still mysterious to the author at the time of writing this dissertation, the job can be done by intensively investigating the algebraic structure of the solutions.

Bibliographical remarks

First, this chapter is built up on the foundation of the denotations and mathematical ideas in the classic textbook by Hirota [14]. Second, it employs the denotations established by Freeman and Nimmo in [25–27] and the presentation from Wen-Xiu Ma in [79]. Last but not least, the results presented here are from the dissemination of the works in [18, 87, 88]. In addition, it should be highlighted that the Wronskian solution for the KdV equation found in [89] has been extended in this work by adding one more free constant parameter. This is especially helpful as it allows to construct the periodic wavetrain or the packet of solitons with non-zero basement value.

3 Amplitude modulation for nonlinear dispersive waves

In this section we employ the variational-asymptotic method to derive the theory of amplitude modulation for wave packets governed by two equations: (i) the Korteweg-de Vries equation, (ii) the scalar Boussinesq equation. We then apply this theory to the amplitude modulations of the train of solitons and of the single positon governed by these wave equations. The modulation solutions are validated by their comparison with the corresponding exact solutions. In practice, for non-integrable equations we must employ the numerical solutions for the comparison purpose. This practice will be consistently used throughout the entire chapter. As a rule of thumb, the modulation equations will be derived starting from the variational principle. We shall take advantage of the results obtained in the supplementary materials regarding the Lagrangian associated with the partial differential equations, yet it is a good practice of recalling them here to escape avoidable distraction.

3.1 A generalization of the modulation equations

Referring to the derivation of Eq. (5.2), the KdV equation can be obtained from the stationarity of the functional

$$I[\eta(x,t)] = \iint \left(-\frac{1}{2}\eta_t\eta_x - \eta_x^3 + \frac{1}{2}\eta_{xx}^2 \right) \mathrm{d}x\mathrm{d}t, \tag{3.1}$$

where the function $\eta(x,t)$ defined by the relation $u = \eta_x$ is interpreted as a potential field. At the first sight we may notice that the formulation of modulation equations presented in the first chapter does not completely fit the current setting of problem. This functional basically differs from the classic one

$$\iint L(u, u_t, u_x)\mathrm{d}x\mathrm{d}t$$

by two points, first of which is the absence of the field variable itself and the second is the companion of its second-order derivative. The natural questions of what influence these changes will make and how we should modify the mathematical treatment to fit the new setting are raised. Perhaps it is the best way to explain some generalization of the theory on an objective example.

Modification of the slowly varying wave packet As we have known, the periodic solution of the KdV equation contains three governing parameters a, m and either β or γ. Thus, it is anticipated that the final modulation equations must govern exactly the same number of slowly varying parameters and that other wave properties can be connected with them via some specific algebraic relations. Now that the periodic solution can be formulated as

$$\eta_x = u(x,t) = k\varphi(\theta(x,t)) + \beta, \quad \theta = kx - \omega t,$$

where β is a constant, θ plays the role of phase and φ is a periodic function with respect to θ. Integrating this equation with respect to x, we obtain

$$\eta = \phi(\theta) + \beta x + f(t), \quad \phi(\theta) = \int^{\theta} \varphi(\theta) \mathrm{d}\theta,$$

where the last integral denotes simply the anti-derivative of a given function and $f(t)$ is an arbitrary function of only time variable. On the other hand, if we try to seek the solution of the equation for η by varying functional (3.1), it will be governed by four parameters. For the modulation purpose this fact allows us to choose $f(t) = -\gamma t$ in our consideration and to rewrite the solution as

$$\eta(x,t) = \phi(\theta) + \chi(x,t), \quad \chi = \beta x - \gamma t.$$

This explanation can be essentially found in [42] by Whitham. According to this analysis, it is compulsory to include one more auxiliary function in the slowly varying wave packet to complete the modulation description provided that we use the variational approach. However, the exclusion of χ in our treatment does not necessarily lead to wrong modulation equations but only to the incomplete description of wave packets. In such circumstance it is obvious that the evolution of β and γ is not taken into account and they are identically equal to zero. This argument will be automatically made clearer after the modulation equations are obtained.

Problem setting It comes to the point we must improve the procedure to take into account the second-order derivative in the Lagrangian. Motivated from the structure of functional (3.1), let us consider the variational problem given by

$$\delta I[\eta(x,t)] = \delta \iint L(\eta_t, \eta_x, \eta_{tt}, \eta_{xt}, \eta_{xx}) \mathrm{d}x \mathrm{d}t = 0, \tag{3.2}$$

where the original field variable is defined by $u = \eta_x$. The Euler-Lagrange equation reads

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \eta_t} + \frac{\partial}{\partial x}\frac{\partial L}{\partial \eta_x} - \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial \eta_{tt}} - \frac{\partial^2}{\partial x \partial t}\frac{\partial L}{\partial \eta_{xt}} - \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial \eta_{xx}} = 0.$$

We look for the extremal of this variational problem in form of a slowly varying wave packet

$$\eta = \varphi(\theta(x,t), x, t) + \chi(x,t), \tag{3.3}$$

with φ being a function of fast variable θ and slow variables x and t. We assume that φ is 2π-periodic with respect to θ and the interpretation of the fast variable θ as the phase remains unchanged. The fundamental hypothesis on the slow variation of the wave number and the frequency is kept. Besides, the derivative $\beta = \chi_x$ accounts for the mean value of u over one θ-period. We compute the partial derivatives of η in accordance with (3.3) as follows

$$\begin{aligned}
\eta_x &= \varphi_\theta \theta_x + \underline{\partial_x \varphi} + \chi_x, \quad \eta_t = \varphi_\theta \theta_t + \underline{\partial_t \varphi} + \chi_t, \\
\eta_{xx} &= \varphi_{\theta\theta} \theta_x^2 + \underline{\varphi_\theta \theta_{xx}} + \underline{2\partial_x \varphi_\theta \theta_x} + \underline{\partial_x^2 \varphi} + \underline{\chi_{xx}}, \\
\eta_{tt} &= \varphi_{\theta\theta} \theta_t^2 + \underline{\varphi_\theta \theta_{tt}} + \underline{2\partial_t \varphi_\theta \theta_t} + \underline{\partial_t^2 \varphi} + \underline{\chi_{tt}}, \\
\eta_{xt} &= \varphi_{\theta\theta} \theta_t \theta_x + \underline{\varphi_\theta \theta_{xt}} + \underline{\partial_t \varphi_\theta \theta_x} + \underline{\partial_x \varphi_\theta \theta_t} + \underline{\partial_{xt} \varphi} + \underline{\chi_{xt}}.
\end{aligned}$$

Additionally, the values $\beta = \chi_x$ and $\gamma = -\chi_t$ are assumed to change slowly in one wavelength and one period. Up to the first approximation, we assume that the values β and γ change so slowly that in one wavelength and one period their derivatives β_x, $\beta_t = -\gamma_x$ and γ_t can be neglected. Note that this assumption is essentially similar to the assumption that k and ω are considered as constants in the strip problem. Taking all these circumstances into account, all the underlined terms are negligible at the first step of VAM and thus the derivatives of η can be approximately replaced by

$$\eta_x = \varphi_\theta \theta_x + \chi_x, \quad \eta_t = \varphi_\theta \theta_t + \chi_t,$$
$$\eta_{xx} = \varphi_{\theta\theta}\theta_x^2, \quad \eta_{xt} = \varphi_{\theta\theta}\theta_x\theta_t, \quad \eta_{tt} = \varphi_{\theta\theta}\theta_t^2.$$

Substituting these formulas into Eq. (3.2), we obtain the approximate functional

$$I_0[\varphi] = \iint L(\varphi_\theta\theta_t + \chi_t, \varphi_\theta\theta_x + \chi_x, \varphi_{\theta\theta}\theta_t^2, \varphi_{\theta\theta}\theta_x\theta_t, \varphi_{\theta\theta}\theta_x^2)\mathrm{d}x\mathrm{d}t.$$

Strip problem Staying consistent with the variational-asymptotic method, the strip problem can be formulated as follows: Find the extremal of the functional

$$I_0[\varphi] = \frac{1}{2\pi}\int_0^{2\pi} L(-\omega\varphi_\theta - \gamma, k\varphi_\theta + \beta, \omega^2\varphi_{\theta\theta}, -\omega k\varphi_{\theta\theta}, k^2\varphi_{\theta\theta})\mathrm{d}\theta$$

among functions $\varphi(\theta)$ satisfying 2π-periodicity conditions

$$\varphi(2\pi) = \varphi(0), \quad \varphi_\theta(2\pi) = \varphi_\theta(0), \quad \varphi_{\theta\theta}(2\pi) = \varphi_{\theta\theta}(0)$$

together with an additional condition characterizing the wave amplitude which will be specified later. In this strip problem k, ω, β, γ are considered as constants. The phase velocity is defined as usual, that is $c = \omega/k$. In order to establish the direct connection with the solution of the original evolution equation, we make a change of dependent variable

$$\phi = k\varphi_\theta + \beta.$$

It is natural to use this new unknown function in the strip problem because it represents the derivative η_x and hence the original solution u. Since this transformation reduces the order of the resulting differential equation, there must be a certain supplementary condition. As function $\varphi(\theta)$ is 2π-periodic, the newly introduced function satisfies the constraint

$$\frac{1}{2\pi}\int_0^{2\pi}\phi\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}(k\varphi_\theta + \beta)\mathrm{d}\theta = \beta. \tag{3.4}$$

Furthermore, it is easy to relate the old variable to the new one by

$$\varphi_\theta = \frac{\phi - \beta}{k}, \quad \varphi_{\theta\theta} = \frac{\phi_\theta}{k}.$$

A standard technique of eliminating this constraint is to introduce the Lagrange multiplier in order to incorporate the constraint into the optimization formulation, leading to the following equivalent variational problem: Find the extremal of the functional

$$I[\phi(\theta)] = \frac{1}{2\pi}\int_0^{2\pi} L(-\gamma - c(\phi - \beta), \phi, \omega c\phi_\theta, -\omega\phi_\theta, k\phi_\theta)\mathrm{d}\theta - \lambda\left(\frac{1}{2\pi}\int_0^{2\pi}\phi\mathrm{d}\theta - \beta\right)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left[L(c\beta - \gamma - c\phi, \phi, \omega c\phi_\theta, -\omega\phi_\theta, k\phi_\theta) - \lambda\phi\right]\mathrm{d}\theta + \lambda\beta$$

among λ and $\phi(\theta)$ satisfying the periodicity conditions

$$\phi(2\pi) = \phi(0), \quad \phi_\theta(2\pi) = \phi_\theta(0).$$

In order to utilize the constraint (3.4), let us split the Lagrangian into three parts as follows

$$L = \Lambda(k, \omega, \beta, \gamma, \phi, \phi_\theta) + Q_1(k, \omega, \beta, \gamma)\phi + Q_2(k, \omega, \beta, \gamma), \tag{3.5}$$

where Λ does not contain the linear term in ϕ which has been put together in the second term $Q_1\phi$. We rewrite the functional as

$$I[\phi(\theta)] = \frac{1}{2\pi}\int_0^{2\pi} [\Lambda(k, \omega, \beta, \gamma, \phi, \phi_\theta) - \lambda\phi]\, \mathrm{d}\theta + Q_1\beta + Q_2 + \lambda\beta. \tag{3.6}$$

The Euler-Lagrange equation of this variational problem reads

$$\frac{\partial}{\partial\theta}\frac{\partial\Lambda}{\partial\phi_\theta} - \frac{\partial\Lambda}{\partial\phi} + \lambda = 0.$$

Multiplying it by ϕ_θ and then integrating with respect to θ, we obtain the first integral in the form

$$\int \frac{\partial}{\partial\theta}\frac{\partial\Lambda}{\partial\phi_\theta}\phi_\theta \mathrm{d}\theta - \int \frac{\partial\Lambda}{\partial\phi}\mathrm{d}\phi + \lambda\phi = h,$$

where h is the integration constant. One useful note is made here that the second term in this equation does not simplify to Λ as the integrand is not the total derivative. In case the original Lagrangian does not contain any cross terms between the first-order derivatives and the second-order derivatives of η, then there exists in Λ no cross terms between ϕ and ϕ_θ. In such circumstance the derivative $\partial\Lambda/\partial\phi_\theta$ does not depend on ϕ so that this first integral can be reduced further to

$$\int \left[\frac{\partial^2\Lambda}{\partial(\phi_\theta)^2}\phi_\theta\right] \mathrm{d}\phi_\theta - \int \frac{\partial\Lambda}{\partial\phi}\mathrm{d}\phi + \lambda\phi = h. \tag{3.7}$$

The periodic solution corresponds to the closed orbit of the phase portrait of this first integral. As an alternative to the parameter h we may use another by specifying the maximum of ϕ as

$$a = \max_{\theta\in[0,2\pi]} \phi(\theta) \tag{3.8}$$

conveying the meaning of the amplitude of wave over one wavelength.

Modulation equations In the next stage the non-uniform wave packet is considered. Then the amplitude defined by Eq. (3.8) may still depend on the slow variables x and t. The wave number k, the frequency ω and the parameters β, γ are treated as slowly varying functions of x and t. The average Lagrangian is defined as before, that is the value of functional (3.6) at its extremal. Then the variational-asymptotic analysis leads to the following average variational problem

$$\delta \iint \mathcal{L}(a, k, \omega, \lambda, \beta, \gamma)\mathrm{d}x\mathrm{d}t = 0,$$

whose Euler-Lagrange equations read

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial a} = 0, \qquad \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial k} = 0, \\ &\frac{\partial \mathcal{L}}{\partial \lambda} = 0, \qquad \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \gamma} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \beta} = 0. \end{aligned} \tag{3.9}$$

The first equation again leads to the nonlinear dispersion relation, whereas the third equation is nothing else but the constraint (3.4). Especially, the Euler-Lagrange equation for θ is called the equation of amplitude modulation.

Shortcut to the formulation of the average Lagrangian It is worth noting that in practice the Lagrangian given by Eq. (3.5) can be obtained at hand after the following substitutions

$$\eta_t \to -(\gamma - c\beta) - c\phi, \quad \eta_x \to \phi, \quad \eta_{xx} \to k\phi_\theta, \quad \eta_{xt} \to -\omega\phi_\theta, \quad \eta_{tt} \to \omega c\phi_\theta. \tag{3.10}$$

Then the average Lagrangian has exactly the same form as Eq. (3.6) keeping in mind that the derivable first integrals of the strip problem must be taken into account.

3.2 Amplitude modulations for the KdV equation

Derivation of the modulation equations As pointed out, the modified theory of modulation can be directly applied to the case of the KdV equation whose associated Lagrangian is

$$L = -\frac{1}{2}\eta_t\eta_x - \eta_x^3 + \frac{1}{2}\eta_{xx}^2, \quad u = \eta_x.$$

Using the substitutions (3.10), we obtain

$$L = \frac{1}{2}(\gamma - c\beta + c\phi)\phi - \phi^3 + \frac{1}{2}k^2\phi_\theta^2 = \frac{1}{2}k^2\phi_\theta^2 + \frac{1}{2}c\phi^2 - \phi^3 + \frac{1}{2}(\gamma - c\beta)\phi = \Lambda + Q_1\phi + Q_2.$$

Thus, the slitting parts are

$$\Lambda = \frac{1}{2}k^2\phi_\theta^2 + \frac{1}{2}c\phi^2 - \phi^3, \quad Q_1 = \frac{1}{2}(\gamma - c\beta), \quad Q_2 = 0.$$

Accordingly, we can write immediately the functional in the strip problem

$$I[\phi(\theta)] = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2}k^2\phi_\theta^2 - U(\phi, c, \lambda)\right]\mathrm{d}\theta + \frac{1}{2}(\gamma - c\beta)\beta + \lambda\beta,$$

where the function of three arguments $U(\phi, c, \lambda)$ is given by

$$U(\phi, c, \lambda) = \phi^3 - \frac{1}{2}c\phi^2 + \lambda\phi.$$

The first integral (3.7) becomes

$$\int k^2\phi_\theta \mathrm{d}\phi_\theta + \phi^3 - \frac{1}{2}c\phi^2 + \lambda\phi = h,$$

or in the shorter form

$$\frac{1}{2}k^2\phi_\theta^2 + U(\phi, c, \lambda) = h. \tag{3.11}$$

This result is consistent with the direct derivation of the first integral from the stationarity of the above functional. Moreover, it suggests us to name the function U 'pseudo-potential' function. Using the definition of amplitude (3.8) and the first integral (3.11), we can write the average Lagrangian as follows

$$\begin{aligned}\mathcal{L} &= \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a}\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}\mathrm{d}\phi - U(a,c,\lambda) + \lambda\beta + \frac{1}{2}(\gamma - c\beta)\beta \\ &= \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} G(\phi,a,c,\lambda)\mathrm{d}\phi - U(a,c,\lambda) + \lambda\beta + H(c,\beta,\gamma),\end{aligned} \tag{3.12}$$

where the energy level h, due to the condition $\max\phi = a$, has been replaced by $U(a, c, \lambda)$, and ϕ_2 is another root of the cubic equation

$$U(a,c,\lambda) - U(\phi,c,\lambda) = (a-\phi)(\phi-\phi_2)(\phi-\phi_3) = 0, \tag{3.13}$$

and the intermediate functions G and H are defined as follows

$$G(\phi,a,c,\lambda) = \sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}, \quad H(c,\beta,\gamma) = \frac{1}{2}(\gamma - c\beta)\beta, \tag{3.14}$$

The roots are ordered according to $a \geq \phi \geq \phi_2 \geq \phi_3$. It is immediately implied that

$$G(\phi_j, a, c, \lambda) = 0, \quad j = 1, 2, 3. \tag{3.15}$$

To facilitate the derivation of the modulation equations, we recall a useful formula summarized in the following statement. Let us consider a function given by an integral of the form

$$I(x) = \int_{g_1(x)}^{g_2(x)} \Gamma(\xi, x)\mathrm{d}\xi,$$

where g_1, g_2 are to differentiable functions and Γ is a differentiable function of two variables. It should be emphasized that the argument x appears not only in the integrand but also in both integral limits. Then its derivative can be calculated according to

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \Gamma\big(g_2(x),x\big)g_2'(x) - \Gamma\big(g_1(x),x\big)g_1'(x) + \int_{g_1(x)}^{g_2(x)}\frac{\partial\Gamma}{\partial x}(\xi,x)\mathrm{d}\xi. \tag{3.16}$$

Now, let us derive the set of modulation equations by computing the involved partial derivatives step by step. Firstly, with the aid of this rule we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a} &= \frac{k\sqrt{2}}{\pi}\big[G(a,a,c,\lambda) - G(\phi_2,a,c,\lambda)\partial_a\phi_2\big] + \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_a(\phi,a,c,\lambda)\mathrm{d}\phi - U_a(a,c,\lambda) \\ &= \left[\frac{k\sqrt{2}}{2\pi}\int_{\phi_2}^{a}\frac{\mathrm{d}\phi}{G(\phi,a,c,\lambda)} - 1\right]U_a(a,c,\lambda),\end{aligned}$$

in which the fact that a and ϕ_2 are the roots of the equation $G(\phi,a,c,\lambda) = 0$ has been used. Thus, the dispersion relation $\partial\mathcal{L}/\partial a = 0$ may be written as

$$\frac{k\sqrt{2}}{2\pi}\int_{\phi_2}^{a}\frac{\mathrm{d}\phi}{G(\phi,a,c,\lambda)} - 1 = 0. \tag{3.17}$$

Next, we compute the partial derivative

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} \frac{U_\lambda(a,c,\lambda) - U_\lambda(\phi,c,\lambda)}{2G(\phi,a,c,\lambda)}\mathrm{d}\phi - U_\lambda(a,c,\lambda) + \beta$$
$$= \left[\frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} \frac{\mathrm{d}\phi}{G(\phi,a,c,\lambda)} - 1\right] U_\lambda(a,c,\lambda) + \beta - \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} \frac{\phi\,\mathrm{d}\phi}{G(\phi,a,c,\lambda)}.$$

Making use of the dispersion relation, we may rewrite the constraint $\partial\mathcal{L}/\partial\lambda = 0$ as

$$\frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} \frac{\phi\,\mathrm{d}\phi}{G(\phi,a,c,\lambda)} - \beta = 0. \tag{3.18}$$

The partial derivatives of U with respect to its arguments can be individually worked out and given by

$$U_a(a,c,\lambda) = 3a^2 - ca + \lambda, \quad U_c(a,c,\lambda) = -\frac{1}{2}a^2, \quad U_\lambda(a,c,\lambda) = a. \tag{3.19}$$

Using equations (3.17), (3.18) and the identities

$$G_\kappa(\phi,a,c,\lambda) = \frac{U_\kappa(a,c,\lambda) - U_\kappa(\phi,c,\lambda)}{2G(\phi,a,c,\lambda)}, \quad \kappa = a, c, \lambda,$$

it is straightforward to deduce some formulas for later use as follows

$$\begin{aligned}
&\frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_a(\phi,a,c,\lambda)\mathrm{d}\phi = \frac{1}{k}U_a(a,c,\lambda),\\
&\frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_c(\phi,a,c,\lambda)\mathrm{d}\phi = F(a,c,\lambda) - \frac{a^2}{2k},\\
&\frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_\lambda(\phi,a,c,\lambda)\mathrm{d}\phi = \frac{1}{k}(a-\beta),
\end{aligned} \tag{3.20}$$

where the intermediate function F is given by

$$F(a,c,\lambda) = \frac{\sqrt{2}}{4\pi}\int_{\phi_2}^{a} \frac{\phi^2\mathrm{d}\phi}{G(\phi,a,c,\lambda)}.$$

In the derivation of identities (3.20) we have used the dispersion relation (3.17). We compute the partial derivatives involved in Eq. $(3.9)_2$ in two steps which are detailed in the following.

Step 1 Using the chain rule of differentiation, we may write

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \omega} &= \left[\frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_c(\phi,a,c,\lambda)\mathrm{d}\phi - U_c(a,c,\lambda) + H_c(c,\beta,\gamma)\right]\frac{\partial c}{\partial \omega},\\
\frac{\partial \mathcal{L}}{\partial k} &= \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G(\phi,a,c,\lambda)\mathrm{d}\phi\\
&\quad + \left[\frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_c(\phi,a,c,\lambda)\mathrm{d}\phi - U_c(a,c,\lambda) + H_c(c,\beta,\gamma)\right]\frac{\partial c}{\partial k}\\
&= \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G(\phi,a,c,\lambda)\mathrm{d}\phi + \frac{\partial \mathcal{L}}{\partial \omega}\left(\frac{\partial c}{\partial k}\Big/\frac{\partial c}{\partial \omega}\right),
\end{aligned} \tag{3.21}$$

Substituting the expressions (3.19), (3.20) into these derivatives, we obtain

$$\frac{\partial \mathcal{L}}{\partial \omega} = F(a,c,\lambda) + \frac{H_c(c,\beta,\gamma)}{k},$$
$$\frac{\partial \mathcal{L}}{\partial k} = \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G(\phi,a,c,\lambda)\mathrm{d}\phi - c\left[F(a,c,\lambda) + \frac{H_c(c,\beta,\gamma)}{k}\right].$$

Step 2 We differentiate $\partial\mathcal{L}/\partial\omega$ and $\partial\mathcal{L}/\partial k$ with respect to t and x, respectively, keeping in mind that a, c, λ are all functions of x and t, to obtain

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \omega} = \frac{\partial F}{\partial a}a_t + \frac{\partial F}{\partial c}c_t + \frac{\partial F}{\partial \lambda}\lambda_t + \frac{\partial}{\partial t}\left[\frac{H_c(c,\beta,\gamma)}{k}\right], \tag{3.22}$$
$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial k} &= \frac{\sqrt{2}}{\pi}\left[G(a,a,c,\lambda)a_x - G(\phi_2,a,c,\lambda)\phi_{2,x}\right]\\ &\quad + \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_a(\phi,a,c,\lambda)\mathrm{d}\phi\, a_x + \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_c(\phi,a,c,\lambda)\mathrm{d}\phi\, c_x\\ &\quad + \frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a} G_\lambda(\phi,a,c,\lambda)\mathrm{d}\phi\, \lambda_x - c_x\left[F(a,c,\lambda) + \frac{H_c(c,\beta,\gamma)}{k}\right]\\ &\quad - c\frac{\partial}{\partial x}\left[F(a,c,\lambda) + \frac{H_c(c,\beta,\gamma)}{k}\right].\end{aligned}$$

Upon using the conditions (3.15), two first terms in the last expression are identically zero. Once again, substitution of Eq. (3.20) into it leads to

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial k} &= \frac{1}{k}U_a(a,c,\lambda)a_x + \left[F(a,c,\lambda) - \frac{a^2}{2k}\right]c_x + \frac{1}{k}(a-\beta)\lambda_x - c\frac{\partial}{\partial x}\left[\frac{H_c(c,\beta,\gamma)}{k}\right]\\ &\quad - c_x\left[F(a,c,\lambda) + \frac{H_c(c,\beta,\gamma)}{k}\right] - c\left[\frac{\partial F}{\partial a}a_x + \frac{\partial F}{\partial c}c_x + \frac{\partial F}{\partial \lambda}\lambda_x\right]\\ &= \frac{1}{k}U_a(a,c,\lambda)a_x - \frac{\partial}{\partial x}\left[c\frac{H_c(c,\beta,\gamma)}{k}\right] - \frac{a^2}{2k}c_x - \frac{1}{k}(\beta-a)\lambda_x\\ &\quad - c\left(\frac{\partial F}{\partial a}a_x + \frac{\partial F}{\partial c}c_x + \frac{\partial F}{\partial \lambda}\lambda_x\right)\end{aligned} \tag{3.23}$$

Subtracting Eq. (3.23) from Eq. (3.22) and rearranging terms appropriately, we finally obtain the equation of amplitude modulation

$$\begin{aligned}&\frac{\partial F}{\partial a}(a_t + ca_x) + \frac{\partial F}{\partial c}(c_t + cc_x) + \frac{\partial F}{\partial \lambda}(\lambda_t + c\lambda_x)\\ &- \left[\frac{\partial}{\partial t}\left(\frac{\beta^2}{k}\right) + \frac{\partial}{\partial x}\left(c\frac{\beta^2}{k}\right)\right] - \frac{1}{k}\left[U_a(a,c,\lambda)a_x - \frac{a^2}{2}c_x - (\beta-a)\lambda_x\right] = 0,\end{aligned} \tag{3.24}$$

in which the derivative $H_c(c,\beta,\gamma) = -\beta^2/2$ has been used. To complete the system of modulation equations, we compute the partial derivatives of $\mathcal{L}$ with respect to β and γ. They are easily computed and given by

$$\frac{\partial \mathcal{L}}{\partial \beta} = \lambda + H_\beta(c,\beta,\gamma), \quad \frac{\partial \mathcal{L}}{\partial \gamma} = H_\gamma(c,\beta,\gamma).$$

Thus, using the definition $(3.14)_2$, the Euler-Lagrange equation for χ, namely

$$[H_\gamma(c,\beta,\gamma)]_t - [\lambda + H_\beta(c,\beta,\gamma)]_x = 0, \tag{3.25}$$

explicitly reads

$$\frac{1}{2}\beta_t - \left(\lambda - c\beta + \frac{1}{2}\gamma\right)_x = 0.$$

In aid of the consistency condition $\beta_t + \gamma_x = 0$ this equation can be cast to

$$\gamma_x = (c\beta - \lambda)_x,$$

which allows us to choose

$$\gamma = c\beta - \lambda. \tag{3.26}$$

Note that the four equations (3.17), (3.18), (3.24), (3.26) are equivalent to the system of equations derived by Whitham in [35].

Amplitude modulation of soliton packets In the limit $\lambda \to 0$ and $h \to 0$ the wave packet becomes a train of solitary waves. For the wave packet consisting of n solitons we know that the solitons cease to interact at large time in such a way that each of them propagates with the individual constant velocity along the line $x/t = \text{const}$. Based on this observation we look for the solution $a = a(x,t)$ of Eq. (3.24) using the following Ansatz for θ and χ

$$\theta(x,t) = p(\xi(x,t)), \quad \chi(x,t) = q(\xi(x,t)), \quad \xi(x,t) = x/t.$$

Differentiating $\theta(x,t)$ and $\chi(x,t)$ in accordance with these Ansatz, we find

$$k = \frac{1}{t}p'(\xi), \quad \omega = \frac{\xi}{t}p'(\xi), \quad c = \xi, \quad \beta = \frac{1}{t}q'(\xi), \quad \gamma = \frac{\xi}{t}q'(\xi).$$

It is easy to see that β and γ from the last expressions satisfy Eq. (3.26), provided $\lambda = 0$. Besides, the following equation

$$\left(\frac{\beta^2}{k}\right)_t + \left(c\frac{\beta^2}{k}\right)_x = 0$$

is then identically fulfilled. Let us prove this statement by another more general claim. Since the fraction

$$\frac{\beta^2}{k} = \frac{q'(\xi)^2}{t\,p(\xi)}$$

can be recognized as a function of the form $h(\xi)/t$, where h is an arbitrary function, we shall show that

$$J = \frac{\partial}{\partial t}\left(\frac{1}{t}h(\xi)\right) + \frac{\partial}{\partial x}\left(\frac{\xi}{t}h(\xi)\right) = 0.$$

Indeed, the left-hand side of this equation is expanded to

$$J = \frac{1}{t}h'(\xi)\xi_t - \frac{1}{t^2}h(\xi) + \frac{1}{t}[h(\xi)\xi_x + \xi h'(\xi)\xi_x] = \frac{1}{t}h'(\xi)(\xi_t + \xi\xi_x).$$

The expression in the last parentheses vanishes and so the statement is justified. Furthermore, if the amplitude is searched among functions of the form

$$a(x,t) = g(\xi(x,t)),$$

the term $\partial F/\partial a(a_t + ca_x)$ vanishes, so Eq. (3.24) reduces to

$$g(\xi) - (6g(\xi) - 2\xi)g'(\xi) = 0.$$

This equation can be treated thoroughly by using the theorem of inverse function and considering the independent variable ξ as the unknown function. Indeed, looking at the equation from a different angle and multiplying both sides by $\xi'(g)$, we may rewrite it as

$$\xi'(g)g + 2\xi(g) - 6g = 0,$$

where the derivative $g'(\xi)$ is assumed nonzero for application of the theorem of inverse function. This equation can be integrated with the standard method of integrating factor to give the final solution

$$\xi(g) = 2g + \frac{C}{g},$$

where C is the integration constant. Treating this equation as a quadratic equation for g, we can recover the functional dependence of g on ξ as follows

$$g_{1,2}(\xi) = \frac{\xi \pm \sqrt{\xi^2 - 8C}}{4}.$$

The principle leading terms of these functions in the limit $\xi \to \infty$ are given by

$$g_1(\xi) \sim \frac{1}{2}\xi, \quad g_2(\xi) \sim 0.$$

All in all, if we deal with the wave at large time and the initial disturbance does not leave much effect on the wave propagation at large time, it is sufficient to choose the modulation solution

$$a(x,t) = \frac{1}{2}\xi(x,t) = \frac{x}{2t}.$$

To see the fulfillment of the dispersion relation at large time, we rewrite Eq. (3.17) in an equivalent form

$$k = \frac{\pi\sqrt{(a-\phi_2)/2}}{mK(m)}, \quad m = \sqrt{\frac{a-\phi_2}{a-\phi_3}}, \tag{3.27}$$

where $K(m)$ is the complete elliptic integral of the first kind. In the limit $\lambda \to 0$, $h \to 0$, the roots ϕ_2 and ϕ_3 go to 0, so $m \to 1$. Provided the derivative $q'(x/t)$ is finite, the left- and right-hand sides of the dispersion relation tend to 0 as $t \to \infty$, so the dispersion relation is asymptotically satisfied at large time.

Amplitude modulation of position We recall here the first-order positon solution of KdV equation given by Eq. (2.60)

$$u(x,t) = \frac{32\kappa^2(\kappa X\cos T - \sin T)\sin T}{(\sin 2T - 2\kappa X)^2}, \tag{3.28}$$

where κ is some positive real number and

$$T = \kappa(x + 4\kappa^2 t), \quad X = x + 12\kappa^2 t.$$

At large time or large coordinate this positon solution obeys the asymptotic formula

$$u(x,t) \sim \frac{4\kappa}{X}\sin 2T.$$

Based on this observation let us seek the asymptotic solution of the equations of amplitude modulation for the positon in the form

$$\theta(x,t) = 2T, \quad a(x,t) = \frac{4\kappa}{X}. \tag{3.29}$$

According to this Ansatz

$$k = 2\kappa, \quad \omega = -8\kappa^3, \quad c = -4\kappa^2.$$

Thus, in contrast to the soliton, the positon moves to the left and changes its form with time since the amplitude depends on X and not on T. Since it is not trivial to find the solution for the function χ so that the system of modulation equations are identically or at least asymptotically fulfilled, further observations could be handy. Investigation of the positon solution from the geometrical point of view suggests that the second zero ϕ_2 of the cubic equation (3.13) obeys the asymptotic law

$$\phi_2 \sim -a \quad \text{as } a \to 0.$$

This observation is best illustrated in Fig. 3.1, where the dashed lines indicate the modulations of two roots $a = \phi_1$ and ϕ_2. More interestingly, this law remains a good approximation

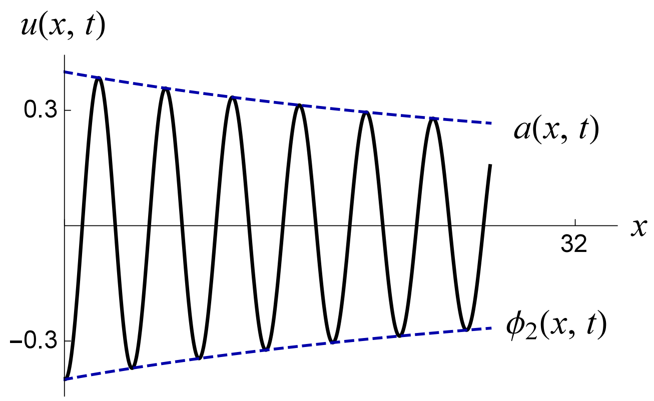


Figure 3.1: The oscillation of the positon solution of the KdV equation with small amplitude.

as long as the amplitude is finite. Now that we expand both sides of Eq. (3.13) as follows

$$U(a,c,\lambda) - U(\phi,c,\lambda) = -\phi^3 + \frac{c}{2}\phi^2 - \lambda\phi + a^3 - \frac{c}{2}a^2 + \lambda a,$$
$$(a-\phi)(\phi+a)(\phi-\phi_3) = -\phi^3 + \phi_3\phi^2 + a^2\phi - a^2\phi_3,$$

and equate the coefficients of these two polynomials with respect to ϕ to obtain

$$\phi_3 = \frac{c}{2}, \quad \lambda = -a^2, \quad -a^2\phi_3 = a^3 - \frac{c}{2}a^2 + \lambda a.$$

What we have done simply reflects the Viet's theorem and if we consider the Viet's equations as a system for only two variables λ and ϕ_3, the last equation should be the direct consequence. It is indeed the case. According to Eq. (3.26), we have

$$4\kappa^2\chi_x - \chi_t = \frac{16\kappa^2}{X^2}.$$

The last equation gives us a hint to seek the function χ as a simple function of one variable X. To this end, let us plug the Ansatz $\chi = g(X)$ into the last equation and simplify it to obtain

$$g'(X) = -\frac{2}{X^2}.$$

Integrating this equation with the zero integration constant, we arrive at the asymptotic solution for the auxiliary function χ as follows

$$\chi(x,t) = \frac{2}{X(x,t)}.$$

Let us show that the three remaining equations (3.17)–(3.24) are satisfied asymptotically in the limit $a \to 0$. Indeed, using the well-known formulas for the elliptic integrals (see [90])

$$\int_{\phi_2}^{a} \frac{\mathrm{d}\phi}{\sqrt{(a-\phi)(\phi-\phi_2)(\phi-\phi_3)}} = \frac{2}{\sqrt{a-\phi_3}}K(m),$$
$$\int_{\phi_2}^{a} \frac{\phi\,\mathrm{d}\phi}{\sqrt{(a-\phi)(\phi-\phi_2)(\phi-\phi_3)}} = \frac{2}{\sqrt{a-\phi_3}}[(a-\phi_3)E(m) + \phi_3 K(m)],$$

where

$$m = \sqrt{\frac{(a-\phi_2)}{(a-\phi_3)}}, \quad \phi_2 = -a, \quad \phi_3 = c/2 = -2\kappa^2,$$

we see that, as a goes to zero, the first and the second integrals are asymptotically equivalent to $\pi/\kappa\sqrt{2}$ and $-\pi a^2/8\sqrt{2}\kappa^3$, respectively. Therefore, the dispersion relation (3.17) and the constraint (3.18) are asymptotically satisfied. It is also easy to see that all terms on the left-hand side of (3.24) approaches zero as a tends to zero. This guarantees the fulfillment of the equation of amplitude modulation in this limit.[1]

Comparison with exact solutions Let us compare the asymptotic solutions obtained previously with the exact solutions of the KdV equation. The previous chapter now comes in use as the simulation of a train of solitons is based on the solution formula (2.21).

Figure 3.2 shows the exact 5-soliton solution and its amplitude modulation described by $x/2t$. From this figure it is seen that, at large time, the curve $x/2t$ can serve as the asymptotic envelope for the amplitude of solitons. Figure 3.3 shows the exact positon solution according to Eq. (3.28) together with its amplitude modulation described by $4\kappa/X(x,t)$. In this simulation we choose $\kappa = 2$, and due to the singularity the plot is restricted to the range $-6 < u(x,t) < 6$. Again, it is seen that the curve $4\kappa/X(x,t)$ can serve as the asymptotic envelope for the amplitude of positon.

[1] For exact integration of the modulation equations we must treat k and ω as true functions of the space-time variables x and t.

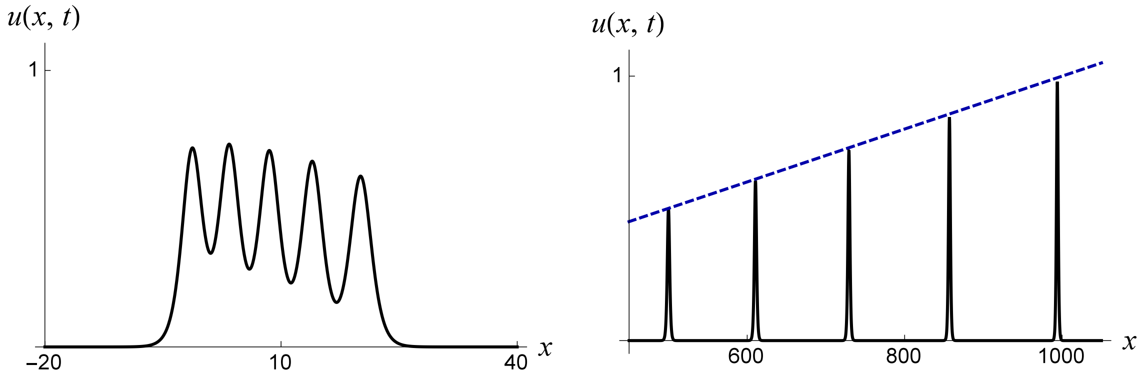


Figure 3.2: A train of solitons (bold line) and the amplitude modulation (dashed line): a) at small time, b) at large time.

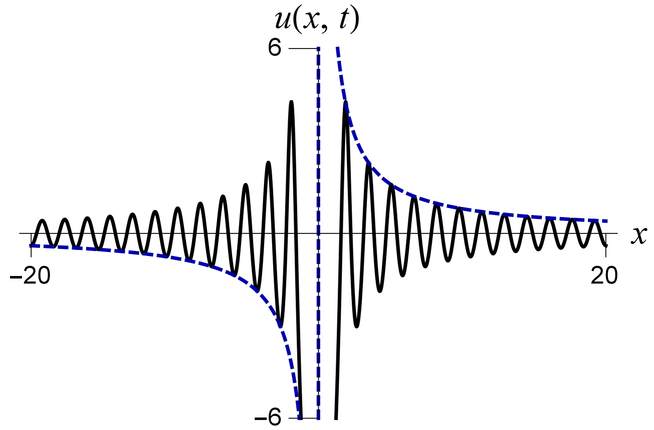


Figure 3.3: A positon solution (bold line) and the amplitude modulation (dashed line): a) at small time, b) at large time.

3.3 Theory of amplitude modulation for the BSQ equation

Derivation of the modulation equations Making use the derivation of the Lagrangian (5.4) in the supplementary materials, we study the BSQ equation

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0$$

by starting with the associated Lagrangian

$$L = \eta_t^2 - \frac{1}{2}\eta_x^2 - 2\eta_x^3 + \frac{1}{2}\eta_{xx}^2, \quad u = \eta_x. \tag{3.30}$$

After the substitutions (3.10), we obtain

$$L = \frac{1}{2}(\gamma - c\beta + c\phi)^2 - \frac{1}{2}\phi^2 - 2\phi^3 + \frac{1}{2}k^2\phi_\theta^2,$$

which can be divided into three parts as follows

$$L = \Lambda + Q_1\phi + Q_2,$$
$$\Lambda = \frac{1}{2}k^2\phi_\theta^2 + \frac{1}{2}c^2\phi^2 - \frac{1}{2}\phi^2 - 2\phi^3, \quad Q_1 = c(\gamma - c\beta), \quad Q_2 = \frac{1}{2}(\gamma - c\beta)^2.$$

Using the obvious identity

$$Q_1\beta + Q_2 = \frac{1}{2}(\gamma^2 - c^2\beta^2),$$

the functional in the strip problem can be written as

$$I[\phi(\theta)] = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2}k^2\phi_\theta^2 - U(\phi, c, \lambda)\right]\mathrm{d}\theta + \frac{1}{2}(\gamma^2 - c^2\beta^2) + \lambda\beta,$$

where the 'pseudo-potential' function U is given by

$$U(\phi, c, \lambda) = 2\phi^3 + \frac{1}{2}(1-c^2)\phi^2 + \lambda\phi.$$

It is straightforward to verify that the first integral derived directly from this strip problem and that from (3.7) produces the same

$$\frac{1}{2}k^2\phi_\theta^2 + U(\phi, c, \lambda) = h.$$

In Fig. 3.4 the phase portrait corresponding to this first integral is shown, in which we shall focus on the closed orbits.

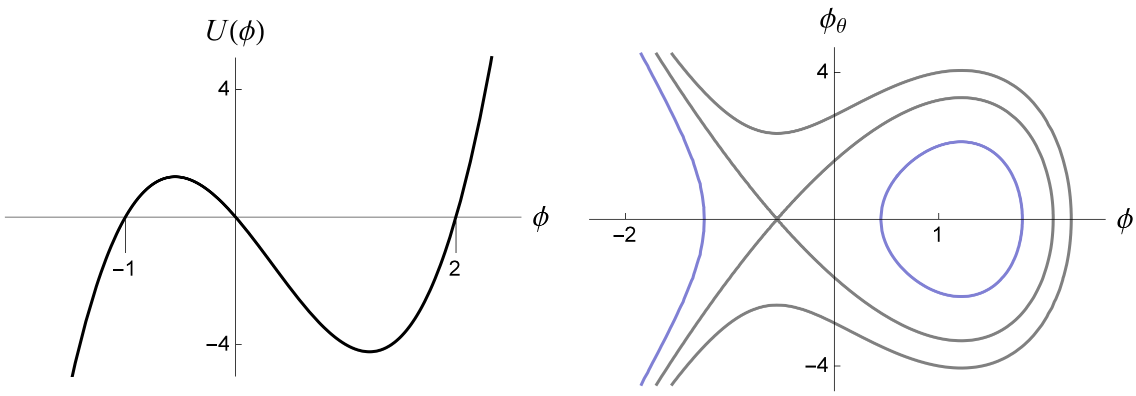


Figure 3.4: Psedo-potential function U (left) and phase portrait of the first integral in the strip problem (right) for the modulation of the BSQ equation. The closed orbit is plotted in blue.

Embedding the definition of amplitude (3.8) into the calculation and using the the first integral, we can write the average Lagrangian as follows

$$\begin{aligned}\mathcal{L} &= \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a}\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}\,\mathrm{d}\phi - U(a,c,\lambda) + \frac{1}{2}(\gamma^2 - c^2\beta^2) + \lambda\beta \\ &= \frac{k\sqrt{2}}{\pi}\int_{\phi_2}^{a} G(\phi,a,c,\lambda)\mathrm{d}\phi - U(a,c,\lambda) + \lambda\beta + H(c,\beta,\gamma),\end{aligned}$$

where the energy level h, due to the condition $\max\phi = a$, has been replaced by $U(a, c, \lambda)$, and ϕ_2 is another zero of the cubic equation

$$U(a,c,\lambda) - U(\phi,c,\lambda) = 2(a-\phi)(\phi-\phi_2)(\phi-\phi_3) = 0,$$

and

$$G(\phi,a,c,\lambda) = \sqrt{U(\phi,c,\lambda) - U(a,c,\lambda)}, \quad H(c,\beta,\gamma) = \frac{1}{2}(\gamma^2 - c^2\beta^2). \tag{3.31}$$

The roots are ordered such that $a \geq \phi \geq \phi_2 > \phi_3$. It is crystal clear that the average Lagrangian for the BSQ case possesses exactly the same algebraic structure as that for the KdV case. Henceforth, without implementation of detailed deduction, the dispersion relation and the constraint can be immediately written in the form

$$\frac{k\sqrt{2}}{2\pi}\int_{\phi_2}^{a}\frac{\mathrm{d}\phi}{G(\phi,a,c,\lambda)}-1=0, \tag{3.32}$$

$$\frac{k\sqrt{2}}{2\pi}\int_{\phi_2}^{a}\frac{\phi\,\mathrm{d}\phi}{G(\phi,a,c,\lambda)}-\beta=0. \tag{3.33}$$

Since the main difference lies only in the partial derivative $U_c(a,c,\lambda)$, it is reasonable to expect that most of derivation in the previous Section can be recycled. Indeed, the system (3.20) remains unaltered except for

$$\frac{\sqrt{2}}{\pi}\int_{\phi_2}^{a}G_c(\phi,a,c,\lambda)\mathrm{d}\phi=\frac{1}{k}U_c(a,c,\lambda)+F(a,c,\lambda),$$

where the interim function F is slightly modified by a factor $c/2$ as follows

$$F(a,c,\lambda)=\frac{c\sqrt{2}}{2\pi}\int_{\phi_2}^{a}\frac{\phi^2\mathrm{d}\phi}{G(\phi,a,c,\lambda)}.$$

These observations and equations (3.22), (3.23) altogether lead to a simple modification on the partial derivatives $(\partial\mathcal{L}/\partial\omega)_t$ and $(\partial\mathcal{L}/\partial k)_x$ as follows

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\omega}&=\frac{\partial F}{\partial a}a_t+\frac{\partial F}{\partial c}c_t+\frac{\partial F}{\partial\lambda}\lambda_t-\frac{\partial}{\partial t}\left(c\frac{\beta^2}{k}\right),\\
\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial k}&=\frac{1}{k}U_a(a,c,\lambda)a_x-\frac{ca^2}{k}c_x+\frac{1}{k}(a-\beta)\lambda_x-c\left(\frac{\partial F}{\partial a}a_x+\frac{\partial F}{\partial c}c_x+\frac{\partial F}{\partial\lambda}\lambda_x\right)\\
&\quad+\frac{\partial}{\partial x}\left(c\frac{c\beta^2}{k}\right).
\end{aligned}$$

Subtracting the last equation from the second last, we arrive at the equation of amplitude modulation

$$\begin{aligned}
&\frac{\partial F}{\partial a}(a_t+ca_x)+\frac{\partial F}{\partial c}(c_t+cc_x)+\frac{\partial F}{\partial\lambda}(\lambda_t+c\lambda_x)\\
&-\left[\frac{\partial}{\partial t}\left(c\frac{\beta^2}{k}\right)+\frac{\partial}{\partial x}\left(c\frac{c\beta^2}{k}\right)\right]-\frac{1}{k}\left[U_a(a,c,\lambda)a_x-\frac{1}{2}a^2(c^2)_x-(\beta-a)\lambda_x\right]=0.
\end{aligned} \tag{3.34}$$

Substituting the definition $(3.31)_2$ into Eq. (3.25), we obtain the last Euler-Lagranage equation in the form

$$\gamma_t=(\lambda-c^2\beta)_x. \tag{3.35}$$

The modulation equations for the BSQ case differ from those for the KdV case not only by the pseudo-potential function U but also by the relation between γ and β. While we have a simple algebraic relation between them in the KdV case, it does not hold true for the BSQ case.

Amplitude modulation of soliton packets For a wavetrain of solitons we shall deal with the modulation equations in the limit $\lambda\to 0$ and $h\to 0$. Once again, the properties of multiple-soliton after collision suggest us to solve Eq. (3.34) using the following Ansatz for θ and χ

$$\theta(x,t) = p(\xi(x,t)), \quad \chi(x,t) = q(\xi(x,t)), \quad \xi(x,t) = x/t.$$

These Ansatz yield

$$k = \frac{1}{t}p'(\xi), \quad \omega = \frac{\xi}{t}p'(\xi), \quad c = \xi, \quad \beta = \frac{1}{t}q'(\xi), \quad \gamma = \frac{\xi}{t}q'(\xi).$$

It is easy to see that β and γ from the last equations satisfy Eq. (3.35) in the limit $\lambda \to 0$. Also, it is clear that the function $c\beta^2/k$ can be once again expressed in the form $h(\xi)/t$, where h is an arbitrary function. Thus, the following identity

$$\frac{\partial}{\partial t}\left(\frac{c\beta^2}{k}\right) + \frac{\partial}{\partial x}\left(c\frac{c\beta^2}{k}\right) = 0$$

is automatically justified by the same statement as in the last Section. The term $\partial F/\partial a(a_t + ca_x)$ suggests that the amplitude might be searched in the form

$$a(x,t) = g(\xi(x,t))$$

so that Eq. (3.34) reduces to

$$[6g^2(\xi) + (1-\xi^2)g(\xi)]g'(\xi) - \xi\, g^2(\xi) = 0.$$

At the first sight this equation looks complicated and almost impossible to solve. However, keeping in mind that the BSQ equation and the KdV equation both have multiple-soliton solutions and their solution structures are analogously comparable, we may suspect that the modulation solutions for the train of solitons have certain similarity in their algebraic structure. In reference with the modulation solution for the KdV case let us try the same strategy of treating this equation with g as independent variable and ξ as dependent variable and transform it to

$$6g^2 + [1 - \xi^2(g)]g - \xi'(g)\xi(g)g^2 = 0,$$

where the derivative $g'(\xi)$ is assumed nonzero for the inverse function $\xi = \xi(g)$ to exist. Under a change of dependent variable $\tau(g) = \xi^2(g)$ it is transformed to

$$\frac{1}{2}\tau'(g)g + \tau(g) - 1 - 6g = 0.$$

It is now easier to integrate this equation to obtain

$$\tau(g) = \xi^2(g) = 1 + 4g + \frac{C}{g^2},$$

where C is an arbitrary integration constant. If we solve this equation as a cubic equation for the functional dependence of g on τ, then the closed form will be sophisticated but won't contain much information. On the other hand, it is known that if we are only interested in the leading terms in the limit $\xi \to \infty$ or equivalently $\tau \to \infty$, the constant C can be set to zero, leaving the asymptotic analysis unchanged. To this end, we solve the last equation with $C = 0$ to end up with the final modulation solution

$$a(x,t) = \frac{1}{4}[\xi(x,t)^2 - 1] = \frac{1}{4}\Big[\Big(\frac{x}{t}\Big)^2 - 1\Big]. \tag{3.36}$$

Note that this asymptotic law agrees with the exact one-soliton solution of the BSQ equation with the amplitude versus velocity relation $a = \frac{1}{4}(c^2 - 1)$. For the wavetrain of non-interacting solitons propagating with the constant velocities along the straight lines $x/t =$ const we anticipate that the amplitude modulation to leading order is $a(x,t) = \frac{1}{4}[(x/t)^2 - 1]$, which confirms Eq. (3.36). However, the method used in this report to derive the equations of amplitude modulation should work in other situations where the exact solution or its first integrals cannot be obtained. Making advantage of the similar structure of the average Lagrangian as in the KdV case, we may guarantee the fulfillment of the dispersion relation at large time using the same argument. To realize this, we rewrite Eq. (3.32) in the equivalent form

$$k = \frac{\pi\sqrt{(a-\phi_2)}}{mK(m)}, \quad m = \sqrt{\frac{a-\phi_2}{a-\phi_3}},$$

where $K(m)$ is the complete elliptic integral of first kind. It is seen immediately that the dispersion relation in this form resembles the counterpart (3.27) in the KdV case and hence our claim is validated.

Amplitude modulation of positon We extract from [88] the single-positon solution of the BSQ equation

$$u(x,t) = \frac{12\sqrt{2}\sin T - \sqrt{2}X\cos T - 17}{(4\sqrt{2}\cos T + X)^2}, \tag{3.37}$$

where

$$T = \frac{1}{4}(\sqrt{3}t + 2x), \quad X = \sqrt{3}t + 3x.$$

At large time it is asymptotically approximated by

$$u(x,t) \sim -\frac{\sqrt{2}}{X}\cos T.$$

This asymptotic rule suggests us to seek the asymptotic solution of the equations of amplitude modulation for the positon in the form

$$\theta(x,t) = T, \quad a(x,t) = \frac{\sqrt{2}}{X},$$

according to which we have

$$k = \frac{1}{2}, \quad \omega = -\frac{\sqrt{3}}{4}, \quad c = -\frac{\sqrt{3}}{2}.$$

By direct inspection of the asymptotic formula for the positon solution, we may impose a condition on the second root as follows

$$\phi_2 \sim -a \quad \text{as } a \to 0. \tag{3.38}$$

In order to find the asymptotic law for the third root ϕ_3, let us study first the dispersion relation and the constraint. Using condition (3.38) and letting $a \to 0$ in Eq. (3.32), we find that

$$\phi_3 \sim -\frac{1}{16} \quad \text{as } a \to 0.$$

Recalling now the constraint (3.33), which can be recast to an equivalent form

$$\frac{k}{\pi\sqrt{a-\phi_3}}[(a-\phi_3)E(m)+\phi_3K(m)]=\beta,$$

we see that β behaves asymptotically in the limit $a\to 0$ according to

$$\beta\sim -4a^2=-\frac{8}{X^2}.$$

The mean value provides a tool to determine the functions χ and γ

$$\chi=\int\beta dx=\frac{8}{3X},\quad \gamma=-\chi_t=\frac{8}{\sqrt{3}X^2},$$

where, based on the exact solution, the arbitrary function of integration depending on t has been set to be zero. With the help of Eq. (3.35) it is ready to compute λ as follows

$$\lambda=\int\lambda_x dx=\int(\gamma_t+c^2\beta_x)dx=-\frac{10}{3X^2},$$

where the integration function in this step has been once again set to zero. It is then directly verified by the asymptotic analysis that the asymptotic formulas for ϕ_2 and ϕ_3 can be obtained from the equation $U(a,c,\lambda)=U(\phi,c,\lambda)$ with c and λ being found above. As the last step, it is also straightforward to check that all terms on the left-hand side of Eq. (3.34) go to zero as a tends to zero.[2]

Comparison with exact solutions of BSQ equation Let us "validate" the obtained modulation solutions by means of visualization comparison. For construction of formula for multiple-soliton solution, it is of ultimate that we can cast the BSQ equation into its bilinear form. The n-soliton solution is obtained following the induction formula (2.21).

Figure 3.5 shows the comparison between the amplitude modulation curve and the 5-soliton solution of BSQ equation simulated numerically at a particular time instant. All the solitons move in one direction to infinity and separate well at large time. It is seen from this Figure that the curve described by Eq. (3.36) can serve as the asymptotic envelope of the soliton packets of the BSQ equation.

In addition, it is completely possible that two packets of solitons move in opposite directions following the mechanism of the BSQ equation. This is further confirmed by the fact that the amplitude of soliton depends quadratically on the phase velocity. The amplitude modulation solution is capable to capture such situation. The whole idea is best explained in Fig. 3.6 in which three solitons move to the left and other three move to the right, all with different velocities and amplitudes.

Last but not least, figure 3.7 shows the exact positon solution (bold line) according to Eq. (3.37), where the plot is restricted to the range $-1/10<u(x,t)<1/10$ due to the singularity, together with its amplitude modulation (dashed line) described by $\sqrt{2}/X(x,t)$. Again, it is seen that the curve $\sqrt{2}/X(x,t)$ can serve as the asymptotic envelope for one single positon.

[2]For exact integration of the modulation equations we must treat k and ω as true functions of the space-time variables x and t.

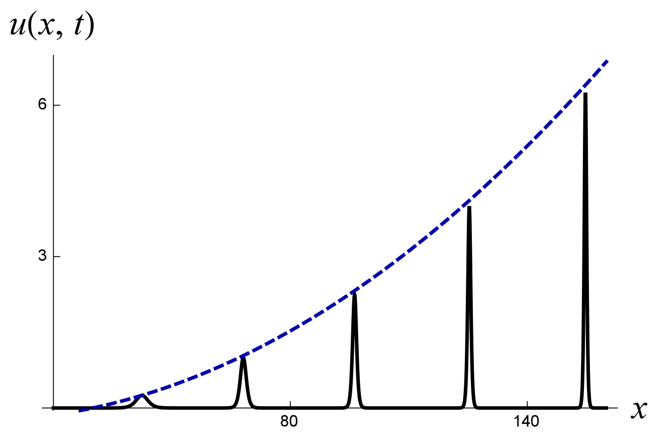


Figure 3.5: Five-soliton solution of the BSQ equation versus its amplitude modulation curve at large time.

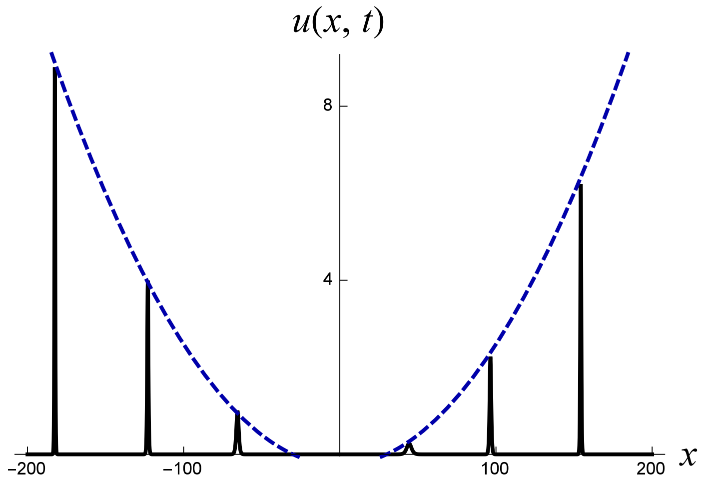


Figure 3.6: Six-soliton solution of the BSQ equation versus its amplitude modulation curve at large time.

Bibliographical remarks

This chapter circulates the materials in the articles [91, 92] with a modification of presentation. The unified approach for both the KdV equation and the BSQ equation reflects the similarity in the amplitude modulation solutions for packets of multiple solitons. In addition, this practice simplifies the derivation and eliminates the unnecessary repetition. For comparison purpose we take the position solutions from [86, 88] and the soliton solutions from [18, 93] (cf. [87]).

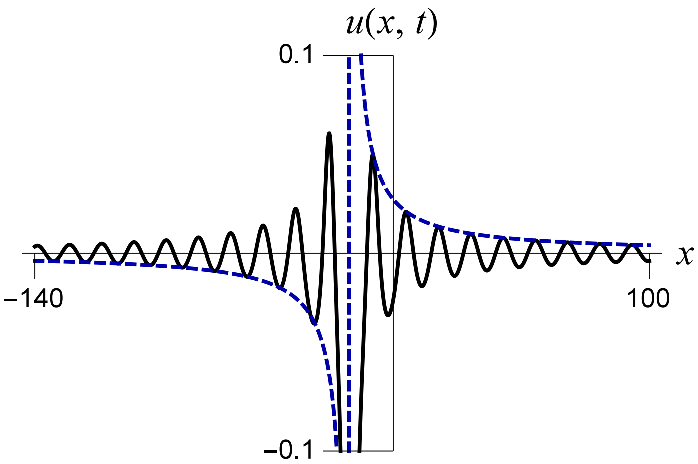


Figure 3.7: A single positon of the BSQ equation and its envelope described by the amplitude modulation solution.

4 Slope modulation for nonlinear dispersive waves

In this chapter we apply the modulation theory with a slight modification to describe the envelope for the slope of a wave packet. We shall present the theory of slope modulation by using the one-dimensional and multi-dimensional sine-Gordon (SG) equations as inspected objects. Since their formulations is in the scope of investigation in the first chapter, the derivation can be immediately conducted. The multi-dimensional SG equation

$$u_{tt} - \Delta u + \sin u = 0, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{4.1}$$

can be degenerated to its 1D version

$$u_{tt} - u_{xx} + \sin u = 0 \tag{4.2}$$

by considering the wave traveling in the x-direction, that is $u = u(x,t)$. However, the multi-dimensional version still possesses several interesting and unexpected attributes. To start with, it is instructive to work with the simple case. In fact, the generalizations can be done in many cases without difficulty.

4.1 Slope modulation for the 1D SG equation

Modulation equations First, it is clear that Eq. (4.2) admits the following Lagrangian density

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \Phi(u), \quad \Phi'(u) = \sin u.$$

It is customary to choose $\Phi(u) = 1 - \cos u$ although the unit constant can be replaced with any other constants. We shall restrict ourselves to the subsonic regime $c^2 < 1$ so that $k^2 - \omega^2 > 0$. We restate the strip problem for our current case: Find the extremal of the functional

$$\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{2}(\omega^2 - k^2)\phi_\theta^2 - (1 - \cos\phi)\right] \mathrm{d}\theta \tag{4.3}$$

among functions $\phi(\theta)$ satisfying the conditions

$$\phi(\theta + 2\pi) = \phi(\theta) + 2\pi, \quad \phi_\theta(\theta + 2\pi) = \phi_\theta(\theta), \tag{4.4}$$

and[1]

$$p = \max_\theta \phi_\theta. \tag{4.5}$$

[1]We choose the “maximal slope” parameter instead of the amplitude in the average Lagrangian.

The preference of fixing the slope of solution rather than the amplitude will be reasoned later. This functional is encountered in the problem of mathematical pendulum with negative mass. The Euler-Lagrange equation of this variational problem describes the mathematical pendulum in the upward position. In order to work on the familiar situation, it is convenient to change its sign, which leaves the Euler-Lagrange equation unchanged, to get

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2}m\phi_\theta^2-(\cos\phi-1)\right]\mathrm{d}\theta,\quad m=k^2-\omega^2.$$

This variational problem possesses an obvious first integral

$$\frac{1}{2}m\phi_\theta^2+(\cos\phi-1)=h.$$

This result completely coincides with Eq. (1.18) if we plug the following expressions

$$\left.\frac{\partial L}{\partial u}\right|_{u\to\phi}=-\sin\phi,\quad \left.\frac{\partial L}{\partial u_t}\right|_{u_t\to-\omega\phi_\theta}=-\omega\phi_\theta,\quad \left.\frac{\partial L}{\partial u_x}\right|_{u_x\to k\phi_\theta}=k\phi_\theta.$$

into it and change the sign.

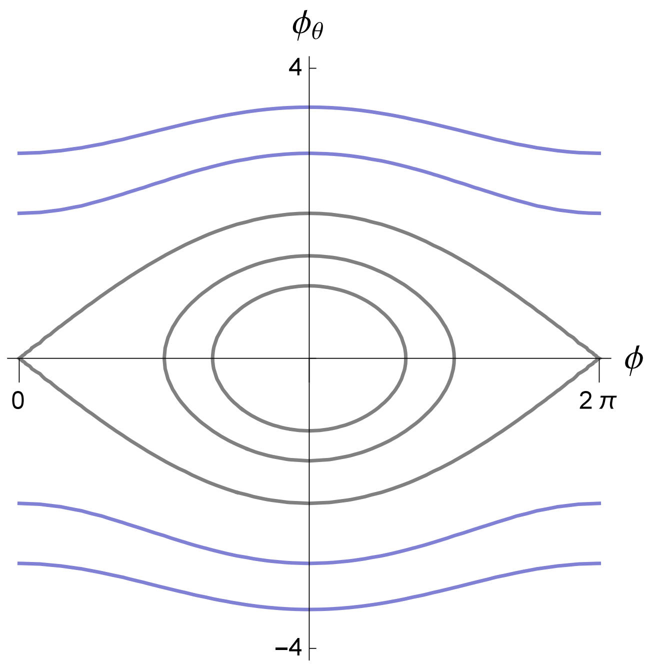


Figure 4.1: Phase portrait of a pendulum with $m=1$.

In Fig. 4.1 the corresponding phase portrait is plotted. Since the kink-type solution can, in principle, escalates from zero to a multiple of 2π, its amplitude cannot be limited to a fixed value 2π. In order to capture this scenario, the supplied energy should be high enough to boost the solution beyond 2π reflecting the open orbits in the phase portrait. If this case is considered, it does not make sense to fix the amplitude whose maximum is not achieved anywhere on one open trajectory. On the other hand, the maximal of slope appears as a good candidate to be fixed on the phase portrait because the slope achieves its maximum over one particular orbit. The periodicity conditions (4.4) are consequently deduced from the solutions corresponding to the open orbits. Using the first integral, we find the solution in terms of elliptic functions and then the average Lagrangian according to

$$\mathcal{L}=\frac{1}{2\pi}\int_0^{2\pi}m\phi_\theta^2\mathrm{d}\theta-h=\frac{1}{2\pi}\int_0^{2\pi}m\phi_\theta\mathrm{d}\phi-h. \tag{4.6}$$

Now, to find the explicit dependence of $\mathcal{L}$ on p and m, we make use of the definition (4.5). Since the maximal slope is achieved at $\theta = \pi$ (see Fig. 4.1), this condition implies that

$$\frac{1}{2}mp^2 - 2 = h. \tag{4.7}$$

We require $h \geq 0$, so $p \geq 2/\sqrt{m}$. Then, from the first integral it follows

$$\mathcal{L}(p,k,\omega) = \frac{\sqrt{2m}}{2\pi}\int_0^{2\pi}\sqrt{h-\cos\phi+1}\,\mathrm{d}\phi - h = \frac{\sqrt{2m}}{2\pi}F(h) - h, \tag{4.8}$$

where $F(h)$ is the function expressed in terms of the complete elliptic integral of the second kind

$$F(h) = 2\left[\sqrt{h}E\left(-\frac{2}{h}\right) + \sqrt{h+2}E\left(\frac{2}{h+2}\right)\right].$$

The dispersion relation $(1.19)_1$ can be written in the form

$$\frac{\sqrt{2m}}{2\pi}F'(h) - 1 = 0. \tag{4.9}$$

To derive the equation of slope modulation, let us compute first the partial derivatives

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial k} &= \frac{\sqrt{2}}{2\pi}\frac{m_k}{2\sqrt{m}}F(h) + \left[\frac{\sqrt{2m}}{2\pi}F'(h) - 1\right]h_k = \frac{\sqrt{2}}{2\pi}\frac{k}{\sqrt{m}}F(h),\\
\frac{\partial\mathcal{L}}{\partial \omega} &= \frac{\sqrt{2}}{2\pi}\frac{m_\omega}{2\sqrt{m}}F(h) + \left[\frac{\sqrt{2m}}{2\pi}F'(h) - 1\right]h_\omega = -\frac{\sqrt{2}}{2\pi}\frac{\omega}{\sqrt{m}}F(h),
\end{aligned} \tag{4.10}$$

where Eq. (4.9) has been used. Then we are ready to compute the partial derivatives

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial k} &= \frac{\sqrt{2}}{2\pi}F(h)\frac{mk_x - km_x/2}{m^{3/2}} + \frac{q}{2m}m_x k + \frac{1}{2}kq_x,\\
\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial \omega} &= -\frac{\sqrt{2}}{2\pi}F(h)\frac{m\omega_t - \omega m_t/2}{m^{3/2}} - \frac{q}{2m}m_t\omega - \frac{1}{2}\omega q_t,
\end{aligned}$$

in which we have used Eq. (4.7) and Eq. (4.9) in the form $F'(h) = 2\pi/\sqrt{2m}$ and $q = p^2$. Subtracting the last equation from the second last, we obtain

$$\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}\left(mk_x + m\omega_t - \frac{km_x}{2} - \frac{\omega m_t}{2}\right) + \frac{q}{m}\left(\frac{1}{2}m_x k + \frac{1}{2}m_t\omega\right) + \frac{1}{2}(kq_x + \omega q_t) = 0.$$

Using the definition $m = k^2 - \omega^2$, the expressions in the parentheses can be reduced further as follows

$$\begin{aligned}
J_1(k,\omega) &= mk_x + m\omega_t - \frac{km_x}{2} - \frac{\omega m_t}{2} = k^2\omega_t - \omega^2 k_x + k\omega\omega_x - k\omega k_t\\
&= k^2\omega_t + 2k\omega\omega_x - \omega^2 k_x,\\
J_2(k,\omega) &= \frac{1}{2}m_x k + \frac{1}{2}m_t\omega = k^2 k_x - k\omega\omega_x + k\omega k_t - \omega^2\omega_t\\
&= k^2 k_x - 2k\omega\omega_x - \omega^2\omega_t,
\end{aligned}$$

where the two last steps are carried out with the aid of the consistency condition $k_t + \omega_x = 0$. Ultimately, we arrive at the equation of slope modulation in the form

$$\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}(k^2\omega_t + 2k\omega\omega_x - \omega^2 k_x) + \frac{q}{m}(k^2 k_x - 2k\omega\omega_x - \omega^2\omega_t) + \frac{1}{2}(kq_x + \omega q_t) = 0. \tag{4.11}$$

Asymptotic solution to the equation of slope modulation From numerous numerical simulations of exact n-kink solutions of the sine-Gordon equation we know that at large time the kinks become non-interacting and propagate along the straight lines $x/t = \text{const.}$ Let us look for the phase in the following form

$$\theta(x,t) = g(\xi(x,t)), \quad \xi(x,t) = x/t.$$

According to this Ansatz, we have

$$\begin{aligned} k &= \frac{1}{t}g'(\xi), \quad \omega = \frac{1}{t}g'(\xi)\xi, \\ k_x = \frac{1}{t^2}g''(\xi), \quad \omega_x &= \frac{1}{t^2}\left[g''(\xi)\xi + g'(\xi)\right], \quad \omega_t = -\frac{1}{t^2}\left[g''(\xi)\xi^2 + 2g'(\xi)\xi\right]. \end{aligned} \tag{4.12}$$

It is now straightforward to check that $J_1(k,\omega) = 0$ by direction substitution, so Eq. (4.11) is reduced to

$$2q(k^2k_x - 2k\omega\omega_x - \omega^2\omega_t) + (k^2 - \omega^2)(kq_x + \omega q_t) = 0, \tag{4.13}$$

Substituting these formulas into Eq. (4.13), we obtain

$$2q[g''(\xi)(t^2 - x^2) - 2g'(\xi)xt] + g'(\xi)t^2(tq_x + xq_t) = 0, \tag{4.14}$$

which is the partial differential equation of first order in terms of $g'(\xi)$ and integrable by the method of characteristics. In the following we demonstrate the method of solution for this equation through three main steps.

Step 1 Let us first recognize the characteristic lines of this equation. Diving it by $g'(\xi)t^3$, we obtain

$$2q\left[\frac{h'(\xi)(1-\xi^2)}{h(\xi)t} - 2\frac{\xi}{t}\right] + \left(q_t\frac{x}{t} + q_x\right) = 0,$$

where the derivative of g has been replaced by a simple function h for convenience of derivation. According to the last term, the characteristic lines are defined by the following ordinary differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{t} = \xi(x,t),$$

whose solutions are implicitly given by

$$t^2 - x^2 = C^2 = \text{const.} \tag{4.15}$$

On the characteristic line we may consider q as a function of one variable which is realized as $Q(x) = q(x,t(x))$. Thus, on one specific line characterized by a certain constant the last equation can be cast down to

$$2Q(x)\left[\frac{h'(\xi)(1-\xi^2)}{h(\xi)t} - 2\frac{\xi}{t}\right] + \frac{\mathrm{d}Q}{\mathrm{d}x} = 0,$$

in which t and ξ must be interpreted as two functions of x according to

$$t(x) = \pm\sqrt{C^2 + x^2}, \quad \xi(x) = \pm\frac{x}{\sqrt{C^2 + x^2}}.$$

Let us work on the half plane corresponding the the “plus” signs of these functions.

Step 2 We solve the last equation by the method of integrating factor. The standard procedure requests us to compute the integral

$$I(x) = \int \left[\frac{h'(\xi)(1-\xi^2)}{h(\xi)t} - 2\frac{\xi}{t} \right] \mathrm{d}x.$$

We consider the integrand and reformulate it as follows

$$J = \frac{h'(\xi)(1-\xi^2) - 2\xi h(\xi)}{h(\xi)t} = \frac{h'(\xi)\xi_x/t^2 - 2h(\xi)/t^3 \times \mathrm{d}t/\mathrm{d}x}{h(\xi)/t^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{h(\xi)}{t^2}\right] \Big/ \left[\frac{h(\xi)}{t^2}\right].$$

Thus, the integrand can be finally rewritten as

$$J = \frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{h(\xi)}{t^2},$$

which yields the integrating factor

$$\exp 2I(x) = \frac{h(\xi(x))^2}{t(x)^4}.$$

Multiplication of the equation for $Q(x)$ by this factor transforms its left-hand side to a total derivative as follows

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Q\frac{h(\xi)^2}{t^4}\right] = 0,$$

and integration of this equation yields

$$Q(x) = q(x, t(x)) = \text{const} \times \frac{t^4}{h(\xi)^2} = \text{const} \times \frac{t^4}{g'(\xi)^2}.$$

Step 3 To recover the solution q as a function of two variables, we may consider t as a free variable by letting the constant in Eq. (4.15) change arbitrarily. Since this constant characterizing the set of characteristic lines, it must depends functionally on the expression $t^2 - x^2$ so that the final solution reads

$$q(x,t) = C(t^2 - x^2)^2 \frac{t^4}{g'(\xi(x,t))^2}.$$

The function of one variable C is nothing else but the function of integration, which is the counter part of the constant of integration resulting from integrating an ordinary differential equation.

Combining all the recent results, the maximal slope has just been obtained as

$$p(x,t) = \sqrt{q(x,t)} = C(t^2 - x^2)\frac{t^2}{g'(\xi(x,t))}. \tag{4.16}$$

As $g(\xi)$ describes the phase, its derivative $g'(\xi)$ can be identified with $2\pi\rho(\xi)$, where $\rho(\xi)$ is the density of kinks, or the number of kinks per unit length.

The unknown function $C(t^2 - x^2)$ should be determined from the dispersion relation (4.9). Using the solution given by (4.16) and formulas (4.12), we obtain

$$h = \frac{1}{2}mp^2 - 2 = \frac{1}{2}(t^2 - x^2)C(t^2 - x^2)^2 - 2.$$

Let say, we are on a particular orbit in the phase portrait. Then the train of kinks can be attained by letting this orbit tend to the separatrix, which can be captured in the limit $h \to 0$. More particularly, the exact fulfillment of the dispersion relation is warranted if h is of the order $m/2$ in this limit. Thus, if the solitons are exclusively under investigation, it follows that

$$C(t^2 - x^2) = \frac{2}{\sqrt{t^2 - x^2}},$$

and the final asymptotic formula for the maximal slope reads

$$p(x,t) = \frac{2t^2}{g'(\xi(x,t))\sqrt{t^2 - x^2}}. \tag{4.17}$$

The asymptotic formula describing the derivative of solution u_x or the slope of solution is therefore given by

$$s(x,t) = p(x,t)k(x,t) = \frac{2t}{\sqrt{t^2 - x^2}} = \frac{2}{\sqrt{1 - (x/t)^2}}. \tag{4.18}$$

An alternative solution method The above presentation follows the original flow of solving the modulation equations and the ultimate asymptotic law (4.18) turns out pretty simple. If we take into account the expression $c = x/t$ for our particular Ansatz, this law gives out the relation

$$s(x,t) = \frac{2}{\sqrt{1 - c(x,t)^2}}.$$

For that reason, it is natural to raise the question whether it is possible extract this kind of information from the equation of slope modulation itself. Let us start here from the reduced equation of slope modulation (4.13) and derive such relation. This equation and the consistency condition $k_t + \omega_x = 0$ can be together interpreted as a system for three parameters k, ω and p. Eliminating one parameter, k or ω, from this system by using the consistency condition, the remaining equation (4.13) can be considered to contain only two left parameters. Instead of these parameters we may choose the slope $s = p\,k$ and the velocity $c = \omega/k$ as the primary variables by certain manipulations. To demonstrate the above arguments, let us sketch some key steps.

To start with, we rewrite Eq. (4.13) as

$$qk(kk_x - \omega\omega_x) + q\omega(kk_t - \omega\omega_t) + \frac{1}{2}(k^2 - \omega^2)(kq_x + \omega q_t) = 0.$$

Using the relation $\omega = c\,k$ and recognizing the expressions in the two sets of parentheses as

$$kk_x - \omega\omega_x = \frac{1}{2}(k^2 - \omega^2)_x, \quad kk_t - \omega\omega_t = \frac{1}{2}(k^2 - \omega^2)_t,$$

this equation can be engendered to

$$\frac{1}{2}qk\left[k^2(1-c^2)\right]_x + \frac{1}{2}k^3(1-c^2)q_x + \frac{1}{2}q\omega\left[k^2(1-c^2)\right]_t + \frac{1}{2}\omega k^2(1-c^2)q_t = 0.$$

The first two terms of the left-hand side is simplified to

$$\begin{aligned}
L_1 &= \frac{1}{2}qk\left[k^2(1-c^2)\right]_x + \frac{1}{2}k^3(1-c^2)q_x \\
&= \frac{1}{2}qk\left[(k^2)_x(1-c^2) + k^2(1-c^2)_x\right] + \frac{1}{2}k^3(1-c^2)q_x \\
&= \frac{1}{2}k\left\{(1-c^2)\left[(k^2)_x q + q_x k^2\right] + qk^2(1-c^2)_x\right\} = \frac{1}{2}k\left[(1-c^2)k^2 q\right]_x.
\end{aligned}$$

The last two terms differ from these by a factor $\omega = c\,k$ and the partial derivative with respect to t, so it is straightforward to bring them to the form

$$L_2 = \frac{1}{2}q\omega\left[k^2(1-c^2)\right]_t + \frac{1}{2}\omega k^2(1-c^2)q_t = \frac{1}{2}\omega\left[(1-c^2)k^2q\right]_t.$$

Substituting L_1 and L_2 into the last equation, dividing both sides by k and using the variable $s^2 = k^2q$, we obtain

$$\left[(1-c^2)s^2\right]_x + c\left[(1-c^2)s^2\right]_t = 0.$$

It is now evident that the obvious solution of this equation is given by a simple algebraic relation

$$(1-c^2)s^2 = \delta^2 \quad \Rightarrow \quad s = \frac{\delta}{\sqrt{1-c^2}},$$

where δ is an arbitrary constant. Once again, the energy level h can be related to this result by reformulating it as

$$h = \frac{1}{2}k^2(1-c^2)p^2 - 2 = \frac{1}{2}(1-c^2)s^2 - 2 = \frac{\delta^2}{2} - 2.$$

In the limit $h \to 0$ we have $\delta^2 = 4$ and then $s = 2/\sqrt{1-c^2}$, which is consistent with the preceding result.

To close this paragraph, we also present here a mathematical reasoning for the Ansatz of the phase variable based on the heuristic condition

$$J_1(k,\omega) = k^2\omega_t + 2k\omega\omega_x - \omega^2k_x = 0.$$

In terms of the phase variable it can be written as

$$J_1(\theta) = -\left(\theta_x^2\theta_{tt} - 2\theta_x\theta_t\theta_{xt} + \theta_t^2\theta_{xx}\right) = 0.$$

We analyze the left-hand side of this equation and rewrite it as follows

$$J_1(\theta) = -\left[\theta_x^3\frac{\theta_{tt}\theta_x - \theta_t\theta_{xt}}{\theta_x^2} + \theta_t^3\frac{\theta_{xx}\theta_t - \theta_x\theta_{xt}}{\theta_t^2}\right] = -\left[\theta_x^3\frac{\partial}{\partial t}\left(\frac{\theta_t}{\theta_x}\right) + \theta_t^3\frac{\partial}{\partial x}\left(\frac{\theta_x}{\theta_t}\right)\right].$$

By setting $\zeta = \theta_t/\theta_x$ and dividing both sides by θ_x^3, the last equation is reduced to

$$\zeta^3(\zeta^{-1})_x + \zeta_t = 0,$$

or equivalently

$$\zeta_t - \zeta\zeta_x = 0.$$

This equation admits a simple solution $\zeta(x,t) = -x/t$, which by definition allows us to write

$$\frac{\theta_t}{\theta_x} = -\frac{x}{t} \quad \Rightarrow \quad \theta_t + \frac{x}{t}\theta_x = 0.$$

The reduced equation for the phase θ can be solved by the method of characteristics to give

$$\theta(x,t) = g(x/t),$$

where g is an arbitrary function of one variable. This eventual solution for θ justifies our heuristic Ansatz mathematically.

Validation of the modulation solution through comparison Let us compare the asymptotic solution with the exact solution of the SG equation. We note that for slowly varying wave packet and to the first approximation, we have

$$u_x \approx \phi_\theta \theta_x = p\,k.$$

Since the maximum of ϕ_θ in one wavelength is chosen to be p, we expect that $s = p\,k$ should serve as the asymptotic envelope for the exact slope of kink-type solution. In Fig. 4.2 the slope of an exact 5-kink solution of the SG equation is plotted as compared to this modulation solution at large time while in Fig. 4.3 the slope of an exact 6-kink solution is plotted in two different frames due to the difficulty of visualization. From these figures it is seen that, at large time, the curve given by Eq. (4.18) can sever as the asymptotic envelope for the slope of kinks.

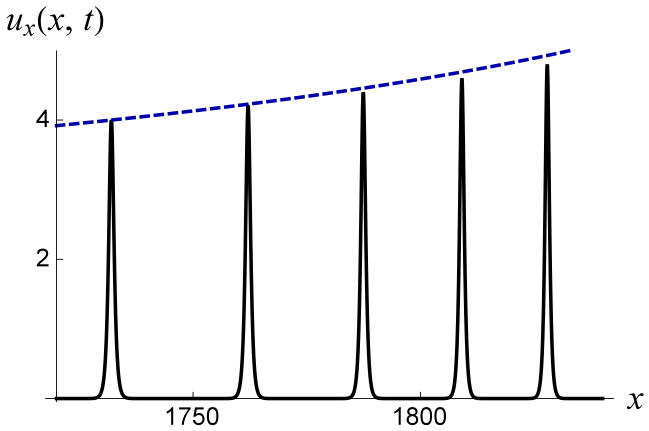


Figure 4.2: Slope of a wave packet of five kinks versus its slope modulation at large time.

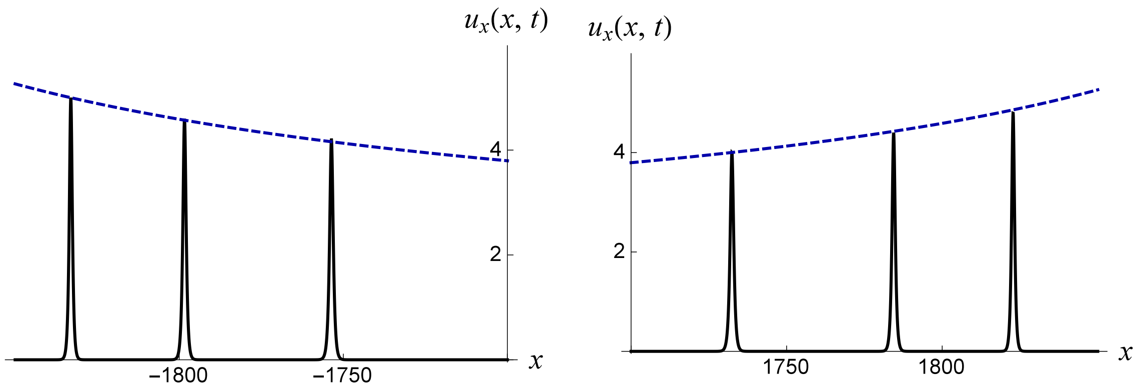


Figure 4.3: Slope of a wave packet of six kinks and its slope modulation are traveling in time. Three kinks move to the minus infinity (left) while the other three to the infinity (right).

4.2 Slope modulation for waves governed by multi-dimensional sine-Gordon equation

It turns out that the modulation equations for the n-dimensional sine-Gordon equation, where $n = 2$ or $n = 3$, can be generalized essentially from the algebraic structure of the average Lagrangian for the one-dimensional version. Let us examine now the Lagrangian associated with the n-dimensional version

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\nabla u \cdot \nabla u - \Phi(u), \quad \Phi(u) = 1 - \cos u.$$

Upon the substitutions

$$u \to \phi, \quad u_t \to -\omega\phi_\theta, \quad \nabla u \to \mathbf{k}\phi_\theta$$

into the Lagrangian, we obtain

$$L = \frac{1}{2}(\omega^2 - \mathbf{k}\cdot\mathbf{k})\phi_\theta^2 - (1 - \cos\phi).$$

With reference to the last Lagrangian after substitution, the new one differs from it by only one constant. To be precise, the constant k^2 is replaced by the scalar product $\mathbf{k}\cdot\mathbf{k}$, where $\mathbf{k} = (k_1, \ldots, k_n)$. Consequently, the strip problem and the first integral do not change their algebraic structure except that the pseudo-mass should be altered to

$$m = \mathbf{k}\cdot\mathbf{k} - \omega^2. \tag{4.19}$$

Before carrying on the procedure, it is practical to examine up to which extent we might keep our previous calculation and how the formulas could be modified. Keeping in mind the new change of definition (4.19), the average Lagrangian (4.8) and henceforth the dispersion relation (4.9) can be apparently kept. Then the derivatives (4.10) remain essentially unchanged except that the wave vector replaces the wave number. That is, we have

$$\frac{\partial\mathcal{L}}{\partial\mathbf{k}} = \frac{\sqrt{2}}{2\pi}\frac{\mathbf{k}}{\sqrt{m}}F(h), \quad \frac{\partial\mathcal{L}}{\partial\omega} = -\frac{\sqrt{2}}{2\pi}\frac{\omega}{\sqrt{m}}F(h).$$

The most significant adaptation is probably the appearance of the equation of slope modulation due to the involvement of the vector-values function. Thus let us repeat the calculation of the partial derivatives of $\partial\mathcal{L}/\partial\mathbf{k}$, $\partial\mathcal{L}/\partial\omega$ with respective $\mathbf{x}$ and t, respectively, in the following

$$\begin{aligned}
\nabla\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{k}} &= \frac{\sqrt{2}}{2\pi}\left[\frac{\nabla\cdot\mathbf{k}}{\sqrt{m}}F(h) - \frac{F(h)}{2m^{3/2}}\mathbf{k}\cdot\nabla m + \frac{F'(h)}{\sqrt{m}}\mathbf{k}\cdot\nabla h\right] \\
&= \frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}\left[m\nabla\cdot\mathbf{k} - \frac{1}{2}\mathbf{k}\cdot\nabla m\right] + \frac{1}{m}\mathbf{k}\cdot\nabla h, \\
\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\omega} &= -\frac{\sqrt{2}}{2\pi}\left[\frac{\omega_t}{\sqrt{m}}F(h) - \frac{F(h)}{2m^{3/2}}\omega m_t + \frac{F'(h)}{\sqrt{m}}\omega h_t\right] \\
&= -\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}\left[m\omega_t - \frac{1}{2}\omega m_t\right] - \frac{1}{m}\omega h_t,
\end{aligned}$$

where the dispersion relation $F'(h) = 2\pi/\sqrt{2m}$ has been inserted. Substituting these derivatives into equation $(1.17)_2$ and using once again the denotation $q = p^2$, we get

$$\begin{aligned}
\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}\left(m\nabla\cdot\mathbf{k} + m\omega_t - \frac{1}{2}\mathbf{k}\cdot\nabla m - \frac{1}{2}\omega m_t\right) + \frac{q}{m}\left(\frac{1}{2}\mathbf{k}\cdot\nabla m + \frac{1}{2}\omega m_t\right) \\
+\frac{1}{2}\left(\mathbf{k}\cdot\nabla q + \omega q_t\right) = 0.
\end{aligned}$$

Using the definition (4.19), it is straightforward to deduce

$$\nabla m = 2(\nabla \mathbf{k}\cdot\mathbf{k} - \omega\nabla\omega), \quad m_t = 2(\mathbf{k}\cdot\mathbf{k}_t - \omega\omega_t),$$

where

$$\nabla\mathbf{k} = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\begin{pmatrix}k_1 & k_2 & k_3\end{pmatrix} = \begin{pmatrix}k_{1,x} & k_{2,x} & k_{3,x}\\ k_{1,y} & k_{2,y} & k_{3,y}\\ k_{1,z} & k_{2,z} & k_{3,z}\end{pmatrix}.$$

Keeping in mind the consistency condition $\mathbf{k}_t + \nabla\omega = 0$, we compute the coefficients of the first two terms in the last equation as follows

$$\begin{aligned}
J_1(\mathbf{k},\omega) &= m\nabla\cdot\mathbf{k} + m\omega_t - \frac{1}{2}\mathbf{k}\cdot\nabla m - \frac{1}{2}\omega m_t\\
&= (\mathbf{k}\cdot\mathbf{k}-\omega^2)\nabla\cdot\mathbf{k} + (\mathbf{k}\cdot\mathbf{k}-\omega^2)\omega_t - \mathbf{k}\cdot(\nabla\mathbf{k}\cdot\mathbf{k}-\omega\nabla\omega) - \omega(\mathbf{k}\cdot\mathbf{k}_t - \omega\omega_t)\\
&= (\mathbf{k}\cdot\mathbf{k})\omega_t - \omega^2\nabla\cdot\mathbf{k} + 2\omega\mathbf{k}\cdot\nabla\omega - \mathbf{k}\cdot\nabla\mathbf{k}\cdot\mathbf{k} + (\mathbf{k}\cdot\mathbf{k})\nabla\cdot\mathbf{k},\\
J_2(\mathbf{k},\omega) &= \frac{1}{2}\mathbf{k}\cdot\nabla m + \frac{1}{2}\omega m_t = \mathbf{k}\cdot(\nabla\mathbf{k}\cdot\mathbf{k}-\omega\nabla\omega) + \omega(\mathbf{k}\cdot\mathbf{k}_t-\omega\omega_t)\\
&= \mathbf{k}\cdot\nabla\mathbf{k}\cdot\mathbf{k} - 2\omega\mathbf{k}\cdot\nabla\omega - \omega^2\omega_t.
\end{aligned}$$

In summary, we have just derived the equation of slope modulation in the form

$$\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}J_1(\mathbf{k},\omega) + \frac{q}{m}J_2(\mathbf{k},\omega) + \frac{1}{2}(\mathbf{k}\cdot\nabla q + \omega q_t) = 0, \tag{4.20}$$

with J_1 and J_2 being given by the above formulas.

Ring-shape solution to the equation of slope modulation For ring-shape solution with $\theta = \theta(r,t)$, the equation of slope modulation becomes

$$\frac{\sqrt{2}}{2\pi}\frac{F(h)}{m^{3/2}}J_1(k,\omega) + \frac{q}{m}J_2(k,\omega) + \frac{1}{2}(kq_r + \omega q_t) = 0, \tag{4.21}$$

where $k = \theta_r$ and

$$\begin{aligned}
J_1(k,\omega) &= k^2\omega_t - \omega^2 k_r + 2\omega k\omega_r + k(k^2-\omega^2)\frac{n_d-1}{r},\\
J_2(k,\omega) &= k^2 k_r - 2\omega k\omega_r - \omega^2\omega_t.
\end{aligned}$$

Based on the result on the slope modulation of the one-dimensional sine-Gordon equation [94], we propose here the heuristic condition

$$J_1(k,\omega) = 0. \tag{4.22}$$

In terms of the phase variable θ, the expression J_1 can be written as

$$J_1(\theta) = -\left[\theta_r^3\frac{\partial}{\partial t}\left(\frac{\theta_t}{\theta_r}\right) + \theta_t^3\frac{\partial}{\partial r}\left(\frac{\theta_r}{\theta_t}\right)\right] + \theta_r^3\left(1-\frac{\theta_t^2}{\theta_r^2}\right)\frac{n_d-1}{r}.$$

Thus, equation (4.22) is reduced to

$$c_t + cc_r + (1-c^2)\frac{n_d-1}{r} = 0,$$

where $c = -\theta_t/\theta_r$ is the phase velocity. We seek its solution in the following form

$$c(r,t) = \lambda(\xi(r,t)), \quad \xi = \frac{t - t_r}{r}, \tag{4.23}$$

where t_r is an arbitrary reference time. Substitution of this Ansatz into the last equation leads to

$$\lambda'(\xi)\left[1 - \xi\lambda(\xi)\right] + (n_d - 1)\left[1 - \lambda(\xi)^2\right] = 0.$$

At the first sight, it is difficult to fully integrate this equation as it is nonlinear in λ. However, it is linear in ξ so that the theorem of inverse function works to its advantage. Accordingly, we may rewrite it as follows

$$(n_d - 1)(1 - \lambda^2)\xi'(\lambda) - \xi(\lambda)\lambda + 1 = 0,$$

which is the non-homogeneous first-order linear ordinary differential equation. Two versions of this equation are

$$(1 - \lambda^2)\xi'(\lambda) - \xi(\lambda)\lambda = -1, \tag{4.24}$$

and

$$2(1 - \lambda^2)\xi'(\lambda) - \xi(\lambda)\lambda = -1, \tag{4.25}$$

corresponding to the two-dimensional and three-dimensional cases, respectively. These two equations can be fully integrated using the standard method of integrating factor. The general solution of equation (4.24) reads

$$\xi_{2\mathrm{D}}(\lambda) = \frac{C - \arcsin\lambda}{\sqrt{1 - \lambda^2}},$$

and for equation (4.25)

$$\xi_{3\mathrm{D}}(\lambda) = \frac{C - \frac{1}{2}\lambda \times {}_2F_1\left[\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, \lambda^2\right]}{\sqrt[4]{1 - \lambda^2}},$$

where C is an arbitrary integration constant and $0 < \lambda < 1$ for real solutions. For the purpose of numerical simulations, let us pose the initial condition $\lambda(0) = 0$, or equivalently $\xi(0) = 0$, for both last equations. This condition implies that the wave stops at the instant $t^* = t_r$ when $\xi = 0$ and the velocity c vanishes just right before it returns according to the negative velocity. This condition clears out the integration constants in both solution formulas, yielding finally

$$\xi_{2\mathrm{D}}(\lambda) = -\frac{\arcsin\lambda}{\sqrt{1 - \lambda^2}}, \quad \xi_{3\mathrm{D}}(\lambda) = -\frac{1}{2}\lambda \times {}_2F_1\left[\frac{3}{4}, 1, \frac{3}{2}, \lambda^2\right]. \tag{4.26}$$

Thus, there are two regimes of wave propagation corresponding to $\xi < 0$ and $\xi > 0$. In the former case $c > 0$, so the waves propagate to infinity and stop at $t = t_r$. In the latter case $c < 0$ and the waves should return to the origin. Note that the phase velocity $c(\xi)$ always lies in the range $(-1, 1)$ and goes to -1 as ξ tends to infinity.

With $c = \lambda(\xi)$ satisfying equation (4.22), the equation of slope modulation for $q = p^2$

$$\frac{q}{m}(k^2 k_r - 2\omega k\omega_r - \omega^2\omega_t) + \frac{1}{2}(kq_r + \omega q_t) = 0$$

in this case is comparable to Eq.(4.13) except for the replacement of the spatial variable x by r. Thus, the same mathematical manipulations as before bring Eq. (4.21) to

$$[(1-c^2)s^2]_r + c[(1-c^2)s^2]_t = 0,$$

where $s = p\,k$. Again, this variable designates precisely the slope of wave packet, or the partial derivative of the solution with respect to r according to

$$u_r = \phi_\theta \theta_r = p\,k.$$

It follows immediately that this equation admits a simple solution

$$(1-c^2)s^2 = mp^2 = \delta^2.$$

As the solution of kink-type arises in the limit $h \to 0$, we have $\delta^2 = 4$ and thus

$$s(r,t) = \frac{2}{\sqrt{1-c(r,t)^2}},$$

where $c(r,t)$ must be computed according to equations (4.23) and (4.26). In this limit the above solution ensures that the dispersion relation (4.9) is asymptotically fulfilled.

Comparison of the modulation solution with the numerical solution For the initial conditions, we employ here the wave packet of five separated kinks as the additive superposition of each kink

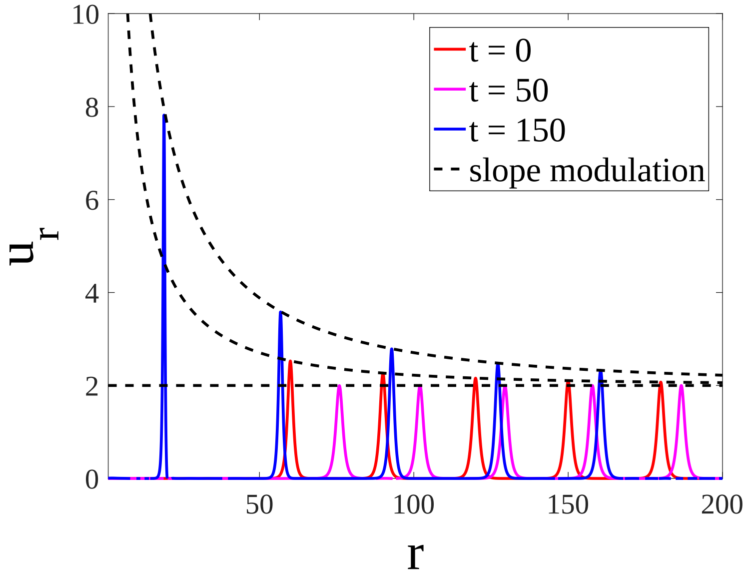
$$u_0(r) = \sum_{j=1}^{5} 4\arctan\left[\exp\left(\frac{r-r_{0j}}{\sqrt{1-c_{0j}^2}}\right)\right],$$

$$v_0(r) = \sum_{j=1}^{5} -\frac{2c_{0j}}{\sqrt{1-c_{0j}^2}}\operatorname{sech}\left(\frac{r-r_{0j}}{\sqrt{1-c_{0j}^2}}\right),$$

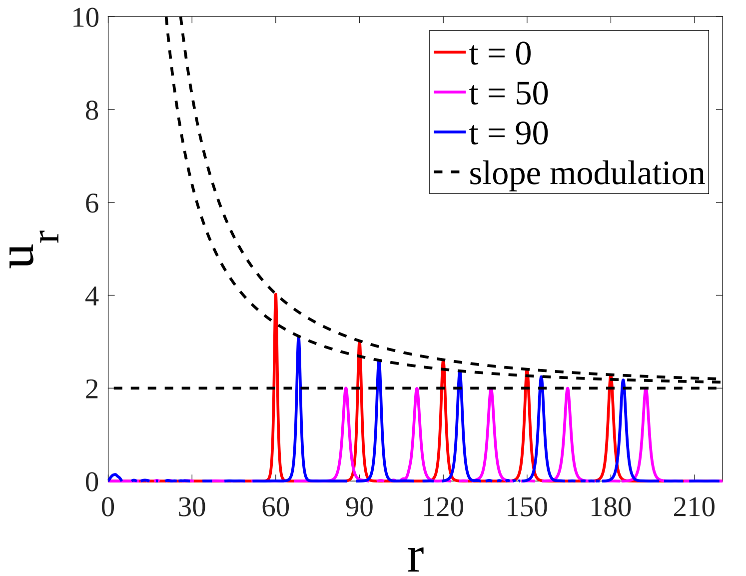
where r_{0j} and c_{0j} are the initial expansion of radius and the initial velocity of each kink, respectively. In Fig. 4.4 the slope of the wave packet of separated five kinks together with its modulation are plotted at different time instants. These numerical results demonstrate the validation of the modulation solutions and vice versa. It is remarkable that the slope modulation s is an even function in ξ. This circumstance can be used to find the slope modulation of multi-kink after the whole train is reflected from the origin and propagates again to infinity. In this case we should take $\xi_+ = -\xi_-$, or $t_+ = -t_- + 2t_r$, where t_- is the time when the first kink reaches the origin, and t_+ is the time when the last kink is about to leave the origin after reflection. Note, however, that if energy dissipation due to the Peierls barriers is taken into account, the kinks must approach some equilibrium configuration after several oscillations.

Bibliographical remarks

This chapter essentially distributes the works in [94–96]. The presentation here elaborates the mathematical ideas and derivation in great detail. In fact, several points have been improved in this report as compared to the earlier publications. For simulation we used the exact N-kink solution given by Hirota in [76] and a numerical scheme proposed by the author in [96].



(a) 2D case.



(b) 3D case.

Figure 4.4: The slope of well separated 5-kink solution and its envelope propagate in time.

5 Supplementary materials

This chapter is to provide the missing link to the main content of the entire report. In fact, it collects my discrete dissemination of work that has not been done before. The material provided here is of absolute necessity for the completeness purposes aiming at a self-studied report.

5.1 Construction of Lagrangian density

As we have seen from previous chapters, to develop the modulation theory for nonlinear dispersive waves using the asymptotic-variational method, the very first ingredient of each problem is the possibility of restating the governing equations in a variational problem. However, an evolution equation does not have to admit a direct variational statement but another particular derivative. The subject has grown up from the beautiful work of Emmy Noether and then actively developed based on the Lie group theory. In order for convenience of later derivation, we provide here a short summary of results from the theory and its application to the evolution equations [97]. We will restrict ourselves to the one-dimensional wave equations so that it suffices to consider smooth functions of spatial-time coordinates $\mathbf{r} = (x, t)$. Let us denote the vector of dependent variables by $\mathbf{q} = (u_1, \dots, u_m)$ and the k^{th} partial derivative of a scalar function $u(x, t)$ by

$$u_\mu = \frac{\partial^k u(x, t)}{\partial x^{j_1} \partial t^{j_2}},$$

where $\mu = (j_1, j_2)$ is a vector of two non-negative integer components satisfying

$$j_1 + j_2 = k.$$

By denotation $P(\mathbf{r}, \mathbf{q})$ we mean a smooth function depending on $\mathbf{r}$ and derivatives of $\mathbf{q}$ with respect to $\mathbf{r}$ up to l^{th} order. Let $\mathbf{P}(\mathbf{r}, \mathbf{q}) = (P_1(\mathbf{r}, \mathbf{q}), \dots, P_n(\mathbf{r}, \mathbf{q}))$ be an n-tuple of smooth functions. The Fréchet derivative of $\mathbf{P} = (P_i)$ is an $n \times m$ differential operator with entries being computed by

$$[D_P]_{ij} = \sum_\mu \frac{\partial P_i}{\partial_\mu q_j} D_\mu, \quad i = 1, \dots, n, \quad j = 1, \dots, m,$$

where D_μ is the total derivative. Its adjoint D_P^* can be, by definition, computed in accordance with

$$[D_P^*]_{ij} = \sum_\mu (-D)_\mu \cdot \frac{\partial P_j}{\partial_\mu q_i},$$

where the differential operator $(-D)_\mu A_\mu$ is interpreted as

$$((-D)_\mu \cdot A_\mu)(B) = \sum_\mu (-D)_\mu (A_\mu B)$$

for all smooth functions $B(\mathbf{r}, \mathbf{q})$ depending on $\mathbf{r}$, $\mathbf{q}$ and the derivatives $\partial_\mu \mathbf{q}$. It is ready to recall a theorem concerning the inverse problem of the variational statement.

Theorem on the construction of Lagrangian Let $\mathbf{P}(\mathbf{r}, \mathbf{q})$ be defined over a vertically star-shaped region. Then the governing system $\mathbf{P} = 0$ admits some variational problem

$$\delta \iint L(\mathbf{r}, \mathbf{q}, \partial_\mu \mathbf{q}) \mathrm{d}x \mathrm{d}t = 0$$

if and only if the Fréchet derivative of the vector-valued function $\mathbf{P}(\mathbf{r}, \mathbf{q})$ is self-adjoint, namely $D_P^* = D_P$. When this is the case, the corresponding Lagrangian can be explicitly constructed using the homotopy formula

$$L(\mathbf{r}, \mathbf{q}, \partial_\mu \mathbf{q}) = \int_0^1 \mathbf{q} \cdot \mathbf{P}(\mathbf{r}, \lambda \mathbf{q}) \mathrm{d}\lambda. \tag{5.1}$$

In the rest of this section we shall be consistent with a simple structure in each paragraph. It will be verified if there is any Lagrangian to give rise of a given evolution equation or a system. If not, an alteration is made on the equation and the same analysis is then repeated. Only until the theorem is fulfilled, the corresponding Lagrangian will be derived.

The Korteweg-de Vries equation Let us rewrite the KdV equation by using the local functional

$$Q(u) = u_t + 6uu_x + u_{xxx} = 0.$$

Its Fréchet derivative is given by

$$D_Q = D_t + 6(u_x + uD_x) + D_x^3 = 0,$$

whose adjoint is equal to

$$D_Q^* = (-D)_t + 6(u_x + (-D)_x u) + (-D)_x^3 = -D_t - 6uD_x - D_x^3 \neq D_Q.$$

Hence, the KdV equation does not admit a direct variational statement. However, we can associate it with a variational principle by letting $u = \eta_x$ and rewriting the equation as

$$P(\eta) = \eta_{xt} + 6\eta_x \eta_{xx} + \eta_{xxxx} = 0.$$

Then it is ready to compute the Fréchet derivative

$$D_P = D_x D_t + 6(\eta_x D_x^2 + \eta_{xx} D_x) + D_x^4,$$

and its adjoint

$$\begin{aligned} D_P^* &= (-D)_x(-D)_t + 6[(-D)_x^2 \eta_x + (-D)_x \eta_{xx}] + (-D)_x^4 \\ &= D_x D_t + 6[(\eta_{xxx} + 2\eta_{xx} D_x + \eta_x D_x^2) - (\eta_{xxx} + \eta_{xx} D_x)] + D_x^4 \\ &= D_x D_t + 6(\eta_{xx} D_x + \eta_x D_x^2) + D_x^4 = D_P. \end{aligned}$$

Thus, D_P is self-adjoint and the Lagrangian can be computed as follows

$$\begin{aligned} L &= \int_0^1 \eta P(\lambda\eta) \mathrm{d}\lambda = \int_0^1 \eta(\lambda \eta_{xt} + 6\lambda^2 \eta_x \eta_{xx} + \lambda \eta_{xxxx}) \mathrm{d}\lambda \\ &= \frac{1}{2} \eta \eta_{xt} + 2\eta \eta_x \eta_{xx} + \frac{1}{2} \eta \eta_{xxxx}. \end{aligned}$$

In order to remove the higher-order derivatives in this Lagrangian, we decompose it into two parts as follows

$$L = -\frac{1}{2}\eta_t\eta_x - \eta_x^3 + \frac{1}{2}\eta_{xx}^2 + \nabla\cdot\mathbf{F}, \quad \mathbf{F} = \left(\frac{1}{2}\eta\eta_{xxx} + \eta\eta_x^2 - \frac{1}{2}\eta_x\eta_{xx}, \frac{1}{2}\eta\eta_x\right).$$

In the above formula we have used the conventional definition of divergence operator acting on the vector field $\mathbf{F}(x,t) = (F_1(x,t), F_2(x,t))$

$$\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial t},$$

and the time variable t is equivalently considered as one space coordinate in our context. Keeping in mind that removing a divergence term in the Lagrangian leaves the Euler-Lagrange equations unchanged, we can safely replace it with

$$L = -\frac{1}{2}\eta_x\eta_t - \eta_x^3 + \frac{1}{2}\eta_{xx}^2. \tag{5.2}$$

The divergence $\nabla\cdot\mathbf{F}$ is actually called null Lagrangian.

The scalar Boussinesq equation We consider specifically the following prototype of the BSQ equation

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0.$$

The corresponding local functional reads

$$Q(u) = u_{tt} - u_{xx} - 12(uu_{xx} + u_x^2) - u_{xxxx},$$

whose Fréchet derivative is given by

$$D_Q = D_t^2 - D_x^2 - 12(u_{xx} + 2u_xD_x + uD_x^2) - D_x^4 = 0.$$

Once again its adjoint is not equal the derivative due to

$$\begin{aligned} D_Q^* &= (-D)_t^2 - (-D)_x^2 - 12[u_{xx} + 2(-D)_xu_x + (-D)_x^2u] - (-D)_x^4 \\ &= D_t^2 - D_x^2 - 12[u_{xx} - 2(u_{xx} + u_xD_x) + u_{xx} + 2u_xD_x + uD_x^2] - D_x^4 \\ &= D_t^2 - D_x^2 - 12uD_x^2 - D_x^4 \neq D_Q. \end{aligned}$$

Despite the fact that the BSQ equation does not have a Lagrangian, we can still find one for its modification by using the same strategy as in the last paragraph. That is, we introduce the potential function by $u = \eta_x$ into the BSQ equation and then integrate it once with respect to x to obtain

$$P(\eta) = \eta_{tt} - \eta_{xx} - 12\eta_x\eta_{xx} - \eta_{xxxx} = 0,$$

in which the integration constant has been chosen to be zero. We compute the Fréchet derivative and its adjoint

$$\begin{aligned} D_P &= D_t^2 - D_x^2 - 12(\eta_{xx}D_x + \eta_xD_x^2) - D_x^4, \\ D_P^* &= (-D)_t^2 - (-D)_x^2 - 12[(-D)_x\eta_{xx} + (-D)_x^2\eta_x] - (-D)_x^4 \\ &= D_t^2 - D_x^2 - 12(-\eta_{xxx} - \eta_{xx}D_x + \eta_{xxx} + 2\eta_{xx}D_x + \eta_xD_x^2) - D_x^4 \\ &= D_t^2 - D_x^2 - 12(\eta_{xx}D_x + \eta_xD_x^2) - D_x^4 = D_P. \end{aligned}$$

The Fréchet derivative is self-adjoint and hence the Lagrangian is computed as follows

$$L = \int_0^1 \eta P(\lambda\eta)\mathrm{d}\lambda = \int_0^1 \eta[\lambda\eta_{tt} - \lambda\eta_{xx} - 6\lambda^2(\eta_x^2)_x - \lambda\eta_{xxxx}]\mathrm{d}\lambda$$
$$= -\left(\frac{1}{2}\eta_t^2 - \frac{1}{2}\eta_x^2 + 2\eta_x^3 + \frac{1}{2}\eta_{xx}^2\right) + \nabla\cdot\mathbf{F},$$

where

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(-\frac{1}{2}\eta\eta_x - \frac{1}{2}\eta\eta_{xxx} + \frac{1}{2}\eta_x\eta_{xx} + \eta\eta_x^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\eta\eta_t\right). \tag{5.3}$$

Removing the divergence term as the null Lagrangian, we may choose the effective Lagrangian as

$$L = \frac{1}{2}\eta_t^2 - \frac{1}{2}\eta_x^2 + 2\eta_x^3 + \frac{1}{2}\eta_{xx}^2. \tag{5.4}$$

The Benjamin-Ono equation As the name suggests, the Benjamin-Ono (BO) equation was firstly derived by Benjamin to describe the internal deep stratified water waves and was later re-investigated by Ono [81, 82]. The equation incorporates a non-local effect due to an integral transformation and is given by

$$u_t + 2uu_x + \mathrm{H}u_{xx} = 0, \tag{5.5}$$

where

$$\mathrm{H}u(x) = \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{u(y)}{y-x}\mathrm{d}y$$

is the Hilbert transform and the dash line over the integral sign indicates the Cauchy principle value. This transform operator possesses many interesting properties, among which we shall need the three followings to derive the corresponding Lagrangian.

1. Hilbert transform is a linear operator

$$\mathrm{H}(\alpha u + \beta v) = \alpha\mathrm{H}u + \beta\mathrm{H}v.$$

2. The Hilbert operator and the differential operator commute

$$\mathrm{H}(u_x) = (\mathrm{H}u)_x.$$

3. The Hilbert transform is an anti-self adjoint operator relative to the duality pairing between $L^p(\mathbb{R})$ and the dual space $L^q(\mathbb{R})$, where p and q are the Hölder conjugates, defined by $1/p + 1/q = 1$, and $1 < p, q < \infty$. If we focus ourselves on the familiar Lebesgue space $L^2(\mathbb{R})$, this property in mathematical formulation means

$$\int_{-\infty}^{\infty} \mathrm{H}u(x)v(x)\mathrm{d}x = -\int_{-\infty}^{\infty} u(x)\mathrm{H}v(x)\mathrm{d}x.$$

We see that the first two terms of Eq. (5.5) bear heavy resemblance to the counterparts of the KdV equation. This suggests us to reuse our “favorite” change of dependent variable $u = \eta_x$ to obtain

$$P(\eta) = \eta_{xt} + 2\eta_x\eta_{xx} + \mathrm{H}\eta_{xxx} = 0.$$

Unfortunately, we cannot apply the given theorem entirely to this equation owning to the last term albeit the rest part does not make any trouble. For this reason let us try to handle this term separately in the following sequence of operations. Multiplying it by the ‘virtual displacement’ $\delta\eta$ and integrate over the who domain, we obtain

$$P_s = \iint \mathrm{H}\eta_{xxx}\,\delta\eta\,\mathrm{d}x\mathrm{d}t.$$

Integrating the last term by parts, taking into account the vanishing boundary conditions at infinity and using the linearity and self-adjointness properties, we can transform it to

$$\begin{aligned} P_s &= -\iint \mathrm{H}\eta_{xx}\delta\eta_x\mathrm{d}x\mathrm{d}t = -\frac{1}{2}\iint \mathrm{H}\eta_{xx}\delta\eta_x\mathrm{d}x\mathrm{d}t - \frac{1}{2}\iint \mathrm{H}\eta_{xx}\delta\eta_x\mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\iint \mathrm{H}\eta_{xx}\delta\eta_x\mathrm{d}x\mathrm{d}t + \frac{1}{2}\iint (\delta\mathrm{H}\eta_x)\eta_{xx}\mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\iint \mathrm{H}\eta_{xx}\delta\eta_x\mathrm{d}x\mathrm{d}t - \frac{1}{2}\iint \delta\mathrm{H}\eta_{xx}\eta_x\mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\delta\left(\iint \mathrm{H}\eta_{xx}\eta_x\mathrm{d}x\mathrm{d}t\right). \end{aligned}$$

Combining this formula with the standard procedure applied to the first part of the above equation, we have eventually established the Lagrangian for the BO equation as follows

$$L = \frac{1}{2}\eta_t\eta_x + \frac{1}{3}\eta_x^3 + \frac{1}{2}\eta_x\mathrm{H}\eta_{xx}, \tag{5.6}$$

where the minus sign has been eliminated for every term.

5.2 A numerical scheme and some theoretical aspects for the cylindrically and spherically symmetric sine-Gordon equations

The two-dimensional sine-Gordon equation in the polar coordinate (r, ϕ) can be rewritten as

$$u_{tt} - \left(\frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\phi\phi}\right) + \sin u = 0,$$

and the three-dimensional sine-Gordon equation in the spherical coordinate (r, θ, ϕ) reads

$$u_{tt} - \left(\frac{2}{r}u_r + u_{rr} + \frac{\sin\theta\, u_{\theta\theta} - \cos\theta\, u_\theta}{r^2\sin\theta} + \frac{u_{\phi\phi}}{r^2\sin^2\theta}\right) + \sin u = 0.$$

In this material we present the results in [**?**] and consider the rotationally and spherically symmetric solutions that are independent of the angular coordinate so that the unknown function depends only on two variables, that is $u = u(r,t)$, and satisfies

$$u_{tt} - \left(\frac{n_d - 1}{r}u_r + u_{rr}\right) + \sin u = 0,$$

where n_d is the number of dimension under consideration. It is clear that the difference between this equation in two- and three-dimensional space is only the constant n_d, so it is sufficient to develop a numerical scheme for the two-dimensional case and then to modify the appropriate coefficient to adapt to the three-dimensional case. For that reason, we shall exclusively derive the formulation in regard with the symmetric two-dimensional sine-Gordon equation with $n_d = 2$

$$u_{tt} - \left(\frac{1}{r}u_r + u_{rr}\right) + \sin u = 0. \tag{5.7}$$

We are going to integrate this equation numerically in a rectangular space-time domain $\Omega = [r_0, R] \times [t_0, T]$, where $r_0 > 0$ is sufficiently small. It is assumed further that u is sufficiently differentiable during the course of derivation. One notices that this equation resembles the SG equation in $(1 + 1)$-dimension except for only the second singular term u_r/r. Equation (5.7) is complemented by the initial conditions of the given profile and velocity

$$\begin{aligned} u(r, t_0) &= u_0(r), \\ u_t(r, t_0) &= v_0(r), \quad \text{for } r_0 \leq r \leq R. \end{aligned} \tag{5.8}$$

For problem with compact domain $\Omega_R = [r_0, R]$ the conditions at both boundaries should be supplemented. Since we are interested in the ring wave of kink-type with the conservative energy over the whole infinite domain, it is reasonable to pose the conditions of no flux at two boundaries

$$u_r(r_0, t) = u_r(R, t) = 0 \quad \text{for } t_0 \leq t \leq T.$$

Rectangular mesh and discretized equation It is the first step to discretize the domain Ω into a rectangular mesh $M = \{r_k, t_n\}_{k,n}$ of points with coordinates

$$(r_k, t_n) = (r_0 + k\Delta r, t_0 + n\Delta t), \quad k = 0, 1, \ldots, N + 1, \quad n = 0, 1, \ldots, M + 1.$$

In this mesh there are $N + 2$ nodes and consequently $N + 1$ steps along the r-axis.

As standardized, the partial differential equation in $(1 + 1)$-dimension is handled in two main stages. That is, the problem is discretized in space and then in time or vice versa. Now, we approximate the spatial partial derivatives with the central-difference formula

$$\begin{aligned} \frac{\partial u}{\partial r}(r_k, t_n) &= \frac{u(r_{k+1}, t_n) - u(r_{k-1}, t_n)}{2\Delta r} + O(\Delta r^2), \\ \frac{\partial^2 u}{\partial r^2}(r_k, t_n) &= \frac{u(r_{k+1}, t_n) - 2u(r_k, t_n) + u(r_{k-1}, t_n)}{\Delta r^2} + O(\Delta r^2). \end{aligned}$$

Evaluating Eq. (5.7) at node (r_k, t_n), substituting the above formulas into the corresponding terms, we obtain

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}t^2}(t_n) - \frac{u_{k+1}^n - u_{k-1}^n}{2r_k\Delta r} - \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta r^2} + \sin u_k^n = 0, \tag{5.9}$$

where $u_k^n = u(r_k, t_n)$ and $u_k(t) = u(r_k, t)$ is considered as the function of time variable at this stage. Obviously, we need to replace the derivative $\mathrm{d}^2/\mathrm{d}t^2$ with an approximant. Let us denote $\mathbf{u} = \mathbf{u}(t)$ a vector-valued function of time variable and expand $\mathbf{u}(t + \Delta t)$ in Taylor series about the time instant t

$$\mathbf{u}(t + \Delta t) = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \frac{\mathrm{d}^m}{\mathrm{d}t^m}\mathbf{u}(t) = \exp(\Delta t\,\mathbf{D})\mathbf{u}(t),$$

Method	(μ,ν)	a_1	b_1	c_1	d_1
I	$(1,1)$	$-1/2$	0	$1/2$	0
II	$(0,2)$	0	0	1	$1/2$

Table 5.1: The coefficients associated with the Padé approximation up to the first order.

where D is the diagonal differential operator defined formally by $\mathrm{D} = \mathrm{diag}(\mathrm{d}/\mathrm{d}t)$. Similarly, the backward expansion is given by

$$\mathbf{u}(t-\Delta t) = \exp(-\Delta t\,\mathrm{D})\mathbf{u}(t).$$

Combining these two expressions, we obtain the following three-time level recurrence relation

$$\mathbf{u}(t+\Delta t) + \mathbf{u}(t-\Delta t) = \left[\exp(\Delta t\,\mathrm{D}) + \exp(-\Delta t\,\mathrm{D})\right]\mathbf{u}(t). \tag{5.10}$$

To obtain a finite expansion of Eq. (5.10), we must replace the matrix-valued exponential term with its appropriate truncation. Following [98], we use the Padé approximation of the form

$$\exp(\Delta t\,\mathrm{D}) \approx (\mathbf{I}_{N+2} + a_1\Delta t\,\mathrm{D} + b_1\Delta t^2\mathrm{D}^2)^{-1}(\mathbf{I}_{N+2} + c_1\Delta t\,\mathrm{D} + d_1\Delta t^2\mathrm{D}^2), \tag{5.11}$$

where $\mathbf{I}_{N+2}$ denotes the identity matrix, a_1, b_1, c_1 and d_1 are the parameters determined specifically according to the type of approximation. In Table 5.1 we list two such ways. It must be highlighted that we shall exclusively use the common word 'method' with the roman number to point out these two ways of Padé approximation and call them method I and method II. Assigning the coefficients given in first row of this table to Eq. (5.11), it is ready to write the expressions $\exp(\pm\Delta t\,\mathrm{D})$ in the explicit form

$$\exp(+\Delta t\,\mathrm{D}) \approx \left(\mathbf{I}_{N+2} - \frac{1}{2}\Delta t\,\mathrm{D}\right)^{-1}\left(\mathbf{I}_{N+2} + \frac{1}{2}\Delta t\,\mathrm{D}\right),$$

$$\exp(-\Delta t\,\mathrm{D}) \approx \left(\mathbf{I}_{N+2} + \frac{1}{2}\Delta t\,\mathrm{D}\right)^{-1}\left(\mathbf{I}_{N+2} - \frac{1}{2}\Delta t\,\mathrm{D}\right).$$

Plugging these two approximants into Eq. (5.10), multiplying both sides by

$$(\mathbf{I}_{N+2} - \frac{1}{2}\Delta t\,\mathrm{D})(\mathbf{I}_{N+2} + \frac{1}{2}\Delta t\,\mathrm{D}),$$

and simplifying it formally, we have just replaced the full form of Eq. (5.10) with its approximation

$$(\mathbf{I}_{N+2} - \frac{1}{4}\Delta t^2\mathrm{D}^2)[\mathbf{u}(t-\Delta t) + \mathbf{u}(t+\Delta t)] = 2(\mathbf{I}_{N+2} + \frac{1}{4}\Delta t^2\mathrm{D}^2)\mathbf{u}(t). \tag{5.12}$$

Repeating these actions in an analogous manner by using the second method, namely (0,2)-Padé approximation, we may write another truncated form of Eq. (5.10) as follows

$$\mathbf{u}(t+\Delta t) + \mathbf{u}(t-\Delta t) = (2\mathbf{I}_{N+2} + \Delta t^2\,\mathrm{D}^2)\mathbf{u}(t). \tag{5.13}$$

Note that this equation can be actually obtained by expanding the left-hand side in the regular Taylor series about the instant t since the second Padé approximation is nothing other than the Taylor polynomial up to the second order.

As the last ingredient for the next step, let us rewrite Eq. (5.9) in the operator form

$$\mathrm{D}^2\mathbf{u}(t) = (\mathbf{A}+\mathbf{B})\mathbf{u}(t) - \mathbf{G}(\mathbf{u}(t)), \tag{5.14}$$

where

$$\mathbf{A} = \frac{1}{2\Delta r}\begin{bmatrix} a_{11} & a_{12} & \dots & & & \\ -\dfrac{1}{r_1} & 0 & \dfrac{1}{r_1} & \dots & & \\ & & & \ddots & & \\ & & & \dots & -\dfrac{1}{r_N} & 0 & \dfrac{1}{r_N} \\ & & & & \dots & a_{N+1,N} & a_{N+1,N+1} \end{bmatrix},$$

$$\mathbf{B} = \frac{1}{\Delta r^2}\begin{bmatrix} b_{11} & b_{12} & \dots & & & \\ 1 & -2 & 1 & \dots & & \\ & & & \ddots & & \\ & & & \dots & 1 & -2 & 1 \\ & & & & \dots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix},$$

in which the unknown coefficients a_{ij}, b_{ij} in the first and last rows should be determined with reference to the boundary conditions, and

$$\mathbf{G}(\mathbf{u}(t)) = \begin{bmatrix}\sin u_0(t) & \dots & \sin u_{N+1}(t)\end{bmatrix}^T.$$

Predictor-corrector scheme It is rather "clever" that we shall not attack the problem directly on the differential equation (5.14) by replacing the derivative with its approximant; our approach here differs more or less from the conventional one. Instead, Eq. (5.14) is supposed to be a valid identity and Eq. (5.12) will be solved in replacement. In the following we shall implement this idea through two steps.

Plugging the "identity" (5.14) into Eq. (5.12), it is transformed to

$$\begin{aligned} &\mathbf{u}(t-\Delta t) - \frac{1}{4}\Delta t^2\left[(\mathbf{A}+\mathbf{B})\mathbf{u}(t-\Delta t) - \mathbf{G}(\mathbf{u}(t-\Delta t)\right] \\ &+ \mathbf{u}(t+\Delta t) - \frac{1}{4}\Delta t^2\left[(\mathbf{A}+\mathbf{B})\mathbf{u}(t+\Delta t) - \mathbf{G}(\mathbf{u}(t+\Delta t)\right] \\ &= 2\mathbf{u}(t) + \frac{1}{2}\Delta t^2\left[(\mathbf{A}+\mathbf{B})\mathbf{u}(t) - \mathbf{G}(\mathbf{u}(t))\right]. \end{aligned}$$

Keeping only the solution vector $\mathbf{u}(t+\Delta t)$ on the left-hand side and bringing all the remaining terms to the other side, we obtain

$$\begin{aligned} \mathbf{u}(t+\Delta t) = \frac{1}{4}\Delta t^2\big\{&(\mathbf{A}+\mathbf{B})\left[\mathbf{u}(t+\Delta t) + 2\mathbf{u}(t) + \mathbf{u}(t-\Delta t)\right] \\ &- \left[\mathbf{G}(\mathbf{u}(t+\Delta t)) + 2\mathbf{G}(\mathbf{u}(\mathbf{t})) + \mathbf{G}(\mathbf{u}(t-\Delta t))\right]\big\} + 2\mathbf{u}(t) - \mathbf{u}(t-\Delta t). \end{aligned} \tag{5.15}$$

This equation is nonlinear in terms of $\mathbf{u}(t+\Delta t)$ and takes the form

$$\mathbf{u}(t+\Delta t) = \mathbf{F}\left[\mathbf{u}(t+\Delta), \mathbf{u}(t), \mathbf{u}(t-\Delta t)\right],$$

whose solution requires an iterative scheme like Newton-Raphson method (see, for example, [99]). Since it is not always simple to deduce the Jacobian matrix of $\mathbf{F}$ with respect to $\mathbf{u}(t+\Delta t)$, and furthermore, is numerically expensive to compute such matrix, we want to avoid it if any possible. That can be done by introducing a predictor and then a corrector which are explained in the following.

Predictor We propose the presumably predicted values $\mathbf{u}(t+\Delta t)$ by using the explicit formula (5.13) with the vector $\mathbf{v}(t+\Delta t)$ being in its position, that is

$$\mathbf{v}(t+\Delta t)+\mathbf{u}(t-\Delta t)=2\mathbf{u}(t)+\Delta t^2\mathrm{D}^2\mathbf{u}(t).$$

With the aid of identity (5.14) this equation can be rewritten as

$$\mathbf{v}(t+\Delta t)=\Delta t^2\left[(\mathbf{A}+\mathbf{B})\mathbf{u}(t)-\mathbf{G}(\mathbf{u}(t))\right]+2\mathbf{u}(t)-\mathbf{u}(t-\Delta t). \tag{5.16}$$

The left-hand side is a predictor vector for $\mathbf{u}(t+\Delta t)$ on the right-hand side of Eq. (5.15). This will be made clear in next paragraph.

Corrector We propose eventually the following solution formula which is usually known as corrector

$$\begin{aligned}\mathbf{u}(t+\Delta t)=\frac{1}{4}\Delta t^2\big\{&(\mathbf{A}+\mathbf{B})\left[\mathbf{v}(t+\Delta t)+2\mathbf{u}(t)+\mathbf{u}(t-\Delta t)\right]\\ &-\left[\mathbf{G}(\mathbf{v}(t+\Delta t))+2\mathbf{G}(\mathbf{u}(t))+\mathbf{G}(\mathbf{u}(t-\Delta t))\right]\big\}+2\mathbf{u}(t)-\mathbf{u}(t-\Delta t).\end{aligned} \tag{5.17}$$

The system (5.16)–(5.17) gives us an alternative tool to the problem of solving Eq. (5.15) for the solution vector at the next time step.

Some notes on implementation

Vector form of the solution formula Notwithstanding simple structure of the final solution formula, the computation cost for actual implementation can be still dramatically reduced if one pays attention on the tri-diagonal structure of the two entering matrices. Apparently, it is unnecessary to carry out the full matrix multiplication but only the scalar multiplication. In order to exploit the special structure of the matrices $\mathbf{A}$ and $\mathbf{B}$, we define the four new vectors as follows

$$\begin{aligned}&\frac{1}{2\Delta r}\mathbf{P}(t)=\mathbf{A}\mathbf{u}(t),\quad \frac{1}{\Delta r^2}\mathbf{Q}(t)=\mathbf{B}\mathbf{u}(t),\\ &\frac{1}{2\Delta r}\mathbf{Y}(t)=\mathbf{A}\mathbf{v}(t),\quad \frac{1}{\Delta r^2}\mathbf{Z}(t)=\mathbf{B}\mathbf{v}(t),\end{aligned}$$

whose components are explicitly calculated according to

$$\begin{aligned}p_k^n&=\frac{u_{k+1}^n-u_{k-1}^n}{r_k},\qquad q_k^n=u_{k+1}^n-2u_k^n+u_{k-1}^n,\\ y_k^{n+1}&=\frac{v_{k+1}^{n+1}-v_{k+1}^{n+1}}{r_k},\quad z_k^{n+1}=v_{k+1}^{n+1}-2v_k^{n+1}+v_{k-1}^{n+1}.\end{aligned} \tag{5.18}$$

Using these definitions, we can rewrite the system (5.16)–(5.17) as

$$\begin{aligned}\mathbf{v}(t+\Delta t)&=4\left[\alpha\,\mathbf{P}(t)+\beta\,\mathbf{Q}(t)\right]-\Delta t^2\mathbf{G}(\mathbf{u}(t))+2\mathbf{u}(t)-\mathbf{u}(t-\Delta t),\\ \mathbf{u}(t+\Delta t)&=\alpha\left[\mathbf{Y}(t+\Delta t)+2\mathbf{P}(t)+\mathbf{P}(t-\Delta t)\right]+\beta\left[\mathbf{Z}(t+\Delta t)+2\mathbf{Q}(t)+\mathbf{Q}(t-\Delta t)\right]\\ &\quad-\frac{\Delta t^2}{4}\left[\mathbf{G}(\mathbf{u}(t+\Delta t))+2\mathbf{G}(\mathbf{u}(t))+\mathbf{G}(\mathbf{u}(t-\Delta t))\right]+2\mathbf{u}(t)-\mathbf{u}(t-\Delta t),\end{aligned}$$

where the coefficients α and β are given by

$$\alpha = \frac{\Delta t^2}{8\Delta r}, \quad \beta = \frac{\Delta t^2}{4\Delta r^2}.$$

For convenience of implementation we write down here the index form of this system

$$\begin{aligned} v_k^{n+1} &= 4(\alpha\, p_k^n + \beta\, q_k^n) - \Delta t^2 \sin u_k^n + 2u_k^n - u_k^{n-1}, \\ u_k^{n+1} &= \alpha(y_k^{n+1} + 2p_k^n + p_k^{n-1}) + \beta(z_k^{n+1} + 2q_k^n + q_k^{n+1}) \\ &\quad - \Delta t^2/4 \times (\sin u_k^{n+1} + 2\sin u_k^n + \sin u_k^{n-1}) + 2u_k^n - u_k^{n-1}. \end{aligned}$$

Treatment of initial conditions In order to compute the solution vector at the second step, we use the initial velocity given by Eq. $(5.8)_2$ and descretize it using the central-difference formula as follows

$$\frac{1}{2\Delta t}\left[\mathbf{u}(t_0 + \Delta t) + \mathbf{u}(t_0 - \Delta t)\right] = \mathbf{v}_0,$$

where $\mathbf{v}_0$ is obtained by evaluating the initial velocity at each node, that is

$$\mathbf{v}_0 = \begin{bmatrix} v_0(r_0) & \dots & v_0(r_{N+1}) \end{bmatrix}^T.$$

Let us reuse the central-difference formula for the time derivative in Eq. (5.14) and evaluate the whole equation at the initial time $t = t_0$ to obtain

$$\frac{1}{\Delta t^2}\left[\mathbf{u}(t_0 + \Delta t) + \mathbf{u}(t_0 - \Delta t) + 2\mathbf{u}(t_0)\right] = (\mathbf{A} + \mathbf{B})\mathbf{u}(t_0) - \mathbf{G}(\mathbf{u}(t_0)).$$

These two equations constitute a system for determining the second-time solution vector

$$\mathbf{u}(t_0 + \Delta t) = \mathbf{u}(t_0) + \Delta t\, \mathbf{v}_0 + 2\left[\alpha\, \mathbf{P}(t_0) + \beta\, \mathbf{Q}(t_0)\right] - \frac{\Delta t^2}{2}\mathbf{G}(\mathbf{u}(t_0)).$$

Stability analysis

We shall investigate the stability of the proposed scheme by examining the predictor and the corrector as two detached objects. To be precise, the predictor $\mathbf{v}$ does not enter the analysis so that we can write the index form of two equations (5.15) and (5.16) as follows

$$\begin{aligned} u_k^{n+1} - 4\alpha\frac{u_{k+1}^n - u_{k-1}^n}{r_k} - 4\beta(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + \Delta t^2 \sin u_k^n - 2u_k^n + u_k^{n-1} &= 0, \\ u_k^{n+1} - \alpha(p_k^{n+1} + 2p_k^n + p_k^{n-1}) - \beta(q_k^{n+1} + 2q_k^n + q_k^{n-1}) & \\ + \frac{\Delta t^2}{4}(\sin u_k^{n+1} + 2\sin u_k^n + \sin u_k^{n-1}) - 2u_k^n + u_k^{n-1} &= 0. \end{aligned} \tag{5.19}$$

Let us study first the predictor formula. Following the Neumann's stability criterion, we introduce the error due to the arithmetical round-off

$$W_k^n = u_k^n - \tilde{u}_k^n = \exp(n\gamma\Delta t)\exp(\mathrm{i}k\lambda\Delta r), \quad \mathrm{i}^2 = -1,$$

where $\tilde{u}_k^n$ is the numerical solution obtained in finite precision arithmetic and λ, γ are the wave number and the frequency of wave under consideration, respectively. Then the following identities can be derived at hand

$$W_{k+1}^{n+\mu} - W_{k-1}^{n+\mu} = 2\mathrm{i}\sin(\lambda\Delta r)\xi^\mu W_k^n,$$
$$W_k^{n+\mu} - 2W_k^{n+\mu} + W_k^{n+\mu} = -4\sin^2\left(\frac{\lambda\Delta r}{2}\right)\xi^\mu W_k^n,$$

where $\xi = \exp(\gamma\Delta t)$ and μ takes the integer values. Substituting $\tilde{u}_k^n$ into Eq. $(5.19)_1$ in replacement of u_k^n, then subtracting the obtained equation from it and using the above identities, we obtain

$$W_k^n\left[\xi - \mathrm{i}\frac{8\alpha}{r_k}\sin(\lambda\Delta r) + 16\beta\sin^2\left(\frac{\lambda\Delta r}{2}\right) + \Delta t^2 S_k^n - 2 + \xi^{-1}\right] = 0,$$

where

$$S_k^n = \sum_{j=1}^{\infty}\frac{1}{(2j+1)!}\left[(u_k^n)^{2j} + (u_k^n)^{2j-1}\tilde{u}_k^n + \cdots + (\tilde{u}_k^n)^{2j}\right].$$

If we assume further that the round-off values are approximate to the numerical ones under an appropriate stability condition and within an allowable error, this series is simplified to

$$S_k^n \approx \sum_{j=1}^{\infty}\frac{(u_k^n)^{2j}}{(2j)!} = \cos u_k^n.$$

By multiplying the last equation by ξ/W_k^n, it is recognized as a quadratic equation

$$\xi^2 - 2\chi\xi + 1 = 0, \quad \chi = 1 - \delta_1 + \mathrm{i}\delta_2,$$

where the newly introduced parameters are given by

$$\delta_1 = 8\beta\sin^2\left(\frac{\lambda\Delta r}{2}\right) + \frac{1}{2}\Delta t^2 S_k^n, \quad \delta_2 = \frac{4\alpha}{r_k}\sin(\lambda\Delta r).$$

The roots read

$$\xi_{1,2} = \chi \pm \sqrt{\chi^2 - 1}.$$

Let us write the modulus of the roots as follows

$$|\xi_{1,2}|^2 = (1-\delta_1)^2 + \delta_2^2 + r \pm 2\sqrt{r}[(1-\delta_1)\cos(\varphi/2) + \delta_2\sin(\varphi/2)], \tag{5.20}$$

where r and φ are the modulus and argument of the complex number $\Delta = \chi^2 - 1$, respectively, and given by

$$r = \sqrt{(\delta_1^2 + \delta_2^2)[(\delta_1 - 2)^2 + \delta_2^2]}, \quad \cos\varphi = \frac{\delta_1^2 - 2\delta_1 - \delta_2^2}{r}, \quad \sin\varphi = \frac{2\delta_2(1-\delta_1)}{r}.$$

In order that the Neumann's criterion $|\xi_{1,2}| \le 1$ is satisfied, we require a stricter inequality

$$(1-\delta_1)^2 + \delta_2^2 + r + 2\sqrt{r}\sqrt{(1-\delta_1)^2 + \delta_2^2} \le 1.$$

In many following case studies we are going to study the propagation of waves at large coordinates and always able to choose the step size so that the parameter δ_2 given by

$$\delta_2 = \frac{\Delta t^2}{4\Delta r}\frac{\sin(\lambda\Delta r)}{r_k} \approx 0.$$

is negligibly small. In such circumstances we may approximate the modulus in Eq. (5.20) by

$$|\xi_{1,2}|^2 \approx |\eta_{1,2}|^2 = (1-\delta_1)^2 + |\delta_1(\delta_1-2)| \pm 2\sqrt{(|\delta_1-2)\delta_1|}(1-\delta_1)\cos(\varphi/2),$$

which is reduced to

$$|\eta_{1,2}|^2 = \begin{cases} 1, & 0 \le \delta_1 \le 2, \\ \left[1-\delta_1 \pm \sqrt{(\delta_1-2)\delta_1}\right]^2, & \text{otherwise.} \end{cases}$$

Thus, for small value $\delta_2 \ll 1$ the numerical scheme is stable if

$$0 \le 8\beta\sin^2\left(\frac{\lambda\Delta r}{2}\right) + \frac{1}{2}\Delta t^2 S_n^k \le 2.$$

In order for these inequalities to be fulfilled, we may require the stricter conditions

$$2\frac{\Delta t^2}{\Delta r^2}\sin^2\left(\frac{\lambda\Delta r}{2}\right) - \frac{1}{2}\Delta t^2 \ge 0, \quad 2\frac{\Delta t^2}{\Delta r^2} + \frac{1}{2}\Delta t^2 \le 2,$$

in which we have used the assumption $S_k^n \approx \cos u_k^n$. Thus, the step sizes in r- and t-directions must be chosen so that

$$\frac{1}{4\sin^2(\lambda\Delta r/2)} \le \frac{1}{\Delta r^2} \le \frac{1}{\Delta t^2} - \frac{1}{4}.$$

Using the analogous argument and the definition (5.18), Eq. $(5.19)_2$ leads to the following quadratic equation

$$(1+\chi)\xi^2 - 2(1-\chi)\xi + 1 + \chi = 0, \quad \chi = 4\beta\sin^2\left(\frac{\lambda\Delta r}{2}\right) + \frac{\Delta t^2}{4}\cos u_k^n - \mathrm{i}\frac{2\alpha}{r_k}\sin(\lambda\Delta r),$$

where the approximation $S_n^k \approx \cos u_n^k$ has been used. Its roots read

$$\xi_{1,2} = \frac{1-\chi \pm \sqrt{-4\chi}}{1+\chi}.$$

By denoting

$$\chi = \delta_1 - \mathrm{i}\delta_2, \quad \delta_1 = 4\beta\sin^2\left(\frac{\lambda\Delta r}{2}\right) + \frac{\Delta t^2}{4}\cos u_k^n, \quad \delta_2 = \frac{2\alpha}{r_k}\sin(\lambda\Delta r),$$

the stability condition can be written as

$$\frac{1+\delta_1(\delta_1-2) + \delta_2^2 + 4\sqrt{\delta_1^2+\delta_2^2} \pm 4(\delta_1^2+\delta_2^2)^{1/4}[(1-\delta_1)\cos\varphi + \delta_2\sin\varphi]}{(1+\delta_1)^2+\delta_2^2} \le 1,$$

where $\varphi = \mathrm{Arg}(-\chi)/2$. We introduce the new variable ψ according to

$$\cos\psi = \frac{1-\delta_1}{\sqrt{(1-\delta_1)^2+\delta_2^2}}, \quad \sin\psi = \frac{\delta_2}{\sqrt{(1-\delta_1)^2+\delta_2^2}},$$

and reduce this inequality further to

$$\sqrt{\delta_1^2+\delta_2^2} \pm (\delta_1^2+\delta_2^2)^{1/4}\sqrt{(1-\delta_1)^2+\delta_2^2}\,\cos(\psi-\varphi) \le \delta_1.$$

Once again, if we take into account the assumption of the negligibly small parameter $\delta_2 \approx 0$, this condition holds "nearly" true. Indeed, a substitution $\delta_2 = 0$ into this condition makes the equality hold true and so does the inequality.

Error analysis

As the global error in the standard finite difference method can be measured by the summation of all the local errors at each step, it suffices to study the local error of both proposed methods. Using the remainder theorem, we are ready to provide the principle parts of the local truncation errors of method I

$$R_1(r,t) = -\frac{\Delta t^2}{6}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u}{\partial t^2} + \frac{1}{4}\Delta r^2\left(\frac{1}{2}\frac{\partial^4 u}{\partial r^4} + \frac{1}{r}\frac{\partial^3 u}{\partial r^3}\right)\right]$$
$$-\frac{\Delta r^2}{6}\left(\frac{1}{2}\frac{\partial^4 u}{\partial r^4} + \frac{1}{r}\frac{\partial^3 u}{\partial r^3}\right) + O(\Delta t^4 + \Delta r^4),$$

and of method II

$$R_2(r,t) = \frac{1}{6}\left[\frac{\Delta t^2}{2}\frac{\partial^4 u}{\partial t^4} - \Delta r^2\left(\frac{1}{2}\frac{\partial^4 u}{\partial r^4} + \frac{1}{r}\frac{\partial^3 u}{\partial r^3}\right)\right] + O(\Delta t^4 + \Delta r^4).$$

Numerical results

To justify the proposed numerical scheme, we illustrate in this section the numerical simulations in accompany with the analytical properties of the solutions.

Conservation of energy It can be justified that the energy of the wave of kink-type is conserved. Multiplying Eq. (4.1) by u_t and using the chain rule of differentiation, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}[u_t^2 + \nabla u \cdot \nabla u + 2(1-\cos u)] - \nabla \cdot (u_t \nabla u) = 0.$$

If the wave under consideration satisfies the boundary conditions

$$u(\mathbf{x} \to \pm\infty, t) = \text{const}, \quad \nabla u(\mathbf{x} \to \pm\infty, t) = 0,$$

in which the notation $\mathbf{x} \to \infty$ means either of the coordinates x, y, z tend to infinity, then integration of this equation over the domain $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{R}^3$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}[u_t^2 + \nabla u \cdot \nabla u + 2(1-\cos u)]\mathrm{d}\mathbf{x} = 0.$$

Note that the last term

$$\int_\Omega \nabla \cdot (u_t \nabla u)\mathrm{d}\mathbf{x} = \int_{\partial\Omega} u_t(\nabla u \cdot \mathbf{n})\mathrm{d}S$$

vanishes in accordance with the divergence theorem and the above boundary conditions. The energy of the rotationally (spherically) symmetric solution $u(r,t)$ is realized in the form

$$E_{2\mathrm{D}}(t) = \int_0^{2\pi}\int_0^\infty \frac{1}{2}\left[u_t^2 + u_r^2 + \frac{1}{r^2}u_\varphi^2 + 2(1-\cos u)\right] r\,\mathrm{d}r\,\mathrm{d}\varphi$$
$$= \pi\int_0^\infty \left[u_t^2 + u_r^2 + 2(1-\cos u)\right] r\mathrm{d}r,$$
$$E_{3\mathrm{D}}(t) = \int_0^{2\pi}\int_0^\pi\int_0^\infty \frac{1}{2}\left[u_t^2 + u_r^2 + \frac{1}{r^2}u_\theta^2 + \frac{1}{r^2\sin^2\theta}u_\phi^2 + 2(1-\cos u)\right] r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= 2\pi\int_0^\infty \left[u_t^2 + u_r^2 + 2(1-\cos u)\right] r^2\mathrm{d}r,$$

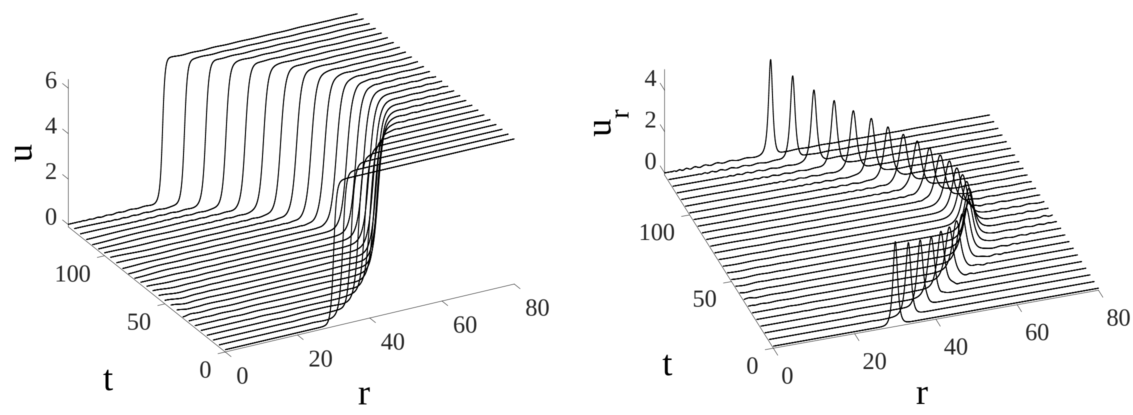
in which we have taken into account that u does not depend on the angular coordinates. Dropping out the constant factors, we shall show in the numerical simulations the conservation of reduced energies computed by the integration over a bounded region, namely

$$E^{(0)}_{\text{2D}}(t) = \int_{r_0}^{R} \left[u_t^2 + u_r^2 + 2(1 - \cos u)\right] r \; \mathrm{d}r,$$
$$E^{(0)}_{\text{3D}}(t) = \int_{r_0}^{R} \left[u_t^2 + u_r^2 + 2(1 - \cos u)\right] r^2 \mathrm{d}r.$$

We want to focus ourselves on the solitary wave whose shape resembles that governed by the 1D sine-Gordon equation, that is

$$u(r,t) = 4 \arctan \Big[\exp \Big(\frac{r - c_0 t - r_0}{\sqrt{1 - c_0^2}} \Big) \Big],$$

where c_0 characterizes the initial velocity of the kink, and r_0 the initial expansion of radius. Note that from now on we abuse the denotation r_0 for this arbitrary constant; it was used as the first node in the r-coordinate. In the first case study we solve the rotationally symmetric sine-Gordon equation using the values $c_0 = \sqrt{3}/2$, $r_0 = 30$. For the 3D case we use the same structure of initial condition but with $c_0 = 9/10$ and $r_0 = 50$. In Fig. 5.1 the numerical solution and its derivative are plotted at different time instants. In Fig. 5.2 and Fig. 5.3 the slopes of solution u_r and the corresponding energies are shown side by side. The small changes in the energy diagrams show that our proposed scheme preserves the energy of waves. The energy of the 3D ring-wave drops fast after the time instant $t_* \approx 100$ in the relative sense will be discussed later.



(a) Solution $u(r,t)$ (b) Derivative $u_r(r,t)$

Figure 5.1: Solitary wave of kink-type propagates in time.

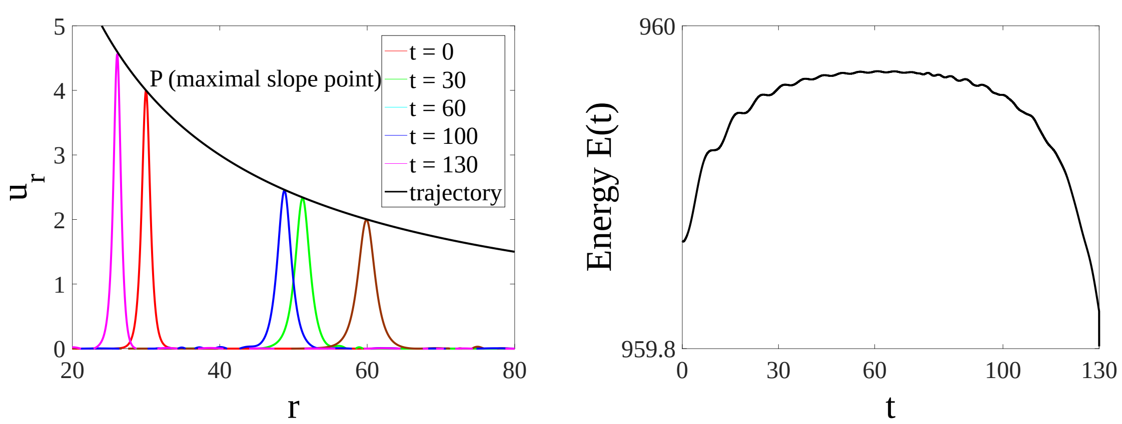
5.2.1 Trajectory of the maximal slope point

We introduce the notion of maximal slope point (see figures 5.2a and 5.3a) by

$$\mathrm{P} = (\gamma(t), u_r(\gamma(t))),$$

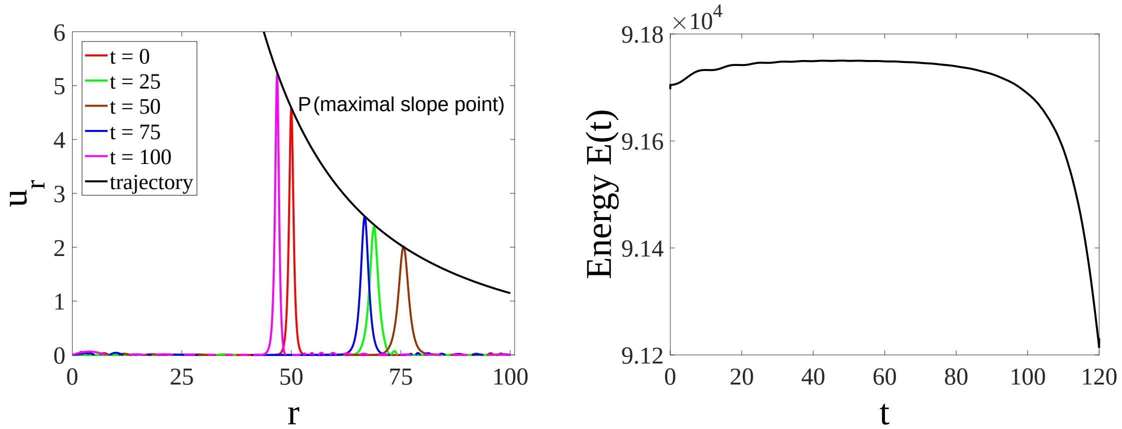
in which $\gamma(t)$ is the coordinate where the derivative u_r achieves its maximum, that is

$$u_r(\gamma(t), t) = \max_r u_r(r,t).$$



(a) Slope of the kink-type solution. (b) Energy tracking.

Figure 5.2: Ring-wave of the two-dimensional sine-Gordon equation.



(a) Slope of the kink-type solution. (b) Energy tracking.

Figure 5.3: Ring-wave of the three-dimensional sine-Gordon equation.

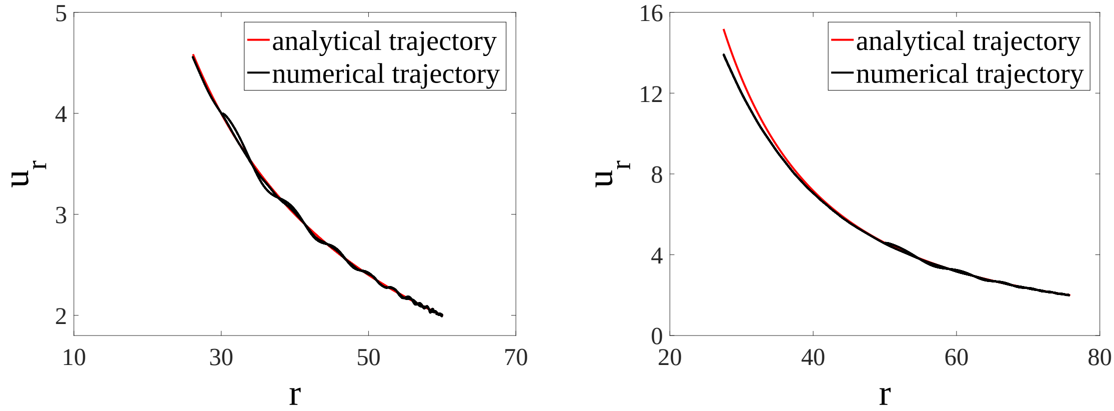
Through various numerical simulations, we observe that each maximal slope point travels on a fixed trajectory that can be described according to the rule

$$f(r) \propto C/r^{n_d-1},$$

where C is a constant depending on the initial expansion of radius r_0 and initial velocity c_0. At the initial time the maximal slope point is situated at the coordinates $(r_0, 2/\sqrt{1-c_0^2})$ and lies on the graph of $f = f(r)$ during its propagation. Thus, we can establish the equation for determining this constant and then express f in the explicit form as follows

$$f_{2D}(r) = \frac{r_0}{r}\frac{2}{\sqrt{1-c_0^2}}, \quad f_{3D}(r) = \left(\frac{r_0}{r}\right)^2 \frac{2}{\sqrt{1-c_0^2}}.$$

Such graphs are plotted in Fig. 5.2a and Fig. 5.3a. To be more convincing, the analytical and numerical trajectories of the maximal slope points in both 2D and 3D cases are illustrated once again in the common coordinate system in Fig. 5.4, which shows excellent agreement. The drop in energy of the 3D ring-wave is reflected by the decrease in height of the maximal slope point as compared to the analytical expected height. We expect that this kind of drop is due to the oscillated disturbance around two tails and the numerical integration using trapezoid rule loses some correctness.



(a) 2D case. (b) 3D case.

Figure 5.4: The maximal slope points "nearly" travel on fixed trajectories depending on the initial kink velocity and initial expansion of radius.

5.2.2 Returning effect and computation of returning times

The kink-type solution exhibits the returning effect with the returning time t_r depending also on r_0 and c_0. Using the fore-mentioned property, we are ready to derive the "pseudo-analytical" expression for t_r, which completely coincides with the counterpart provided in [100]. By observing the shape of several numerical solutions, it is reasonable to assume the solution in the form

$$u(r,t) = 4\arctan\left[\exp\left(\frac{r-\gamma(t)}{\sqrt{1-\gamma'(t)^2}}\right)\right],$$

according to which the slope is easily computed as follows

$$u_r(r,t) = \frac{2}{\sqrt{1-\gamma'(t)^2}}\operatorname{sech}\left[\frac{r-\gamma(t)}{\sqrt{1-\gamma'(t)^2}}\right].$$

Since the maximal slope point with the coordinates

$$P = \left(\gamma(t), \frac{2}{\sqrt{1-\gamma'(t)^2}}\right)$$

runs on the graph of $f = f(r)$, we arrive at the equation

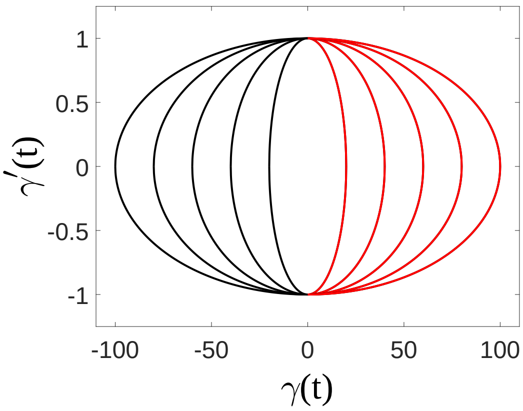
$$\frac{2}{\sqrt{1-\gamma'(t)^2}} = \left[\frac{r_0}{\gamma(t)}\right]^{n_d-1}\frac{2}{\sqrt{1-c_0^2}}.$$

This equation must be complemented by the initial conditions

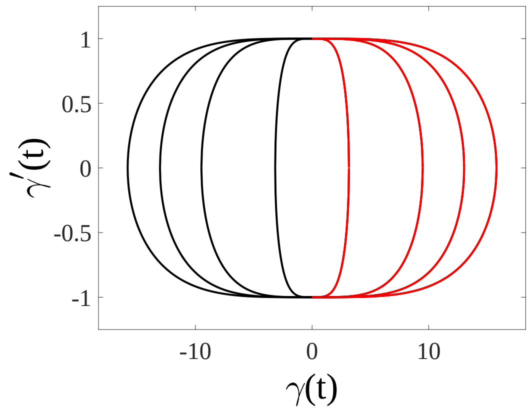
$$\gamma(0) = r_0, \quad \gamma'(0) = c_0. \tag{5.21}$$

The second condition is posed for the purpose of reminding the initial velocity but it does not play a role in the formulation of the first-order differential equation. This condition is a direct consequence of equation the above equation and the first initial condition indeed. We transform this equation to a more familiar form

$$\gamma(t)^{2n_d-2} + R^2\gamma'(t)^2 = R^2, \quad R = \frac{r_0^{n_d-1}}{\sqrt{1-c_0^2}}. \tag{5.22}$$



(a) 2D case: $\gamma(t)^2 + R^2\gamma'(t)^2 = R^2$.



(b) 3D case: $\gamma(t)^4 + R^2\gamma'(t)^2 = R^2$.

Figure 5.5: Phase portrait corresponding to the first-order ordinary differential equation for $\gamma(t)$.

It is clearly seen from the phase portrait of this equation in Fig. 5.5 that the periodic solution $\gamma(t)$ exists. We shall restrict our attention on the red curves in the half plane $\gamma > 0$ of the coordinate system.

For the rotationally symmetric sine-Gordon equation with $n_d = 2$, we have

$$\gamma(t)^2 + R^2\gamma'(t)^2 = R^2, \quad R = \frac{r_0}{\sqrt{1-c_0^2}}, \tag{5.23}$$

which is nothing else but the familiar equation of an ellipse. Thus, the harmonic solution exists and is given by

$$\gamma(t) = r_0\left[\cos\left(\frac{t}{R}\right) + \frac{c_0}{\sqrt{1-c_0^2}}\sin\left(\frac{t}{R}\right)\right].$$

The returning time is determined by the condition of vanishing velocity

$$\gamma'(t_r) = \frac{r_0}{R}\left[\frac{c_0}{\sqrt{1-c_0^2}}\cos\left(\frac{t_r}{R}\right) - \sin\left(\frac{t_r}{R}\right)\right] = 0,$$

which can be resolved for

$$t_r = \frac{r_0}{\sqrt{1-c_0^2}}\arctan\frac{c_0}{\sqrt{1-c_0^2}}.$$

Thus, the wave packet starts to return about the coordinate γ_r given by

$$\gamma_r = \gamma(t_r) = \frac{r_0}{\sqrt{1-c_0^2}}. \tag{5.24}$$

Now we consider the 3D case and equation (5.22) with $n_d = 3$ becomes

$$\gamma(t)^4 + R^2\gamma'(t)^2 = R^2, \quad R = \frac{r_0^2}{\sqrt{1-c_0^2}}. \tag{5.25}$$

This equation can be fully integrated to give

$$\gamma(t) = \frac{r_0}{\sqrt[4]{1-c_0^2}}\,\mathrm{sn}\left[\frac{\sqrt[4]{1-c_0^2}}{r_0}t + \mathrm{sn}^{-1}\left(\sqrt[4]{1-c_0^2}, -1\right), -1\right],$$

where initial condition $(5.21)_1$ has been taken into account and sn is the Jacobi elliptic function. The velocity of the maximal slope point is then given by

$$\gamma'(t) = \text{cn}(\psi, -1)\text{dn}(\psi, -1), \quad \psi(t, r_0, c_0) = \frac{\sqrt[4]{1-c_0^2}}{r_0}t + \text{sn}^{-1}\left(\sqrt[4]{1-c_0^2}, -1\right).$$

Solving equation $\gamma'(t_r) = 0$ for the positive root, we obtain the returning time as follows

$$t_r = \frac{r_0}{\sqrt[4]{1-c_0^2}}\left[\mathrm{K}(-1) - \mathrm{F}\left(\arcsin\sqrt[4]{1-c_0^2}, -1\right)\right],$$

where F and K are the elliptic integral of the first kind, and the complete elliptic integral of the first kind, respectively and we have used the following identities

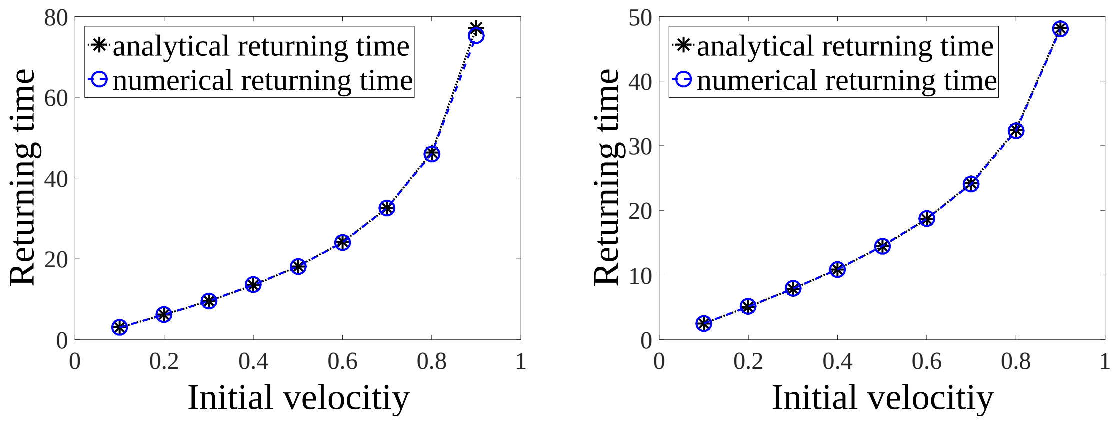
$$\mathrm{K}(-1) = (\text{cn} \times \text{dn})^{-1}(0, -1), \quad \text{sn}^{-1}(\phi, m) = \mathrm{F}(\arcsin\phi, m).$$

Consequently, we can also compute the maximum expanding radius where the kink starts to return in accordance with

$$\gamma_r = \gamma(t_r) = \frac{r_0}{\sqrt[4]{1-c_0^2}}. \tag{5.26}$$

The "returning radii" γ_r in equations (5.24) and (5.26) can be deduced directly from equations (5.23) and (5.25) taking into account the condition $\gamma'(t_r) = 0$.

In Fig. 5.6 the analytical and numerical returning times are compared for different values of initial velocities using one single initial expansion of radius. The comparison shows once again astonishingly good agreement.



(a) 2D case. (b) 3D case.

Figure 5.6: Comparison between the analytical and numerical returning times.

Bibliographical remarks

The Benjamin-Ono equation can be classified as equation of Rossby waves [101]. The modulation theory of the Benjamin-Ono equation is not presented in this report as it has been fully developed taking into account the modulations of multi-phase periodic solutions. However, we show the derivation of the Lagrangian for this equation as a link to the below references. The modulation equations were obtained by using an IST-based averaging technique proposed by Dobrokhotov and Krichever [102]. The paper had unfortunately laid quite unnoticed until a group of Whitham's followers put the theory into action by using the modulation equations to compute the evolution of three initial conditions possessing distinguishable properties [103]. Not so long later, the theory was applied to model the Morning Glory of the Gulf of Carpentaria in Australia by Porter and Smyth [104]. The numerical scheme is taken from the work in [96].

Summary

This work is concerned with the Whitham modulation theory for nonlinear dispersive waves by using the variational-asymptotic method (VAM). The averaging technique explained in this report falls into the category of multi-scale methods. The small parameter lies in the initial condition in that the variations of the wave characteristics such as wavenumber, wave frequency, mean height of the wave are on much longer length and time scales than the local oscillation of the wavetrain. In such scenarios we can effectively describe the evolution of the wavetrain by its wave characteristics by using an appropriate system of modulation equations governing these characteristics. So we recognize two scales in our problem: one corresponds to the fast oscillation of the phase variable of the field variable and the other to the slow variations of the wave characteristics. The fundamental assumption for the application of this theory is that these two scales must be well-separated.

For the mathematical realization we base our formulation on the variational-asymptotic analysis. First, a typical uniform periodic wavetrain of the governing equation is allowed to vary slowly in its wave characteristics so that the slowly varying wave packet consists of the fast variable, namely the phase, and the slow variables, namely the space-time coordinates, in its argument. Due to the assumption of well-separated scales we may decompose the domain under consideration into infinitely many strips to obtain the same optimization problem in each strip which is called strip problem. By intuitive argument we see that the uniform periodic wavetrain can be obtained in this variational problem in which we have considered the wave parameters as frozen in each strip. By doing so we have averaged out the fast oscillation of the field variable, leaving the functional defined in the strip problem dependent only on the parameters of interest. After obtaining the stationary point of the strip problem, we can compute the average Lagrangian. Then the variational-asymptotic analysis enables us to construct the average variational problem involving only these wave parameters. In the ultimate aim, we obtain the governing equations for the wave characteristics as the Euler equations for the average variational problem. In general, it is not easy to integrate the Whitham modulation equations exactly. However, for the train of solitary waves it is possible to resolve its amplitude modulation as the simple wave solution, that is a function of the similarity variable x/t. This similarity solution is not always valid, but it works in all our cases. The comparison between the asymptotic modulation solution and the exact or numerical solutions validates the theory from the practical point of view.

In addition, the dissertation presents three direct methods for the equations of nonlinear dispersive waves: the modified homogeneous balance method, the Hirota direct method and the Wronskian technique. The results not only provide a means of graphical comparison but also contribute new findings to the existing ones in nonlinear wave problems. More precisely, we revisited the well-known wave equations and extended the range of existing solutions. Last but not least, a predictor-corrector scheme is proposed to deal with a wide class of Klein-Gordon equations. The numerical scheme can be essentially applied to the wave equations involving the second-order derivatives in time in almost the same fashion.

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